

Systematic uncertainties of predictions for p_T spectrum of photons from Z/γ^* at LHC

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1. I was told that there are concerns for p_T spectrum of photons in ATLAS for $\mu\mu\gamma$ final state systems of invariant mass close to the Z mass.
2. Distributions are not public and I am not member of Atlas. I know only that predictions obtained Sherpa and with the help of Photos differ much less, than the two from the data.
3. That is not the most convenient precondition for the talk, especially over internet.
4. **But I understand, one can not present something what is not ready.**
5. To be watched: · QED FSR · QED ISR and IFI · EW corrections
· **bremsstrahlung and orientation frames** · quark level initial states PDFs p_T spectra
etc.
6. **I will bet for the source of theory-experiment discrepancy.**

To do ...

1. isolate simulation segment, the source of inconsistency
2. I understand that: · QED ISR and IFI · quark level initial states PDFs p_T spectra, were checked already and they do not pose the problem
3. I will concentrate:
4. **QED FSR:** this is the main source of photons.
5. **EW corrections:** because this sometimes bring large effects
6. **Frames orientation for iterfaces:** because I was surprized myself with numerical importance of the seemingly equivalent variants.

Web page of Photos: <http://photospp.web.cern.ch/photospp/>

Final state bremsstrahlung

- Can be separated from other parts of radiative corrections and form separate segment.
- Its description must take into account emission of detectable photons. They need to be explicitly generated. Because of infrared singularities exponentiation is required to take into account terms formally of $\mathcal{O}(\alpha_{QED}^3)$ and higher.
- Fortunately, since more than half a century, scheme of Yennie Frautchi and Suura is available and enables to resum dominant parts of contributions to all orders. Perturbation expansion is not compromised. Reordering of terms is needed.
- Our KKMC MC Monte Carlo S. Jadach, B. F. L. Ward, and Z. Was *Comput.Phys.Commun.* **130** (2000) 260, is build on that principle.
Several version of matrix element are implemented, of the first and second order:
EEX1, EEX2, CEEX1, CEEX2
For the generation algorithm conformal symmetry of the phase space and eikonal part of matrix elements was of fundamental importance.

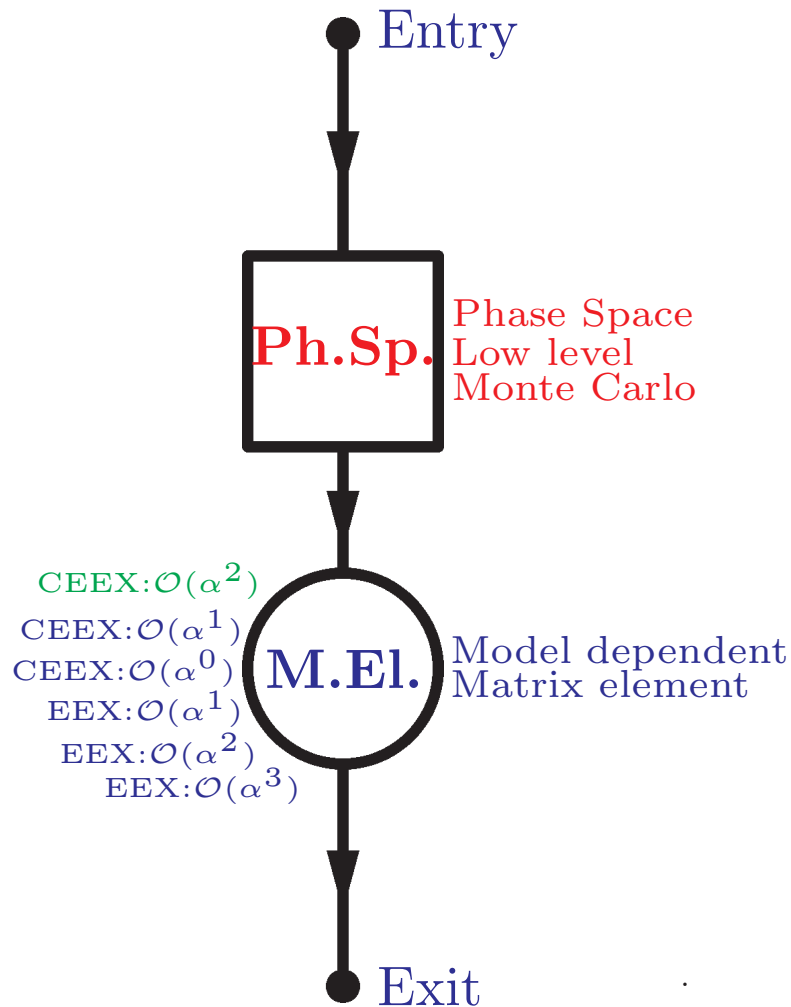
Full phase space coverage is assured for ISR, FSR and IFI contributions.

– Program was fundamental for LEP applications. Now is used for precision simulations for high statistics/precision Belle experiments. Version for LHC applications is available too.

– Offers wealth of benchmarks of the cutting edge precision level. For LHC applications too. Even if initial state strong interaction need to be discussed.

- Sherpa Monte Carlo implementation of final state QED bremsstrahlung relies on similar approach.
- Photos Monte Carlo for bremsstrahlung in decays (N. Davidson, T. Przedzinski, Z. Was, Comput. Phys. Commun. 199 (2016) 86) algorithm differ. Instead of conformal symmetry, iterative solution was devised for projection from crude distribution built on tangent space where photons are treated independently and follow poissonian distribution to the real phase-space. At this step matrix elements are added as well as $\lambda^{1/2}(a, b, c)$ factors of phase space.
- Exact and full coverage of phase-space for emitted photons is assured.

Textbook principle “matrix element \times full phase space”



- Phase-space Monte Carlo module producing “raw events”.
- Library of “models” provides input for tests of physics approximations
- **Phase-space generation must be treated with the great care. Any simplification lead sooner or later to massive progress slow down.**
- Extra benefits: statistically correlated samples useful for tracking small effect,

- The fully differential distribution from MUSTRAAL Nucl.Phys. B202 (1982) 63 (used also in KORALZ Comput.Phys.Commun. 66 (1991) 276 for single FSR photon mode) $e^+(p^+)e^-(p^-) \rightarrow \mu^+(q^+)\mu^-(q^-)\gamma(k)$ reads:

$$X_f = \frac{Q'^2 \alpha (1 - \Delta)}{4\pi^2 s} s^2 \left\{ \frac{1}{(k'_+ k'_-)} \left[\frac{d\sigma_B}{d\Omega}(s, t, u') + \frac{d\sigma_B}{d\Omega}(s, t', u) \right] \right\}$$

- Here:

$$\begin{aligned} s &= 2p_+ \cdot p_-, & s' &= 2q_+ \cdot q_-, \\ t &= 2p_+ \cdot q_+, & t' &= 2p_+ \cdot q_-, \\ u &= 2p_+ \cdot q_-, & u' &= 2q_- \cdot q_+, \\ k'_\pm &= q_\pm \cdot k, & x_k &= 2E_\gamma / \sqrt{s} \end{aligned}$$

- The Δ term is responsible for final state mass dependent terms, p_+, p_-, q_+, q_-, k denote four-momenta of incoming positron, electron beams, outgoing muons and bremsstrahlung photon.

- The above formula, where one can clearly see separation into emission factor and born like factor is valuable because:
 - phase space matrix element factors separation
 - it is fully differential
 - appropriate phase space parametrization of old school geometrical approach.
- YFS exponentiation as perturbation expansion re-ordering, universality of dominant terms
- lots of other results, I will refer to some of those later
- The most important was to assure, proper matching of phase space and matrix elements
- in context of approximations used for higher orders too.

Born cross-section, for $q\bar{q} \rightarrow Z/\gamma^* \rightarrow \ell^+\ell^-$ can be expressed as:

$$\frac{d\sigma_{Born}^{q\bar{q}}}{d\cos\theta}(s, \cos\theta, p) = (1+\cos^2\theta)F_0(s) + 2\cos\theta F_1(s) - p[(1+\cos^2\theta)F_2(s) + 2\cos\theta F_3(s)] \quad (1)$$

p denotes polarization of the outgoing leptons, and form-factors read:

$$\begin{aligned} F_0(s) &= \frac{\pi\alpha^2}{2s} [q_f^2 q_\ell^2 \cdot \chi_\gamma^2(s) + 2 \cdot \chi_\gamma(s) \text{Re}\chi_Z(s) q_f q_\ell v_f v_\ell + |\chi_Z^2(s)|^2 (v_f^2 + a_f^2)(v_\ell^2 + a_\ell^2)], \\ F_1(s) &= \frac{\pi\alpha^2}{2s} [2\chi_\gamma(s) \text{Re}\chi(s) q_f q_\ell v_f v_\ell + |\chi^2(s)|^2 2v_f a_f 2v_\ell a_\ell], \\ F_2(s) &= \frac{\pi\alpha^2}{2s} [2\chi_\gamma(s) \text{Re}\chi(s) q_f q_\ell v_f v_\ell + |\chi^2(s)|^2 (v_f^2 + a_f^2) 2v_\ell a_\ell], \\ F_3(s) &= \frac{\pi\alpha^2}{2s} [2\chi_\gamma(s) \text{Re}\chi(s) q_f q_\ell v_f v_\ell + |\chi^2(s)|^2 (v_f^2 + a_f^2) 2v_\ell a_\ell], \end{aligned} \quad (2)$$

$\cos\theta$ denotes angle between incoming quark and outgoing lepton in the rest frame of outgoing leptons. That is rather simple spherical harmonics of the second order.

Why is it of interest?

1. Condition: $s_W^2 = 1 - m_W^2/m_Z^2$ is important for some gauge cancellations, in case of multileg processes, but at the same time bring inconsistencies with measurements:
2. either m_W must be off by many experimental errors
3. or electroweak observables such as A_{FB} or P_τ by 50 % of their measurable values.
4. Nonetheless such on mass shell scheme is used by many programs of importance for QCD phenomenology.
5. **If possible, technical solutions using calculation of weights are of interest...**
6. ... at least for some time. Is it possible?

Mustraal frame

[18] F. A. Berends, R. Kleiss, and S. Jadach, *Comput. Phys. Commun.* **29** (1983) 185–200.

Mustraal: Monte Carlo for $e^+ e^- \rightarrow \mu^+ \mu^- (\gamma)$

$$s = 2p_+ \cdot p_-, \quad t = 2p_+ \cdot q_+, \quad u = 2p_+ \cdot q_- \\ s' = 2q_+ \cdot q_-, \quad t' = 2p_- \cdot q_-, \quad u' = 2p_- \cdot q_+$$

$$\sigma_{\text{hard}} = \int d\tau (X_i + X_f + X_{\text{int}}),$$

The explicit forms of the three terms in σ_{hard} read:

$$X_i = \frac{Q^2 \alpha}{4\pi^2 s} \frac{1 - \Delta}{k_+ k_-} s'^2 \left[\frac{d\sigma^B}{d\Omega}(s', t, u) + \frac{d\sigma^B}{d\Omega}(s', t', u') \right], \quad (3.4)$$

$$X_f = \frac{Q'^2 \alpha}{4\pi^2 s} \frac{1 - \Delta'}{k'_+ k'_-} s^2 \left[\frac{d\sigma^B}{d\Omega}(s, t, u') + \frac{d\sigma^B}{d\Omega}(s, t', u) \right], \quad (3.5)$$

$$X_{\text{int}} = \frac{QQ'\alpha}{4\pi^2 s} W \frac{\alpha^2}{2ss'} \left[(u^2 + u'^2 + t^2 + t'^2) \tilde{f}(s, s') + \frac{1}{2}(u^2 + u'^2 - t^2 - t'^2) \tilde{g}(s, s') \right] \\ + \frac{QQ'\alpha^3}{4\pi^2 s} \frac{(s - s') M \Gamma}{k_+ k_- k'_+ k'_-} \epsilon_{\mu\nu\rho\sigma} p_+^\mu p_-^\nu q_+^\rho q_-^\sigma \left[\tilde{E}(s, s')(t^2 - t'^2) + \tilde{F}(s, s')(u^2 - u'^2) \right], \quad (3.6)$$

Resulting optimal frame used to minimise higher order corrections from initial state radiation in $e^+e^- \rightarrow Z/\gamma^* \rightarrow \mu \mu$ for algorithms of genuine EW corrections implementation in LEP time Monte Carlo's like Koral Z.

- We can see that distribution is a stochastic sum of Born-like distributions with coefficients which are *positive*. **But it is only QED!**

What are the Limitations and Perspectives for case of QCD jets:

- E. Mirkes and J. Ohnemus, “Angular distributions of Drell-Yan lepton pairs at the Tevatron: Order $\alpha - s^2$ corrections and Monte Carlo studies,” PRD **51** (1995) 4891
- R. Kleiss, “Inherent Limitations in the Effective Beam Technique for Algorithmic Solutions to Radiative Corrections,” Nucl. Phys. B **347**, 67 (1990).

If jets are present definition of angles θ, ϕ (**also QED FSR**), for effective Born becomes an **issue**. But we have $\alpha_s^2 \sim 0.01$ corrections only, to spherical harmonics independently of the choice (p_T transverse momentum of $\tau\tau$ -pair, Y rapidity):

$$\frac{d\sigma}{dp_T^2 dY d\cos\theta d\phi} = \frac{3}{16\pi} \frac{d\sigma^{U+L}}{dp_T^2 dY} [(1 + \cos^2\theta) + 1/2 A_0(1 - 3\cos^2\theta) + A_1 \sin(2\theta) \cos\phi + 1/2 A_2 \sin^2\theta \cos(2\phi) + A_3 \sin\theta \cos\phi + A_4 \cos\theta + A_5 \sin^2\theta \sin(2\phi) + A_6 \sin(2\theta) \sin\phi + A_7 \sin\theta \sin\phi] \quad (3)$$

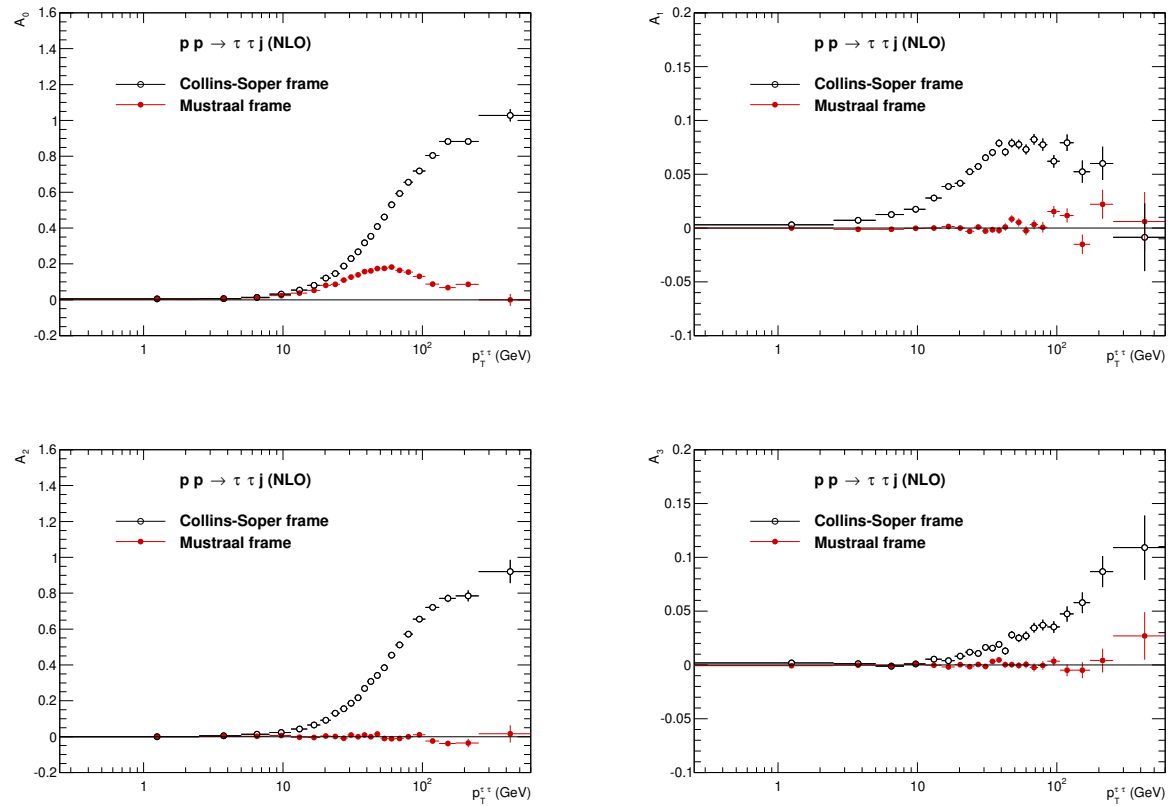


Figure 1: The A_i coefficients of Eq. (3) calculated in Collins-Soper (black) and in Mustraal (red) frames for $pp \rightarrow \tau\tau j$ (NLO) process generated with Powheg+MiNLO. From **Eur.Phys.J. C76 (2016) 473**.

Note that for complete QCD Z+1jet NLO plus MiNLO pattern remained!

We can write amplitude for Born with EW loop corrections, $ME_{Born+EW}$, as:

$$\begin{aligned}
 ME_{Born+EW} &= [\bar{u}\gamma^\mu v g_{\mu\nu} \bar{v}\gamma^\nu u] \cdot (q_e \cdot q_f) \cdot \Gamma_{V\Pi} \cdot \frac{\chi_\gamma(s)}{s} \\
 &+ [\bar{u}\gamma^\mu v g_{\mu\nu} \bar{v}\gamma^\nu u \cdot (v_e \cdot v_f \cdot vv_{ef}) + \bar{u}\gamma^\mu v g_{\mu\nu} \bar{v}\gamma^\nu \gamma^5 u \cdot (v_e \cdot a_f) \\
 &+ \bar{u}\gamma^\mu \gamma^5 v g_{\mu\nu} \bar{v}\gamma^\nu u \cdot (a_e \cdot v_f) + \bar{u}\gamma^\mu \gamma^5 v g_{\mu\nu} \bar{v}\gamma^\nu \gamma^5 u \cdot (a_e \cdot a_f)] \frac{\chi_Z(s) Z_{V\Pi}}{s}
 \end{aligned} \tag{4}$$

One has to take into account, the angle dependent double-vector coupling extra correction, which breaks structure of the couplings into ones associated with Z boson production and decay:

$$\begin{aligned}
 vv_{ef} = & \frac{1}{v_e \cdot v_f} [(2 \cdot T_3^e)(2 \cdot T_3^f) - 4 \cdot q_e \cdot s_W^2 \cdot K_f(s, t) - 4 \cdot q_f \cdot s_W^2 \cdot K_e(s, t) \\
 & + (4 \cdot q_e \cdot s_W^2)(4 \cdot q_f \cdot s_W^2) K_{ef}(s, t)] \frac{1}{\Delta^2}
 \end{aligned} \tag{5}$$

further terms are straightforward:

$$\begin{aligned}
 v_e &= (2 \cdot T_3^e - 4 \cdot q_e \cdot s_W^2 \cdot K_e(s, t)) / \Delta \\
 v_f &= (2 \cdot T_3^f - 4 \cdot q_f \cdot s_W^2 \cdot K_f(s, t)) / \Delta \\
 a_e &= (2 \cdot T_3^e) / \Delta \\
 a_f &= (2 \cdot T_3^f) / \Delta
 \end{aligned}
 \tag{6}$$

The form-factors $K_e(s, t)$, $K_f(s, t)$ are functions of two Mandelstam invariants (s, t) due to the WW and ZZ box contributions.

Vacuum polarisation corrections $\Gamma_{V\Pi}$ to γ propagator are expressed as:

$$\Gamma_{V\Pi} = \frac{1}{2 - (1 + \Pi_{\gamma\gamma})}
 \tag{7}$$

Normalisation correction $Z_{V\Pi}$ to Z-boson propagator is expressed as

$$Z_{V\Pi} = \rho_{e,f}(s, t)
 \tag{8}$$

- Bremsstrahlung and initial state jet emissions do not break structure of Born-like cross section convoluted with emission distributions and can not be the source of observed differences
- **HOWEVER:** orientation of required reference frames used for matching QED FSR with other simulation segments may have an impact
- One can also expect, that in some calculations for consistency of perturbative fixed order calculations for M.E. for final states with photons lower order results for EW calculations will be used, e.g. $\sin^2\theta_W$ on mass shell rather than effective one
- **Possibly sizable numerical consequences for p_T spectra.**
- One may have perfect predictions for $l^+l^-n\gamma$ system as a whole, its internal details too, but incorrect distribution of e.g. p_T for the hardest photon. (then for lepton too) .

More input is needed to identify what could have went wrong and to be sure if it is on the theory simulation side.

Check p_T spectrum of leptons too.

Thank you

Extra slides attached for possible discussion

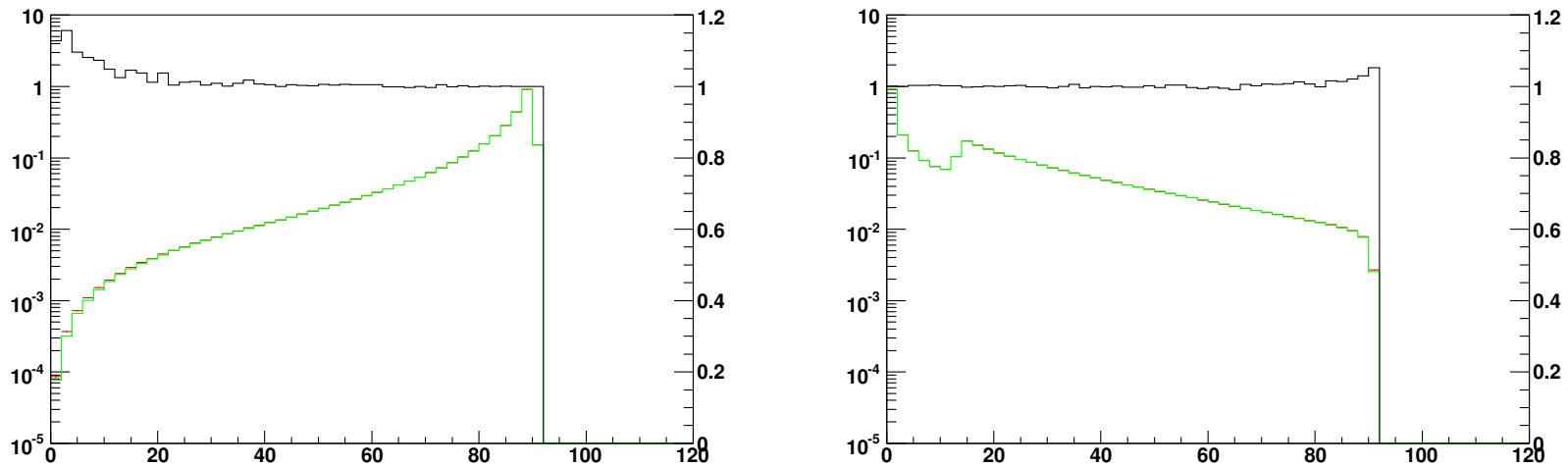


Figure 4: Comparisons of improved PHOTOS with multiple photon emission and KKMC with second order matrix element and exponentiation. In the left frame the invariant mass of the $\mu^+ \mu^-$ pair; $SDP=0.0000249$. In the right frame the invariant mass of the $\mu^- \gamma$ pair; $SDP=0.0000203$. The fraction of events with hard photon was $16.0824 \pm 0.004\%$ for KKMC and $16.0688 \pm 0.004\%$ for PHOTOS.

