

# SPECTROSCOPIC FACTORS: OBSERVABILITY AND MEASURABILITY

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# OUTLINE

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## DEFINITION OF SPECTROSCOPIC FACTORS

Let us consider a composite bound nuclear system  $a$ , which can be divided into two subsystems (fragments)  $b$  and  $c$ . Define an overlap function (OF) (or overlap integral)  $I(\vec{r})$  ( $\hbar = c = 1$ ):

$$I_{abc}(\vec{r}) = \int \psi_b^+(\tau_b) \psi_c^+(\tau_c) \psi_a(\tau_b, \tau_c, \vec{r}) d\tau_b d\tau_c, \quad (1)$$

where  $\psi_i(\tau_i)$  is the inner wave function of system  $i$  depending on inner coordinates  $\tau_i$  and  $\vec{r}$  is the radius-vector connecting the centers of mass of  $b$  and  $c$ .

In fact,  $I(\vec{r})$  is the projection of the wave function of  $a$  onto the  $b + c$  channel. It is common knowledge that OFs appear as important parts of matrix elements within various approaches to describing nuclear reactions. The partial-wave expansion of  $I(\vec{r})$  is as follows

$$I_{abc}(\vec{r}) = \sum_{l m_l m_s} i^l (J_b M_b J_c M_c | s m_s) (l m_l s m_s | J_a M_a) Y_{l m_l}(\vec{r}/r) I_{abc; l s}(r), \quad (2)$$

where  $I_{abc}(l; r)$  is the radial OF,  $l$  and  $s$  are the channel orbital momentum and the channel spin, respectively.

Note that  $I_{abc}(l; r)$  is not normalized to unity. Its norm is related to the spectroscopic factor (SF)  $S_{abc;l}$ :

$$S_{abc;l} = N \int_0^\infty |I_{abc}(l; r)|^2 r^2 dr. \quad (3)$$

Here  $N$  is a combinatorial factor accounting for identity of nucleons. Within the isospin formalism

$$N = \frac{A_a!}{A_b!A_c!}, \quad (4)$$

where  $A_i$  is the mass number of nucleus  $i$ . In particular, if one of fragments  $b$  or  $c$  is a nucleon,  $N = A_a$ . If, within the shell model, a nuclear wave function is written in the form of the Slater determinant, then the SF defined according to (3) equals 1. The factor  $N^{1/2}$  is often included in the definition of an OF.

The SF determines the weight of the  $b + c$  configuration in nucleus  $a$ . SFs have long been used in the analysis of nuclear reactions. Note that the definitions of the OF (1) and SF (3) can be easily generalized to the case of the splitting of a nucleus into three or more fragments.

Recently, in the analysis of nuclear reactions, asymptotic normalization coefficients are increasingly employed, which determine the asymptotic form of radial overlap functions:

$$I_l(r)|_{r \rightarrow \infty} \approx C_l \frac{W_{-\eta, l+1/2}(2\kappa r)}{r} \approx C_l \frac{\exp(-\kappa r)}{(2\kappa r)^\eta}, \quad (5)$$

where  $W_{\alpha, \beta}(z)$  is the Whittaker function,  $Z_i e$  is the charge of fragment  $i$ ,  $\kappa$ ,  $\eta = Z_b Z_c e^2 \mu / \kappa$ , and  $\mu$  are the wave number, the Coulomb (Sommerfeld) parameter, and the reduced mass for the bound state  $a$  in the  $b + c$  channel, respectively. Indices  $a, b, c, s$  are omitted in Eq.(5) and in what follows for brevity.

The quantity  $C_l$  in Eq. (5) is called the asymptotic normalization coefficient (ANC).

The ANC for the  $a \rightarrow b + c$  channel determines the probability of the  $b + c$  configuration in nucleus  $a$  at distances greater than the radius of the nuclear interaction. ANCs determine the cross sections of peripheral nuclear reactions, in particular, reactions between charged particles at low energies. The most important type of such reactions is represented by astrophysical nuclear reactions.

The ANC is expressed through the residue of the elastic  $b + c$  scattering amplitude at the pole corresponding to the binding energy of  $a$  in the  $b + c$  channel.

Both ANCs and SFs are nuclear characteristics. It should be emphasized that an ANC, in contrast to a SF, is an on-shell quantity, similar to a phase shift and binding energy. The SF is an off-shell quantity, similar to a wave function.

Note that in some, albeit rare cases, the asymptotics of  $I_{abc}(l; r)$  may differ from that of Eq. (5) (the so-called anomalous asymptotics) (L.B., *Yad. Fiz.* 1981 34, 865; L.B., A.M.Mukhamedzhanov, R.Yarmukhamedov, *Eur. Phys. J. A* 2013 49, 108).

