Interaction of a relativistic particle with an electron cloud

G. Iadarola

Many thanks to:
R. De Maria, L. Giacomel, E. Metral, N. Mounet,
Y. Papaphilippou, K. Paraschou and G. Rumolo

Outline

- Introduction
- Boosted reference frame
- Charges and currents in the boosted frame
- Fields and potentials in the boosted frame
- Potentials equations in the lab frame
- Kicks on the beam particles
- Recipe for the e-cloud kick
- Hamiltonian and symplecticity
- Where did the magnetic field go?

When simulating the **electron cloud formation and its effects of the beam** some **simplifying assumptions** are "traditionally" made:

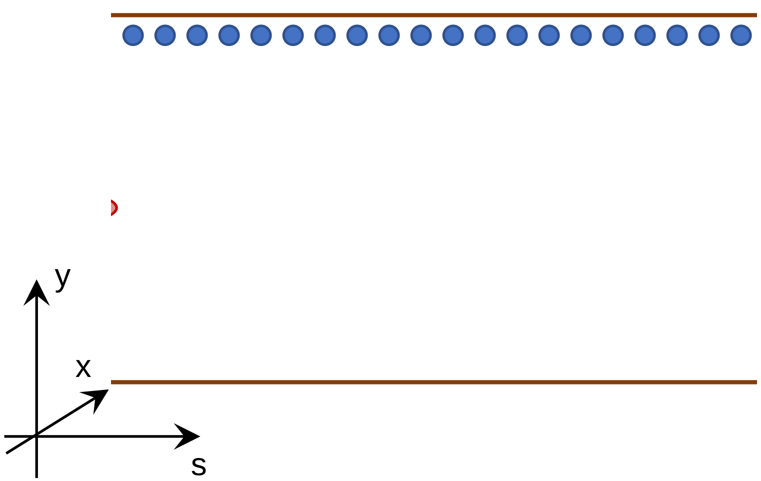
1. The **magnetic field** generated by the electron motion is **neglected**
2. The electric field is calculated using **equations of electrostatics** (Poisson) instead of the full electromagnetic problem (Maxwell)
3. Instead of a full 3D problem we solve a **2D Poisson problem** at a certain accelerator section
4. Only **transverse forces are applied**, but no longitudinal kicks

Qualitative arguments can be found for some of these choices (in literature and in "oral tradition") but, to my knowledge, no systematic derivation was available

In a document recently published (CERN-ACC-NOTE-2019-0033) I try to develop a **consistent mathematical treatment** that addresses these points.

I will summarize it in the following

We start from **a simple cartoon** that illustrates what we will then try to do more rigorously in formulas.



- We **start from a longitudinally uniform distribution of electrons** initially at rest
- Electrons are **attracted towards** a passing bunch

General idea

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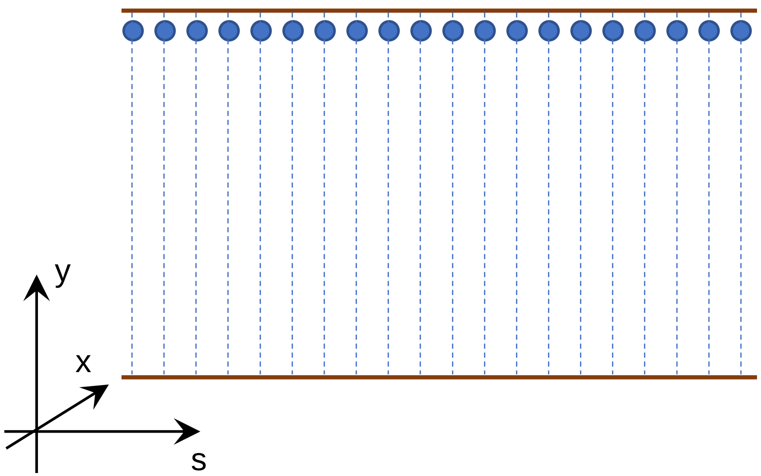

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- The motion at **different longitudinal sections is identical**, except for a **delay** defined by the bunch speed:

$$\Delta t = \frac{\Delta s}{v_{\mathrm{bunch}}}$$

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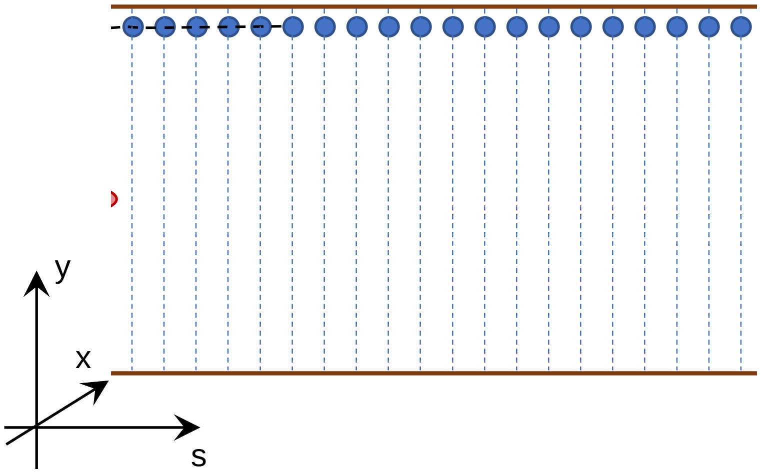

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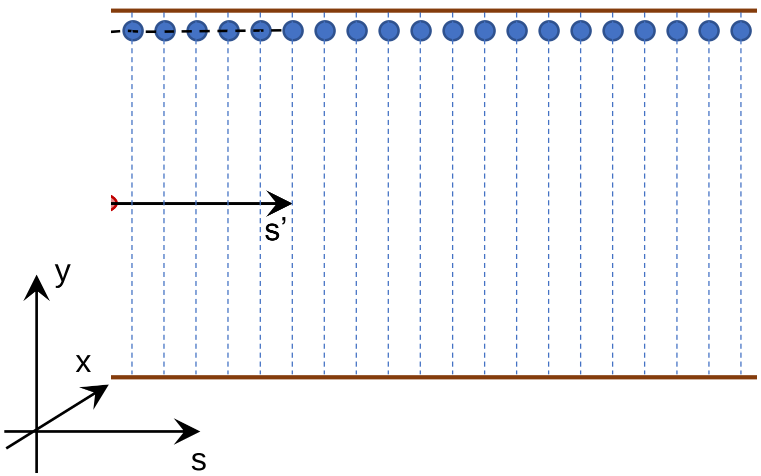

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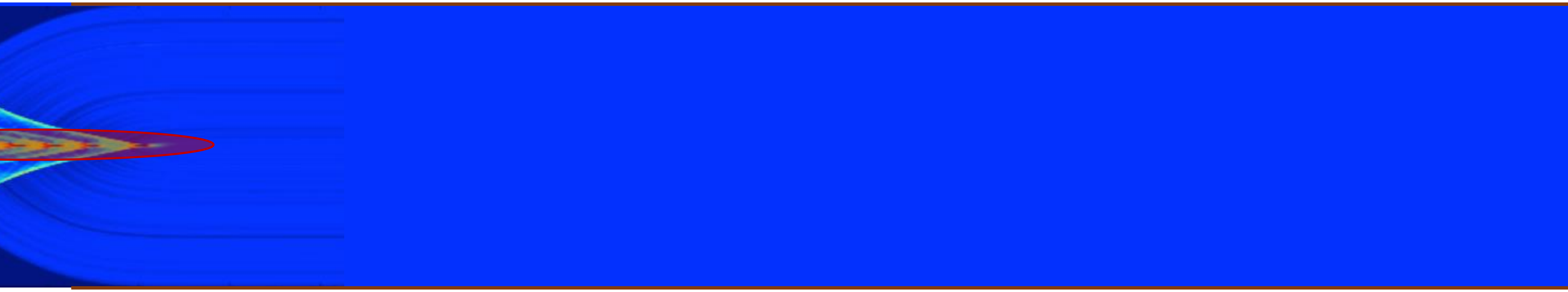
$$\Delta t = \frac{\Delta s}{v_{\text{bunch}}}$$