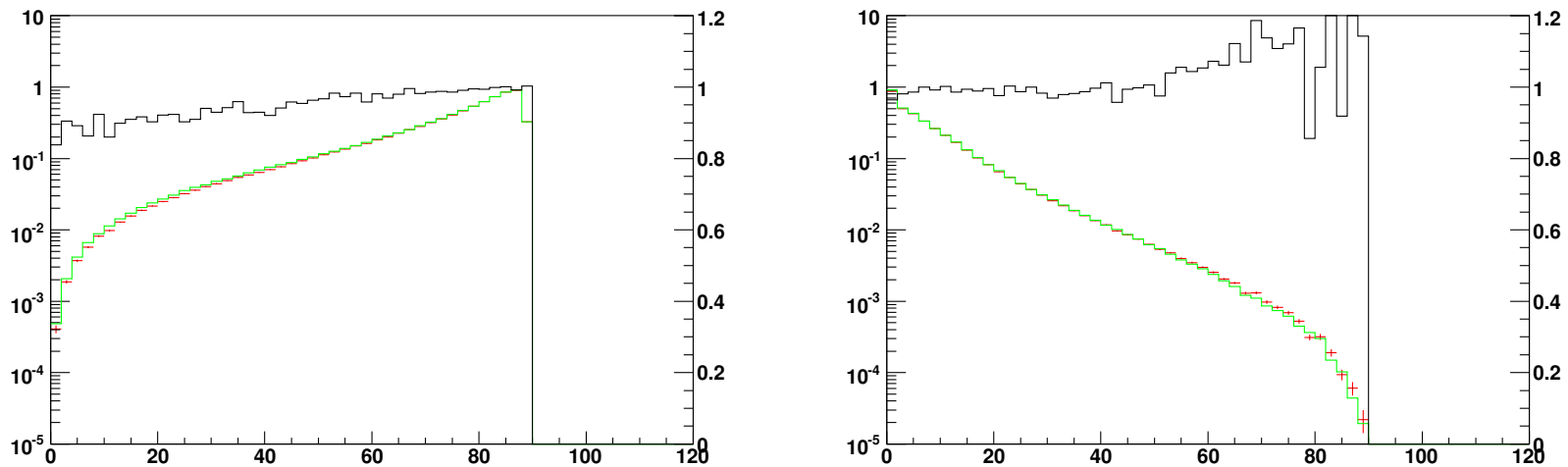


Figure 6: Comparisons of improved PHOTOS with multiple photon emission and KKMC with second order matrix element and exponentiation. In the left frame the invariant mass of the $\mu^+ \mu^-$ pair; $SDP= 0.00142$. In the right frame the invariant mass of the $\gamma\gamma$; $SDP=0.00293$. The fraction of events with two hard photons was $1.2659 \pm 0.0011\%$ for KKMC and $1.2868 \pm 0.0011\%$ for PHOTOS.

Goal: use n -body phase space as source of random numbers for $n+1$

After some meanders in 80's PHOTOS use orthodox Lorentz-invariant phase space (*Lips*). **It must be presented.**

$$\begin{aligned}
 dLips_{n+1}(P) &= \\
 & \frac{d^3 k_1}{2k_1^0 (2\pi)^3} \cdots \frac{d^3 k_n}{2k_n^0 (2\pi)^3} \frac{d^3 q}{2q^0 (2\pi)^3} (2\pi)^4 \delta^4 \left(P - \sum_1^n k_i - q \right) \\
 &= d^4 p \delta^4 (P - p - q) \frac{d^3 q}{2q^0 (2\pi)^3} \frac{d^3 k_1}{2k_1^0 (2\pi)^3} \cdots \frac{d^3 k_n}{2k_n^0 (2\pi)^3} (2\pi)^4 \delta^4 \left(p - \sum_1^n k_i \right) \\
 &= d^4 p \delta^4 (P - p - q) \frac{d^3 q}{2q^0 (2\pi)^3} dLips_n(p \rightarrow k_1 \dots k_n).
 \end{aligned}$$

Integration variables, the four-vector p , compensated with $\delta^4(p - \sum_1^n k_i)$, and another integration variable M_1 compensated with $\delta(p^2 - M_1^2)$ are introduced.

Exact Phase Space \mathcal{F} parametrization

$$dLips_{n+1}(P \rightarrow k_1 \dots k_n, k_{n+1}) = dLips_n^{+1 \text{ tangent}} \times W_n^{n+1},$$

$$dLips_n^{+1 \text{ tangent}} = dk_\gamma d \cos \theta d\phi \times dLips_n(P \rightarrow \bar{k}_1 \dots \bar{k}_n),$$

$$\{k_1, \dots, k_{n+1}\} = \mathbf{T}(k_\gamma, \theta, \phi, \{\bar{k}_1, \dots, \bar{k}_n\}). \quad (9)$$

1. One can verify that if $dLips_n(P)$ was exact, then this formula lead to exact parametrization of $dLips_{n+1}(P)$
2. Practical implementation: Take the configurations from n-body phase space.
3. Turn it back into some coordinate variables.
4. construct new kinematical configuration from all variables.
5. **Forget about temporary $k_\gamma \theta \phi$. From now on, only weight and four vectors count.**
6. A lot depend on \mathbf{T} . Options depend on matrix element: must tangent at singularities. Simultaneous use of several \mathbf{T} is possible and necessary/convenient if more than one charge is present in final state.

Phase Space: (main formula)

If we choose

$$G_n : M_{2\dots n}^2, \theta_1, \phi_1, M_{3\dots n}^2, \theta_2, \phi_2, \dots, \theta_{n-1}, \phi_{n-1} \rightarrow \bar{k}_1 \dots \bar{k}_n \quad (10)$$

and

$$G_{n+1} : k_\gamma, \theta, \phi, M_{2\dots n}^2, \theta_1, \phi_1, M_{3\dots n}^2, \theta_2, \phi_2, \dots, \theta_{n-1}, \phi_{n-1} \rightarrow k_1 \dots k_n, k_{n+1} \quad (11)$$

then

$$\mathbf{T} = G_{n+1}(k_\gamma, \theta, \phi, G_n^{-1}(\bar{k}_1, \dots, \bar{k}_n)). \quad (12)$$

The ratio of the Jacobians form the phase space weight W_n^{n+1} for the transformation. Such solution is universal and valid for any choice of G 's. However, G_{n+1} and G_n has to match matrix element, otherwise algorithm will be inefficient (factor 10^{10} ...).

In case of PHOTOS G_n 's

$$W_n^{n+1} = k_\gamma \frac{1}{2(2\pi)^3} \times \frac{\lambda^{1/2}(1, m_1^2/M_{1\dots n}^2, M_{2\dots n}^2/M_{1\dots n}^2)}{\lambda^{1/2}(1, m_1^2/M^2, M_{2\dots n}^2/M^2)}, \quad (13)$$

Phase Space: (multiply iterated)

By iteration, we can generalize formula (9) and add l particles:

$$\begin{aligned}
 dLips_{n+l}(P \rightarrow k_1 \dots k_n, k_{n+1} \dots k_{n+l}) &= \frac{1}{l!} \prod_{i=1}^l \left[dk_{\gamma_i} d \cos \theta_{\gamma_i} d\phi_{\gamma_i} W_{n+i-1}^{n+i} \right] \\
 &\times dLips_n(P \rightarrow \bar{k}_1 \dots \bar{k}_n), \\
 \{k_1, \dots, k_{n+l}\} &= \mathbf{T}(k_{\gamma_l}, \theta_{\gamma_l}, \phi_{\gamma_l}, \mathbf{T}(\dots, \mathbf{T}(k_{\gamma_1}, \theta_{\gamma_1}, \phi_{\gamma_1}, \{\bar{k}_1, \dots, \bar{k}_n\}) \dots).
 \end{aligned} \tag{14}$$

Note that variables $k_{\gamma_m}, \theta_{\gamma_m}, \phi_{\gamma_m}$ are used at a time of the m -th step of iteration only, and are not needed elsewhere in construction of the physical phase space; the same is true for invariants and angles $M_{2\dots n}^2, \theta_1, \phi_1, \dots, \theta_{n-1}, \phi_{n-1} \rightarrow \bar{k}_1 \dots \bar{k}_n$ of (10,11), which are also redefined at each step of the iteration. Also intermediate steps require explicit construction of temporary $\bar{k}'_1 \dots \bar{k}'_n \dots \bar{k}'_{n+m}$, statistical factor $\frac{1}{l!}$ added.

We have got **exact distribution of weighted** events over $n + l$ body phase space.

Phase Space: Summary.

- We use n-body phase-space as source of random numbers for the sub-set of variables used in n+l phase spaces.
- The number of added particles l , is phase space parameter too.
 - original phase space events were stripped off from matrix elements and phase space jacobians, effectively used as random numbers.
 - that is why n+l phase space could be covered without any approximations.
- Any new matrix element could be introduced ...
- ... but if to a good approximation all followed factorization structure, one could obtain efficient algorithm for the replacement of n-body decay with n+l body configuration of l extra photons.

Phase Space: Trivial link to evolution equations used in benchmarks

- if $c=0$, $\lambda^{1/2}(a, b, c) := \sqrt{a^2 + b^2 + c^2 - 2ab - 2ac - 2bc} = |a - b|$
- If eg. in 3 body phase space invariant mass M of pair is used and third particle is massless then its energy k in rest frame: $k = \frac{S - M^2}{2\sqrt{S}}$.
- That is all what is needed from phase space jacobians for picture of structure functions. Rest belong to matrix elements.
- Side remark: This simplification is **not** used in PHOTOS, but it provides good way for getting benchmarks and can be used to show that PHOTOS sums up those terms which contribute to leading logarithms (of bremsstrahlung) to all orders.

Phase Space Formula: multi-channels.

Often MC algorithm has to be split into branches. In the most general case, when n different parametrizations of the phase space with different orderings of particles are in use, the cross section can be written as follows:

$$d\Gamma_X = \sum_{\lambda=1}^n \int_0^1 \prod_{i=1}^m dx_i P_\lambda \left[\sum_{\delta=1}^n P_\delta J_\delta^{-1}(q_1(\lambda, x_i), \dots, q_k(\lambda, x_i)) \right]^{-1} \times |M|^2.$$

In the above formula the four-momenta $q_i(\lambda, x_i)$ are calculated from the random numbers x_i according to the parametrization of the phase space of type λ . The Jacobians J_δ have to be calculated for all parametrizations of the phase space at the point q_i ; P_λ denotes the probability of choosing the parametrization of type λ in the generation, λ thus takes^a a role of an additional discrete variable in the generation. Numerical values of probabilities P_λ do not affect the final distributions, but only the efficiency of the generation.

^aBut not δ .

Phase Space case of complex singularity structure

- Several G_{n+1} can be used simultaneously (branching of the generation algorithm).
- Each G_{n+1} can be used presample distinct singularities.
- The price: W_n^{n+1} is more complicated but all remain exact.
- **HOWEVER:** We have observed that while matching Jacobians for the two branches related to collinear singularity of photons along direction of l^+ and l^+ (in Z decay) approximation must be used if more than one photon is present in final state. Otherwise solution become inconsistent. Phase space is not iterative, whereas matrix element for multiphoton state is obtained by iteration.
- **AVOID INCONSISTENCY:** in expanding manifold curvature: must be the same for phase space and Matrix Element. Phase space is manifold, Matrix element squared – bilinear form on it. Truncation of perturbative expansion or iterative solutions mean truncation in powers of Ricci tensor, this has to be consistent. **Multichannel phase space is not iterative, single branch algorithm we explained before is that is OK for expansion and exact phases space remain. I have learned that hard way.**

Phase Space: (multiply iterated)

We have generalized formula phase space formula to case of l particles added:

$$\begin{aligned}
 dLips_{n+l}(P \rightarrow k_1 \dots k_n, k_{n+1} \dots k_{n+l}) &= \frac{1}{l!} \prod_{i=1}^l \left[dk_{\gamma_i} d \cos \theta_{\gamma_i} d\phi_{\gamma_i} W_{n+i-1}^{n+i} \right] \\
 &\times dLips_n(P \rightarrow \bar{k}_1 \dots \bar{k}_n), \\
 \{k_1, \dots, k_{n+l}\} &= \mathbf{T}(k_{\gamma_l}, \theta_{\gamma_l}, \phi_{\gamma_l}, \mathbf{T}(\dots, \mathbf{T}(k_{\gamma_1}, \theta_{\gamma_1}, \phi_{\gamma_1}, \{\bar{k}_1, \dots, \bar{k}_n\}) \dots).
 \end{aligned} \tag{15}$$

Now we have to start talking about matrix elements: Our relation between n and $n+l$ body phase space is motivated by cancellation of infrared singularities. It provides kind of **triangulation**. Measure defining distance between points from manifolds of distinct no. of particles. Such phase space points are close if they differ by presence of soft photons only.

Experimental user attention necessary. Can 1 GeV photon be ignored or only 0.1 MeV one.

We will move now from **exact distribution** of **weighted** events over $n + l$ body phase space to case where l is parameter too, but all remain exact!

Crude Ddistribution and crude matrix element

If we add **arbitrary** factors $f(k_{\gamma_i}, \theta_{\gamma_i}, \phi_{\gamma_i})$ and sum over l we obtain:

$$\sum_{l=0} \exp(-F) \frac{1}{l!} \prod_{i=1}^l f(k_{\gamma_i}, \theta_{\gamma_i}, \phi_{\gamma_i}) dLips_{n+l}(P \rightarrow k_1 \dots k_n, k_{n+1} \dots k_{n+l}) =$$

$$\sum_{l=0} \exp(-F) \frac{1}{l!} \prod_{i=1}^l \left[f(k_{\gamma_i}, \theta_{\gamma_i}, \phi_{\gamma_i}) dk_{\gamma_i} d \cos \theta_{\gamma_i} d\phi_{\gamma_i} W_{n+i-1}^{n+i} \right] \times$$

$$dLips_n(P \rightarrow \bar{k}_1 \dots \bar{k}_n), \tag{16}$$

$$\{k_1, \dots, k_{n+l}\} = \mathbf{T}(k_{\gamma_l}, \theta_{\gamma_l}, \phi_{\gamma_l}, \mathbf{T}(\dots, \mathbf{T}(k_{\gamma_1}, \theta_{\gamma_1}, \phi_{\gamma_1}, \{\bar{k}_1, \dots, \bar{k}_n\}) \dots),$$

$$F = \int_{k_{min}}^{k_{max}} dk_{\gamma} d \cos \theta_{\gamma} d\phi_{\gamma} f(k_{\gamma}, \theta_{\gamma}, \phi_{\gamma}). \leftarrow \text{KLN good start}$$

- The **olive** parts of rhs. alone, give crude distribution over tangent space (orthogonal set of variables k_i, θ_i, ϕ_i). We restrict phase space by k_{min} (typically 10^{-6} but not by k_{max}).

- Factors f must be integrable over tangent space. Regulators of singularities necessary.
- We requested

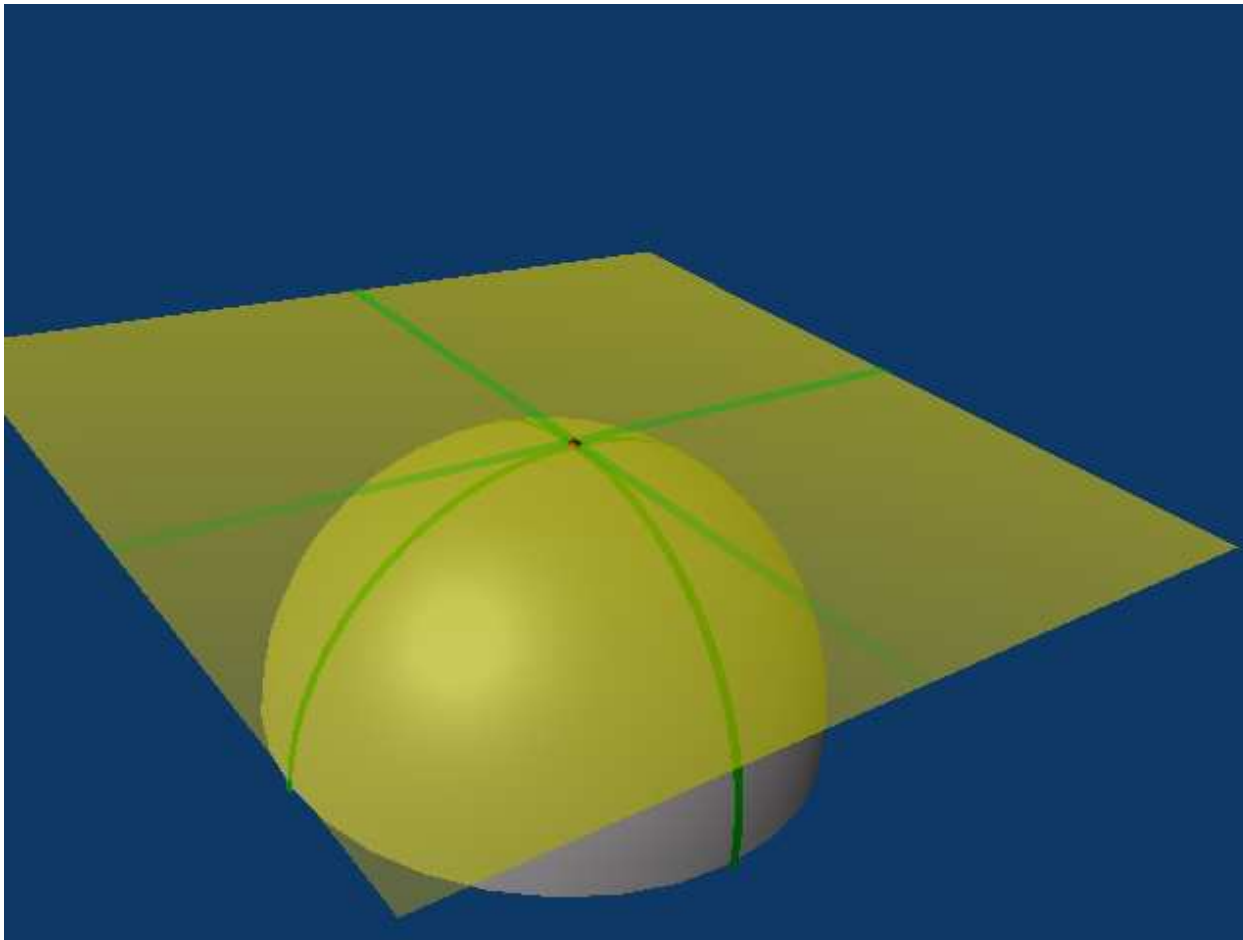
$$\sigma_{tangent} = 1 = \sum_{l=0} \exp(-F) \frac{1}{l!} \prod_{i=1}^l \left[f(k_{\gamma_i}, \theta_{\gamma_i}, \phi_{\gamma_i}) dk_{\gamma_i} d \cos \theta_{\gamma_i} d\phi_{\gamma_i} \right]$$

Because of Kinoshita Lee Nauenberg theorem we know that exact calculation of the cross section give result of similar property. We get another element of PHOTOS basis.

- For precise predictions, real emission and virtual corrections need to be calculated and their factorization properties analyzed.
- Choice of f must be synchronized so correcting weights for matrix elements do not fluctuate wildly.

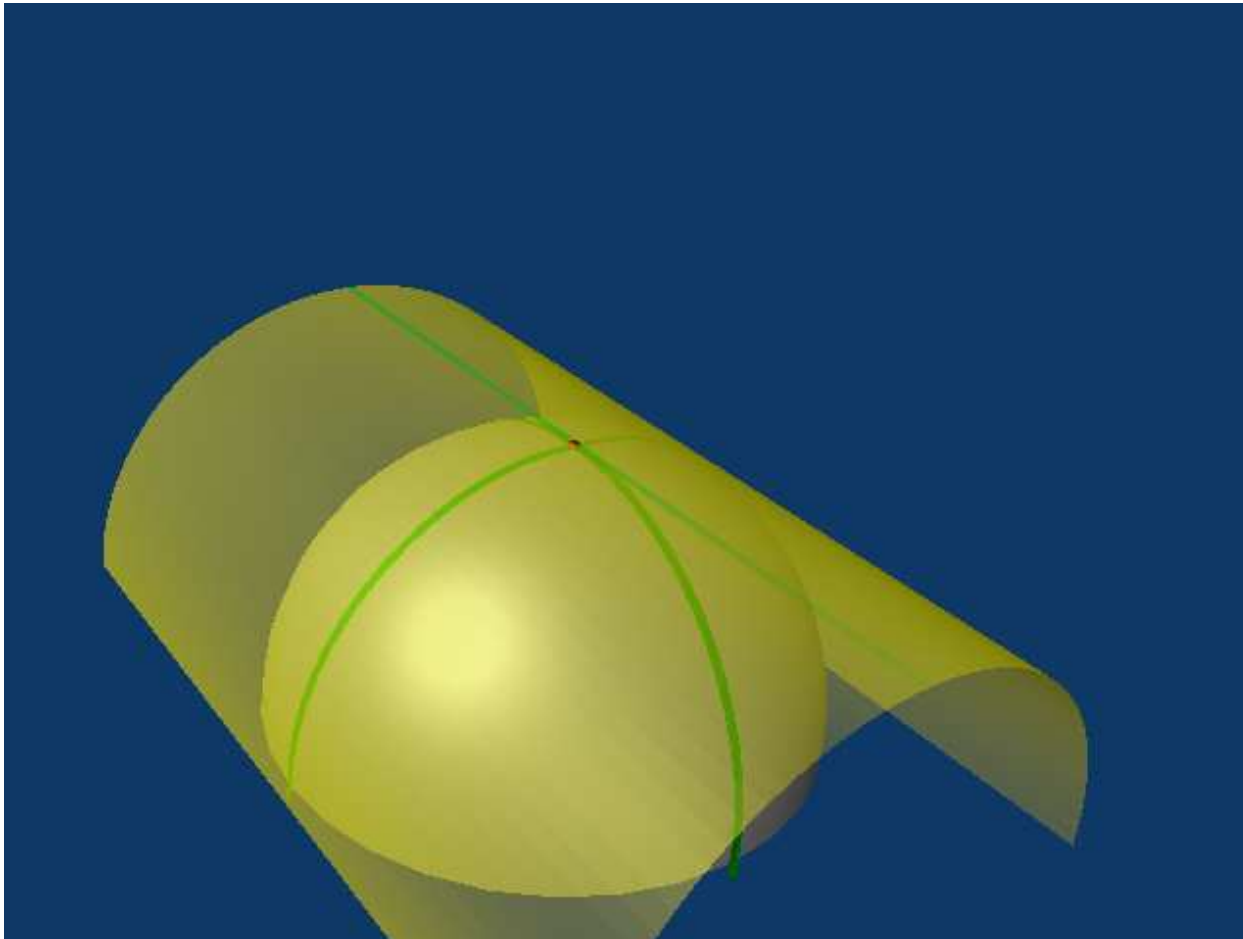
Heuristic CW complexes

We define our crude distribution over yellow space (surface=1) (represented by sum of: red point, green lines and flat yellow square), then we do projections using transformation T and matrix elements.