# SPECTROSCOPIC FACTORS AND NUCLEAR REACTIONS

SFs were first introduced in the analysis of direct one-nucleon transfer reactions, such as  $(p, d)$ ,  $(d, t)$ ,  $({}^3\text{He}, d)$ , and  $({}^3\text{He}, \alpha)$  within the DWBA. These reactions were intensely used and continue to be used to determine SFs corresponding to the detachment of a nucleon from a nucleus. Other processes can be used for this purpose as well, such as a direct radiation capture or electron-induced knockout (e.g.,  ${}^{16}\text{O}(e, e'p)$  or  ${}^{12}\text{C}(e, e'p)$ ).

**Note that the DWBA is evidently an approximation procedure which does not have any serious theoretical justification. In this regard, it is necessary to point out that SFs are absent in expressions of exact nuclear reaction theory. Moreover, they do not appear in the original expressions for the DWBA matrix elements.**



SFs arise as a result of the following traditional replacement of multiparticle OFs appearing in expressions for the DWBA matrix elements by single-particle quantities:

$$I_l(r) \longrightarrow S_l^{1/2} \varphi_l(r), \quad (6)$$

where  $\varphi_l$  is a solution for a two-body potential  $V(r)$ .  $\varphi_l$  is normalized to unity. Moreover, the so-called one-particle ANC  $b_l$  is often introduced into consideration, which characterizes the asymptotic behavior of  $\varphi_l$  and is related to the true ANC  $C_l$  (5) as follows

$$C_l = S_l^{1/2} b_l. \quad (7)$$

Traditionally, the Woods-Saxon potential with standard geometric parameters  $r_0$  and  $a$  is used as the potential  $V(r)$ . The depth of the potential is fitted to the binding energy of nucleus  $a$  in the  $b + c$  channel (the so-called well-depth procedure).

To determine the SF  $S_I$ , the experimental differential cross section is equated to the theoretical one in the following form:

$$\left(\frac{d\sigma(\theta)}{d\Omega}\right)_{exp} = S_I \left(\frac{d\sigma(\theta)}{d\Omega}\right)_{DWBA}, \quad (8)$$

where  $(d\sigma(\theta)/d\Omega)_{DWBA}$  is the DWBA cross section calculated using Eq. (6) with  $S_I = 1$ . Then SF  $S_I$  is found by comparing left-hand and right-hand sides of (8) in the region of best agreement of theoretical and experimental angular distributions (usually within the main peak). For transfer reactions, the product of two SFs is included in (8), but one of them may be known (for example, for  $d$  or  $t$ ).

However, the correctness and accuracy of the replacement procedure (6) is not at all obvious. This replacement is based on the assumption that the radial dependence of functions  $I_l(r)$  and  $\varphi_l(r)$  is the same for all  $r$  values and that they differ only in normalization. In fact, this assumption has no serious justification.

For peripheral reactions, only the asymptotic part of the OF contributes to the DWBA matrix element. This part is represented by the universal form (5). Hence the corresponding cross section is parametrized in terms of ANCs rather than in terms of SFs and it is impossible to extract SFs' values from experimental data.

This is the case, in particular, for most astrophysical nuclear reactions which are peripheral due to the Coulomb barrier and low collision energy. The role of ANCs in nuclear astrophysics was first emphasized in [Mukhamedzhanov and Timofeyuk \*Sov. J. Nucl. Phys.\* 1990 51, 431](#) where it was noted that the ANC determines the overall normalization of peripheral radiative capture reactions.

Peripheral nuclear reactions are an appropriate tool for determining experimental ANC values, but not SFs.

If a reaction is not peripheral, then the interior part of the OF contributes to the DWBA matrix element and the problem becomes less transparent. In this case, the DWBA cross section is not determined only by the ANC, but this does not mean at all that it is proportional to the corresponding SF.

As a result, the SF values extracted from the analysis of the various transfer reactions at different energies within the DWBA are characterized by a considerable spread.

This spread is especially large in the case of SFs for the removal of composite objects, say,  $\alpha$ -particles. For example, from Table II of the work [N. Oulebsir et al., Phys. Rev. C 85, 035804 \(2012\)](#) it follows that the SF values for the removal of an  $\alpha$ -particle from  $^{16}\text{O}$  excited states obtained in different works by analyzing reactions  $^{12}\text{C}(^7\text{Li}, t)^{16}\text{O}$  and  $^{12}\text{C}(^6\text{Li}, d)^{16}\text{O}$  at various energies may differ by more than an order of magnitude.