- There is a **wave of "falling" electrons** that translates rigidly with the bunch
- In the **reference system of the bunch**, the electron charge and current density **distributions are stationary** (they do not depend on time):

$$\rho'(x', y', s', t') = \rho'(x', y', s')$$
$$\mathbf{J}'(x', y', s', t') = \mathbf{J}'(x', y', s')$$

General features

A **realistic case** is “slightly” more complicated
but the **features discussed before are all there**



Let’s try to go through the steps discussed before in a more rigorous way

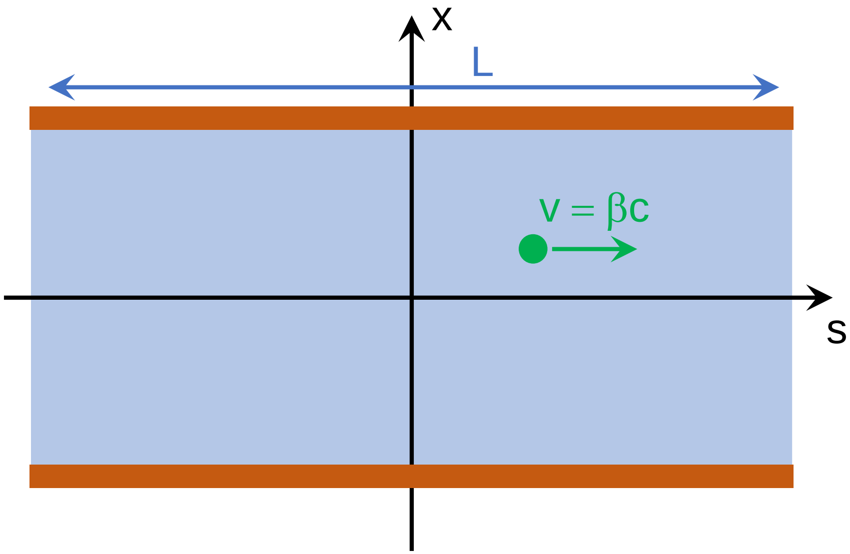
The bunch moves practically the speed of light → we will have to use special relativity

Hypotheses:

- **Indefinitely long, perfectly conducting pipe** (we call *s* the direction defined by the pipe)
- A **bunch** travels along *s* with speed βc
- The **initial distribution of the electrons is homogenous along s** (it can depend on x,y)
- The **electron motion is purely transverse** (for simplicity – but it is possible to generalize)
- **We know the electron motion at s=0 as a function of time**, defined by:

Charge density: $\rho_0 \left(x, y, t\right)$ Current density: $\mathbf{J}_0 \left(x, y, t\right)$

with: $\hat{\mathbf{i}}_s \cdot \mathbf{J}_0 = 0$



Goal:

We want to compute the **kick (change in momenta) on a beam particle** within the bunch due to the interaction with a **portion of the e-cloud having length L**

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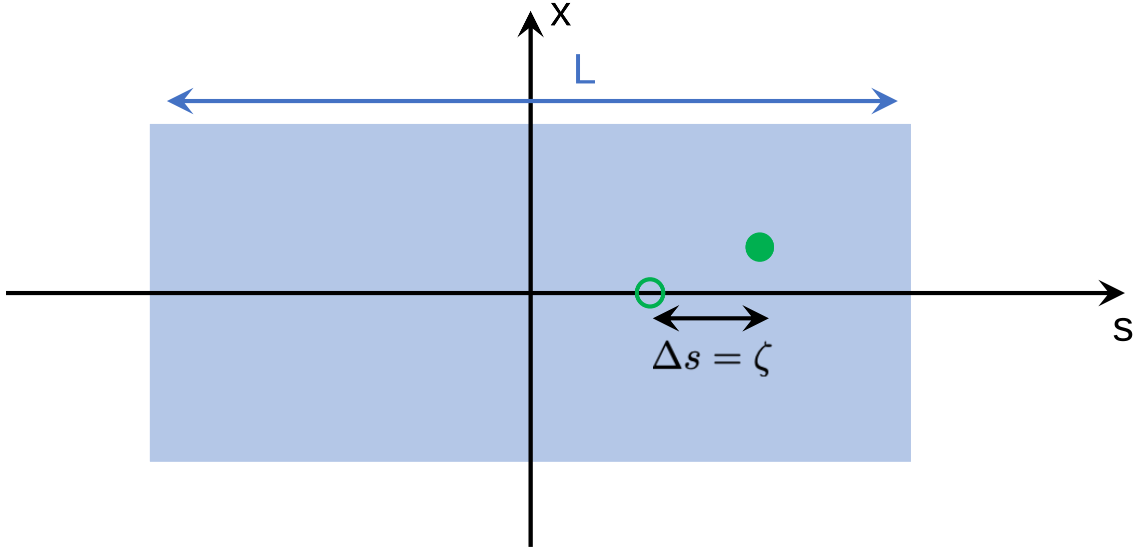
Particle motion:

Synchronous particle: $s\left(t\right) = \beta c t$

Generic particle: $s(t) = \beta c t + \zeta$

Time taken by the particle to cross the e-cloud: $T = \dfrac{L}{\beta c}$



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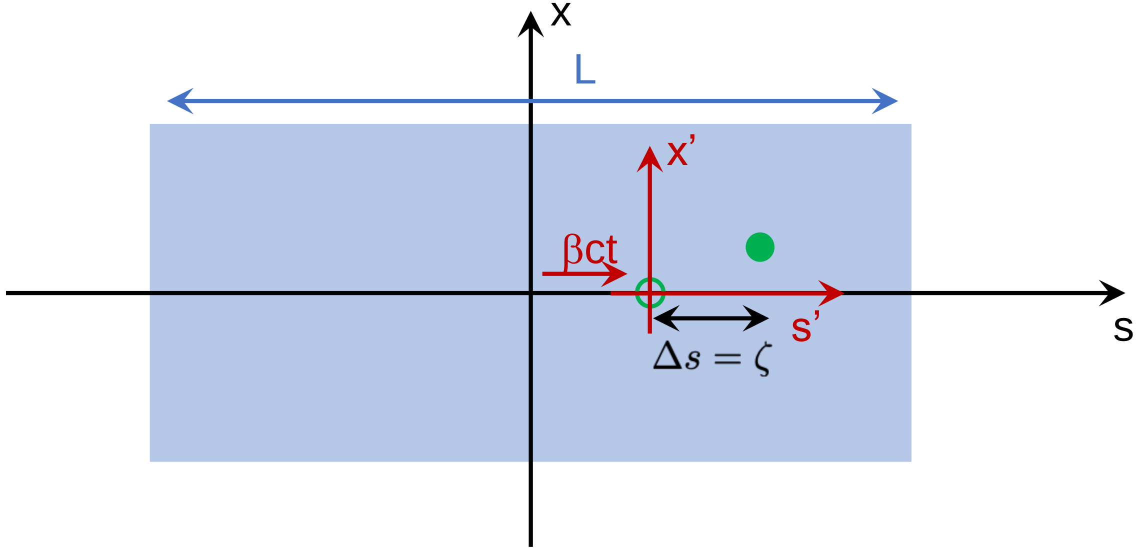


We introduce a **"boosted" reference frame (x', y', s', t')** moving with the reference particle.