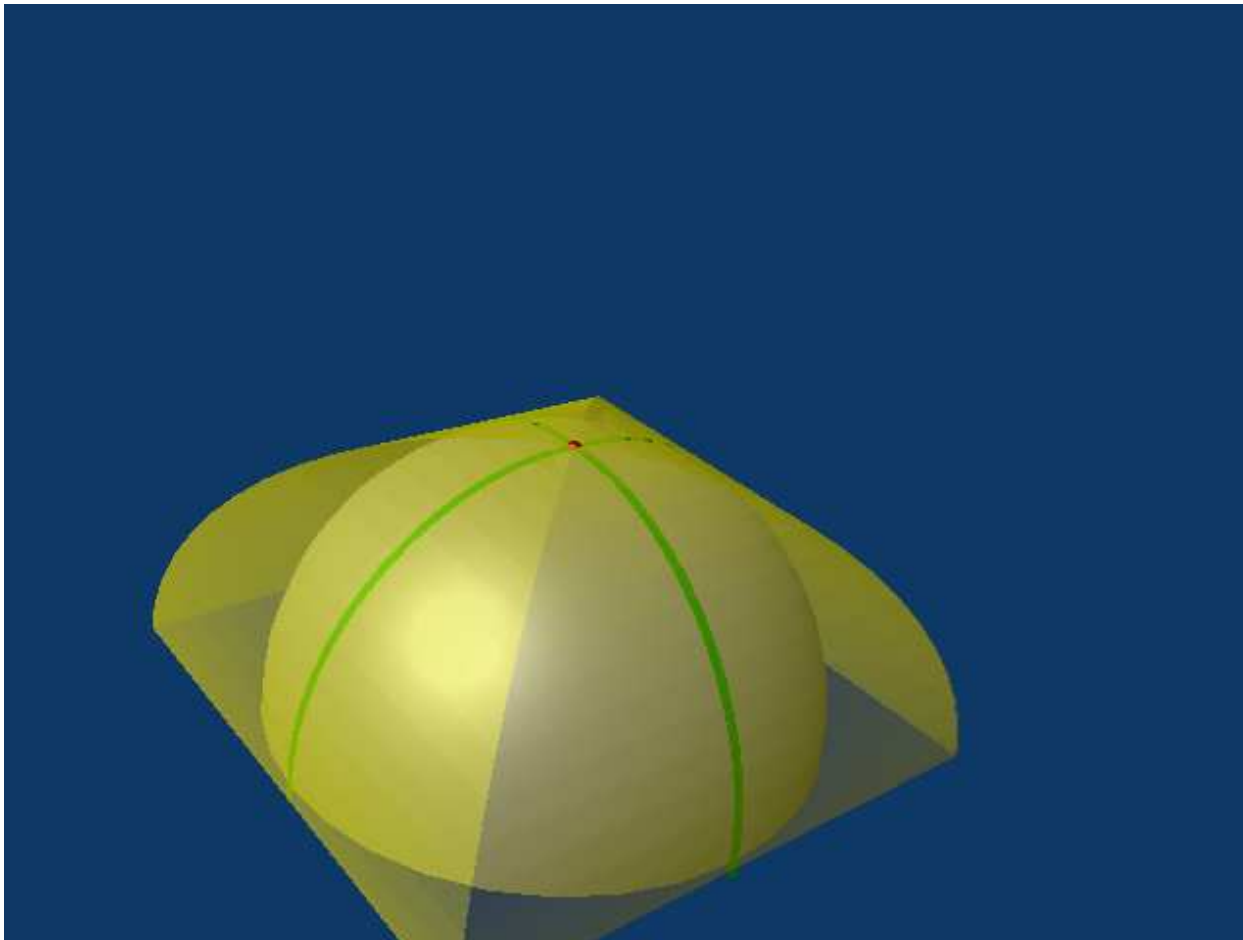
Heuristic CW complexes projection step 1

We project in steps,
relative measure of point and lines on cylinder is larger than in previous step, overall
measure remain 1.



Heuristic CW complexes projection step 2

Final distribution does not match the exact one, solely because approximation in matrix elements, phase space is exact.



Crude distribution of quality → matrix elements can be introduced

- we defined relation of some n - and $(n + l)$ -body ph. space param.
- resulting triangulation (CW-complexes) of phase space parametrizations match structure of singularities.
- CW-complexes for exact space and tangent space match
- to achieve that we used properties of factorization as known in QED, guessed in scalar QED, **also tried for QCD?**
- infrared singularity being within perturbative domain was a bonus.
- we studied (1982-2008) properties of cross sections and later also spin amplitudes: distinct processes.
- formulas for virtual corrections more often than real emissions could be taken from other people works in unchanged form.
- M.E. content of CEEX-YFS exponentiation is the basis → **benchmarks too.**

Matrix Element (starting point):

- Directly starting from Feynman rules one can calculate spin amplitude for any QED/QCD process.
- The case of $Z \rightarrow l^+ l^- \gamma$ is the backbone of PHOTOS design
- single photon amplitude (momentum k_1 polarization e_1 fermion spinors $u(p)$ and $v(q)$ dropped):

$$I = \mathcal{J} \left[\left(\frac{p \cdot e_1}{p \cdot k_1} - \frac{q \cdot e_1}{q \cdot k_1} \right) \right] - \left[\frac{1}{2} \frac{\not{\epsilon}_1 \not{k}_1}{p \cdot k_1} \right] \mathcal{J} + \mathcal{J} \left[\frac{1}{2} \frac{\not{\epsilon}_1 \not{k}_1}{q \cdot k_1} \right]$$

these three gauge invariant parts appear in other processes too

Universal pre-property for factorizations of any sorts... must be deciphered

- The fully differential distribution from MUSTRAAL (used also in KORALZ for single photon mode) reads:

$$X_f = \frac{Q'^2 \alpha (1 - \Delta)}{4\pi^2 s} s^2 \left\{ \frac{1}{(k'_+ k'_-)} \left[\frac{d\sigma_B}{d\Omega}(s, t, u') + \frac{d\sigma_B}{d\Omega}(s, t', u) \right] \right\}$$

- Here:

$$\begin{aligned} s &= 2p_+ \cdot p_-, & s' &= 2q_+ \cdot q_-, \\ t &= 2p_+ \cdot q_+, & t' &= 2p_+ \cdot q_-, \\ u &= 2p_+ \cdot q_-, & u' &= 2q_- \cdot q_+, \\ k'_\pm &= q_\pm \cdot k, & x_k &= 2E_\gamma / \sqrt{s} \end{aligned}$$

- The Δ term is responsible for final state mass dependent terms, p_+ , p_- , q_+ , q_- , k denote four-momenta of incoming positron, electron beams, outgoing muons and bremsstrahlung photon.

- after trivial manipulation it can be written as:

$$X_f = \frac{Q'^2 \alpha (1 - \Delta)}{4\pi^2 s} s^2 \left\{ \frac{1}{(k'_+ + k'_-)} \frac{1}{k'_-} \left[\frac{d\sigma_B}{d\Omega}(s, t, u') + \frac{d\sigma_B}{d\Omega}(s, t', u) \right] + \frac{1}{(k'_+ + k'_-)} \frac{1}{k'_+} \left[\frac{d\sigma_B}{d\Omega}(s, t, u') + \frac{d\sigma_B}{d\Omega}(s, t', u) \right] \right\}$$

- In PHOTOS the following expression is used in universal application (AP adj.):

$$X_f^{PHOTOS} = \frac{Q'^2 \alpha (1 - \Delta)}{4\pi^2 s} s^2 \left\{ \frac{1}{k'_+ + k'_-} \frac{1}{k'_-} \left[(1 + (1 - x_k)^2) \frac{d\sigma_B}{d\Omega} \left(s, \frac{s(1 - \cos \Theta_+)}{2}, \frac{s(1 + \cos \Theta_+)}{2} \right) \right] \frac{(1 + \beta \cos \Theta_\gamma)}{2} + \frac{1}{k'_+ + k'_-} \frac{1}{k'_+} \left[(1 + (1 - x_k)^2) \frac{d\sigma_B}{d\Omega} \left(s, \frac{s(1 - \cos \Theta_-)}{2}, \frac{s(1 + \cos \Theta_-)}{2} \right) \right] \frac{(1 - \beta \cos \Theta_\gamma)}{2} \right\}$$

where : $\Theta_+ = \angle(p_+, q_+)$, $\Theta_- = \angle(p_-, q_-)$

$\Theta_\gamma = \angle(\gamma, \mu^-)$ are defined in (μ^+, μ^-) -pair rest frame

The matrix element weight

- weight for exact matrix element is easy to implement $WT = X_f / X_f^{PHOTOS}$
- also factor $\Gamma^{total} / \Gamma^{Born} = 1 + \frac{3}{4} \frac{\alpha}{\pi}$ defines first order weight, it depends on virtual corrections if non leading mass terms are kept.
- $WT = \frac{X_f}{X_f^{PHOTOS}} \frac{\Gamma^{Born}}{\Gamma^{total}}$

The differences of X_f and X_f^{PHOTOS} are important

- Without process dependent weight PHOTOS is universal and can be combined with any generator rather easily.
- Photos weight becomes process independent at a cost of the approximation.
- Virtual corrections for decays like $K^+ \rightarrow \pi^+ \pi^- l^+ \nu_l$ more complicated than overall constant.

Matrix Element (anything in common?):

- We have seen nice properties of matrix element squared which were factorizing into Born-like distribution and photon factor.
- It was shown many years ago by Ronald Kleiss that such property of distributions does not hold beyond first order!
- Dead end? Not really, just complex weights^a
- single photon amplitude again:

$$I = \mathcal{J} \left[\left(\frac{p \cdot e_1}{p \cdot k_1} - \frac{q \cdot e_1}{q \cdot k_1} \right) \right] - \left[\frac{1}{2} \frac{\not{\epsilon}_1 \not{k}_1}{p \cdot k_1} \right] \mathcal{J} + \mathcal{J} \left[\frac{1}{2} \frac{\not{\epsilon}_1 \not{k}_1}{q \cdot k_1} \right]$$

three gauge invariant parts, first is eikonal, other for collinear configuration along q and p

We look for these parts in higher order amplitudes

^aAlso: samples at different level of sophistication can be correlated up to NLO level. That is enough for most of experimental techniques, precision of correlated programs can be higher.

Matrix Element (double emission):

- The structure of exact spin amplitude for single emission looks promising.
- How does it translate to distributions?
- Does it extend to other processes, interactions? Scalar QED QCD as well?
- Does it extend to higher orders?
- Can one decipher anything without enforcing some phase space conditions?
- To identify the building blocks we have used gauge invariance, and we have used also segments localized at lower order.
- For tree diagrams gauge invariance mean in practice that replacement $k \rightarrow e$ set expression to zero
- Virtual corrections add complication because of regularization schemes, we will skip that now.

Exact Matrix Element: $e^+ e^- \rightarrow \nu_\mu \bar{\nu}_\mu \gamma \gamma$ explicitly;

- Expressions are valid for any current J ,
- For complete amplitude add fermionic fields, eg. $\bar{u}(p)$ and $v(q)$; 1-st/2-nd photon momenta/polarizations are: k_1/k_2 e_1/e_2 .

$$I_1^{\{1,2\}} = \frac{1}{2} J \left(\frac{p \cdot e_1}{p \cdot k_1} - \frac{q \cdot e_1}{q \cdot k_1} \right) \left(\frac{p \cdot e_2}{p \cdot k_2} - \frac{q \cdot e_2}{q \cdot k_2} \right) \quad \text{eikonal}$$

$$I_{2l}^{\{1,2\}} = -\frac{1}{4} \left[\left(\frac{p \cdot e_1}{p \cdot k_1} - \frac{q \cdot e_1}{q \cdot k_1} \right) \frac{\not{\epsilon}_2 \not{k}_2}{p \cdot k_2} + \left(\frac{p \cdot e_2}{p \cdot k_2} - \frac{q \cdot e_2}{q \cdot k_2} \right) \frac{\not{\epsilon}_1 \not{k}_1}{p \cdot k_1} \right] J \quad \beta_1$$

$$I_{2r}^{\{1,2\}} = \frac{1}{4} J \left[\left(\frac{p \cdot e_1}{p \cdot k_1} - \frac{q \cdot e_1}{q \cdot k_1} \right) \frac{\not{k}_2 \not{\epsilon}_2}{q \cdot k_2} + \left(\frac{p \cdot e_2}{p \cdot k_2} - \frac{q \cdot e_2}{q \cdot k_2} \right) \frac{\not{k}_1 \not{\epsilon}_1}{q \cdot k_1} \right] \quad \beta_1$$

$$I_3^{\{1,2\}} = -\frac{1}{8} \left(\frac{\not{\epsilon}_1 \not{k}_1}{p \cdot k_1} J \frac{\not{k}_2 \not{\epsilon}_2}{q \cdot k_2} + \frac{\not{\epsilon}_2 \not{k}_2}{p \cdot k_2} J \frac{\not{k}_1 \not{\epsilon}_1}{q \cdot k_1} \right) \quad \text{start for } \beta_2 \dots$$

$$I_{4p}^{\{1,2\}} = \frac{1}{8} \frac{1}{p \cdot k_1 + p \cdot k_2 - k_1 \cdot k_2} \left(\frac{\not{\epsilon}_1 \not{k}_1 \not{\epsilon}_2 \not{k}_2}{p \cdot k_1} + \frac{\not{\epsilon}_2 \not{k}_2 \not{\epsilon}_1 \not{k}_1}{p \cdot k_2} \right) \not{J}$$

$$I_{4q}^{\{1,2\}} = \frac{1}{8} \not{J} \frac{1}{q \cdot k_1 + q \cdot k_2 - k_1 \cdot k_2} \left(\frac{\not{k}_2 \not{\epsilon}_2 \not{k}_1 \not{\epsilon}_1}{q \cdot k_1} + \frac{\not{k}_1 \not{\epsilon}_1 \not{k}_2 \not{\epsilon}_2}{q \cdot k_2} \right)$$

$$I_{5pA}^{\{1,2\}} = \frac{1}{2} \not{J} \frac{k_1 \cdot k_2}{p \cdot k_1 + p \cdot k_2 - k_1 \cdot k_2} \left(\frac{p \cdot e_1}{p \cdot k_1} - \frac{k_2 \cdot e_1}{k_2 \cdot k_1} \right) \left(\frac{p \cdot e_2}{p \cdot k_2} - \frac{k_1 \cdot e_2}{k_1 \cdot k_2} \right)$$

$$I_{5pB}^{\{1,2\}} = -\frac{1}{2} \not{J} \frac{1}{p \cdot k_1 + p \cdot k_2 - k_1 \cdot k_2} \left(\frac{k_1 \cdot e_2 k_2 \cdot e_1}{k_1 \cdot k_2} - e_1 \cdot e_2 \right)$$

$$I_{5qA}^{\{1,2\}} = \frac{1}{2} \not{J} \frac{k_1 \cdot k_2}{q \cdot k_1 + q \cdot k_2 - k_1 \cdot k_2} \left(\frac{q \cdot e_1}{q \cdot k_1} - \frac{k_2 \cdot e_1}{k_2 \cdot k_1} \right) \left(\frac{q \cdot e_2}{q \cdot k_2} - \frac{k_1 \cdot e_2}{k_1 \cdot k_2} \right)$$

$$I_{5qB}^{\{1,2\}} = -\frac{1}{2} \not{J} \frac{1}{q \cdot k_1 + q \cdot k_2 - k_1 \cdot k_2} \left(\frac{k_1 \cdot e_2 k_2 \cdot e_1}{k_1 \cdot k_2} - e_1 \cdot e_2 \right)$$

$$I_{6B}^{\{1,2\}} = -\frac{1}{4} \frac{k_1 \cdot k_2}{p \cdot k_1 + p \cdot k_2 - k_1 \cdot k_2} \left[+ \left(\frac{p \cdot e_1}{p \cdot k_1} - \frac{k_2 \cdot e_1}{k_1 \cdot k_2} \right) \frac{\not{\epsilon}_2 \not{k}_2}{p \cdot k_2} + \left(\frac{p \cdot e_2}{p \cdot k_2} - \frac{k_1 \cdot e_2}{k_1 \cdot k_2} \right) \frac{\not{\epsilon}_1 \not{k}_1}{p \cdot k_1} \right] \not{J}$$

$$I_{7B}^{\{1,2\}} = -\frac{1}{4} \not{J} \frac{k_1 \cdot k_2}{q \cdot k_1 + q \cdot k_2 - k_1 \cdot k_2} \left[+ \left(\frac{q \cdot e_1}{q \cdot k_1} - \frac{k_2 \cdot e_1}{k_1 \cdot k_2} \right) \frac{\not{k}_2 \not{\epsilon}_2}{q \cdot k_2} + \left(\frac{q \cdot e_2}{q \cdot k_2} - \frac{k_1 \cdot e_2}{k_1 \cdot k_2} \right) \frac{\not{k}_1 \not{\epsilon}_1}{q \cdot k_1} \right]$$

- for the **exponentiation** we have used **separation** into 3 parts only. It is **crystal clear**, also in case of contributions with t -channel W , was very useful for KKMC,
- for PHOTOS kernel, parts $I_3^{\{1,2\}}$, $I_{4p}^{\{1,2\}}$, $I_{4q}^{\{1,2\}}$ were studied separately as well.
- In fact older works on spin amplitudes were used E. Richter-Was Z.Phys.C64:227-240,1994, Z.Phys.C61:323-340,1994.
- Clearly visible but not used for PHOTOS further separation of β_2 terms ...
- Presented above properties of spin amplitudes were used for PHOTOS design to make a choice of phase space parametrization and iteration of consecutive emission kernels that respect numerically as much as possible results of second order amplitudes. Also one want to remain consistent with NLO and exponentiation to all orders.

I had to present project foundation

Important to know: homework completed

Somebody around should be always able to return to details

Important when precision thresholds are crossed

->.<-

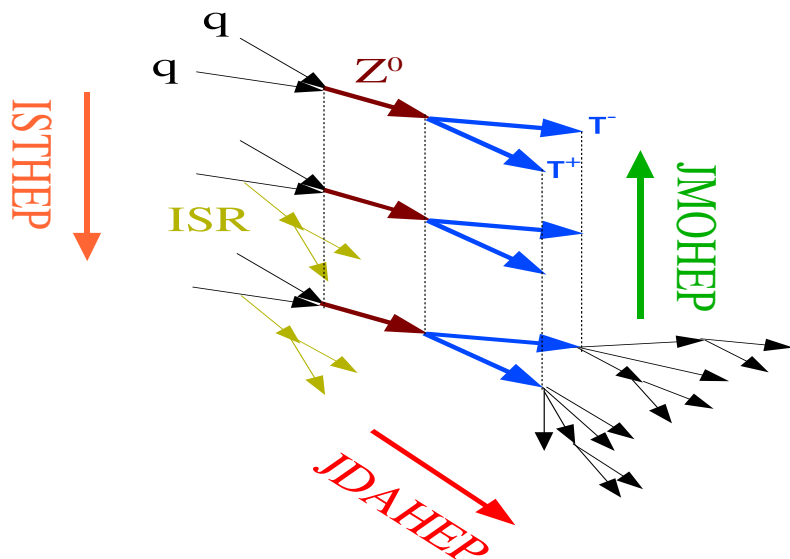
Presentation

- PHOTOS (first versions by E. Barberio, B. van Eijk, Z. W., P. Golonka) is available since 1989, to simulate the effect of radiative corrections in decays.
- Full events combining complicated tree structure of production and subsequent decays have to be fed into PHOTOS, usually with the help of HEPEVT event record of F77 (beta version working with HepMC event record of C++ exist).
- This is often source of technical difficulties as standard is often overruled. Also numerical precision may be an issue.
- At every event decay branching, PHOTOS intervene. With certain probability extra photon may be added and kinematics of other particles adjusted.
- PHOTOS works on four-momenta double precision four-vectors. This creates some 'fun' with numerical stability.
- I will not talk about those time consuming aspects and my frustrations ...

Main References

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- P. Golonka and Z. Was, Eur. Phys. J. C **45** (2006) 97, *ibid*, C50:53-62,2007
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- G. Nanava, Z. Was, Eur.Phys.J.C51:569-583,2007 [Scalar QED and room for Form factors](#)
- G. Nanava, Z. Was, Q. Xu Eur.Phys.J. C70 (2010) 673 [Scalar QED for vector state decay, W decays](#)
- N. Davidson, T. Przedzinski, Z. Was, Comput.Phys.Commun. 199 (2016) 86 [C++](#),
[HepMC, bremsstrahlung of pairs](#)
- S. Antropov, A. Arbuzov, R. Sadykov, Z. Was, Acta Phys.Polon. B48 (2017) 1469
[example of comparison/external tests](#)

Problems With Event Record Content



1. Hard process
2. with shower
3. after hadronization
4. Event record overloaded with physics beyond design → grammar problems.
5. That is potential problem for PHOTOS, TAUOLA, MC-TESTER, ...

Precision: complication arise with spin state of eg. Z⁰

Preparation of analytic benchmarks

- One of the necessary steps was to verify, that once PHOTOS activated, the lepton spectra will be reproduced as far as the LL corrections to required order.
- Formal solution of QED evolution equation can be written as:

$$D(x, \beta_{ch}) = \delta(1-x) + \beta_{ch} P(x) + \frac{1}{2!} \beta_{ch}^2 \{P \times P\}(x) + \frac{1}{3!} \beta_{ch}^3 \{P \times P \times P\}(x) + \dots \quad (17)$$

where $P(x) = \delta(1-x)(\ln \varepsilon + 3/4) + \Theta(1-x-\varepsilon) \frac{1}{x} (1+x^2)/(1-x)$
and $\{P \times P\}(x) = \int_0^1 dx_1 \int_0^1 dx_2 \delta(x - x_1 x_2) P(x_1) P(x_2)$.