A new combined method for extracting SFs from transfer reactions was proposed in the works [A.M. Mukhamedzhanov et al., Phys. Rev. C 72, 017602 \(2005\)](#); [77, 015601 \(2008\)](#).

Within this method, the matrix element of the non-peripheral reaction is written as the sum of contributions from internal and external parts of the OF. The external part is parametrized in terms of the ANC and the internal one in terms of the SF. The value of the ANC is supposed to be known. It can be taken from other reactions or from the same reaction at lower energy when it becomes peripheral. For the applicability of this method, the external contribution must be non-negligible.

This method was applied to the analysis of several  $(d, p)$  reactions and made it possible to improve the accuracy of extracted SFs.

However, the problem of low accuracy of SFs extracted from transfer reactions remains.

Note that the extraction of SFs from the data on radiation capture or electron-induced knockout reactions gives rise to the same problems associated with the approximation of OFs, as with the use of transfer reactions. In addition, electron-induced knockout reactions cannot be used to obtain information on exotic short-lived nuclei, the study of which is of the greatest interest at present.

It can be concluded that the existing methods for determining SFs from experimental data do not allow obtaining their values with the sufficient accuracy.

In addition, there is a more general and more principal problem associated with the determination of SFs including their microscopic calculations.

## OBSERVABILITY OF SPECTROSCOPIC FACTORS

The concept of an observable quantity (or just an observable) has been discussed for a long time in quantum physics, and this discussion continues at present. An observable (or a measurable) quantity  $M$  is determined by the  $S$  matrix. Therefore,  $M$  should be invariant under various transformations that do not change the  $S$  matrix.

Consider an arbitrary unitary transformation (UT)  $U$ , ( $U^+ = U^{-1}$ ), which does not change the norm of a wave function:  $\langle U\Psi|U\Psi\rangle = \langle\Psi|\Psi\rangle$ . The transformed matrix element of a Hamiltonian can be written in two forms

$$\langle U\Psi_1|H|U\Psi_2\rangle \equiv \langle\Psi_1|\tilde{H}|\Psi_2\rangle, \quad \tilde{H} = U^{-1}HU. \quad (9)$$

Two Hamiltonians related by an UT are called equivalent if they lead to the same  $S$  matrix.



All equivalent potentials should lead to the same phase shifts, the same binding energies, and the same ANCs since all these quantities are determined by the  $S$  matrix.

The question arises: Is the existence of different equivalent potentials consistent with the conclusions of the inverse scattering theory? According to this theory, if the scattering phase shift  $\delta_l(E)$  is known for  $0 \leq E < \infty$ , as well as the binding energies and the ANCs for all bound states in a given channel, then the one single interaction potential that describes the given system can be restored.

The answer to the above question is simple: This conclusion applies only to local potentials, while UTs applied to local potentials generally lead to the appearance of nonlocal terms. A nonlocal potential should not be considered exotic. Modern  $NN$  potentials contain nonlocal terms. An optical potential describing the interaction of composite systems, strictly speaking, is also nonlocal.

In the work [W.N.Polyzou, Phys. Rev. C 82, 014002 \(2010\)](#) the necessary and sufficient conditions were derived for the invariance of the  $S$  matrix under the UT of the Hamiltonian. In nuclear theory, the UT of Hamiltonians is used to treat the short-range repulsive core of the  $NN$ -interaction. In particular, in [Feldmeier et al., Nucl. Phys. A 632, 61 \(1998\)](#) the unitary correlation operator method (UCOM) was suggested which shifts the nucleons away from each other whenever their positions are within the repulsive core. This method effectively makes the  $NN$  potential softer. It clearly follows from that work that applying the UCOM to the local potential leads to a non-locality of the potential at short distances.

The quantities, which change under the UT conserving the  $S$  matrix, are traditionally considered as non-observables. The deuteron  $D$ -state probability  $P_D$  can serve as a well-known example of such quantities (see, e.g., Friar-1979, Amado-1979, Furnsdahl-2010).  $P_D$  changes under UTs of nuclear forces which do not change the other characteristics of a deuteron, such as the binding energy, quadrupole and magnetic moments, and the asymptotic  $D/S$  ratio. Of course,  $P_D \neq 0$ , otherwise the deuteron would have no quadrupole moment. However, experimental data allow a change of  $P_D$  in a wide range from 3% to 9%.

$P_D$  is actually a SF for the  $D$ -state part of the deuteron wave function.

Note that, strictly speaking, nuclear forces derived on the basis of the QCD (e.g. via the EFT) are also non-observables.

In the work [A.M.Mukhamedzhanov, A.S.Kadyrov, Phys. Rev. C, 051601 \(2010\)=MK](#) the authors drew attention to the existence of finite-range UTs which change the short-range behavior of wave functions but do not affect their asymptotics. An example of such UT is the aforementioned UCOM. Since phase shifts and ANCs are determined by the asymptotic behavior of wave functions, their values do not change under finite-range UTs, in contrast to the SF.