The two systems are related by Lorentz transformations:

Direct:

$$ct'=\gamma\left(ct-\beta s\right)$$
$$s'=\gamma\left(s-\beta ct\right)$$

$$\gamma=\frac{1}{\sqrt{1-\beta^2}}$$

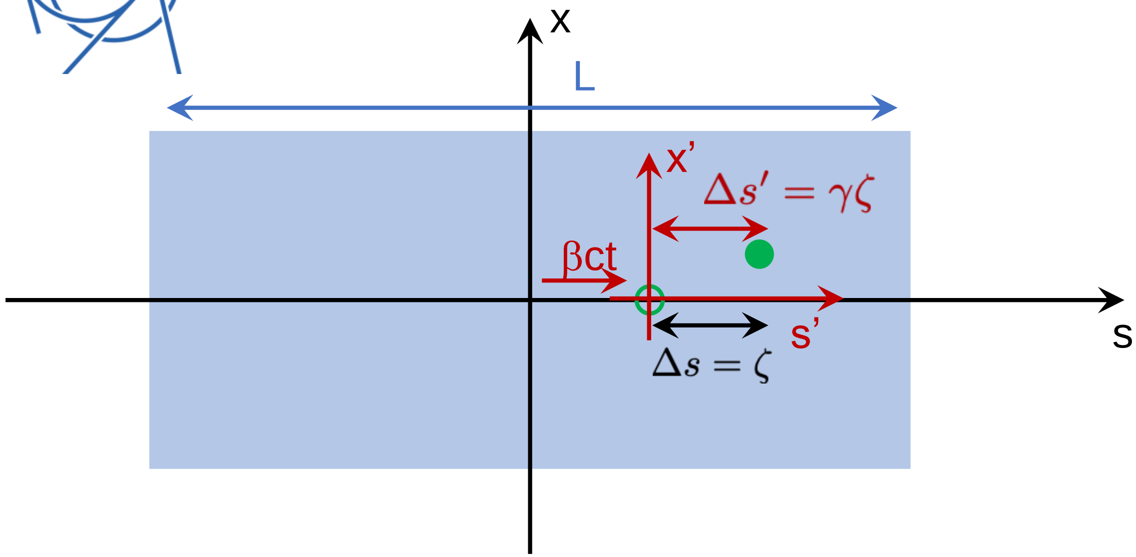
Inverse:

$$ct=\gamma\left(ct'+\beta s'\right)$$
$$s=\gamma\left(s'+\beta ct'\right)$$

Transverse coordinates are invariant:

$$x=x' \qquad y=y'$$

Transform particle coordinates



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Inverse:

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$$s = \gamma\left(s' + \beta ct'\right)$$

with: $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$

We transform the particle motion:

Lab frame: $s(t) = \beta ct + \zeta$

$$\gamma\left(s' + \beta \cancel{ct'}\right) = \beta\gamma\left(\cancel{ct'} + \beta s'\right) + \zeta$$

$$\gamma\left(1-\beta^2\right)s' = \zeta$$

In the boosted frame: $\boxed{s' = \gamma\zeta}$

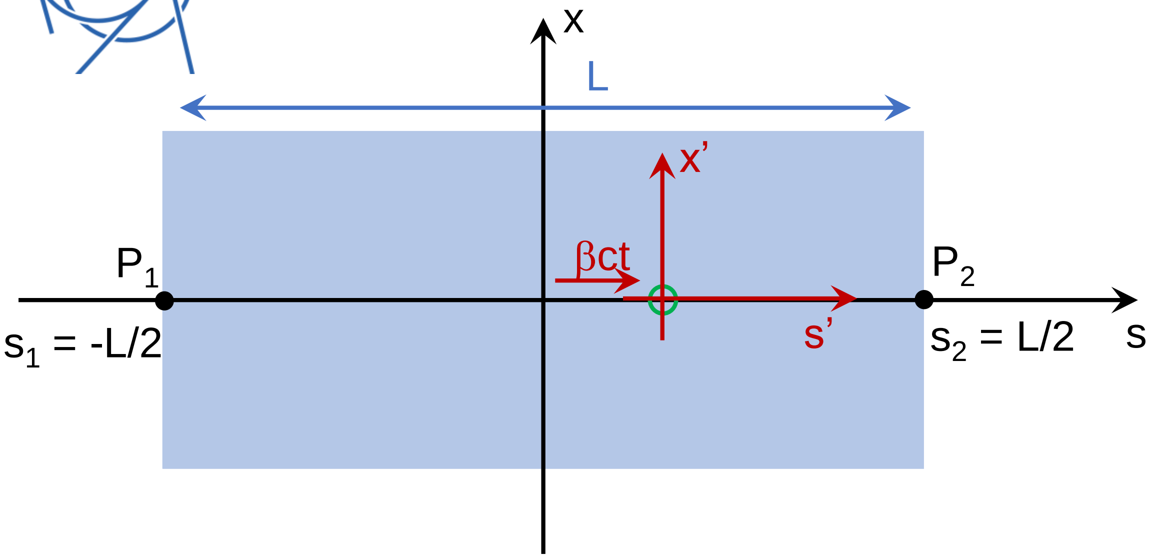
1. In the boosted frame, **beam particles are at rest**

   → In the boosted frame the **magnetic field of the electrons has no effect** on them

2. In the boosted frame **distances** among particles **are increases by a factor gamma** (the bunch becomes longer in the boosted frame)

3. There is a **mapping between s' and ζ** (which independent of time)

Transform e-cloud volume



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$$ct' = \gamma\,(ct - \beta s)$$

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Inverse:

$$ct = \gamma\,(ct' + \beta s')$$

$$s = \gamma\,(s' + \beta ct')$$

with: $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$

We call P_1 and P_2 the edges of the e-cloud region.
In the lab frame they are at rest:

$$s_1(t) = -\frac{L}{2} \qquad s_2(t) = -\frac{L}{2}$$

We transform them:

$$\gamma\,(s_1' + \beta ct') = -\frac{L}{2} \qquad \gamma\,(s_2' + \beta ct') = \frac{L}{2}$$

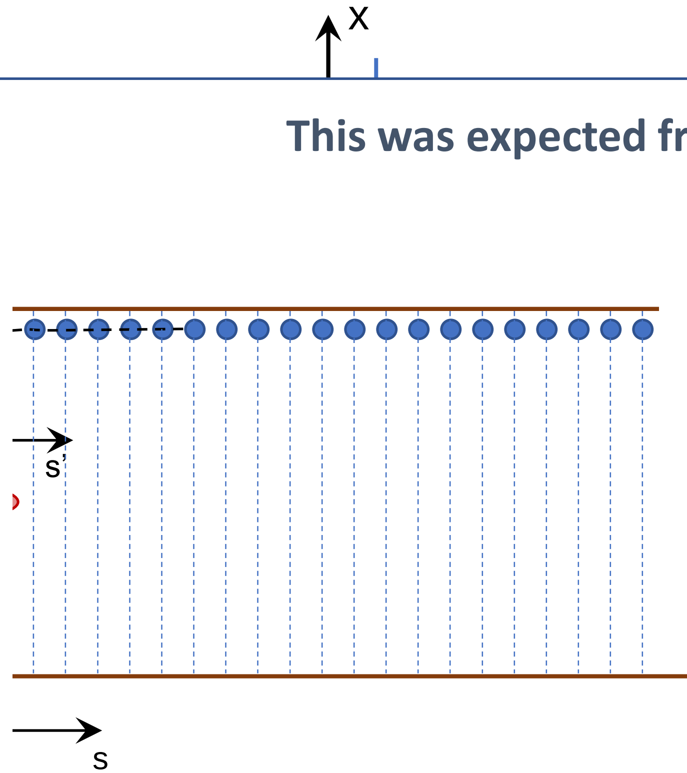
Reordering

$$s_1'\,(t') = -\frac{L}{2\gamma} - \beta ct' \qquad s_2'\,(t') = \frac{L}{2\gamma} - \beta ct'$$

The cloud travels along s' with speed -βc

Direct:

This was expected from our intuitive picture



We transform it: $\gamma\left(s_1' + \beta c t'\right) = -\frac{L}{2}$ $\quad \gamma\left(s_2' + \beta c t'\right) = \frac{L}{2}$

Reordering $\quad s_1'\left(t'\right) = -\frac{L}{2\gamma} - \beta c t' \qquad s_2'\left(t'\right) = \frac{L}{2\gamma} - \beta c t'$

