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Low energy theorem for $\gamma \to 3\pi$:

Surface terms against $\pi - a_1$ mixing. *

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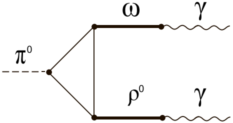
* based on: Osipov, Khalifa, Hiller, e-print 2001.00901 [hep-ph], PRD 101, 034012 (2020)

(a)



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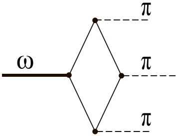

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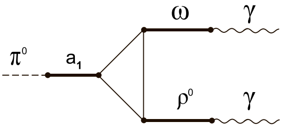

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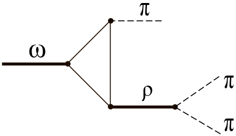

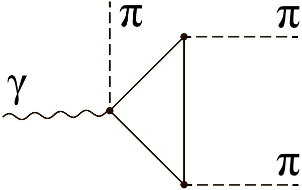

Overview

- Short history and motivation

- FOCUS: surface terms (ST) (arbitrary regularization dependent parameteres) in

- $\pi_0 \to \gamma\gamma$, $a_1 \to \rho\gamma$, $a_1 \to \omega\gamma$

- $\gamma \to 3\pi$ violation of the low energy theorem (LET) for conventional πa_1 diagonalization.

- restoration of LET with gauge covariant diagonalization.

- Complete VMD fails in the anomalous sector.

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Short history and motivation

- Low energy theorem (LET) of current algebra

Adler, Lee, Treiman, Zee (PRD '71), Terentiev (JETP '71), Aviv, Zee (PRD '72)

$$F^{\pi} = e f_{\pi}^{2} F^{3\pi} \tag{1}$$

- $F_{\pi^0 \to \gamma\gamma} = F^{\pi}$ and $F_{\gamma \to \pi^+ \pi^0 \pi^-} = F^{3\pi}$ both taken at vanishing momenta of mesons.

- Wess-Zumino (WZ) (PLB '71): effective action describes all effects of QCD anomalies in low-energy processes with photons and Goldstone bosons.

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- The WZ action gives correct predictions for a set of low-energy processes, e.g., $\pi^0 \to \gamma\gamma$, $\gamma \to 3\pi$ without any reference to the massive vector mesons.

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Inclusion of spin-1 states must not change the predictions of the WZ action.

Questions:

- Is the phenomenological successful concept of vector meson dominance (VMD) still applicable?

- Inclusion of axial vector mesons induce $\pi - a_1$ mixing. How to deal with it?

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– Fujiwara et al. (Prog. Theor. Phys. '85): complete VMD is not valid in either $\pi^0 \to \gamma\gamma$ or $\gamma \to 3\pi$ process.

–Gasiorovicz, Geffen (Rev. Mod. Phys. '69), Volkov, Osipov (JINR '85): The mixing affects hadronic amplitudes.

–Wakamatsu (Ann Phys. '89): reports on a recurrent problem in well known models, such as massive Yang-Mills, the hidden symmetry model, or the NJL model due to $\pi - a_1$ mixing:

violation of LETs involving anomalous processes such as $\gamma \to 3\pi$, $K^+K^- \to 3\pi$.

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- The extension to the case with spin-1 mesons is not unique:

1-Kaymakcalan, Rajeev, Schechter (PRD '84): In the massive Yang-Mills approach the chiral $U(3)_R \times U(3)_L$ group is gauged.

$\rightarrow$ Must take Bardeen's form of the anomaly, Bardeen (PR '69). Problem: the global chiral $U(3)_R \times U(3)_L$ symmetry is broken, even if the external gauge fields are absent.

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2- Fujiwara et al. (Prog. Theor. Phys. '85) avoid this problem: vector mesons are identified as dynamical gauge bosons of the hidden local $U(3)_V$ symmetry.

The WZ action gets an anomaly-free term with vector mesons (homogeneous solution of the inhomogeneous linear differential equation known as the Wess-Zumino consistency condition)

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3-This approach has been generalized (14 new terms) to include the axial vector mesons by Kaiser, Meissner (NPA '90) and is free from the πa_1-mixing effects by construction.