- In the LL contributing regions, phase space Jacobian's of PHOTOS trivialize (CPC 1994). The solution above is reproduced by PHOTOS in a straightforward manner, for each of the outgoing charged lines.
- But it is only a limit! **PHOTOS treat phase space exactly and covers all corners.**

- In a similar way (simplifying phase space Jacobians and dropping parts of ME) one can get convinced that distribution of soft photons is as should be for exclusive exponentiation.

- So far we were discussing building blocks, but how does it work in practice?
- Avalanche of numerical results...
- **Important:** We could see that kernel simplified with respect to NLO is sufficient for sub-permille precision. → Much easier to use.
- MC-TESTER by P. Golonka, N. Davidson, T. Przedzinski, Z. Was is used for tests. Idea is to generate histograms of all possible invariant masses which can be constructed from final state momenta.
- One can select events, for example only photons of energy above 1 GeV will be considered as final state.
- On one frame distribution from two program is printed (in logarithmic scale) and their ration (in linear scale).
- No of events of distinct (in the selected way) final states is printed too.

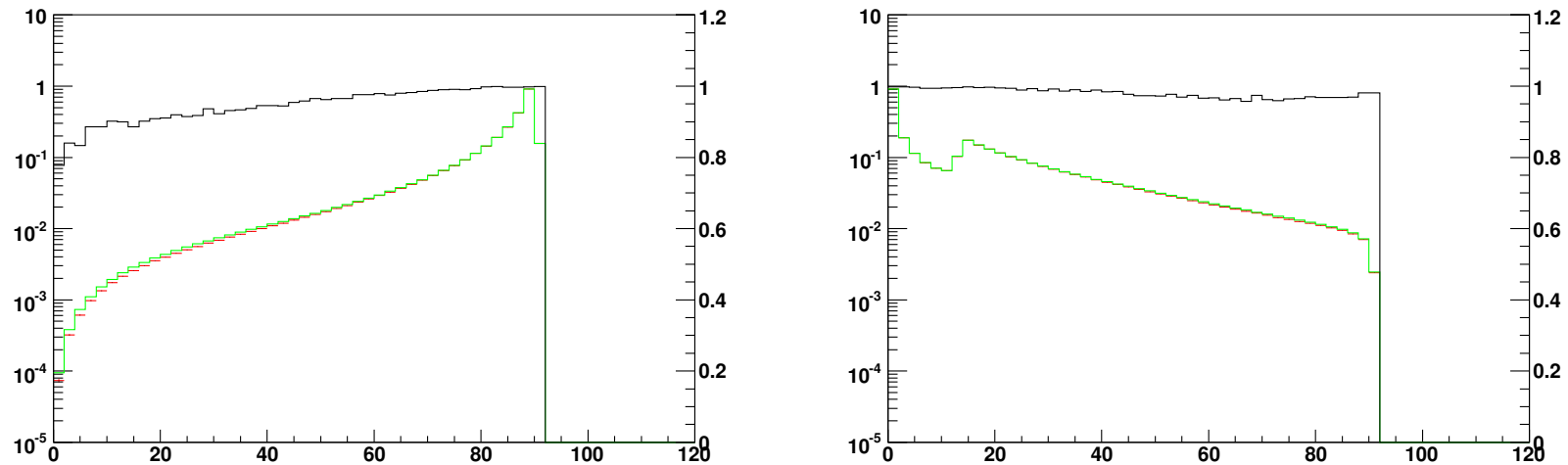


Figure 1: Comparison of standard PHOTOS and KORALZ for single photon emission. In the left frame the invariant mass of the $\mu^+ \mu^-$ pair; $SDP=0.00534$. In the right frame the invariant mass of $\mu^- \gamma$; $SDP=0.00296$. The histograms produced by the two programs (logarithmic scale) and their ratio (linear scale, black line) are plotted in both frames. The fraction of events with hard photon was $17.4863 \pm 0.0042\%$ for KORALZ and $17.6378 \pm 0.0042\%$ for PHOTOS.

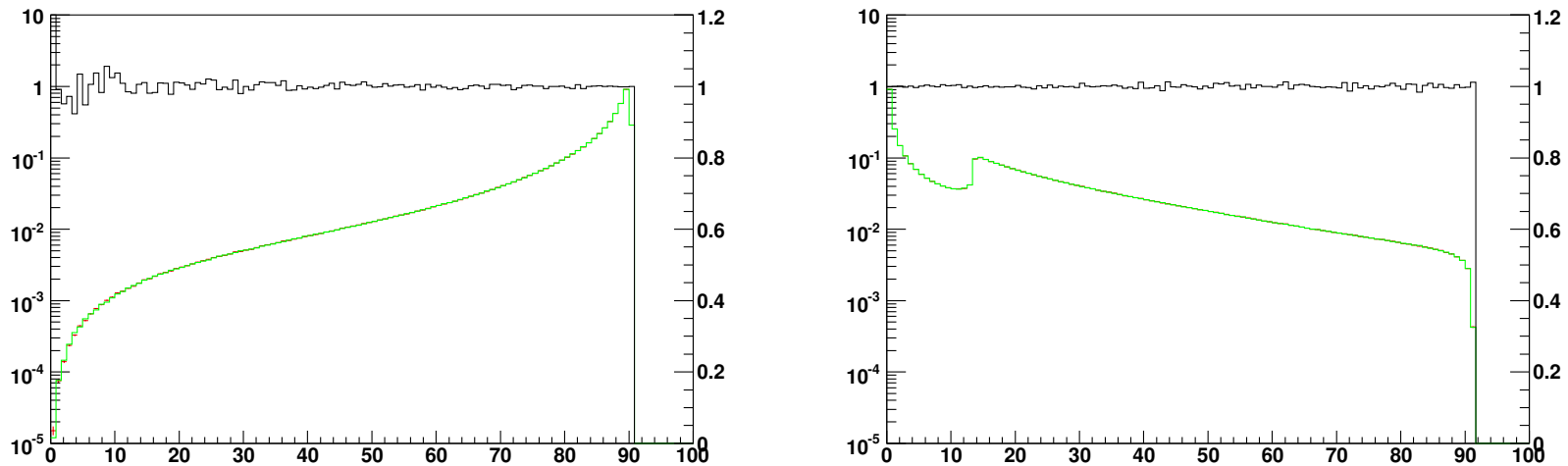


Figure 2: Comparisons of improved PHOTOS and KORALZ for single photon emission. In the left frame the invariant mass of the $\mu^+ \mu^-$ pair. In the right frame the invariant mass of $\mu^- \gamma$ pair is shown. In both cases differences between PHOTOS and KORALZ are below statistical error. The fraction of events with hard photon was $17.4890 \pm 0.0042\%$ for KORALZ and $17.4926 \pm 0.0042\%$ for PHOTOS.

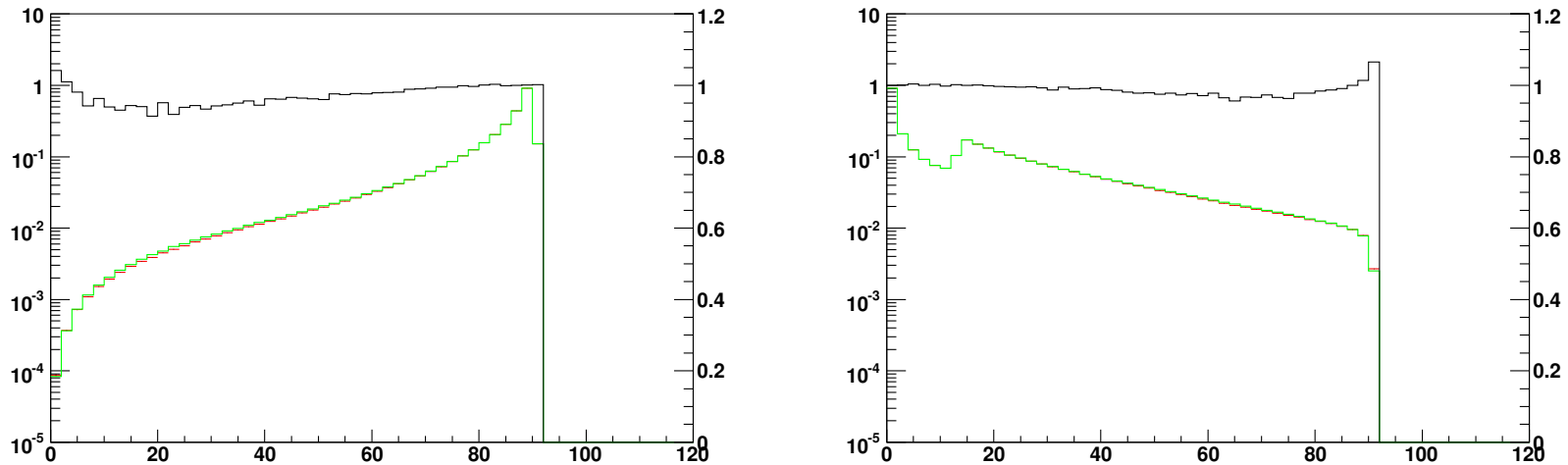


Figure 3: Comparison of standard PHOTOS with multiple photon emission and KKMC with second order matrix element and exponentiation. In the left frame the invariant mass of the $\mu^+ \mu^-$ pair; $SDP=0.00409$. In right frame the invariant mass of the $\mu^- \gamma$ pair; $SDP=0.0025$. The pattern of differences between PHOTOS and KKMC is similar to the one of Fig 1. The fraction of events with hard photon was $16.0824 \pm 0.0040\%$ for KKMC and $16.1628 \pm 0.0040\%$ for PHOTOS.

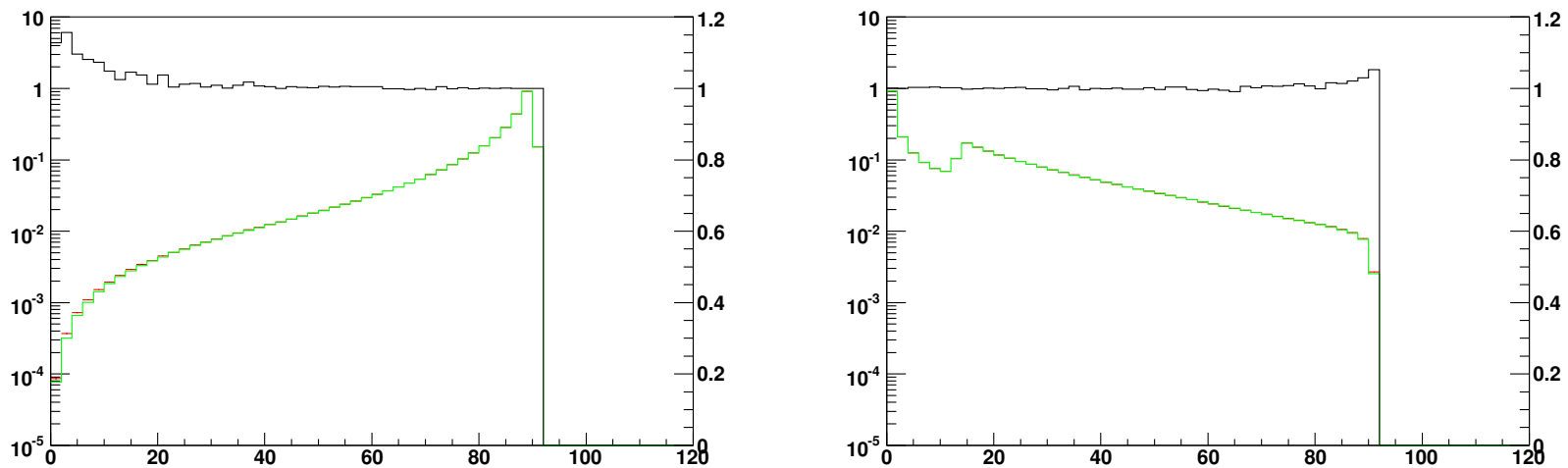


Figure 4: Comparisons of improved PHOTOS with multiple photon emission and KKMC with second order matrix element and exponentiation. In the left frame the invariant mass of the $\mu^+ \mu^-$ pair; $SDP=0.0000249$. In the right frame the invariant mass of the $\mu^- \gamma$ pair; $SDP=0.0000203$. The fraction of events with hard photon was $16.0824 \pm 0.004\%$ for KKMC and $16.0688 \pm 0.004\%$ for PHOTOS.

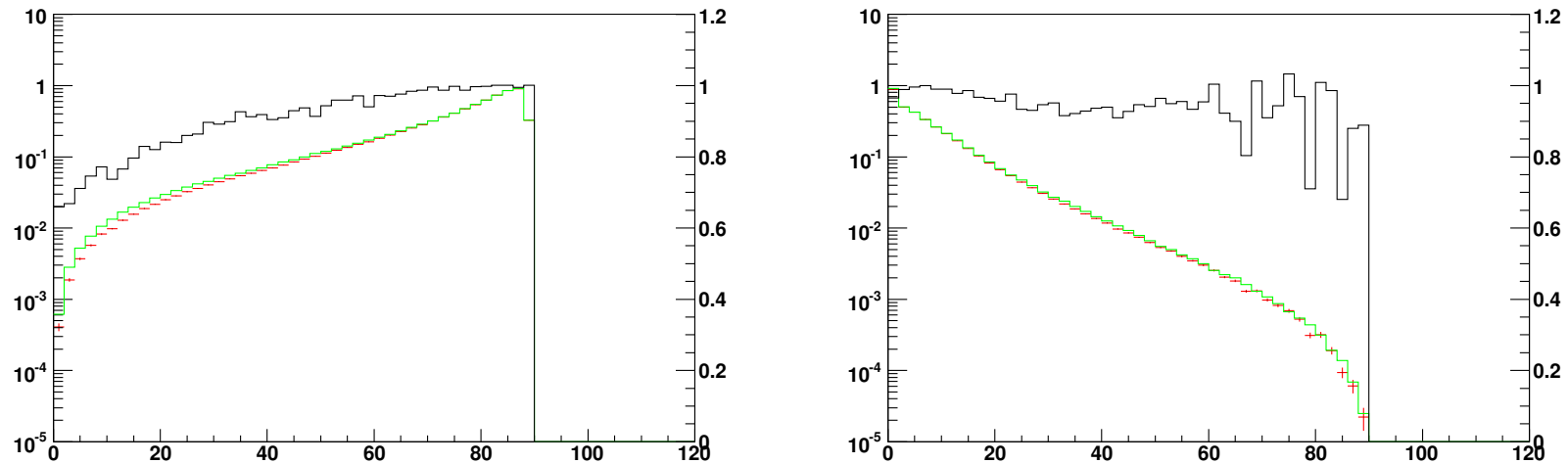


Figure 5: Comparisons of standard PHOTOS with multiple photon emission and KKMC with second order matrix element and exponentiation. In the left frame the invariant mass of the $\mu^+ \mu^-$ pair; $SDP= 0.00918$. In the right frame the invariant mass of the $\gamma\gamma$ pair; $SDP=0.00268$. The fraction of events with two hard photons was $1.2659 \pm 0.0011\%$ for KKMC and $1.2952 \pm 0.0011\%$ for PHOTOS.

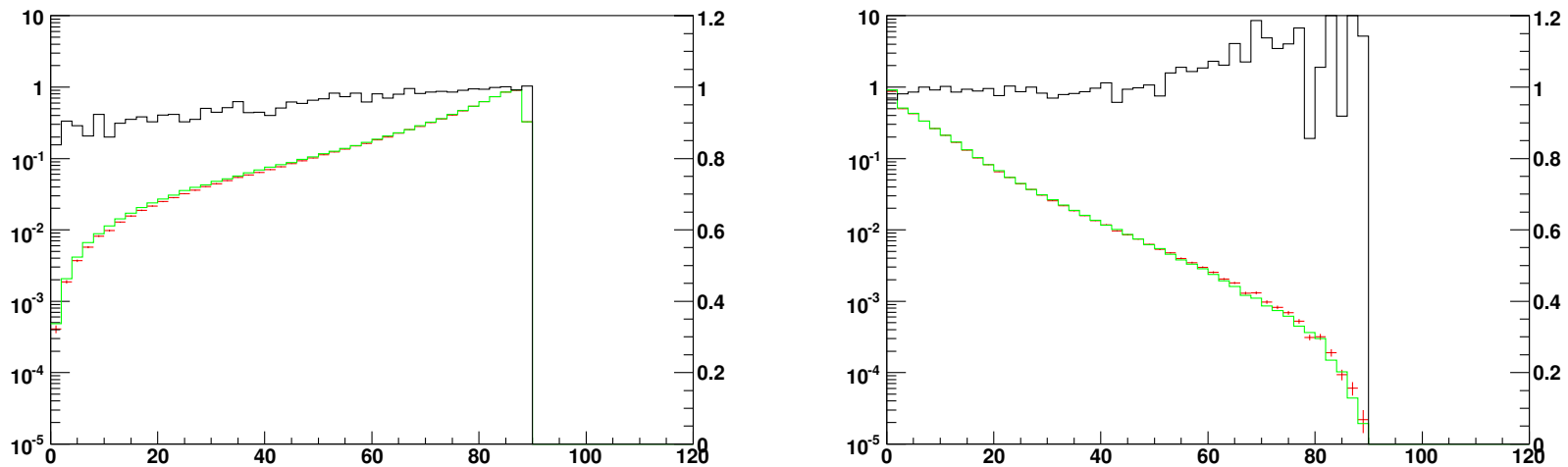


Figure 6: Comparisons of improved PHOTOS with multiple photon emission and KKMC with second order matrix element and exponentiation. In the left frame the invariant mass of the $\mu^+ \mu^-$ pair; $SDP= 0.00142$. In the right frame the invariant mass of the $\gamma\gamma$; $SDP=0.00293$. The fraction of events with two hard photons was $1.2659 \pm 0.0011\%$ for KKMC and $1.2868 \pm 0.0011\%$ for PHOTOS.

Again Matrix Element for Z decay:

- Our discussion of structure for double emission amplitudes was started from the details of the single photon one
- The same is true for amplitudes of other processes. We have to check if they are indeed similar to these of Z decay.
- Only if the structure match we can expect our discussion of multiemission to apply as well:

$$I = I^A + I^B + I^C$$

$$I = \mathcal{J} \left[\left(\frac{p \cdot e_1}{p \cdot k_1} - \frac{q \cdot e_1}{q \cdot k_1} \right) \right] - \left[\frac{1}{2} \frac{\not{\epsilon}_1 \not{k}_1}{p \cdot k_1} \right] \mathcal{J} + \mathcal{J} \left[\frac{1}{2} \frac{\not{\epsilon}_1 \not{k}_1}{q \cdot k_1} \right]$$

three gauge invariant parts, I^A is eikonal; I^B, I^C carry collinear contrib from p and q

QED for $W \rightarrow l\nu_l\gamma$: I^A , I^B and non-leading

$$\begin{aligned}
 M_{\lambda, \lambda_\nu, \lambda_l}^\sigma(k, Q, p_\nu, p_l) &= \left[\frac{Q_l}{2k \cdot p_l} b_\sigma(k, p_l) - \frac{Q_W}{2k \cdot Q} (b_\sigma(k, p_l) + b_\sigma(k, p_\nu)) \right] B_{\lambda_l, \lambda_\nu}^\lambda(p_l, Q, p_\nu) \\
 &+ \frac{Q_l}{2k \cdot p_l} \sum_{\rho=\pm} U_{\lambda_l, \rho}^\sigma(p_l, m_l, k, 0, k, 0) B_{\rho, -\lambda_\nu}^\lambda(k, Q, p_\nu) \\
 &- \frac{Q_W}{2k \cdot Q} \sum_{\rho=\pm} \left(B_{\lambda_l, -\rho}^\lambda(p_l, Q, k) U_{-\rho, -\lambda_\nu}^\sigma(k, 0, k, 0, p_\nu, 0) \right. \\
 &\quad \left. + U_{\lambda_l, \rho}^\sigma(p_l, m_l, k, 0, k, 0) B_{\rho, -\lambda_\nu}^\lambda(k, Q, p_\nu) \right), \tag{18}
 \end{aligned}$$

$$\begin{aligned}
 B_{\lambda_1, \lambda_2}^\lambda(p_1, Q, p_2) &\equiv \frac{g}{2\sqrt{2}} \bar{u}(p_1, \lambda_1) \hat{\epsilon}_W^\lambda(Q) (1 + \gamma_5) v(p_2, \lambda_2), \\
 U_{\lambda_1, \lambda_2}^\sigma(p_1, m_1, k, 0, p_2, m_2) &\equiv \bar{u}(p_1, \lambda_1) \hat{\epsilon}_\gamma^\sigma(k) u(p_2, \lambda_2), \\
 \delta_{\lambda_1 \lambda_2} b_\sigma(k, p) &\equiv U_{\lambda_1, \lambda_2}^\sigma(p, m, k, 0, p, m), \tag{19}
 \end{aligned}$$

Q_l and Q_W are the electric charges of the fermion l and the W boson, respectively, in units of the positron charge, $\epsilon_\gamma^\sigma(k)$ and $\epsilon_W^\lambda(Q)$ denote respectively the polarization vectors of the photon and the W boson. An expression of the function $U_{\lambda_1, \lambda_2}^\sigma$ in terms of the massless spinors.