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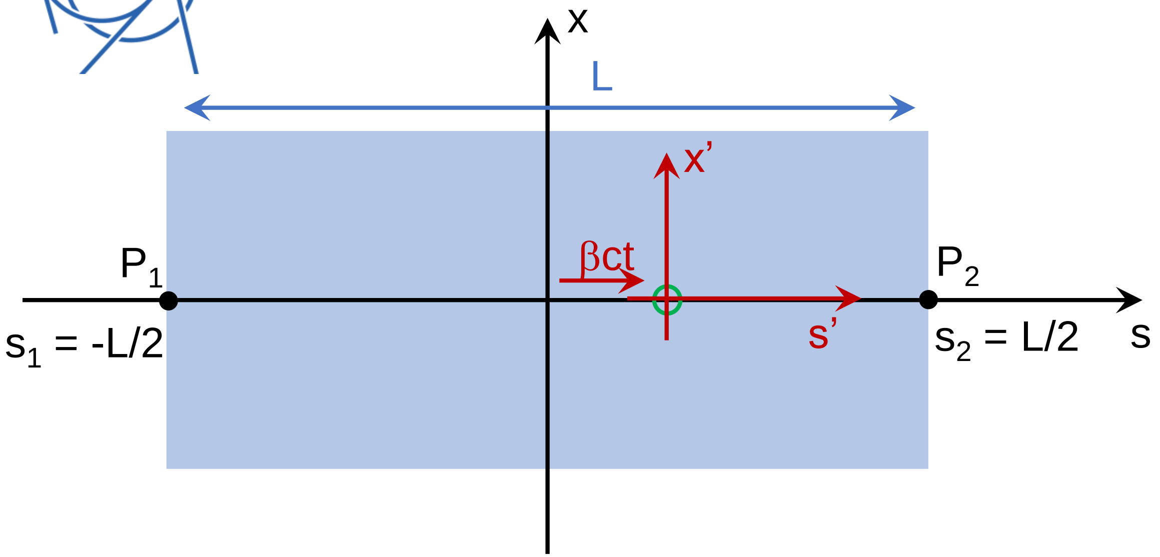

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with: $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$

In the lab frame: $$s_1(t) = -\frac{L}{2} \qquad s_2(t) = -\frac{L}{2}$$

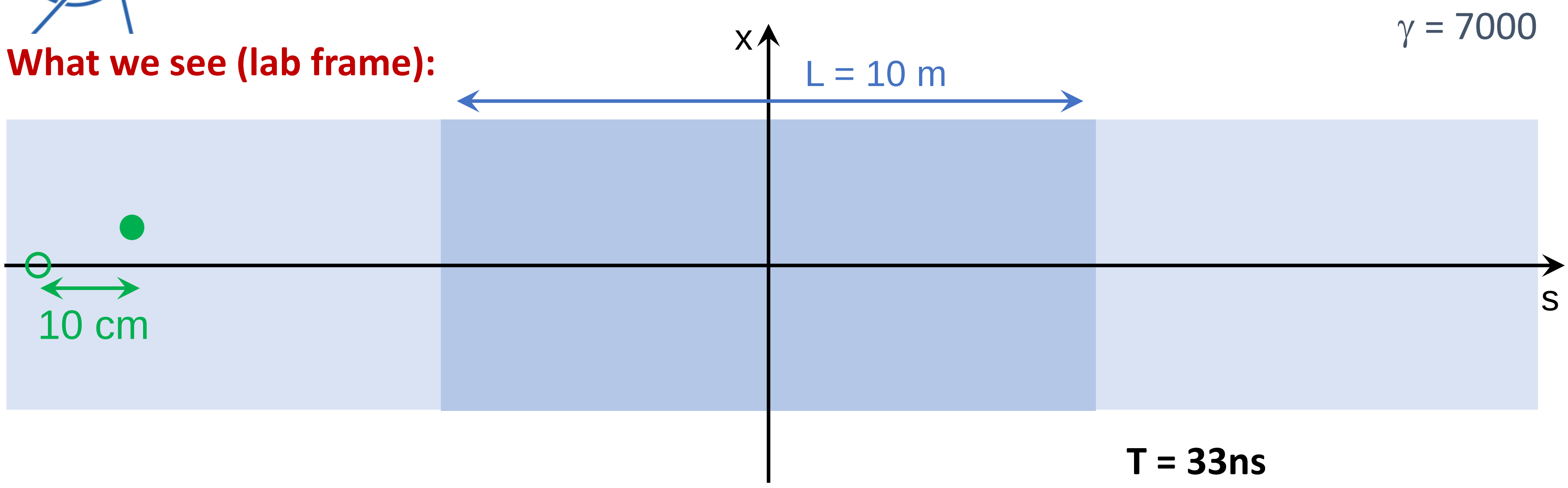
In the boosted frame: $$s_1'\,(t') = -\frac{L}{2\gamma} - \beta ct' \qquad s_2'\,(t') = \frac{L}{2\gamma} - \beta ct'$$

We compute the **length of the cloud in the boosted** frame:

$$L' = s_2'\,(t') - s_1'\,(t') = \frac{L}{\gamma}$$

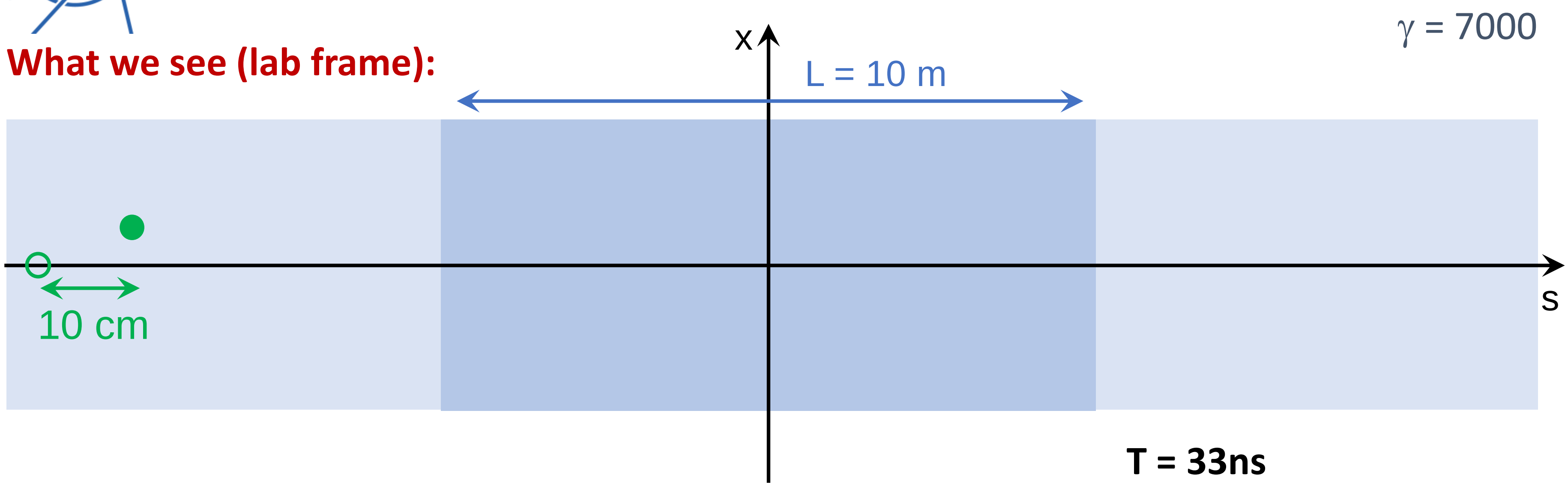
The e-cloud gets geometrically shorter

What we see (lab frame):

