QUANTUM SPEED LIMITS FOR TIME EVOLUTION OF A SYSTEM SUBSPACE*

Alexander K. Motovilov

Bogoliubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research
Dubna, Russia

LXX International conference "NUCLEUS – 2020"
September 17, 2020

---

*Based on a joint work with Sergio Albeverio.

Introduction

In Q.M., a ***quantum speed limit*** is a ***bound on the minimal time*** required for a quantum system to evolve between two (given) distinguishable states.

For shortness, we will assume that the measurement units are chosen in such a way that

$$\hbar = 1.$$

Let us consider an isolated Q.M. system with a Hamiltonian H, which is supposed to be a time-independent self-adjoint operator in the Hilbert space $\mathfrak{H}$. So that the vectors of $\mathfrak{H}$ are considered as possible (pure) states of the system. Evolution of a state vector $\psi(t) \in \mathfrak{H}$, $t \in \mathbb{R}$, is governed by the Schrödinger equation

$$\mathrm{i}\frac{d}{dt}\psi = H\psi, \tag{1}$$

$$\psi(t)\Big|_{t=t_0} = \psi_0, \tag{2}$$

where $\psi_0 \in \mathfrak{H}$ is an initial state. (Surely, it is required that $\psi(t) \in \mathrm{Dom}(H)$ for any t under consideration :-).

Let $t_0 = 0$. Then the solution to (1), (2) is given by

$$\psi(t) = U(t)\psi_0, \qquad \text{where } U(t) = \mathrm{e}^{-\mathrm{i}Ht}, \quad t \in \mathbb{R}; \tag{3}$$

the operators $U(t)$, $t \in \mathbb{R}$, form a strongly continuous unitary group.

Studies of quantum speed limits originate from the very basic question:

How fast can the isolated system with the Hamiltonian H evolve to a state orthogonal to its initial state ψ_0?

The importance of this question is obvious in many respects. Probably, the very latest motivation comes from quantum information theory and quantum computing.

Known answers to the above basic question — lower bounds for the orthogonalization time $T_\perp$:

Mandelshtam–Tamm inequality (1945)

$$T_\perp \geq \frac{\pi}{2\,\Delta E}, \tag{4}$$

Margolis–Levitin inequality (1998)

$$T_\perp \geq \frac{\pi}{2\,\delta E}, \tag{5}$$

where

$$\Delta E = \sqrt{\langle H^2\psi_0, \psi_0\rangle - \langle H\psi_0, \psi_0\rangle^2} \quad \text{and} \quad \delta E = \langle H\psi_0, \psi_0\rangle - \min\big(\mathrm{spec}(H)\big) \tag{6}$$

are the energy spread (dispersion) for the state ψ_0 and the average energy for this state measured relative to the lower bound of H.

Both inequalities recall the uncertainty relation for energy and time but are very different in the essence since these inequalities are related not to the standard deviation in the measurement of t but to the well-founded time for a given state to evolve into an orthogonal state.

Fleming bound (1973)

$$T_\theta \geq \frac{\theta}{\Delta E}, \tag{7}$$

where T_θ is the time moment at which the acute angle

$$\angle\big(\psi_0, \psi(t)\big) := \arccos|\langle\psi_0, \psi(t)\rangle|$$

between the vectors ψ_0 and $\psi(t)$ reaches the value of $\theta \in (0, \pi/2]$.

The Mandelshtam-Tamm bound is a particular case of the Fleming bound for $\theta = \frac{\pi}{2}$.

All the three bounds (4), (5), and (7) have been proven to be sharp.

Notice that the Mandelshtam-Tamm bound has been rediscovered several times by different researchers. Also, there are generalizations of this bound to the evolution of mixed states and more detail estimates for particular classes of quantum-mechanical evolutionary problems.

Our results: Bounds for the speed of the subspace evolution

We are concerned not with a single state but with a whole (possibly infinite-dimensional) subspace spanned by the system states that are subject to the Schrödinger evolution. That is, we consider a subspace $\mathfrak{P}_0 \subset \mathfrak{H}$ every vector of which is the subject to the Schrödinger evolution (1), (2), that is,

$$\mathrm{i}\frac{d}{dt}\psi = H\psi, \tag{8}$$

$$\psi(t)\Big|_{t=t_0} = \psi_0, \quad \psi_0 \in \mathfrak{P}_0. \tag{9}$$

For simplicity (and sometimes not only for simplicity), the Hamiltonian H **is** assumed to be **a bounded operator**.

Given $t \geq 0$, by $\mathfrak{P}(t)$ we will denote the subspace of $\mathfrak{H}$ spanned by the values $\psi(t)$ of the vector functions that solve (8), (9) for various $\psi_0 \in \mathfrak{P}_0$. So that we deal with a path $\mathfrak{P}(t)$, $t \geq 0$, in the set of all subspaces of the Hilbert space $\mathfrak{H}$. Or (and this is the same) with the path

$$P(t), \quad t \geq 0, \qquad \mathrm{Ran}\big(P(t)\big) = \mathfrak{P}(t), \tag{10}$$

of the orthogonal projections $P(t)$ in $\mathfrak{H}$ onto the respective subspaces $\mathfrak{P}(t)$.