Scalar QED: $B \rightarrow K\pi$ decays – pure I^A

- The one-loop QED correction to the decay width can be represented as the sum of the Born contribution with the contributions due to virtual loop diagrams and soft and hard photon emissions.

$$d\Gamma^{\text{Total}} = d\Gamma^{\text{Born}} \left\{ 1 + \frac{\alpha}{\pi} \left[\delta^{\text{Soft}}(m_\gamma, \omega) + \delta^{\text{Virt}}(m_\gamma, \mu_{UV}) \right] \right\} + d\Gamma^{\text{Hard}}(\omega)$$

- where for **Neutral meson decay channels**, hard photon contribution:

$$d\Gamma^{\text{Hard}} = |A^{\text{Born}}|^2 4\pi\alpha \left(q_1 \frac{k_1 \cdot \epsilon}{k_1 \cdot k_\gamma} - q_2 \frac{k_2 \cdot \epsilon}{k_2 \cdot k_\gamma} \right)^2 dLips_3(P \rightarrow k_1, k_2, k_\gamma)$$

- for **Charged meson decay channels**, hard photon contribution:

$$d\Gamma^{\text{Hard}} = |A^{\text{Born}}|^2 4\pi\alpha \left(q_1 \frac{k_1 \cdot \epsilon}{k_1 \cdot k_\gamma} - q \frac{P \cdot \epsilon}{P \cdot k_\gamma} \right)^2 dLips_3(P \rightarrow k_1, k_2, k_\gamma)$$

Scalar QED for $\gamma^ \rightarrow \pi^+ \pi^- \gamma$: I^A and non-leading*

- This case is different, because of spin structure. One can not make spin of initial state out of internal spin of outgoing particles.

$$H^\mu = \frac{e^2 F_{2\pi}(p^2)}{p^2} \left\{ (q_1 + k - q_2)^\mu \frac{q_1 \cdot \epsilon^*}{q_1 \cdot k} + (q_2 + k - q_1)^\mu \frac{q_2 \cdot \epsilon^*}{q_2 \cdot k} - 2\epsilon^{*\mu} \right\}$$

- As in case of Z decay one can separate spin amplitude into gauge invariant parts ($C = \frac{e^2 F_{2\pi}(p^2)}{p^2}$):

$$H_I^\mu = C (q_1 - q_2)^\mu \left(\frac{q_1 \cdot \epsilon^*}{q_1 \cdot k} - \frac{q_2 \cdot \epsilon^*}{q_2 \cdot k} \right), H_{II}^\mu = C \left(k^\mu \left(\frac{q_1 \cdot \epsilon^*}{q_1 \cdot k} + \frac{q_2 \cdot \epsilon^*}{q_2 \cdot k} \right) - 2\epsilon^{*\mu} \right), \quad (20)$$

- This can be improved with the following change:

$$H_{I'}^\mu = C \left((q_1 - q_2)^\mu + k^\mu \frac{q_2 \cdot k - q_1 \cdot k}{q_2 \cdot k + q_1 \cdot k} \right) \left(\frac{q_1 \cdot \epsilon^*}{q_1 \cdot k} - \frac{q_2 \cdot \epsilon^*}{q_2 \cdot k} \right), \quad (21)$$

$$H_{II'}^\mu = C \left(\frac{k^\mu}{q_2 \cdot k + q_1 \cdot k} (q_1 \cdot \epsilon^* + q_2 \cdot \epsilon^*) - \epsilon^{*\mu} \right). \quad (22)$$

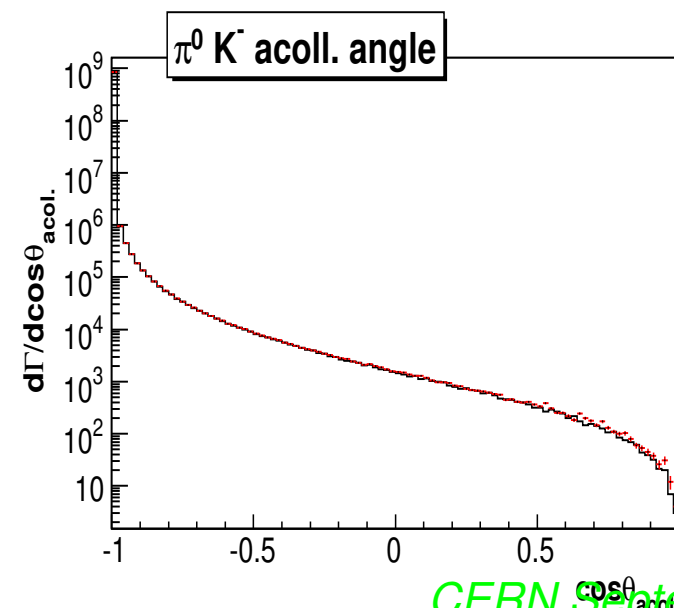
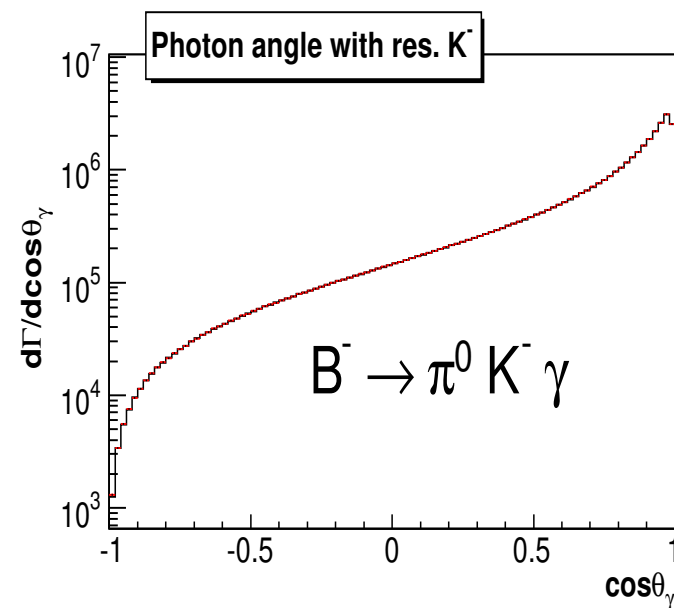
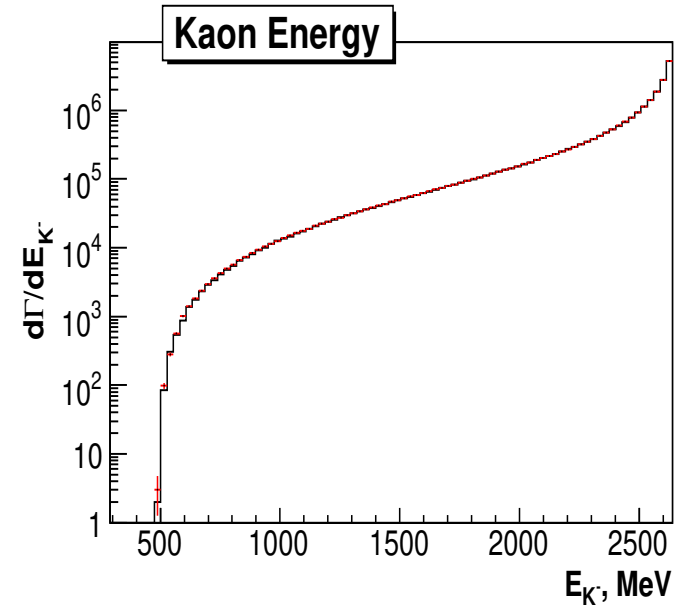
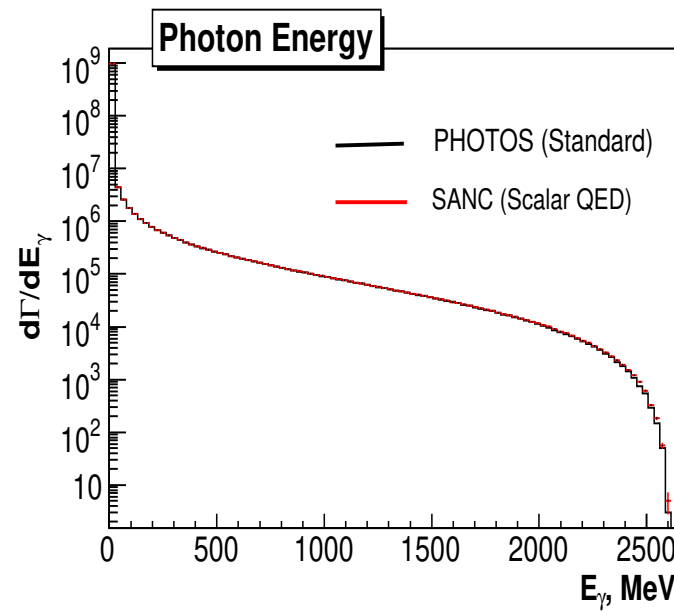
- In the second case non-eikonal term is free of collinear logarithm, but is non trivial and contributes 0.2 % to total rate, thus can be numerically studied!

Matrix Element (anything useful seen?):

- We have seen that in all cases terms I^A , I^B , I^C appear
- These are the only ones which carry soft or collinear contributions
- That is why universal weight of Photos could be defined.
- That is also why the solution defined from Z amplitudes work for other processes as well
- part of NLO proof.
- Tests confirm that NLO complete kernel (process dependent) is not necessary even for sub-permille precision level \rightarrow good for users.

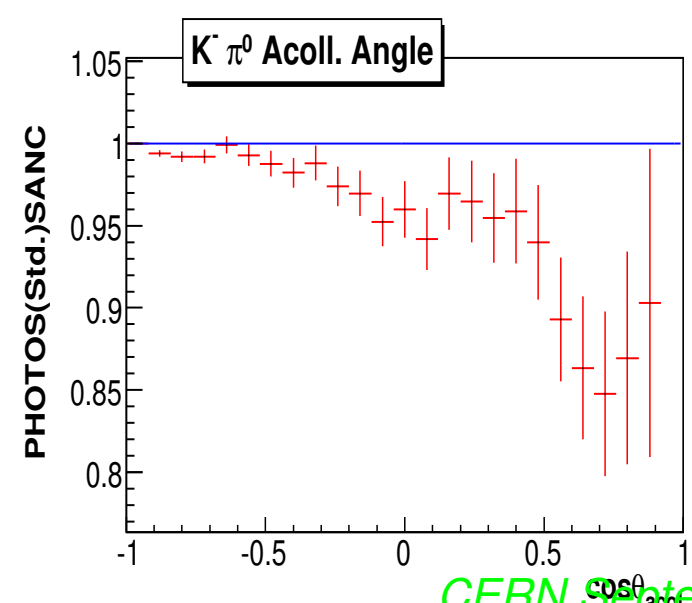
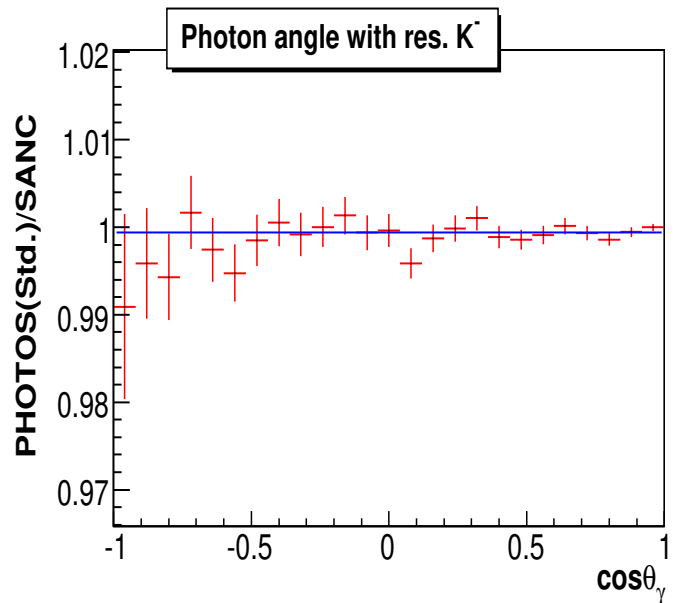
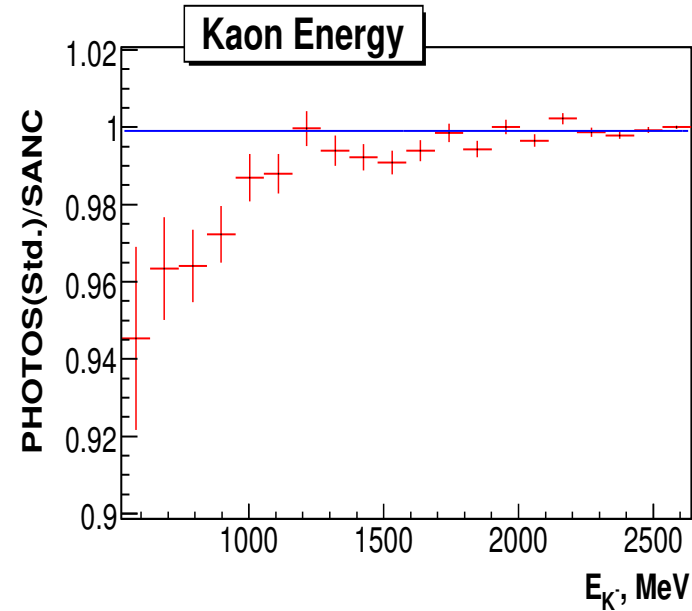
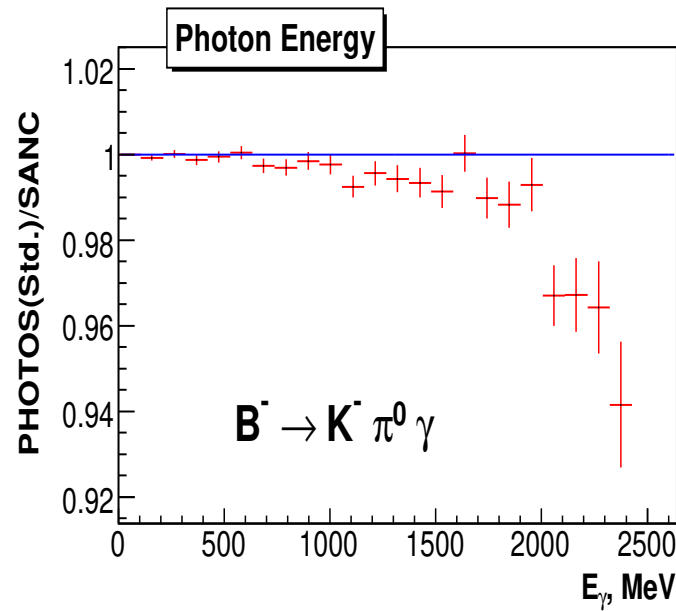
4) Other processes single photon ME , from questionable models but precisely. 61

$B^- \rightarrow \pi^0 K^-$: standard PHOTOS looks good. but ...



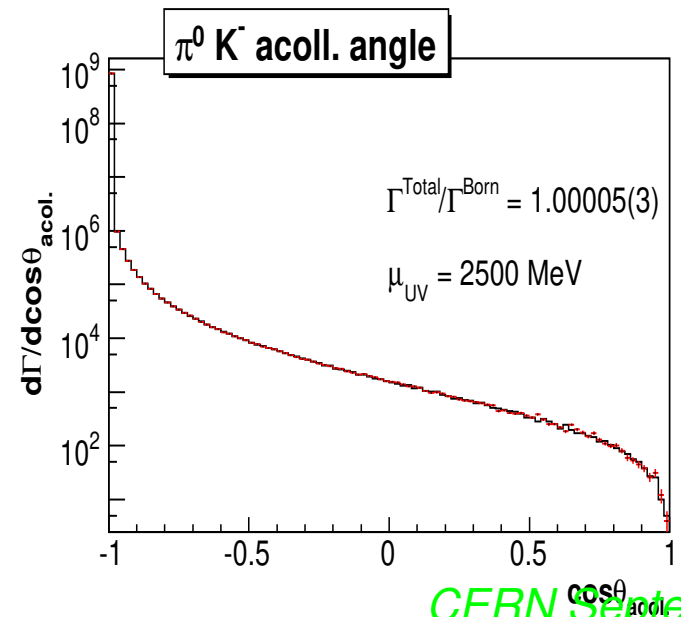
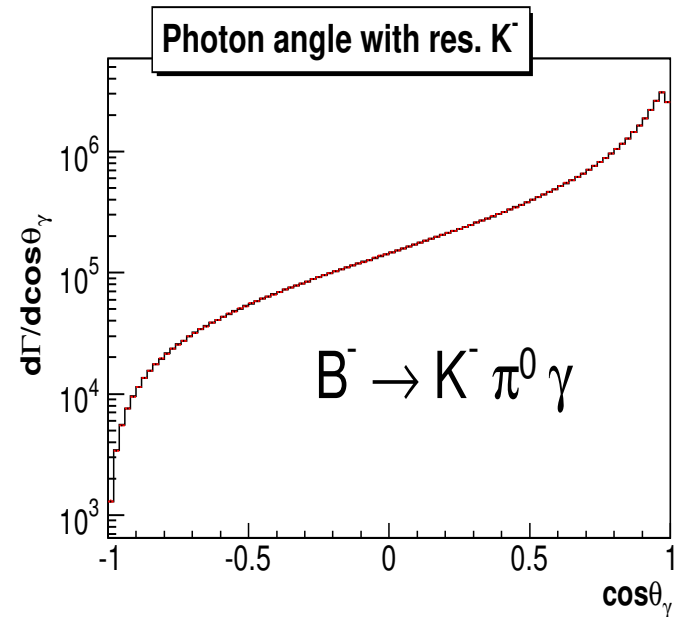
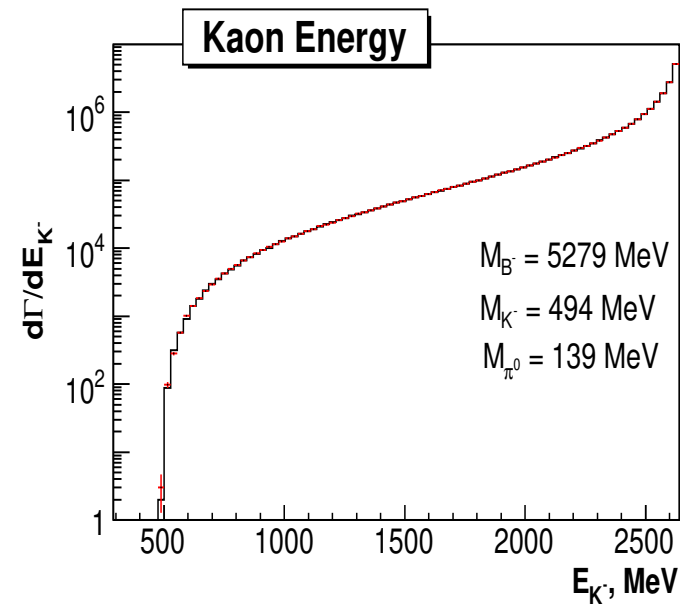
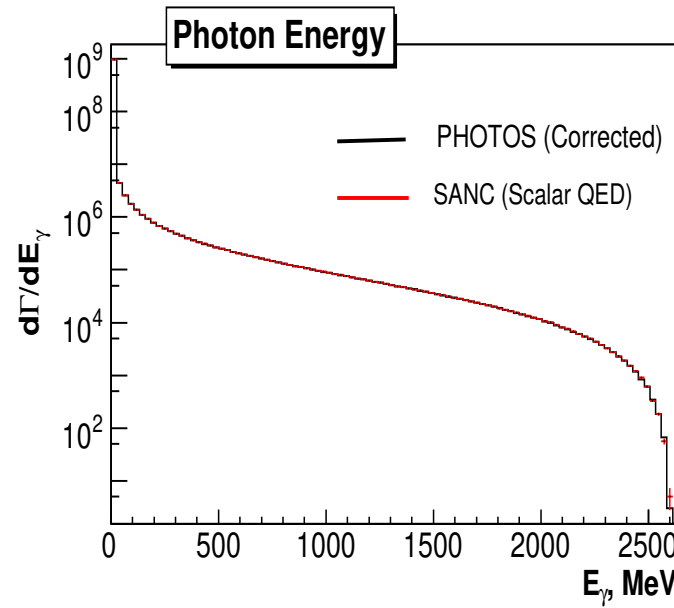
4) Other processes single photon ME , from questionable models but precisely. 62

$B^- \rightarrow \pi^0 K^-$ · standard PHOTOS not perfect

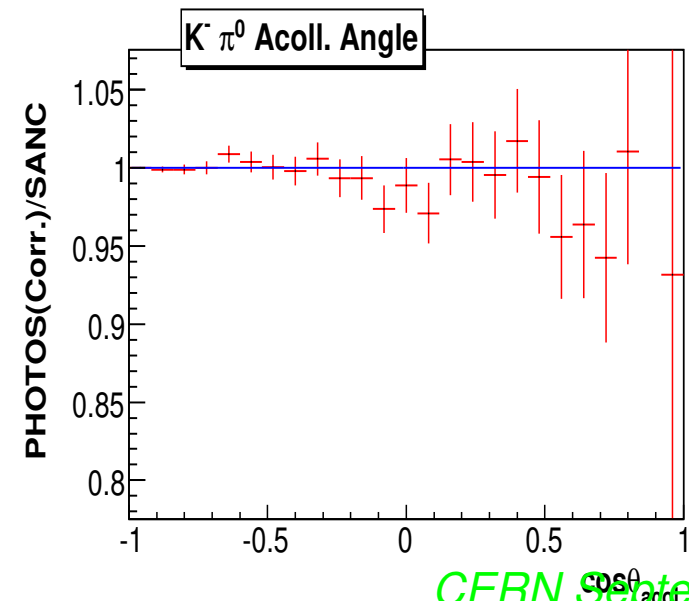
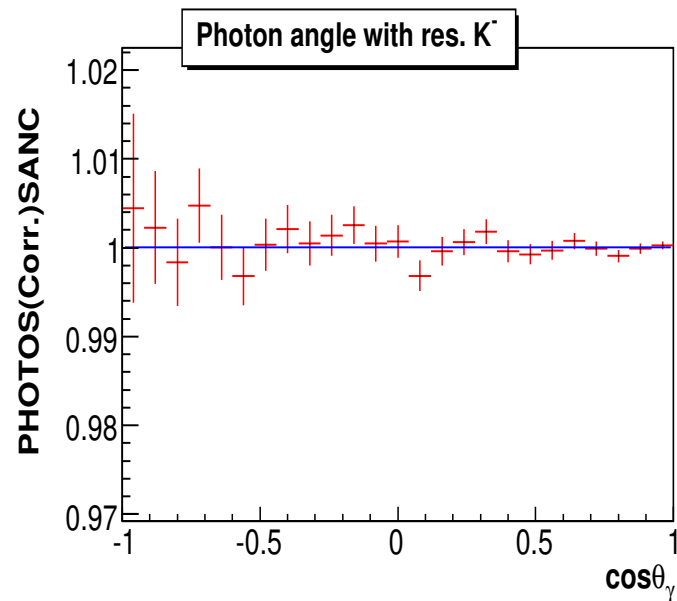
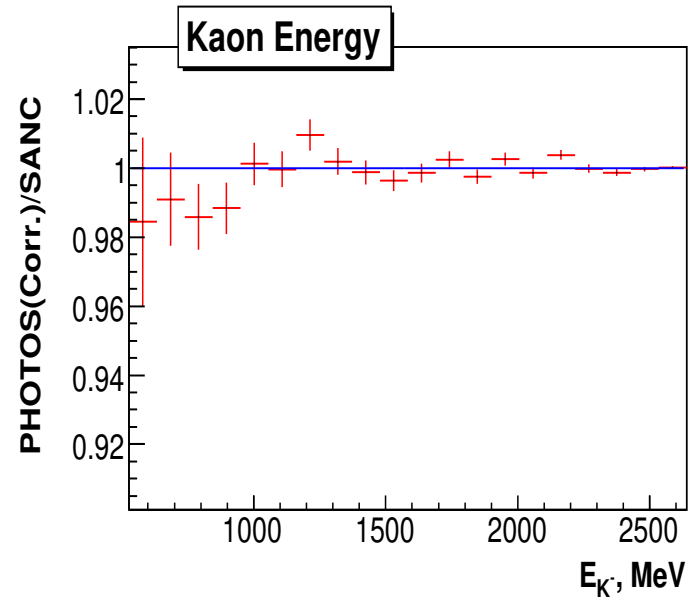
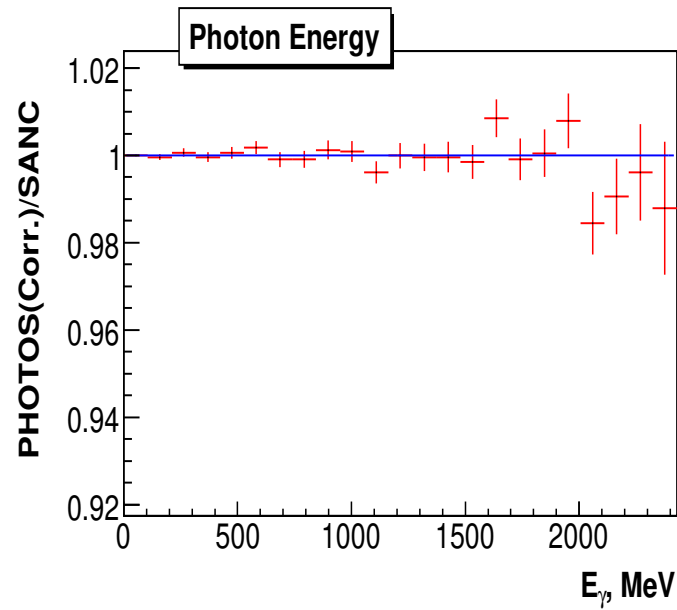