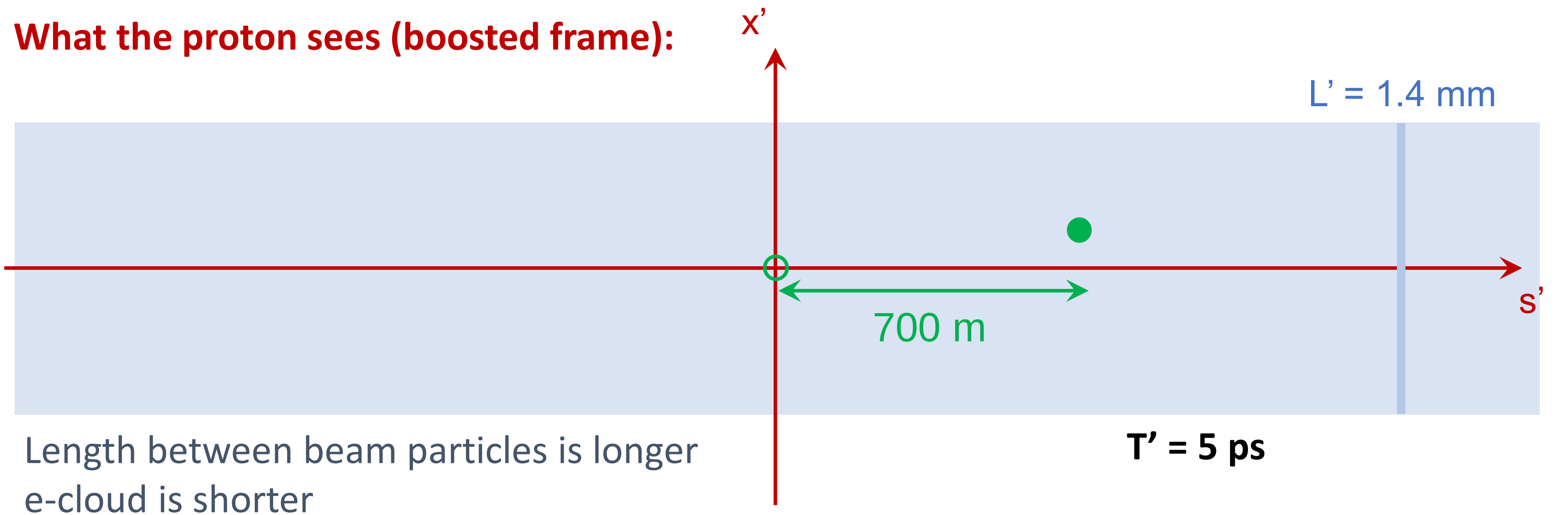


Visually

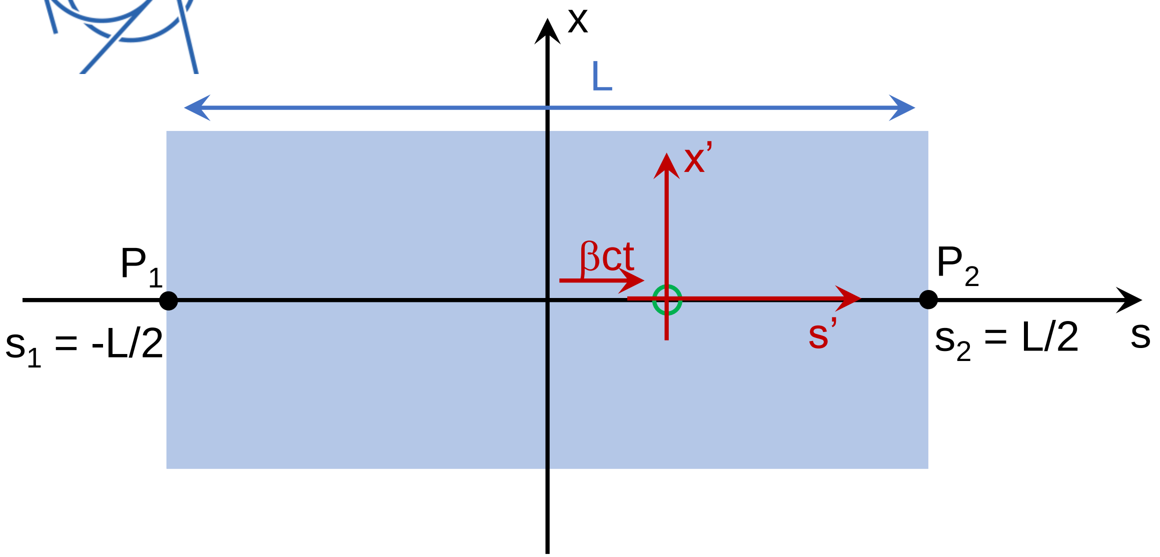
What we see (lab frame):



What the proton sees (boosted frame):



- Length between beam particles is longer
- e-cloud is shorter
- Interaction lasts less time



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with: $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$

In the boosted frame: $s_1'\,(t') = -\dfrac{L}{2\gamma} - \beta ct'$ $\quad s_2'\,(t') = \dfrac{L}{2\gamma} - \beta ct'$

At what t' do P_1 and P_2 pass by s'=0? $t_1' = -\dfrac{L}{2\beta c\gamma}$ $\quad t_2' = \dfrac{L}{2\beta c\gamma}$

The interaction lasts:

$$\boxed{T' = t_2' - t_1' = \frac{L}{\gamma\beta c} = \frac{T}{\gamma}}$$

The interaction is γ times quicker

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The sources of the electromagnetic field

As the structure is indefinite, **if we know the electron dynamics at s=0 we know it everywhere**:

$$\rho\left(x,y,s,t\right)=\rho\left(x,y,0,t-\frac{s}{\beta c}\right)$$
$$\mathbf{J}\left(x,y,s,t\right)=\mathbf{J}\left(x,y,0,t-\frac{s}{\beta c}\right)$$

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The sources have the form of a **travelling wave**

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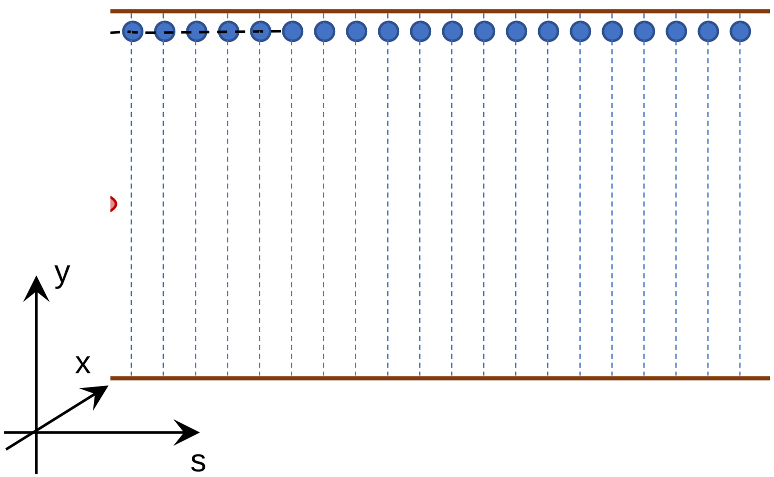
The sources have the form of a **travelling wave**

We saw it in our simple picture:



Charges and currents in the boosted frame

The quantities $(c\rho, J_x, J_y, J_s)$ form a **Lorentz 4-vector**

$$\boxed{\begin{aligned} ct' &= \gamma\,(ct - \beta s) \\ s' &= \gamma\,(s - \beta ct) \end{aligned}}$$

The **obey to Lorentz transformations**:

$$\begin{aligned} c\rho' &= \gamma\,(c\rho - \beta J_s) \\ J'_s &= \gamma\,(J_s - \beta c\rho) \end{aligned}$$

J_x and J_y are invariant

We assumed $J_s = 0$:

$$\boxed{\begin{aligned} &\rho' = \gamma\rho \\ &J'_s = -\gamma\beta c\rho = -\beta c\rho' \end{aligned}}$$

The cloud is **γ times more dense**

J'_s is non-zero and **proportional to ρ'**

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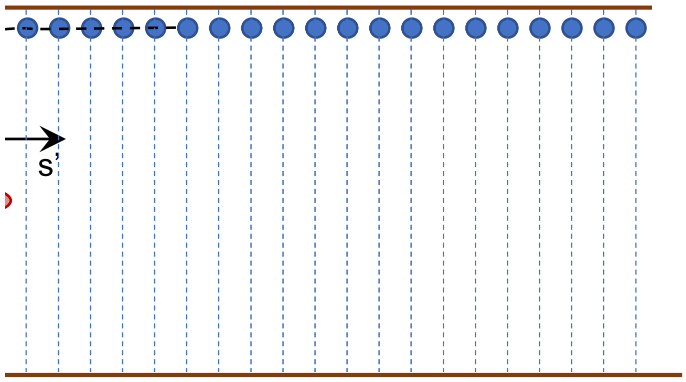
Again, this is consistent with our intuitive picture, where electrons are indeed moving along s'



$$\text{Before we have found:} \quad \rho' = \gamma\rho \qquad \rho(x,y,s,t) = \rho_0\left(x,y,t-\frac{s}{\beta c}\right)$$

Combining them we obtain:
$$\rho'(x,y,s,t) = \gamma\rho_0\left(x,y,t-\frac{s}{\beta c}\right)$$

which is still defined w.r.t. to the lab frame coordinates
→ we need to transform also the coordinates

$$ct = \gamma(ct' + \beta s') \qquad s = \gamma(s' + \beta ct')$$

$$t - \frac{s}{\beta c} = \left(\cancel{\gamma t'} + \gamma\frac{\beta}{c}s'\right) - \left(\frac{\gamma}{\beta c}s' + \cancel{\gamma t'}\right) = -\frac{\gamma s'}{\beta c}\left(1-\beta^2\right) = -\frac{s'}{\beta\gamma c}$$