The authors clearly showed that the exact reaction amplitudes (direct transfer, breakup, and electron-induced disintegration) are invariant under finite-range UTs and that these processes cannot be used as a tool to determine SFs.

The assertion that the SF is a non-observable was made in a number of other works. ([Furnstahl-2010](#), [Jennings-2011](#)).

However, attempts to extract or to calculate SFs continue.

In the work MK, the action of the UT on an OF was considered using the example of the nucleon detachment  $(A + 1) \rightarrow A + N$ . It was supposed in MK that the UT acts separately on internal coordinates of  $A$  and on the radius-vector  $\vec{r}$  connecting  $A$  and  $N$ .

For generality, consider the arbitrary case of the  $a \rightarrow b + c$  process. Consider the OF  $I(\vec{r})$ .

$$I(\vec{r}) = \int \psi_b^+(\tau_b) \psi_c^+(\tau_c) \psi_a(\tau_b, \tau_c, \vec{r}) d\tau_b d\tau_c = \langle \psi_b \psi_c | \psi_a \rangle. \quad (10)$$

By analogy with MK, we assume that UT  $U$  acts separately on internal coordinates of fragments  $b$  and  $c$  and on the radius  $\vec{r}$ , that is

$$U = U_b U_c U_r. \quad (11)$$

Thus under the UT

$$I(\vec{r}) \longrightarrow \tilde{I}(\vec{r}) = \langle U_b \psi_b U_c \psi_c | U_b U_c U_r \psi_a \rangle = \langle \psi_b \psi_c | U_b^{-1} U_c^{-1} U_b U_c U_r \psi_a \rangle. \quad (12)$$

As a result,

$$\tilde{I}(\vec{r}) = \langle \psi_b \psi_c | U_r \psi_a \rangle \neq I(\vec{r}) = \langle \psi_b \psi_c | \psi_a \rangle. \quad (13)$$

It follows from (13) that the OF is non-invariant under the UT. This conclusion is rather obvious since the OF can be considered as a generalized wave function and wave functions change under the UT.

Now let's see what happens with the SF  $S$  under the UT. Since the UT  $U_r$  acts only on  $\vec{r}$ , one can write

$$\tilde{I}(\vec{r}) = \langle \psi_b \psi_c | U_r \psi_a \rangle = U_r \langle \psi_b \psi_c | \psi_a \rangle = U_r I(\vec{r}). \quad (14)$$

Therefore the transformed total SF  $\tilde{S}$  is (the constant antisymmetrization factor  $N$  is omitted for brevity)

$$\tilde{S} = \langle \tilde{I}(\vec{r}) | \tilde{I}(\vec{r}) \rangle = \langle U_r I(\vec{r}) | U_r I(\vec{r}) \rangle = \langle I(\vec{r}) | U_r^{-1} U_r I(\vec{r}) \rangle = \langle I(\vec{r}) | I(\vec{r}) \rangle = S. \quad (15)$$

Eq. (15) means that the SF is invariant under arbitrary UT. This conclusion is contrary to what is written in MK as well as in some other papers. The only explanation of this result is that in the general case the separability condition (11) is not valid. Apparently, it can be satisfied only when fragments  $b$  and  $c$  are far apart, that is, for large values of  $r$ . In the field theory, this assertion is known as the cluster decomposition property.

Note that the validity of (11) at large  $r$  guarantees the invariance of the  $S$  matrix under short-range UTs.

## CONCLUSIONS

1. Spectroscopic factors are absent in the rigorous theory of nuclear reactions. They appear only in an approximate version of the DWBA, which has no serious theoretical justification and is essentially a convenient method for approximate modeling of experimental data on direct nuclear reactions. Even within this approach, the accuracy of the spectroscopic factors extracted from the experiment is low.
2. Spectroscopic factors are off-shell quantities. They are not determined by the  $S$  matrix unlike on-shell quantities, such as phase shifts, binding energies, ANCs, etc.



3. Spectroscopic factors are not invariant under the unitary transformations conserving the  $S$  matrix. Therefore, they are ill-defined non-observables, which can only be defined within a special convention, like a particular form of the nuclear Hamiltonian which is used to derive or calculate them.

4. It follows from the foregoing that determining spectroscopic factors from experimental data is of rather limited value.

5. Spectroscopic factors can be calculated in the framework of specific microscopic approaches. However, comparing the results of such calculations with the phenomenological values of spectroscopic factors is unlikely to provide any significant information.

*Thank You for Patience*