4) Other processes single photon ME , from questionable models but precisely. 63

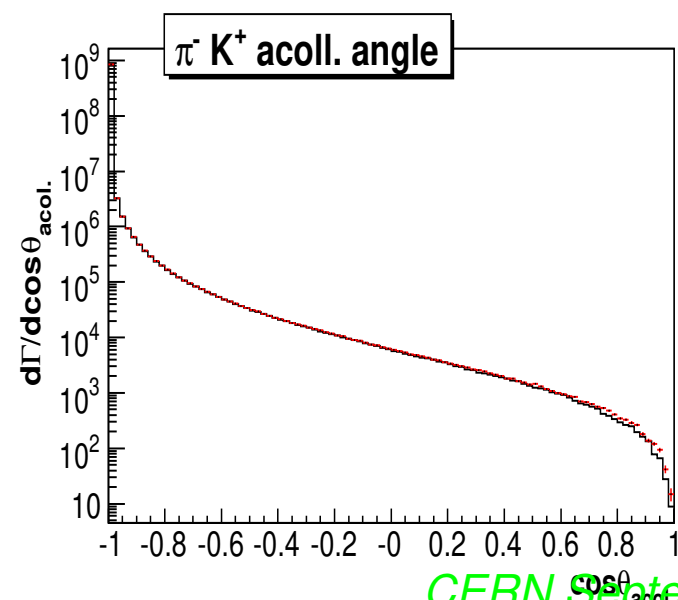
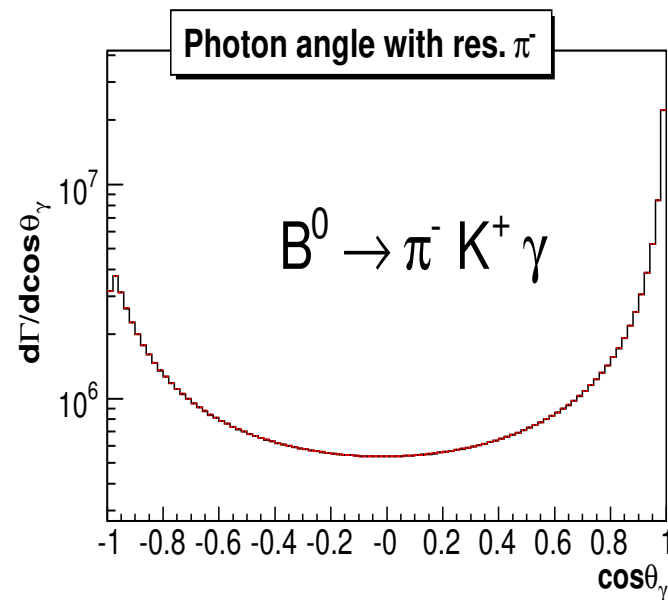
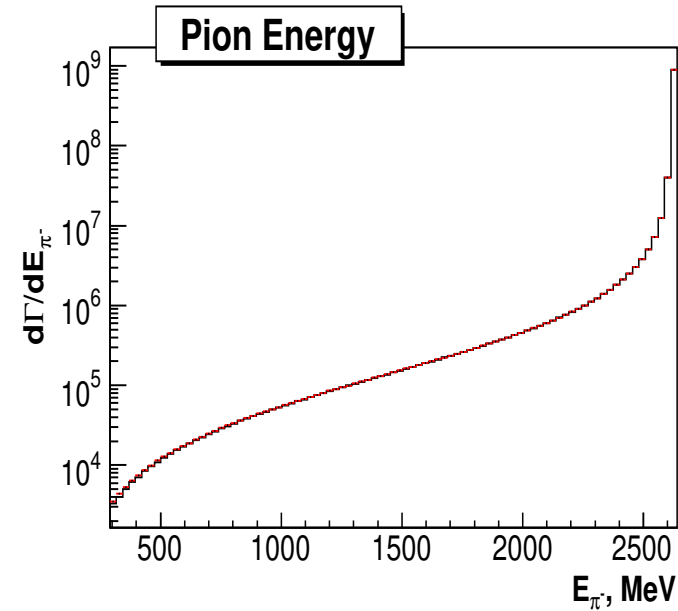
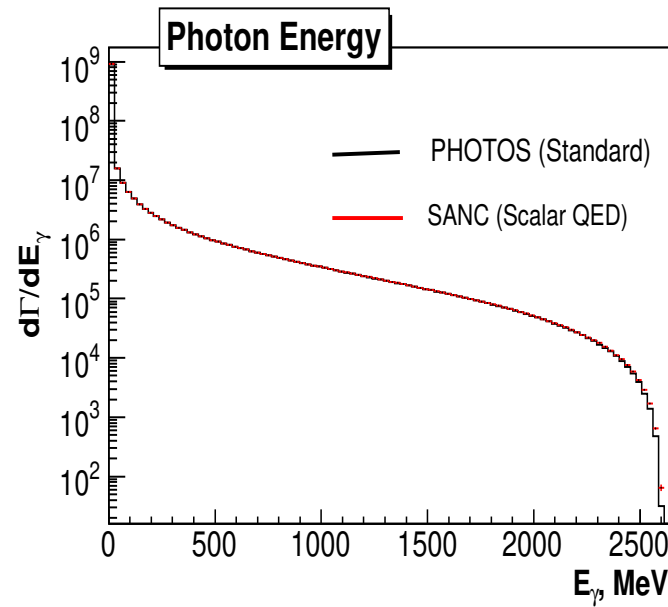
$B^- \rightarrow \pi^0 K^- \cdot NI \ O \ improved \ PHOTOS$ Looks good



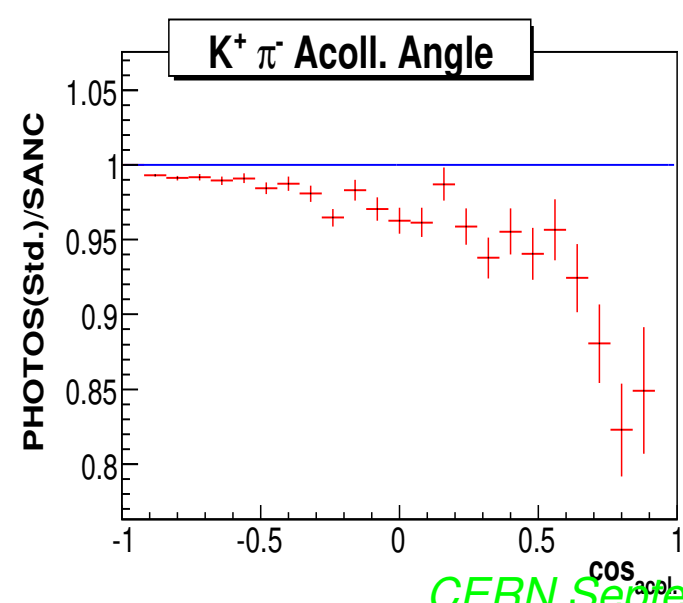
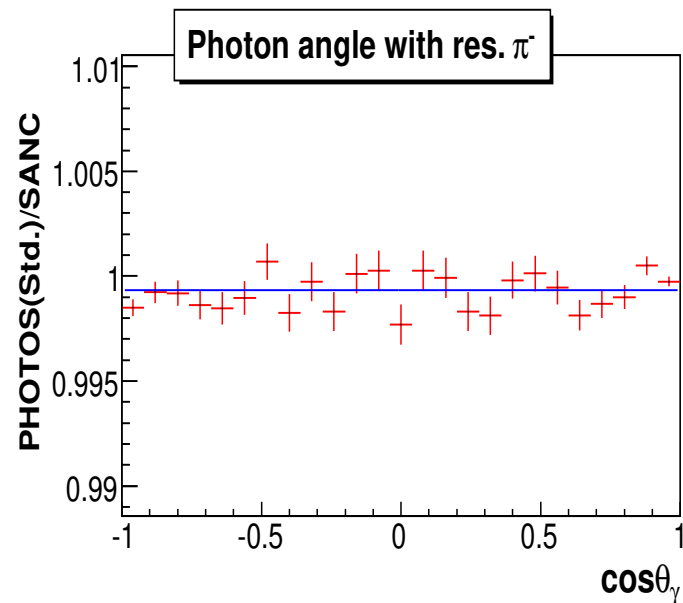
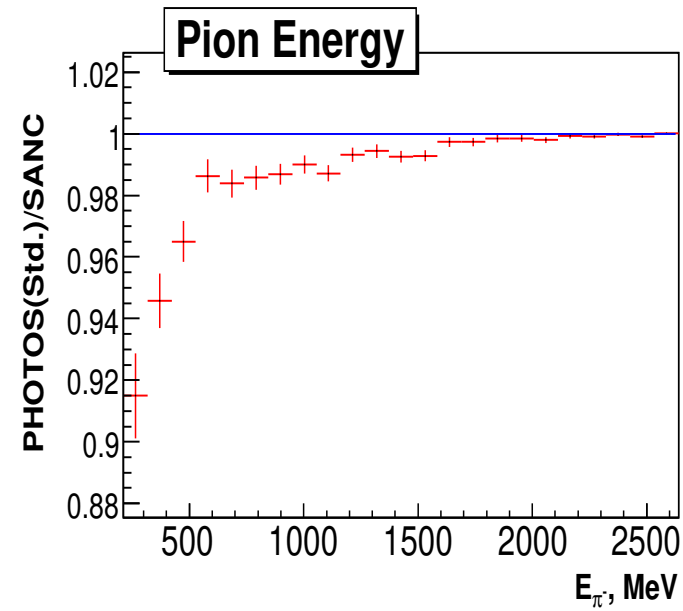
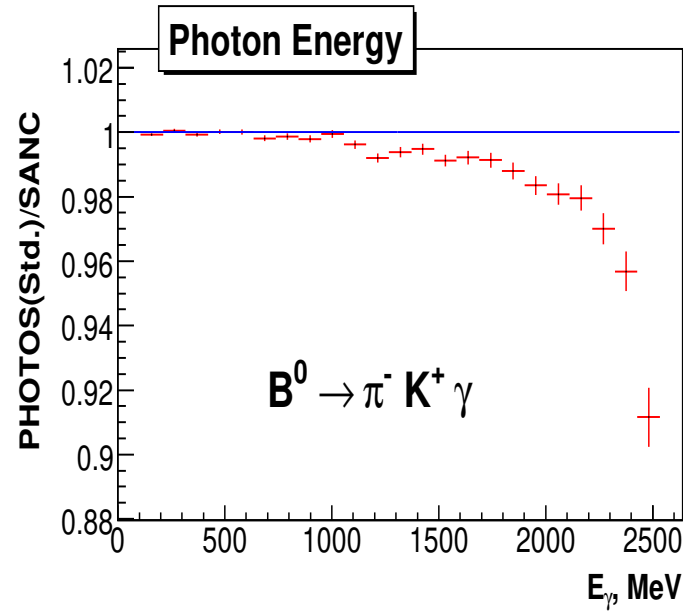
$B^- \rightarrow \pi^0 K^-$: NLO improved PHOTOS ... and is good.



$B^0 \rightarrow \pi^- K^+ : \text{standard PHOTOS}$ Looks good ...

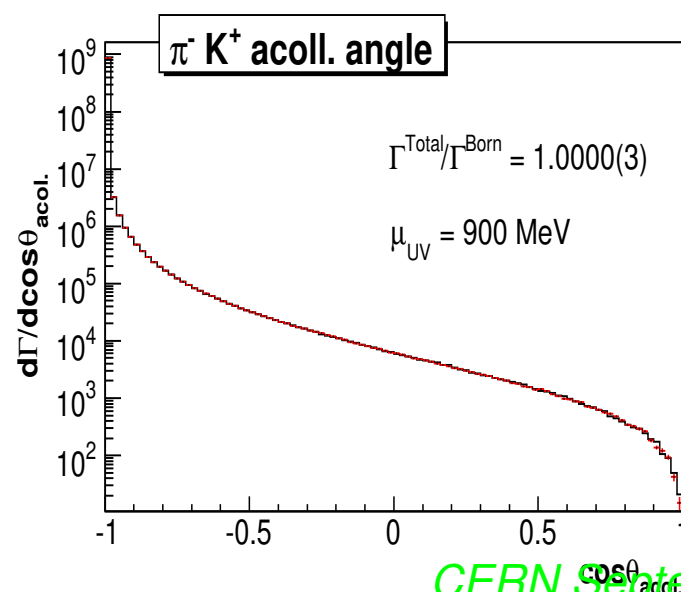
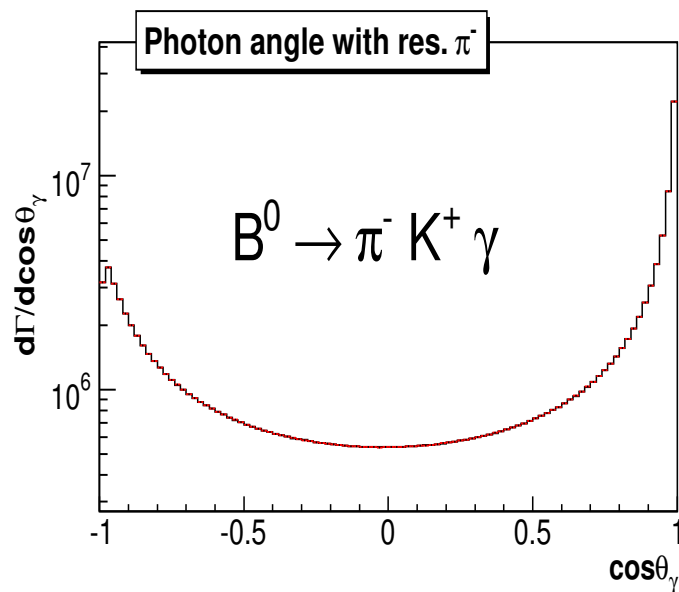
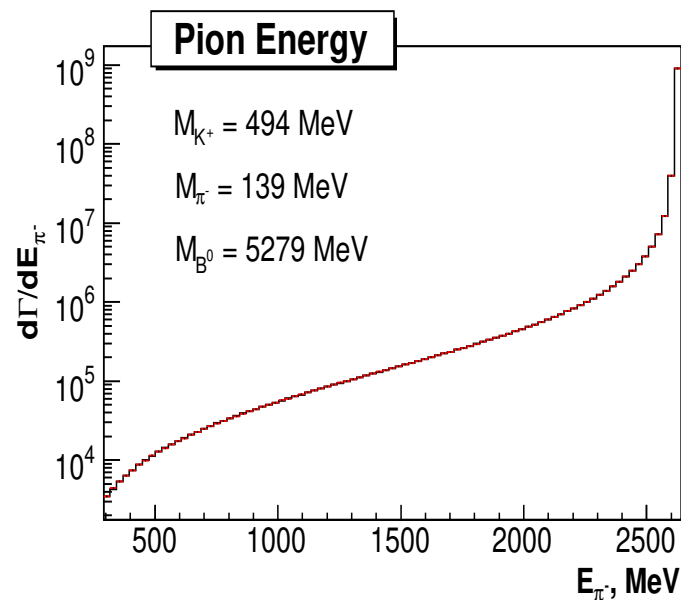
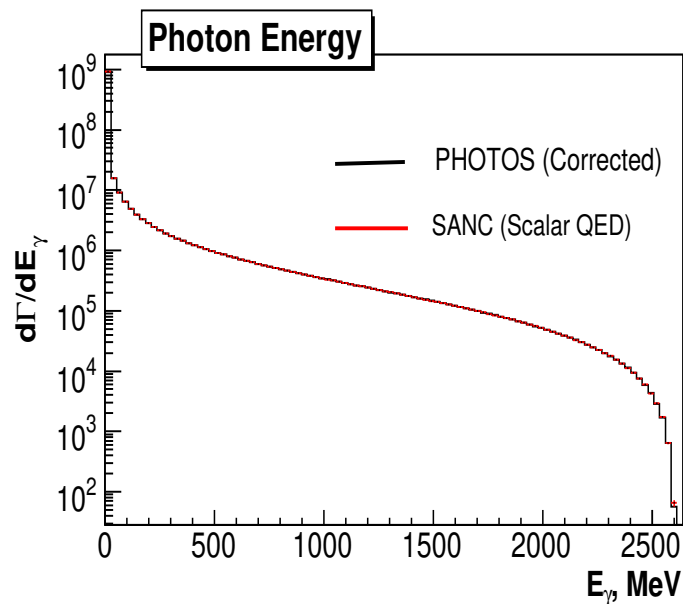


$B^0 \rightarrow \pi^- K^+ \gamma$: standard PHOTOS ... but not perfect.



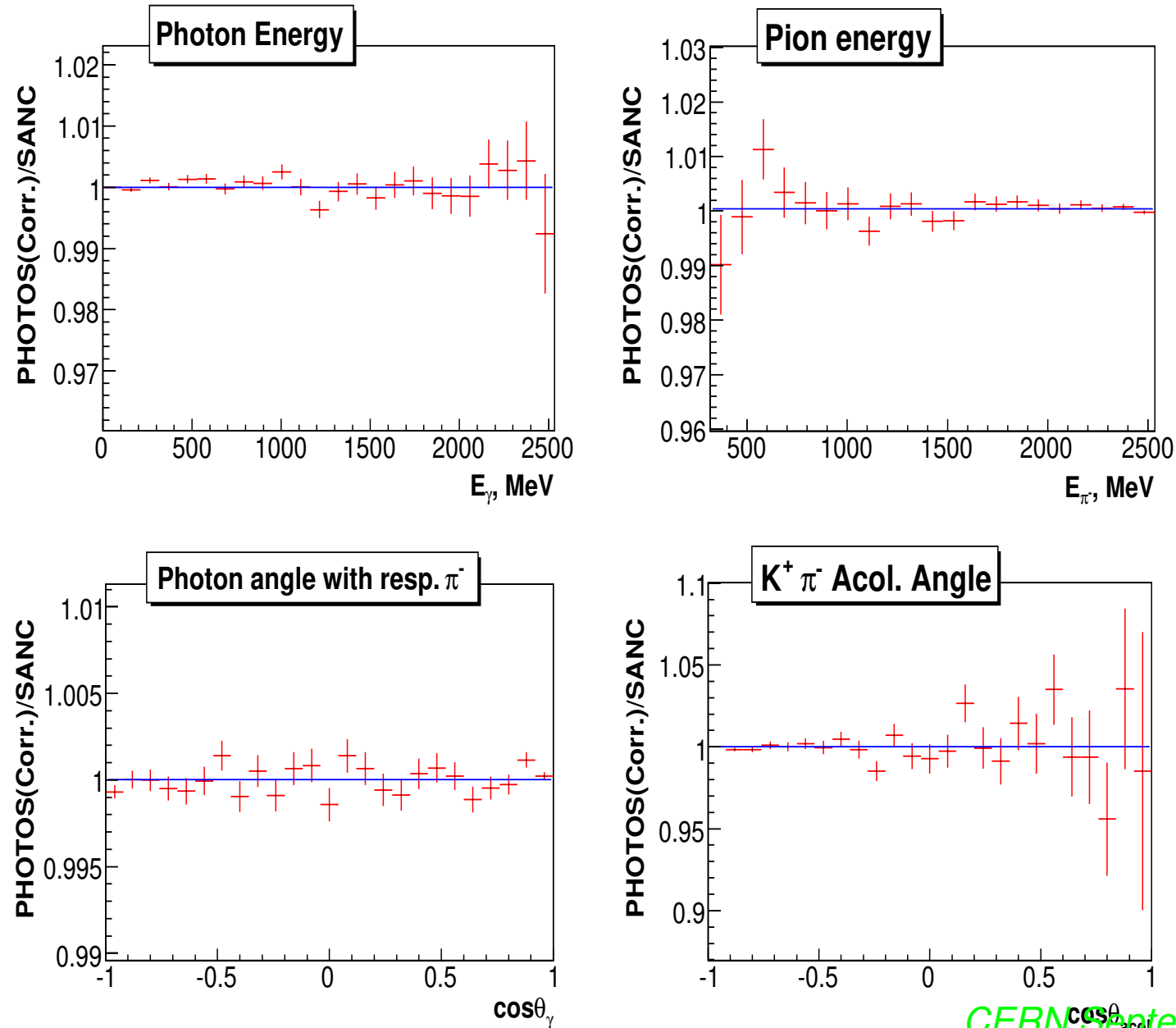
4) Other processes single photon ME , from questionable models but precisely. 67

$B^0 \rightarrow \pi^- K^+ : NLO \text{ improved PHOTOS}$ Looks good ...



4) Other processes single photon ME , from questionable models but precisely. 68

$B^0 \rightarrow \pi^- K^+$; NLO improved PHOTOS ... also perfect !