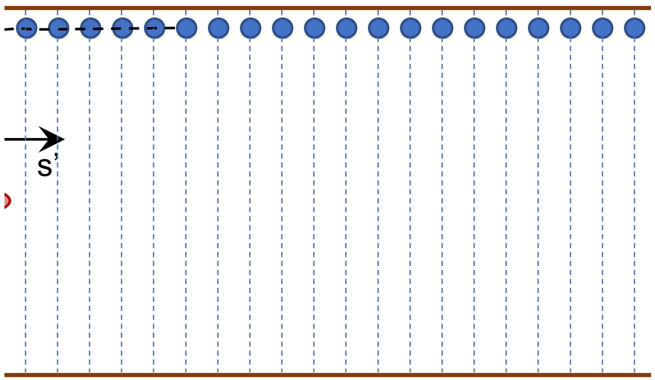
In the boosted frame the charge density can be written as:

$$\boxed{\rho'(x',y',s',t') = \gamma\rho_0\left(x',y',-\frac{s'}{\gamma\beta c}\right)}$$

It does not depend on time → it is stationary!

Charges and currents in the boosted frame

Again, as expected from our intuitive picture



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Fields and potentials in the boosted frame

We need to solve a **Maxwell's equations with stationary sources**:

$$\nabla' \times \mathbf{E'} = -\cancel{\frac{\partial \mathbf{B'}}{\partial t'}}$$

$$\nabla' \cdot \mathbf{E'} = \frac{\rho'}{\varepsilon_0}$$

$$\nabla' \times \mathbf{B'} = \mu_0 \mathbf{J'} + \cancel{\varepsilon_0 \mu_0 \frac{\partial \mathbf{E'}}{\partial t'}}$$

$$\nabla' \cdot \mathbf{B'} = 0$$

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Fields and potentials in the boosted frame

We need to solve a **Maxwell's equations with stationary sources**:

Electrostatics:

$$\nabla' \times \mathbf{E'} = 0$$

$$\nabla' \cdot \mathbf{E'} = \frac{\rho'}{\varepsilon_0} \qquad \text{(A)}$$

As the electric field is irrotational we can introduce a **scalar potential**:

$$\mathbf{E'} = -\nabla'\phi' \qquad \text{(B)}$$

Magnetostatics:

$$\nabla' \times \mathbf{B'} = \mu_0 \mathbf{J'} \qquad \text{(C)}$$

$$\nabla' \cdot \mathbf{B'} = 0$$

As the magnetic field is solenoidal we can introduce a **vector potential**:

$$\mathbf{B'} = \nabla' \times \mathbf{A}' \qquad \text{(D)}$$

We impose the **Lorentz gauge**:

$$\nabla' \cdot \mathbf{A}' + \cancel{\frac{1}{c^2}\frac{\partial \phi'}{\partial t'}} = 0$$

ϕ is stationary

$$\nabla' \cdot \mathbf{A}' = 0$$

$$\nabla' \times \mathbf{B'} = \nabla' \times \left(\nabla' \times \mathbf{A}'\right)$$

$$= \nabla'\left(\cancel{\nabla' \cdot \mathbf{A}'}\right) - \nabla'^2 \mathbf{A}'$$

Combining (A) and (B):

$$\boxed{\nabla'^2 \phi' = -\frac{\rho'}{\varepsilon_0}}$$

Both potential satisfy Poisson's equation in the boosted frame

Combining (C) and (D):

$$\boxed{\nabla'^2 \mathbf{A}' = -\mu_0 \mathbf{J'}}$$

Fields and potentials in the boosted frame

We need to solve a set of **Poisson's equations**:

Electrostatics: $$\nabla'^2 \phi' = -\frac{\rho'}{\varepsilon_0}$$

Magnetostatics: $$\nabla'^2 \mathbf{A}' = -\mu_0 \mathbf{J}'$$

The s component of the equation in the vector potential reads: $$\nabla'^2 A'_s = -\mu_0 J'_s$$

Transforming the sources to the boosted frame we had found: $$J'_s = -\beta c \rho'$$

Combining: $$\nabla'^2 A'_s = \mu_0 \beta c \rho' = \frac{\beta \rho'}{\varepsilon_0 c}$$

Re-arranging: $$\nabla'^2 \left(-\frac{c}{\beta} A'_s\right) = -\frac{\rho'}{\varepsilon_0}$$

Comparing against the equation in ϕ': $$\boxed{A'_s = -\frac{\beta}{c}\phi'}$$

The longitudinal component of A' is proportional to ϕ'

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Potentials in the lab frame

$$\boxed{A'_s = -\frac{\beta}{c}\phi'}$$

$$\boxed{\begin{aligned} ct' &= \gamma\left(ct - \beta s\right) \\ s' &= \gamma\left(s - \beta ct\right) \end{aligned}}$$

The scalar and vector potential form a **Lorentz 4-vector**

We can Lorentz transform them

$$A_s = A'_s + \beta\frac{\phi'}{c} \qquad A_s = 0$$

$$\phi = \gamma\left(\phi' + \beta c A'_s\right) \qquad \phi = \gamma(1-\beta^2)\phi' = \frac{\phi'}{\gamma}$$

In the **lab frame**:

$$A_s = 0$$

- The vector potential has no longitudinal component

$$\phi = \frac{\phi'}{\gamma}$$

- The scalar potential is simply proportional to the scalar potential in the boosted frame

Equation in the scalar potential for the lab frame

In the boosted frame we have a Poisson equation:

$$\boxed{\nabla'^2\phi' = -\frac{\rho'}{\varepsilon_0}}$$

$$\frac{\partial^2\phi'}{\partial x'^2} + \frac{\partial^2\phi'}{\partial y'^2} + \frac{\partial^2\phi'}{\partial s'^2} = -\frac{\rho'(x',y',s')}{\varepsilon_0}$$

We know how to transform the charge density:

$$\boxed{\rho'\left(x',y',s',t'\right) = \gamma\rho_0\left(x',y',-\frac{s'}{\gamma\beta c}\right)}$$

$$\frac{\partial^2\phi'}{\partial x'^2} + \frac{\partial^2\phi'}{\partial y'^2} + \frac{\partial^2\phi'}{\partial s'^2} = -\frac{\gamma\rho_0(x',y',-\frac{s'}{\gamma\beta c})}{\varepsilon_0}$$

From $\rho_0\left(x,y,t\right)$ we define:

$$\tilde{\rho}_0\left(x,y,\zeta\right) = \rho_0\left(x,y,-\frac{\zeta}{\beta c}\right)$$

where $\zeta = -\beta ct$ is the position along the bunch that is passing at a certain t

$$\frac{\partial^2\phi'}{\partial x'^2} + \frac{\partial^2\phi'}{\partial y'^2} + \frac{\partial^2\phi'}{\partial s'^2} = -\frac{\gamma\tilde{\rho}_0\left(x',y',\frac{s'}{\gamma}\right)}{\varepsilon_0}$$

We have found before: $\boxed{\phi = \frac{\phi'}{\gamma}}$

$$\cancel{\gamma}\left(\frac{\partial^2\phi}{\partial x'^2} + \frac{\partial^2\phi}{\partial y'^2} + \frac{\partial^2\phi}{\partial s'^2}\right) = -\frac{\cancel{\gamma}\tilde{\rho}_0\left(x',y',\frac{s'}{\gamma}\right)}{\varepsilon_0}$$

Equation in the scalar potential for the lab frame

In the very beginning we had found:

$$\boxed{s' = \gamma\zeta}$$

$$\frac{\partial^2\phi}{\partial x'^2} + \frac{\partial^2\phi}{\partial y'^2} + \frac{\partial^2\phi}{\partial s'^2} = -\frac{\tilde{\rho}_0\left(x', y', \frac{s'}{\gamma}\right)}{\varepsilon_0}$$

Lab-frame equation for the scalar potential

$$\boxed{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{1}{\gamma^2}\frac{\partial^2\phi}{\partial \zeta^2} = -\frac{\tilde{\rho}_0\left(x, y, \zeta\right)}{\varepsilon_0}}$$