It is well known (and this is elementarily verified) that the path $P(t)$ is a (unique) solution to the Cauchy problem

$$\mathrm{i}\frac{d}{dt}P = [P,H], \tag{11}$$

$$P(t)\Big|_{t=t_0} = P_0, \tag{12}$$

where $[P,H] := PH - HP$ is the commutator of $P = P(t)$ and H; P_0 denotes the orthogonal projection onto the initial subspace $\mathfrak{H}_0$. The solution to (11), (12) is explicitly given by

$$P(t) = U(t)P_0U(t)^* = \mathrm{e}^{-\mathrm{i}Ht}P_0\mathrm{e}^{\mathrm{i}Ht}. \tag{13}$$

It is well known that the set of all orthogonal projections in the Hilbert space $\mathfrak{H}$ (and hence the set of all subspaces of $\mathfrak{H}$) is a metric space with distance given by the standard operator norm,

$$\rho(Q_1, Q_2) := \|Q_1 - Q_2\|, \qquad \rho(\mathfrak{Q}_1, \mathfrak{Q}_2) := \rho(Q_1, Q_2),$$

where Q_1, Q_2 are arbitrary orthogonal projections and $\mathfrak{Q}_1$, $\mathfrak{Q}_2$, their ranges.

It is, however, much less known that there is another natural metric on the set of all orthogonal projections in/ all the subspaces of the Hilbert space $\mathfrak{H}$. The distance is defined by

$$\vartheta(\mathfrak{Q}_1, \mathfrak{Q}_2) := \vartheta(Q_1, Q_2) := \arcsin(\|Q_1 - Q_2\|). \tag{14}$$

That (14) is a metric has been first proven in 1993 by Lawrence Brown. An alternative (and, we think, somewhat simpler) proof may be found in our joint paper with Sergio Albeverio (2013).

The quantity $\vartheta(\mathfrak{Q}_1, \mathfrak{Q}_2)$ is called the ***maximal angle*** between the subspaces $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$.

Remark The concept of maximal angle between subspaces is traced back to Krein, Krasnoselsky, and Milman (1948). Assuming that $(\mathfrak{Q}_1, \mathfrak{Q}_2)$ is an ordered pair of subspaces with $\mathfrak{Q}_1 \neq \{0\}$, they applied the notion of the (relative) maximal angle between $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ to the number

$$\sin\varphi(\mathfrak{Q}_1, \mathfrak{Q}_2) = \sup_{x\in\mathfrak{Q}_1,\, \|x\|=1} \operatorname{dist}(x, \mathfrak{Q}_2).$$

If both $\mathfrak{Q}_1 \neq \{0\}$ and $\mathfrak{Q}_2 \neq \{0\}$ then

$$\vartheta(\mathfrak{Q}_1, \mathfrak{Q}_2) = \max\bigl\{\varphi(\mathfrak{Q}_1, \mathfrak{Q}_2), \varphi(\mathfrak{Q}_2, \mathfrak{Q}_1)\bigr\}.$$

Unlike $\varphi(\mathfrak{Q}_1, \mathfrak{Q}_2)$, the maximal angle $\vartheta(\mathfrak{Q}_1, \mathfrak{Q}_2)$ is always symmetric with respect to the interchange of the arguments $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$. Furthermore,

$$\varphi(\mathfrak{Q}_2, \mathfrak{Q}_1) = \varphi(\mathfrak{Q}_1, \mathfrak{Q}_2) = \vartheta(\mathfrak{Q}_1, \mathfrak{Q}_2) \quad \text{whenever } \|Q_1 - Q_2\| < 1.$$

Theorem 1. Suppose that T_θ is a time moment for which the maximal angle between the initial subspace $\mathfrak{P}_0$ and a subspace in the path $\mathfrak{P}(t)$, $t \geq 0$, reaches the value of θ, $0 < \theta \leq \frac{\pi}{2}$, that is,

$$\vartheta\big(\mathfrak{P}_0, \mathfrak{P}(T_\theta)\big) = \theta. \tag{15}$$

Then the following inequality holds:

$$T_\theta \geq \frac{\theta}{\Delta E_{\mathfrak{P}_0}}, \tag{16}$$

where

$$\Delta E_{\mathfrak{P}_0} := \sup_{\psi \in \mathfrak{P}_0,\, \|\psi\|=1} \big(\langle H^2\psi, \psi\rangle - \langle H\psi, \psi\rangle^2\big)^{1/2}$$

Remark. The bound (15) is sharp since it is sharp already in the one-dimensional case (where (15) turns into the Fleming bound for the speed of a state evolution).

Lemma. For the maximal energy dispersion $\Delta E_{\mathfrak{P}_0}$ on the subspace $\mathfrak{P}_0$ one always has the (optimal) bound

$$\Delta E_{\mathfrak{P}_0} \leq \frac{E_{\max}(H) - E_{\min}(H)}{2}, \tag{17}$$

where the subspace $\mathfrak{P}_0$ is arbitrary subspace of $\mathfrak{H}$ and

$$E_{\min}(H) = \min\bigl(\mathrm{spec}(H)\bigr) \quad \text{and} \quad E_{\max}(H) = \max\bigl(\mathrm{spec}(A)\bigr)$$

are the upper and lower bounds of the spectrum of H, respectively.

Corollary. Assume that Ω is a non-negative number and let $\mathscr{B}_\Omega(\mathfrak{H})$ be the set of all bounded self-adjoint operators H in $\mathfrak{H}$ such that

$$E_{\max}(H) - E_{\min}(H) \leq \Omega.$$

Then

$$\inf_{H \in \mathscr{B}_\Omega(\mathfrak{H})} T_\theta(H) \geq \frac{2\theta}{\Omega}, \tag{18}$$

where $T_\theta(H)$ is a time moment for which the maximal angle between the initial subspace $\mathfrak{P}_0$ and a subspace in the path $\mathfrak{P}(t)$, $t \geq 0$, reaches the value of $\theta \leq \frac{\pi}{2}$.