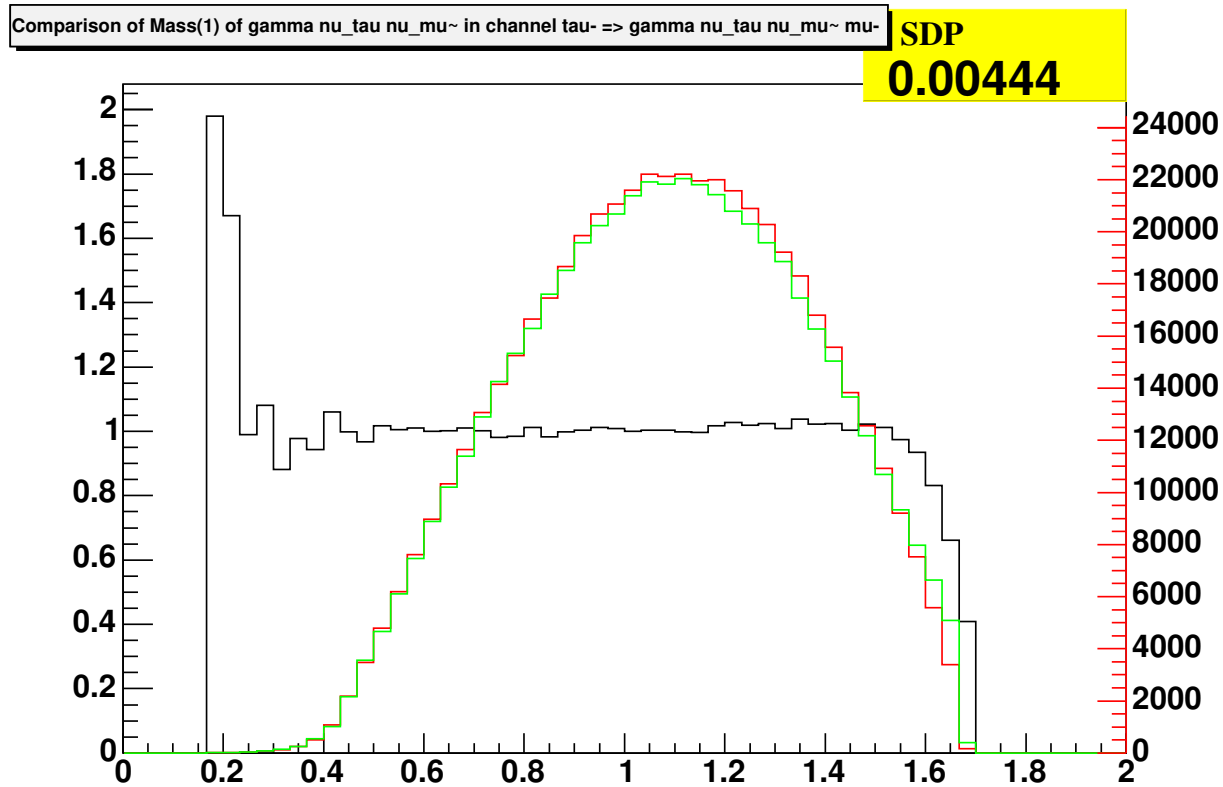
Other processes

Results for W , H , and $\gamma^* \rightarrow \pi^+ \pi^-$ are of similar quality.

Tests for $K^\pm \rightarrow l^\pm \nu \pi^+ \pi^-$ are going on. In general for more than two body decays effort of tests was never matching the one presented earlier.

$\tau \rightarrow l\nu\bar{\nu}(\gamma)$ PHOTOS vs TAUOLA

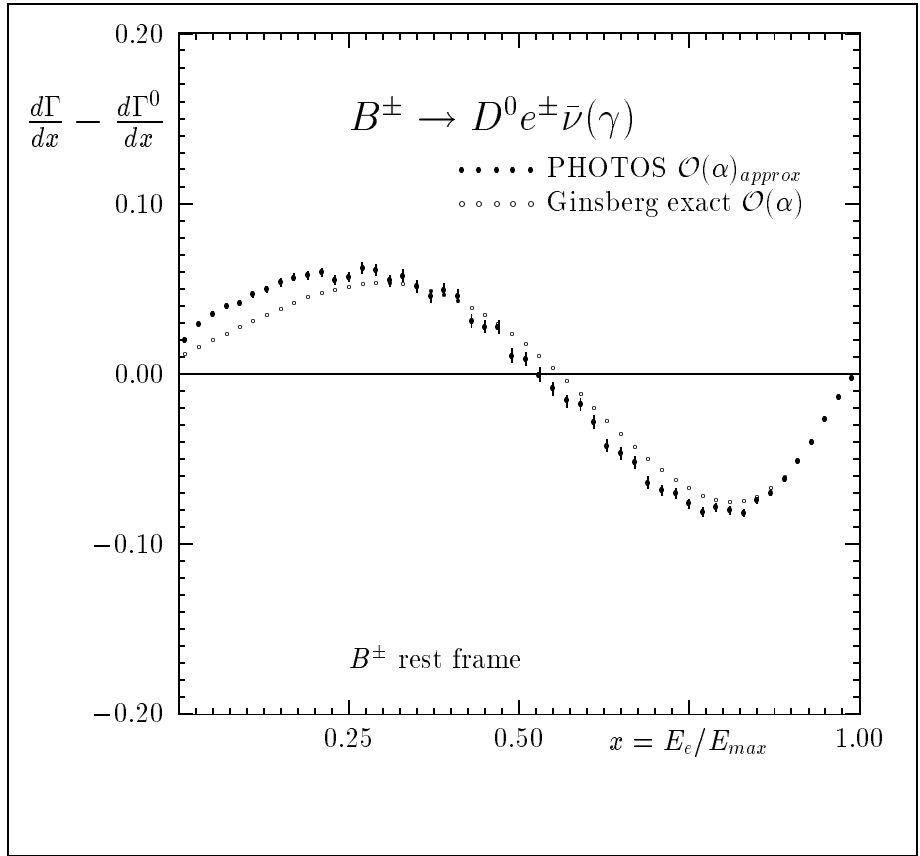
Plot of worst agreement for the channel. Distribution of $\gamma\nu_\tau\nu_\mu$ system mass is shown .



Also the fraction of events with photon above threshold agrees better than permille level.

In TAUOLA complete matrix element, comparison test PHOTOS approximations and design.

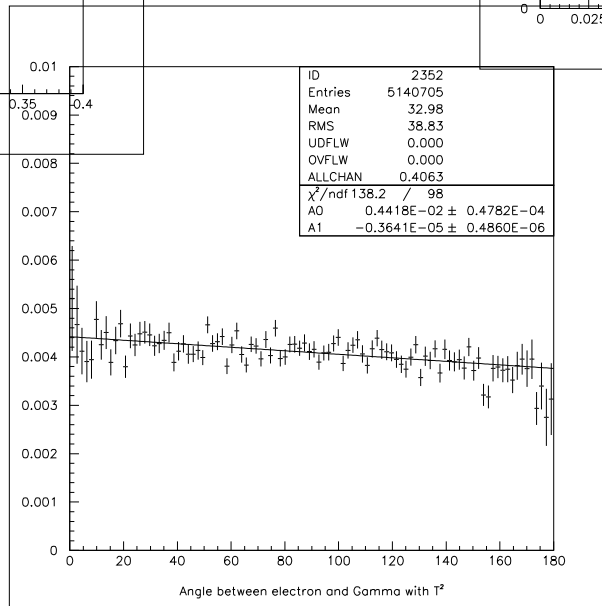
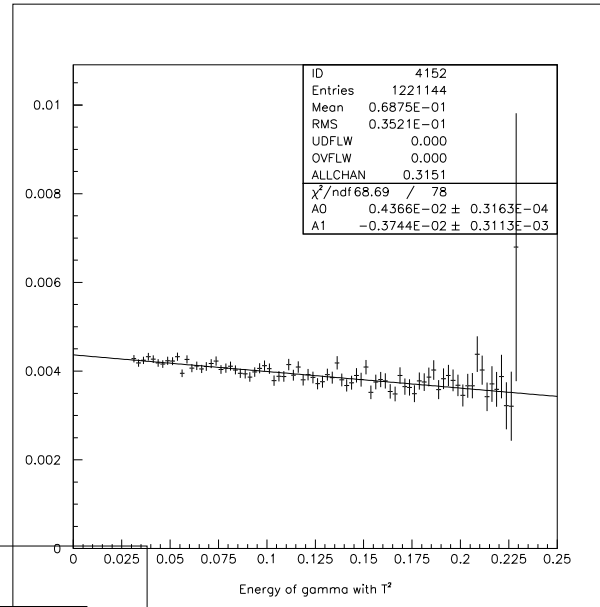
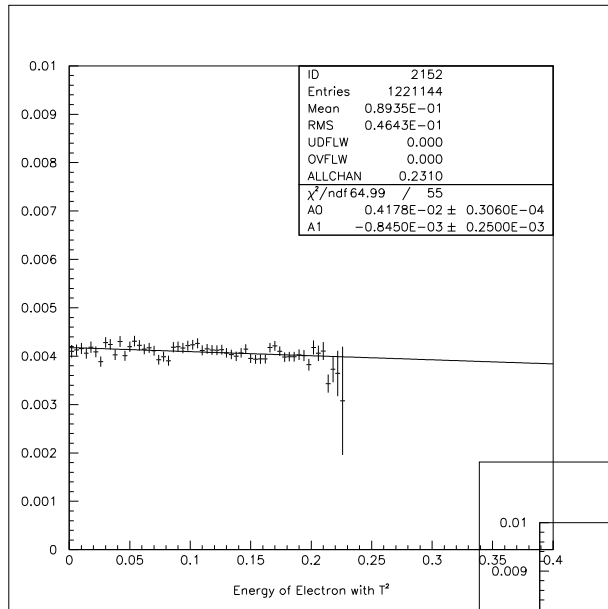
Phys. Lett, B 303 (1993) 163-169



Radiative correction to the decay rate ($d\Gamma/dx - d\Gamma^0/dx$) for $B^\pm \rightarrow D^0 e^\pm \bar{\nu}(\gamma)$ in the B^\pm rest frame. Open circles are from the exact analytical formula [2], points with the marked statistical errors from PHOTOS applied to JETSET 7.3. A total of 10^7 events have been generated. The results are given in units of $(G_\mu^2 m_B^5 / 32\pi^3) N_\eta |V_{cb}|^2 |f_+^D|^2$, where $N_\eta = \eta^5 \int_0^1 x^2 (1-x)^2 / (1-\eta x) dx$ and $\eta = 1 - m_D^2/m_B^2$.

- “QED bremsstrahlung in semileptonic B and leptonic τ decays” by E. Richter-Was.
- agreement up to 1%
- disagreement in the low- x region due to missing sub-leading terms
- study performed in 1993.

$K \rightarrow \pi e \nu(\gamma)$ PHOTOS w/Interf vs Gasser



This was OK in 2005

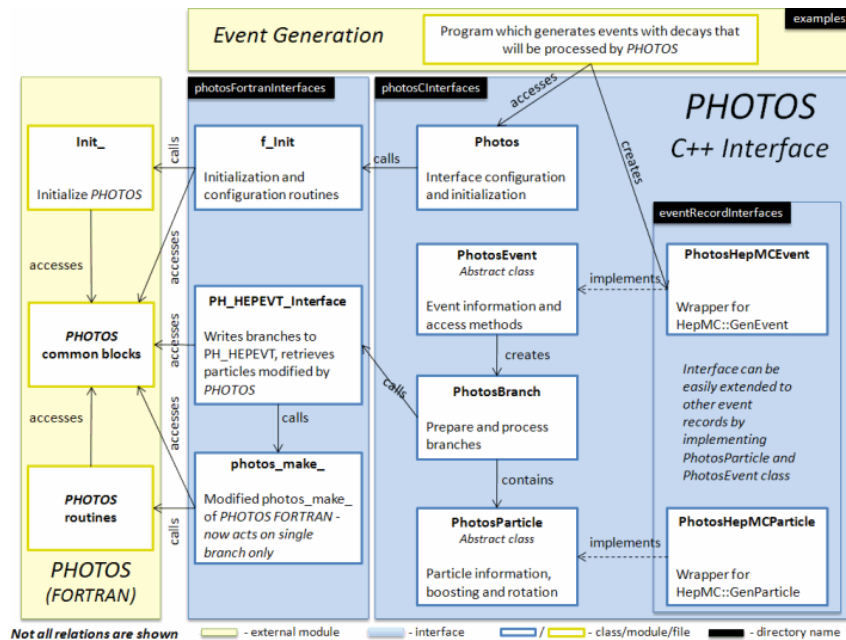
but it is not systematic work.

Events with and without photon:

$R = \frac{\Gamma_{K_{e3}\gamma}}{\Gamma_{K_{e3}}}$	PHOTOS	GASSER
	%	%
$5 < E_\gamma < 15 \text{ MeV}$	2.38	2.42
$15 < E_\gamma < 45 \text{ MeV}$	2.03	2.07
$\Theta_{e,\gamma} > 20$	0.876	0.96

courtesy of NA48 and Prof. L.Litov

This results can be obtained starting from PHOTOS version 2.13.



- Version in C++ is available and complete.
- Physicswise already more sophisticated than F77, eg. with emission of light fermion pairs is.
- Better access to decaying particle frame. Necessary for installation of process dependent matrix elements.
- HepMC event format is a standard, HEPEVT interface exists still.
- documentation completed.
- Regularly updated web page: <http://photospp.web.cern.ch/photospp/> with tar balls of program versions.

- It was a pleasure to give a talk on the project which was started when I was regularly visiting Paris (from Marseille). At that time nobody was dreaming that it could become so popular and for long! Potential was underestimated too. **TODAY:**
- Program is based on exact multiphoton phase space and matrix element
- Only for some channels, kernels of exact first order matrix elements installed.
- Spin amplitudes were essential in program development.
- In construction we rely on properties of factorization and those properties of QED matrix elements which lead to YFS exponentiation. Studies of double emission amplitudes were necessary. Eg. $e^+e^- \rightarrow \nu_e\bar{\nu}_e\gamma\gamma$, EPJC C44 (2005) 489 paper, emissions from fermions and vectors studied.
- Program can be used as a precision tool. Benchmarks better than 0.1 % .
- Program works as after-burner. This is advantageous, events generated by other programs can be modified. It is a challenge too. Numerical stability and correctness of PHOTOS operation depends on quality of events read in.
- Implementation for FORTRAN HEPEVT and C++ HepMC.

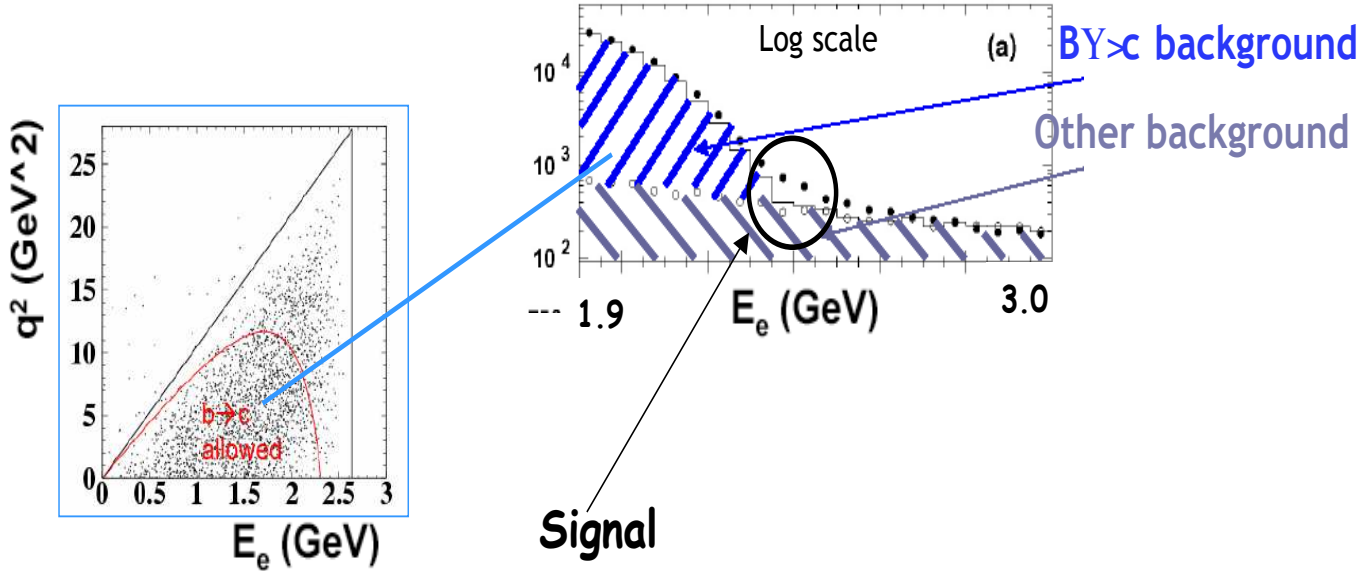
- For some meson decay channels matrix elements are implemented exactly.
- Often in itself this does not improve precision

-*-*-*

- but **opens the gate** for the data adjusted form-factors •
- **That is your activity of the forthcoming years** •
- **that is the message of my talk** •

Extra transparencies

electron endpoint



Transparency of E. Barberio

Extra or other old talks transparencies

Systematic uncertainty:
 Model dependent signal efficiencies
 $B_{Xc} \ell Y$ background estimation

Phys. Lett. B 621, 28 (2005)

$$\Delta Br(X_u 1\nu) = \frac{N(X_u 1\nu)}{2N_{BB} \epsilon_{MC}}$$

$$|V_{ub}| = \sqrt{\frac{(1 + \delta_{rad}) \times \Delta Br(X_u 1\nu)}{\tau_B} \frac{1}{R}}$$

1. we are using **STANDARD and FORMAL** parametrizations of Lorentz group. One can express it with the help of consecutive boosts and rotations.
2. Convenient for Monte Carlo event construction!
3. For the definition of coordinate system in the P -rest frame the \hat{x} and \hat{y} axes of the laboratory frame boosted to the rest frame of P can be used. The orthogonal right-handed system can be constructed with their help in a standard way.
4. We choose polar angles θ_1 and ϕ_1 defining the orientation of the four momentum \bar{k}_2 in the rest frame of P . In that frame \bar{k}_1 and \bar{k}_2 are back to back^a, see fig. (2).
5. The previous two points would complete the definition of the two-body phase space, if both \bar{k}_1 and \bar{k}_2 had no measurable spin degrees of freedom visualizing themselves e.g. through correlations of the secondary decay products' momenta. Otherwise we need to know an additional angle ϕ_X to complete the set of Euler angles defining the relative orientation of the axes of the P rest-frame system with the coordinate system used in the rest-frame of \bar{k}_2 (and possibly also of \bar{k}_1), see fig. (3).

^aIn the case of phase space construction for multi-body decays \bar{k}_2 should read as a state representing the sum of all decay products of P but \bar{k}_1 .

6. If both rest-frames of \bar{k}_1 and \bar{k}_2 are of interest, their coordinate systems are oriented with respect to P with the help of θ_1, ϕ_1, ϕ_X . We assume that the coordinate systems of \bar{k}_1 and \bar{k}_2 are connected by a boost along the \bar{k}_2 direction, and in fact share axes: $z' \uparrow\downarrow z'', x' \uparrow\uparrow x'', y' \uparrow\downarrow y''$.
7. For the three-body phase space: We take the photon energy k_γ in P rest frame. We calculate: photon, k_1 and k_2 energies, all in $k_1 + k_2$ frame.
8. We use the angles θ, ϕ , in the rest-frame of the $k_1 + k_2$ pair: angle θ is an angle between the photon and k_1 direction (i.e. $-z''$). Angle ϕ defines the photon azimuthal angle around z'' , with respect to x'' axis (of the k_2 rest-frame), see fig. (4).
9. If all k_1, k_2 and $k_1 + k_2$ rest-frames exist, then the x -axes for the three frames are chosen to coincide. It is OK, all frames connected by boosts along z'' see fig. (4).
10. To define orientation of k_2 in P rest-frame coordinate system, and to complete construction of the whole event, we will re-use Euler angles of \bar{k}_2 : ϕ_X, θ_1 and ϕ_1 (see figs. 5 and 6), defined again of course in the rest frame of P .

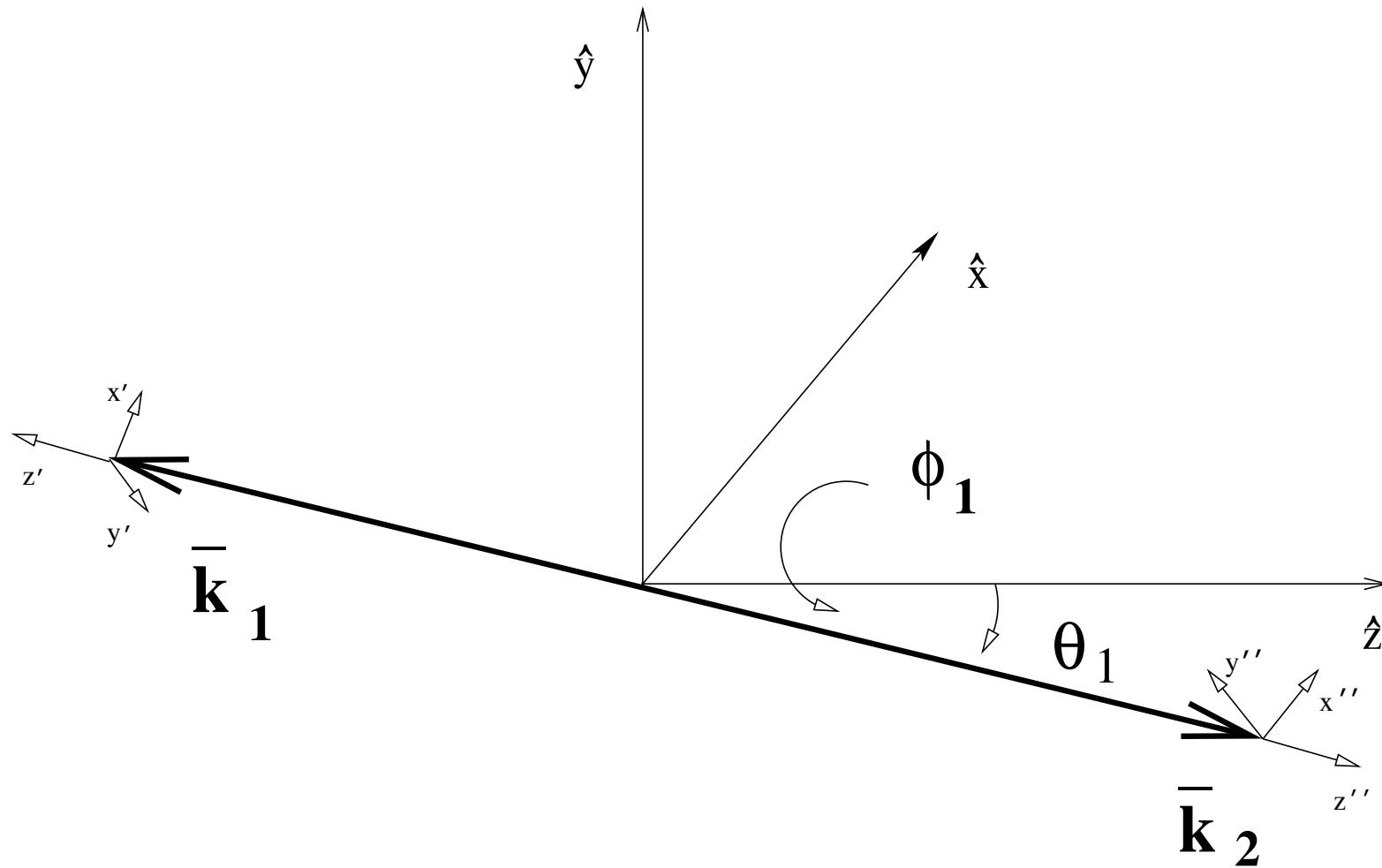


Figure 2: The angles θ_1 , ϕ_1 defined in the rest-frame of P and used in parametrization of two-body phase-space.