For **large γ** this can be approximated by a **2D Poisson equation**

$$\boxed{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -\frac{\tilde{\rho}_0\left(x, y, \zeta\right)}{\varepsilon_0}}$$

which is exactly what we solve in PyECLOUD, HEADTAIL, and similar codes

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Transverse kick on the beam particle

Transverse momenta are invariant: $\Delta P_x = \Delta P'_x = qE'_xT'$

$$\boxed{\mathbf{E'} = -\nabla'\phi'}$$

$$E'_x = -\frac{\partial \phi'}{\partial x}$$

We found that $\boxed{\phi = \frac{\phi'}{\gamma}}$

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$$\boxed{\mathbf{E'} = -\nabla' \phi'}$$

$$E'_x = -\frac{\partial \phi'}{\partial x} = -\gamma \frac{\partial \phi}{\partial x}$$

We found that $\boxed{\phi = \frac{\phi'}{\gamma}}$

Before we had found the interaction time $\boxed{T' = \frac{L}{\gamma \beta c} = \frac{T}{\gamma}}$

$$\Delta P_x = -\frac{qL}{\beta c} \frac{\partial \phi}{\partial x}(x, y, \zeta)$$

Transverse kick due to the e-cloud interaction

Normalizing to the momentum of the reference particle:

$$\boxed{\Delta p_x = \frac{\Delta P_x}{P} = -\frac{qL}{m \gamma \beta^2 c^2} \frac{\partial \phi}{\partial x}(x, y, \zeta)}$$

This is exactly what we usually apply in HEADTAIL, PyECLOUD-PyHEADTAIL (which had been derived by just assuming longitudinal derivatives to be zero)

Longitudinal kick on the particle

In the boosted frame the particle is at rest before the interaction:

$$P'_s = qE'_sT'$$

$$\boxed{\mathbf{E'} = -\nabla'\phi'}$$

$$E'_s = -\frac{\partial\phi'}{\partial s'}$$

We found that $\boxed{\phi = \frac{\phi'}{\gamma}}$

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We found that $\boxed{\phi = \frac{\phi'}{\gamma}}$ $\boxed{s' = \gamma\zeta}$

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We found that $\boxed{\phi = \frac{\phi'}{\gamma}}$ $\boxed{s' = \gamma\zeta}$

$$\boxed{T' = \frac{L}{\gamma\beta c} = \frac{T}{\gamma}}$$

Longitudinal kick in the boosted frame

$$\boxed{P'_s = qE'_sT' = -\frac{qL}{\gamma\beta c}\frac{\partial\phi}{\partial\zeta}}$$

To apply it in a tracking code we need the corresponding change in $\delta = \Delta P/P$ in the lab frame

Longitudinal kick on the particle

We want the relation between ΔP/P (lab frame) and P_s'

$$\boxed{P_s' = qE_s'T' = -\frac{qL}{\gamma\beta c}\frac{\partial\phi}{\partial\zeta}}$$

$$(\mathcal{E}/c, P_x, P_y, P_s)$$

is a Lorentz 4-vector,
where $\mathcal{E}$ is the total energy

$$\frac{\mathcal{E}}{c} = \gamma\left(\frac{\mathcal{E}'}{c} + \beta P_s'\right)$$

$$\mathcal{E}' = \sqrt{m^2c^4 + c^2\left(P_s'^2 + P_x'^2 + P_y'^2\right)} \simeq mc^2\left(1 + \frac{P_s'^2 + P_x'^2 + P_y'^2}{2m^2c^2}\right)$$

$$\mathcal{E} = c\gamma\left(mc\left(1 + \frac{P_s'^2 + P_x'^2 + P_y'^2}{2m^2c^2}\right) + \beta P_s'\right)$$

Longitudinal kick on the particle

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Neglecting second order terms

$$\mathcal{E} = c\gamma\left(mc\left(1 + \frac{P_s'^2 + P_x'^2 + P_y'^2}{2m^2c^2}\right) + \beta P_s'\right) \simeq mc^2\gamma\left(1 + \beta\frac{P_s'}{mc}\right)$$

Relative energy change: $$\frac{\Delta\mathcal{E}}{\mathcal{E}_0} = \frac{\mathcal{E} - mc^2\gamma}{mc^2\gamma} = \frac{\beta}{mc}P_s'$$

Longitudinal kick on the particle

We want the relation between $\Delta P/P$ (lab frame) and P_s'

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$$\ldots{}^2 + P_x'^2 + P_y'^2) \simeq mc^2\left(1 + \frac{P_s'^2 + P_x'^2 + P_y'^2}{2m^2c^2}\right)$$

Neglecting second order terms

$$\ldots\frac{\ldots{}^2 + P_x'^2 + P_y'^2}{2m^2c^2}\right) + \beta P_s'\right) \simeq mc^2\gamma\left(1 + \beta\frac{P_s'}{mc}\right)$$

$$\frac{dP}{P} = \frac{dP}{d\beta}\frac{d\beta}{d\mathcal{E}}\frac{d\mathcal{E}}{\mathcal{E}}\frac{\mathcal{E}}{P}$$

$$\frac{d\mathcal{E}}{d\beta} = mc^2\beta\gamma^3 \qquad \frac{P}{\mathcal{E}} = \frac{\beta}{c} \qquad \frac{dP}{d\beta} = mc\gamma^3$$

Relative energy change: $$\frac{\Delta\mathcal{E}}{\mathcal{E}_0} = \frac{\mathcal{E} - mc^2\gamma}{mc^2\gamma} = \frac{\beta}{mc}P_s'$$

Longitudinal kick on the particle

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$$P_s' = qE_s'T' = -\frac{qL}{\gamma\beta c}\frac{\partial\phi}{\partial\zeta}$$

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Neglecting second order terms

$$\cdots \frac{\cdots'^2_s + P_x'^2 + P_y'^2}{2m^2c^2}\right) + \beta P_s'\right) \simeq mc^2\gamma\left(1 + \beta\frac{P_s'}{mc}\right)$$

$$\frac{dP}{P} = \frac{dP}{d\beta}\frac{d\beta}{d\mathcal{E}}\frac{d\mathcal{E}}{\mathcal{E}}\frac{\mathcal{E}}{P} = \frac{1}{\beta^2}\frac{d\mathcal{E}}{\mathcal{E}}$$

$$\frac{d\mathcal{E}}{d\beta} = mc^2\beta\gamma^3 \qquad \frac{P}{\mathcal{E}} = \frac{\beta}{c} \qquad \frac{dP}{d\beta} = mc\gamma^3$$

Relative energy change: $$\frac{\Delta\mathcal{E}}{\mathcal{E}_0} = \frac{\mathcal{E} - mc^2\gamma}{mc^2\gamma} = \frac{\beta}{mc}P_s'$$

$$\frac{\Delta P}{P_0} = \frac{1}{\beta^2}\frac{\Delta\mathcal{E}}{\mathcal{E}_0}$$

Longitudinal kick on the particle

We want the relation between ΔP/P (lab frame) and P_s'

$$\boxed{P_s' = qE_s'T' = -\frac{qL}{\gamma\beta c}\frac{\partial\phi}{\partial\zeta}}$$

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$$\mathcal{E}' = \sqrt{m^2c^4 + c^2\left(P_s'^2 + P_x'^2 + P_y'^2\right)} \simeq mc^2\left(1 + \frac{P_s'^2 + P_x'^2 + P_y'^2}{2m^2c^2}\right)$$

Neglecting second order terms

$$\mathcal{E} = c\gamma\left(mc\left(1 + \frac{P_s'^2 + P_x'^2 + P_y'^2}{2m^2c^2}\right) + \beta P_s'\right) \simeq mc^2\gamma\left(1 + \beta\frac{P_s'}{mc}\right)$$

Relative energy change: $\displaystyle \frac{\Delta\mathcal{E}}{\mathcal{E}_0} = \frac{\mathcal{E} - mc^2\gamma}{mc^2\gamma} = \frac{\beta}{mc}P_s'$

$$\frac{\Delta P}{P_0} = \frac{1}{\beta^2}\frac{\Delta\mathcal{E}}{\mathcal{E}_0} = \frac{1}{\beta mc}P_s'$$