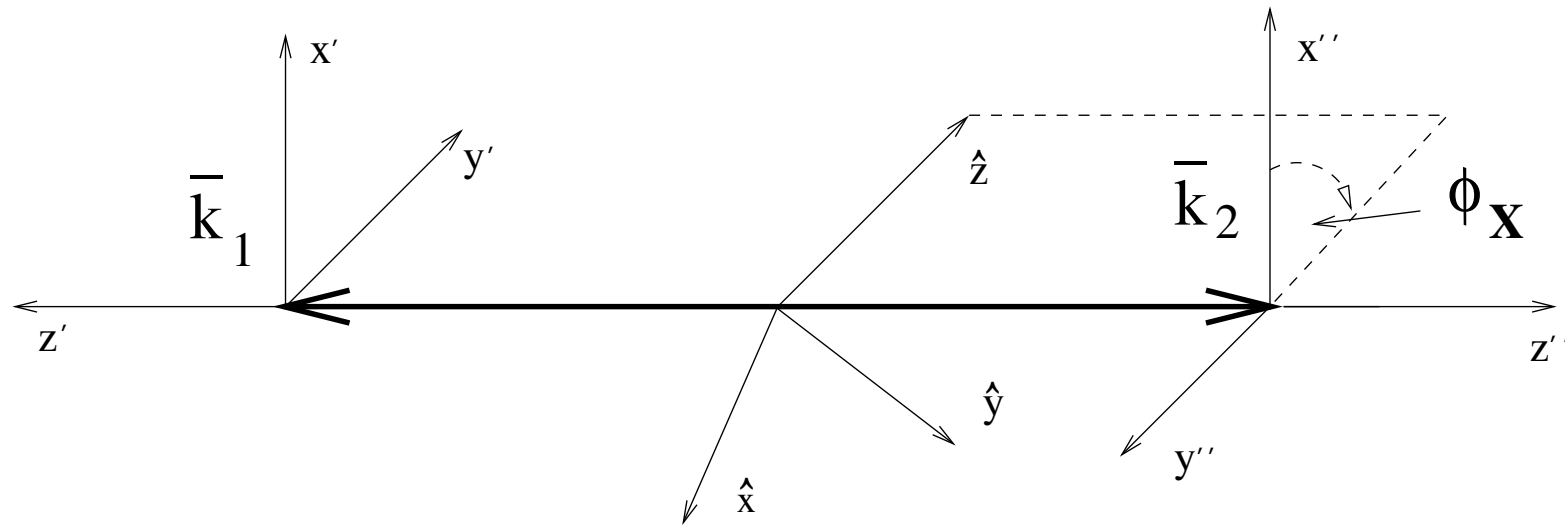


Figure 3: Angle ϕ_X is also defined in the rest-frame of P as an angle between (oriented) planes spanned on: (i) \bar{k}_1 and \hat{z} -axis of the P rest-frame system, and (ii) \bar{k}_1 and x'' -axis of the \bar{k}_2 rest frame. It completes definition of the phase-space variables if internal orientation of \bar{k}_1 system is of interest. In fact, Euler angle ϕ_X is inherited from unspecified in details, parametrization of phase space used to describe possible future decay of \bar{k}_2 (or \bar{k}_1).

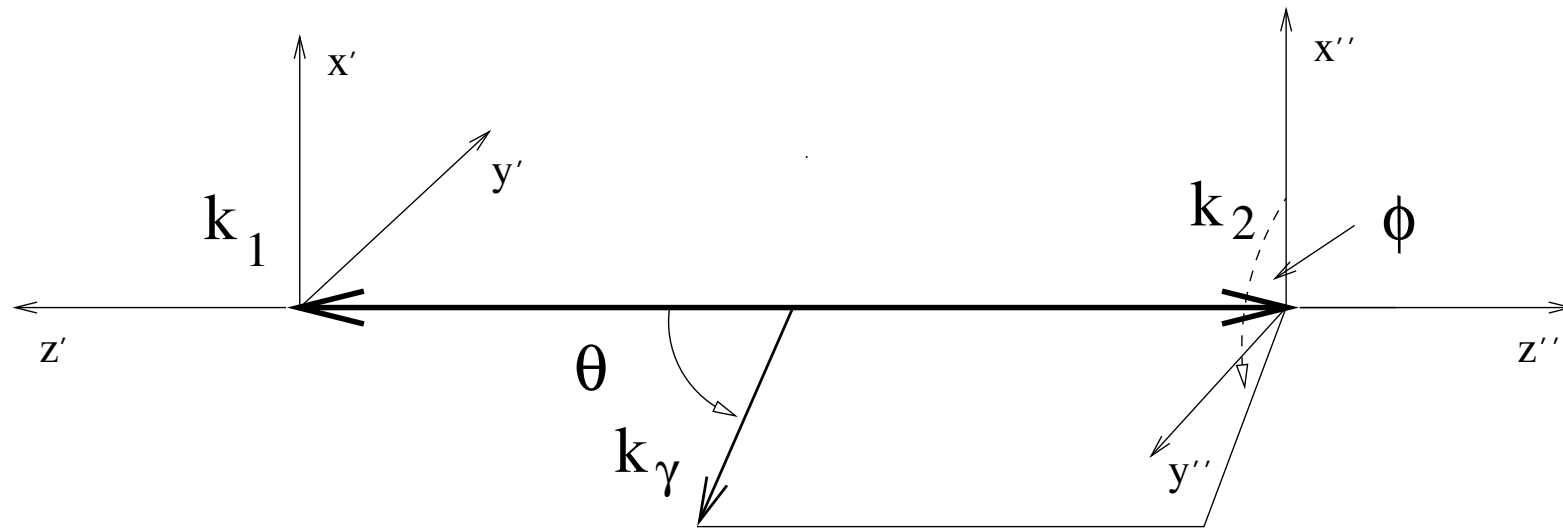


Figure 4: The angles θ , ϕ are used to construct the four-momentum of k_γ in the rest-frame of $k_1 + k_2$ pair (itself not yet oriented with respect to P rest-frame). To calculate energies of k_1 , k_2 and photon, it is enough to know m_1 , m_2 , M and photon energy k_γ of the P rest-frame.

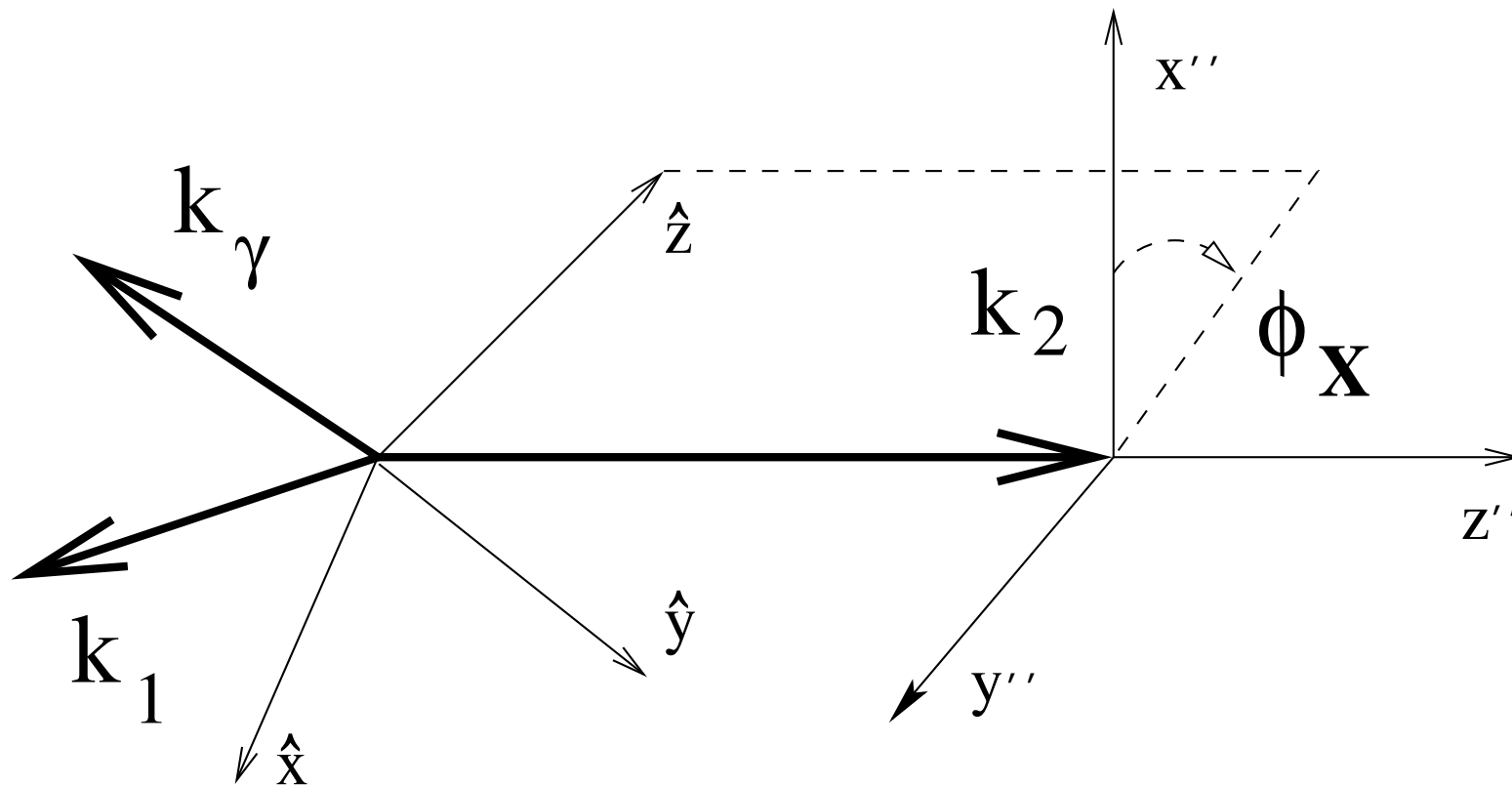


Figure 5: Use of angle ϕ_x in defining orientation of k_1 , k_2 and photon in the rest-frame of P . At this step only the plane spanned on P frame axis \hat{z} and k_2 is oriented with respect to $k_2 \times x''$ plane.

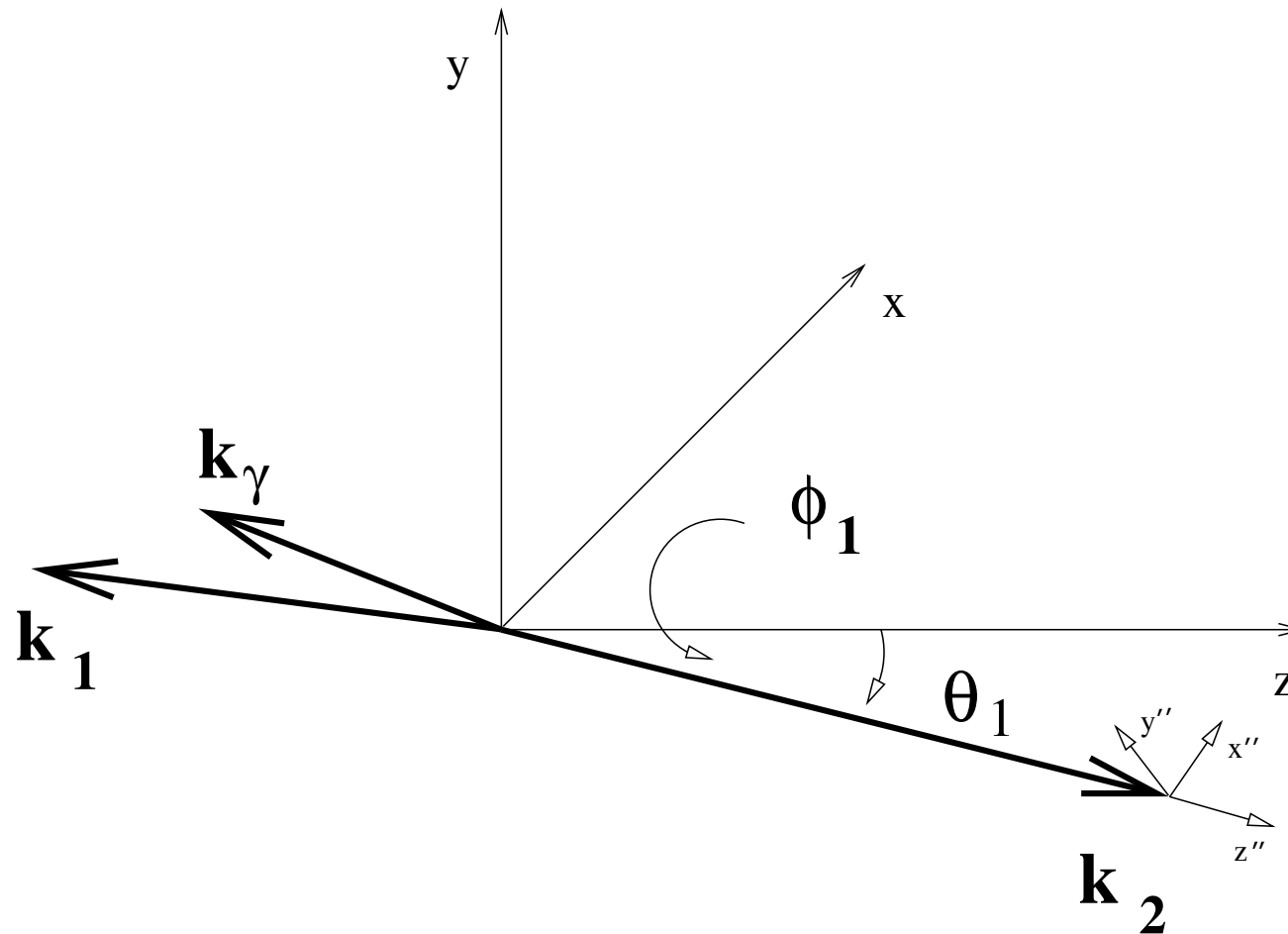


Figure 6: Final step in event construction. Angles θ_1 , ϕ_1 are used. The final orientation of k_2 coincide with this of \bar{k}_2 .

Matrix Element: $q\bar{q} \rightarrow Jgg$ - part proportional to $T^A T^B$ fermion spinors dropped

$$I_{lr}^{(1,2)} = \left(\frac{p \cdot e_1}{p \cdot k_1} - \frac{k_2 \cdot e_1}{k_2 \cdot k_1} - \frac{\not{\epsilon}_1 \not{k}_1}{2p \cdot k_1} \right) \not{J} \left(\frac{\not{k}_2 \not{\epsilon}_2}{2q \cdot k_2} + \frac{k_1 \cdot e_2}{k_1 \cdot k_2} - \frac{q \cdot e_2}{q \cdot k_2} \right)$$

$$I_{ll}^{(1,2)} = \frac{p \cdot k_2}{p \cdot k_1 + p \cdot k_2 - k_1 \cdot k_2} \left(\frac{p \cdot e_1}{p \cdot k_1} - \frac{k_2 \cdot e_1}{k_2 \cdot k_1} - \frac{\not{\epsilon}_1 \not{k}_1}{2p \cdot k_1} \right) \left(\frac{p \cdot e_2}{p \cdot k_2} - \frac{k_1 \cdot e_2}{k_1 \cdot k_2} - \frac{\not{\epsilon}_2 \not{k}_2}{2p \cdot k_2} \right) \not{J}$$

$$I_{rr}^{(1,2)} = \not{J} \frac{q \cdot k_1}{q \cdot k_1 + q \cdot k_2 - k_1 \cdot k_2} \left(\frac{q \cdot e_1}{q \cdot k_1} - \frac{k_2 \cdot e_1}{k_2 \cdot k_1} - \frac{\not{k}_1 \not{\epsilon}_1}{2q \cdot k_1} \right) \left(\frac{q \cdot e_2}{q \cdot k_2} - \frac{k_1 \cdot e_2}{k_1 \cdot k_2} - \frac{\not{k}_2 \not{\epsilon}_2}{2q \cdot k_2} \right)$$

$$I_e^{(1,2)} = \not{J} \left(1 - \frac{p \cdot k_2}{p \cdot k_1 + p \cdot k_2 - k_1 \cdot k_2} - \frac{q \cdot k_1}{q \cdot k_1 + q \cdot k_2 - k_1 \cdot k_2} \right) \left(\frac{k_1 \cdot e_2}{k_1 \cdot k_2} - \frac{k_2 \cdot e_1}{k_1 \cdot k_2} - \frac{e_1 \cdot e_2}{k_1 \cdot k_2} \right)$$

Remainder:

$$I_p^{(1,2)} = -\frac{1}{4} \frac{1}{p \cdot k_1 + p \cdot k_2 - k_1 \cdot k_2} \left(\frac{\not{\epsilon}_1 \not{k}_1 \not{\epsilon}_2 \not{k}_2 - \not{\epsilon}_2 \not{k}_2 \not{\epsilon}_1 \not{k}_1}{k_1 \cdot k_2} \right) \not{J}$$

$$I_q^{(1,2)} = -\frac{1}{4} \not{J} \frac{1}{q \cdot k_1 + q \cdot k_2 - k_1 \cdot k_2} \left(\frac{\not{k}_1 \not{\epsilon}_1 \not{k}_2 \not{\epsilon}_2 - \not{k}_2 \not{\epsilon}_2 \not{k}_1 \not{\epsilon}_1}{k_1 \cdot k_2} \right)$$

Matrix Element: $q\bar{q} \rightarrow Jgg$ - part proportional to $T^B T^A$ fermion spinors dropped

$$I_{lr}^{(2,1)} = \left(\frac{p \cdot e_2}{p \cdot k_2} - \frac{k_1 \cdot e_2}{k_1 \cdot k_2} - \frac{\not{\epsilon}_2 \not{k}_2}{2p \cdot k_2} \right) \not{J} \left(\frac{\not{k}_1 \not{\epsilon}_1}{2q \cdot k_1} + \frac{k_2 \cdot e_1}{k_2 \cdot k_1} - \frac{q \cdot e_1}{q \cdot k_1} \right)$$

$$I_{ll}^{(2,1)} = \frac{p \cdot k_1}{p \cdot k_2 + p \cdot k_1 - k_2 \cdot k_1} \left(\frac{p \cdot e_2}{p \cdot k_2} - \frac{k_1 \cdot e_2}{k_1 \cdot k_2} - \frac{\not{\epsilon}_2 \not{k}_2}{2p \cdot k_2} \right) \left(\frac{p \cdot e_1}{p \cdot k_1} - \frac{k_2 \cdot e_1}{k_2 \cdot k_1} - \frac{\not{\epsilon}_1 \not{k}_1}{2p \cdot k_1} \right) \not{J}$$

$$I_{rr}^{(2,1)} = \not{J} \frac{q \cdot k_2}{q \cdot k_2 + q \cdot k_1 - k_2 \cdot k_1} \left(\frac{q \cdot e_2}{q \cdot k_2} - \frac{k_1 \cdot e_2}{k_1 \cdot k_2} - \frac{\not{k}_2 \not{\epsilon}_2}{2q \cdot k_2} \right) \left(\frac{q \cdot e_1}{q \cdot k_1} - \frac{k_2 \cdot e_1}{k_2 \cdot k_1} - \frac{\not{k}_1 \not{\epsilon}_1}{2q \cdot k_1} \right)$$

$$I_e^{(2,1)} = \not{J} \left(1 - \frac{p \cdot k_1}{p \cdot k_2 + p \cdot k_1 - k_2 \cdot k_1} - \frac{q \cdot k_2}{q \cdot k_2 + q \cdot k_1 - k_2 \cdot k_1} \right) \left(\frac{k_2 \cdot e_1}{k_2 \cdot k_1} - \frac{k_1 \cdot e_2}{k_2 \cdot k_1} - \frac{e_2 \cdot e_1}{k_2 \cdot k_1} \right)$$

$$I_p^{(2,1)} = -\frac{1}{4} \frac{1}{p \cdot k_2 + p \cdot k_1 - k_2 \cdot k_1} \left(\frac{\not{\epsilon}_2 \not{k}_2 \not{\epsilon}_1 \not{k}_1 - \not{\epsilon}_1 \not{k}_1 \not{\epsilon}_2 \not{k}_2}{k_2 \cdot k_1} \right) \not{J}$$

$$I_q^{(2,1)} = -\frac{1}{4} \not{J} \frac{1}{q \cdot k_2 + q \cdot k_1 - k_2 \cdot k_1} \left(\frac{\not{k}_2 \not{\epsilon}_2 \not{k}_1 \not{\epsilon}_1 - \not{k}_1 \not{\epsilon}_1 \not{k}_2 \not{\epsilon}_2}{k_2 \cdot k_1} \right)$$

For QCD we have separation too; 12 gauge invariant parts

- Terms like

$$\left(\frac{p \cdot e_1}{p \cdot k_1} - \frac{k_2 \cdot e_1}{k_2 \cdot k_1} - \frac{\cancel{e}_1 \cancel{k}_1}{2p \cdot k_1} \right) \quad A$$

once integrated over part of phase space give Atarelli-Parisi kernel

- Terms

$$\frac{q \cdot k_1}{q \cdot k_1 + q \cdot k_2 - k_2 \cdot k_1} \quad B$$

if combined with phase space Jacobians can be used to redefine fermionic fields from $v(q)$ to $v(q - k_2)$ for example. **Term of such type appeared already in scalar QED (normalization of hadronic current).**

- No QCD prototype, for amplitudes \rightarrow my paper with A. van Hameren.
- For studies of amplitudes used in PHOTOS see my papers with: P. Golonka, G. Nanava (Z W H decays), G. Nanava Qinjun Xy (B γ^* decays), soon $K^\pm \rightarrow l^\pm \nu \pi^+ \pi^-$.