Change in normalized longitudinal momentum

$$\boxed{\frac{\Delta P}{P_0} = -\frac{qL}{m\gamma\beta^2 c^2}\frac{\partial\phi}{\partial\zeta}}$$

at the moment not included in PyEC-PyHT

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Recipe to apply the e-cloud kick

1. We have an **electron pinch defined by the current density** at the section s=0

$\rho_0(x,y,t)$ or equivalently as a function of the position along the bunch $\zeta = -\beta c t$ $\tilde{\rho}_0\left(x,y,\zeta\right) = \rho_0\left(x,y,-\frac{\zeta}{\beta c}\right)$

2. We **compute the scalar potential** solving:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{1}{\gamma^2}\frac{\partial^2\phi}{\partial \zeta^2} = -\frac{\tilde{\rho}_0\left(x,y,\zeta\right)}{\varepsilon_0}$$

or for γ>>1 $$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -\frac{\tilde{\rho}_0\left(x,y,\zeta\right)}{\varepsilon_0}$$ (2D Poisson equation)

2. **Apply kick** to the particles

$$p_x \mapsto p_x - \frac{qL}{P_0\beta c}\frac{\partial\phi}{\partial x}(x,y,\zeta)$$

$$p_y \mapsto p_y - \frac{qL}{P_0\beta c}\frac{\partial\phi}{\partial y}(x,y,\zeta)$$

$$\delta \mapsto \delta - \frac{qL}{P_0\beta c}\frac{\partial\phi}{\partial \zeta}(x,y,\zeta)$$

The interaction with an e-cloud can be modelled by the following **non-linear map**

$$p_x \mapsto p_x - \frac{qL}{P_0\beta c}\frac{\partial\phi}{\partial x}(x,y,\zeta)$$

$$p_y \mapsto p_y - \frac{qL}{P_0\beta c}\frac{\partial\phi}{\partial y}(x,y,\zeta)$$

$$\delta \mapsto \delta - \frac{qL}{P_0\beta c}\frac{\partial\phi}{\partial \zeta}(x,y,\zeta)$$

The **Hamiltonian of the transformation** can be easily written:

$$H = \frac{qL}{P_0\beta c}\phi\left(x,y,\zeta\right)\delta(s)$$

$$\frac{dp_x}{ds} = -\frac{\partial H}{\partial x}$$
$$\frac{dp_y}{ds} = -\frac{\partial H}{\partial y}$$
$$\frac{d\delta}{ds} = -\frac{\partial H}{\partial \zeta}$$
$$\frac{dx}{ds} = \frac{\partial H}{\partial p_x}$$
$$\frac{dy}{ds} = \frac{\partial H}{\partial p_y}$$
$$\frac{d\zeta}{ds} = \frac{\partial H}{\partial \delta}$$

As it can be derived from an Hamiltonian,

the map is symplectic

(important for long-term tracking)

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Where did the magnetic field go?

$$\boxed{\begin{aligned} p_x &\mapsto p_x - \frac{qL}{P_0\beta c}\frac{\partial\phi}{\partial x}(x,y,\zeta) \\ p_y &\mapsto p_y - \frac{qL}{P_0\beta c}\frac{\partial\phi}{\partial y}(x,y,\zeta) \\ \delta &\mapsto \delta - \frac{qL}{P_0\beta c}\frac{\partial\phi}{\partial \zeta}(x,y,\zeta) \end{aligned}}$$

An apparent contradiction:

- In the lab frame the **electrons are moving and generate a magnetic field**.
  - → We expect this magnetic field to induce a force on the beam particle
- Still in the equations of the kick **we don't see anything that looks like a (v x B) term**
- In our calculation **we never use J_x and J_y which are the sources of the magnetic field**

Let's try to understand why…

Where did the magnetic field go?

Let's write explicitly the Lorentz force in the lab frame: $$\mathbf{F} = q\left(\mathbf{E} + \beta c\,\hat{\mathbf{i}}_s \times \mathbf{B}\right)$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \qquad (\mathbf{E} \neq -\nabla\phi)$$

In the lab frame the fields are not stationary
→ the electric field is not irrotational!

The Lorentz force can be written as:

$$\mathbf{F} = q\left(\underbrace{-\nabla\phi}_{\substack{\text{Irrorational}\\ \text{part of } \mathbf{E}}} \underbrace{-\frac{\partial \mathbf{A}}{\partial t}}_{\substack{\text{Non-irrotational}\\ \text{part of } \mathbf{E}}} + \beta c\,\boxed{\hat{\mathbf{i}}_s \times \underbrace{(\nabla \times \mathbf{A})}_{\mathbf{B}\text{ field}}}\right)$$

We focus on
this term

Where did the magnetic field go?

With a bit of patience (see note) it is possible to prove this vector identity:

$$\hat{\mathbf{i}}_s \times (\nabla \times \mathbf{A}) = \nabla A_s - \frac{\partial \mathbf{A}}{\partial s}$$

Before we have found: $\boxed{A_s = 0}$

$$\hat{\mathbf{i}}_s \times (\nabla \times \mathbf{A}) = -\frac{\partial \mathbf{A}}{\partial s}$$

In the lab frame the potentials propagate along s (together with the sources):

$$\mathbf{A}(x, y, s, t) = \mathbf{A}_0 \left(x, y, \underbrace{t - \frac{s}{\beta c}}_{\tau}\right)$$

$$\frac{\partial \mathbf{A}}{\partial s} = \frac{\partial \mathbf{A}_0}{\partial \tau}\frac{\partial \tau}{\partial s} = -\frac{1}{\beta c}\frac{\partial \mathbf{A}_0}{\partial \tau}$$
$$= -\frac{1}{\beta c}\frac{\partial \mathbf{A}_0}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{1}{\beta c}\frac{\partial \mathbf{A}}{\partial t}$$

$$\frac{\partial \mathbf{A}}{\partial s} = -\frac{1}{\beta c}\frac{\partial \mathbf{A}}{\partial t}$$

$$\boxed{\beta c\, \hat{\mathbf{i}}_s \times (\nabla \times \mathbf{A}) = \frac{\partial \mathbf{A}}{\partial t}}$$

Where did the magnetic field go?

We just found $$\boxed{\beta c\,\hat{\mathbf{i}}_s \times (\nabla \times \mathbf{A}) = \frac{\partial \mathbf{A}}{\partial t}}$$

$$\mathbf{F} = q\left(\underbrace{-\nabla\phi}_{\substack{\text{Irrorational} \\ \text{part of } \mathbf{E}}} \quad \underbrace{\cancel{-\frac{\partial \mathbf{A}}{\partial t}}}_{\substack{\text{Non-irrotational} \\ \text{part of } \mathbf{E}}} \quad \cancel{+\beta c\,\hat{\mathbf{i}}_s \times \underbrace{(\nabla \times \mathbf{A})}_{\mathbf{B}\text{ field}}} \right)$$

$$\boxed{\mathbf{F} = -q\nabla\phi}$$

The term due to magnetic field is cancelled exactly by term due the non irrotational part of the electric field!

This is a consequence of the fact that **we are probing the field with a particle that is moving in the same direction of the electron wave and with the same speed**

These conditions are verified for the beam particles but are **not verified for the electrons themselves** → in principle the **electrons do feel the magnetic field**

- The effect is **expected to be small** as the electrons are relatively slow
- **Confirmed by numerical tests** performed by L. Giacomel (to be presented at e-cloud meeting on 20 Aug)

- The **forces acting on a beam particle due to an electron cloud**, can be **conveniently calculated in a boosted reference frame** moving rigidly with the beam.
- In such a reference frame, **charge and current densities are stationary:**
  - Electric and magnetic fields are solution of an **electrostatic and a magnetostatic problem** respectively.
- The **force** acting on the bunch (in the lab frame) is **proportional to the gradient of the scalar potential** and is therefore **irrotational**.
  - This happens since the force due to the **non-irrotational component of the electric field** is **cancelled exactly by the magnetic field term**.
- For a **relativistic beam** the scalar potential can be calculated with good approximation as the solution of a **2D Poisson problem**.
- The **Hamiltonian** of the resulting transformation **can be written as a function of the position coordinates**, showing that the map is **symplectic.**