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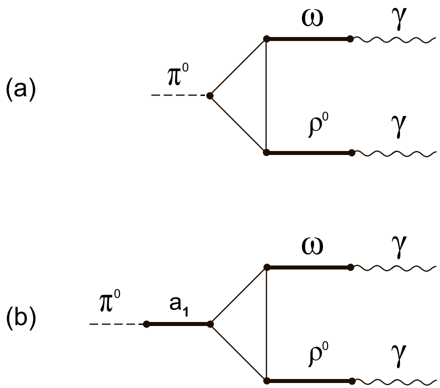
We want to focus on the precise handling of πa_1 mixing in hadron models, involving quark degrees of freedom

and the mechanism that suppresses its effects.

- Osipov, Khalifa, Hiller (PRD '20): We calculate the Low-energy amplitudes $\pi^0 \to \gamma\gamma$ and $\gamma \to 3\pi$ in the framework the Nambu-Jona-Lasinio (NJL) model with spin-1 states.

The πa_1-mixing in the $\pi^0 \to \gamma\gamma$ decay

This process can be solely described by the VMD-type graph (a)



Graphs describing the $\pi^0 \to \gamma\gamma$ decay in the NJL model.

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Contribution (a) given by Lagrangian density WZ (PLB '71), Witten (NPB '83)

$$\mathcal{L}_{\pi\gamma\gamma} = -\frac{1}{8} F^{\pi} \pi^0 \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}, \quad F^{\pi} = \frac{N_c e^2}{12\pi^2 f_\pi}, \quad (2)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad f_\pi = 93\,\text{MeV} \;.$$

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- Recall: in the NJL model one can switch to spin-1 variables without direct photon-quark coupling, as described in the VMD picture.

- $\mathcal{L}_{\pi\gamma\gamma}$ follows from the direct calculation of the $\pi^0\omega\rho$ quark triangle at leading order of a derivative expansion.

- This yields the current-algebra result
$\Gamma(\pi^0 \to \gamma\gamma) = 7.1\,\text{eV}$

Experiment: 7.9 eV.

Contribution due to $\pi - a_1$ mixing

In the NJL model one also has diagram (b), an anomalous AVV quark-loop amplitude

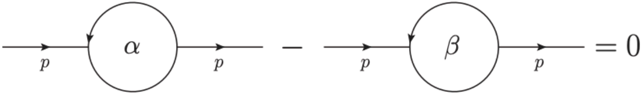
$$\Gamma^{\sigma\mu\nu}(q,p) = -i\frac{N_c g_\rho^3}{16\pi^2} e^{\sigma\mu\nu\alpha}(\chi + p - q)_\alpha + \ldots, \quad (3)$$

$g_\rho \simeq \sqrt{12\pi}$ is the coupling of the $\rho \to \pi\pi$ decay

q, p: outgoing 4-momenta of ω and ρ
χ : arbitrary momentum
σ, μ, ν: Lorentz indices of a_1, ω, ρ.

Surface Terms

- Definition of Momentum Rounting Invariance



$$\int \frac{d^d k}{(2\pi)^d}\left[f(k+\alpha,p) - f(k+\beta,p)\right] = 0, \quad d \in \mathbb{N}$$

$$\int \frac{d^d k}{(2\pi)^d}[exp(\alpha_\sigma \frac{\partial}{\partial k_\sigma}) - exp(\beta_\sigma \frac{\partial}{\partial k_\sigma})]f(k,p) = 0$$

- Considering $f(k, p)$ to be linearly divergent

$$f(k, p) = f_{lin}(k, p) + f_{log}(k, p) + f_{fin}(k, p)$$

- Only the first term violates MRI

$$(\alpha_\sigma - \beta_\sigma) \int \frac{d^d k}{(2\pi)^d} \frac{\partial}{\partial k_\sigma} f_{lin}(k, p) = 0$$

.

$\Gamma^{\sigma\mu\nu}(q,p)$ is finite, resulting from a difference of two linearly divergent amplitudes.

Due to linear divergence, a shift of the integration momentum

$$k_\alpha \to k_\alpha + \chi_\alpha$$

in the quark loop yields essential ambiguity, embodied in arbitrary value of χ.

Parametrize

$$\chi_\alpha = (c_1 - 1)p_\alpha + (c_2 + 1)q_\alpha; \quad c_1,\ c_2 \text{ dimensionless.}$$

Use Ward identities (WI) to fix c_1, c_2 .

For $\pi^0 \to \gamma\gamma$ decay, VMD transitions $\omega \to \gamma$ and $\rho^0 \to \gamma$ require transversality of $\Gamma^{\sigma\mu\nu}$

$$q_\mu \Gamma^{\sigma\mu\nu}(q,p) = 0, \quad p_\nu \Gamma^{\sigma\mu\nu}(q,p) = 0, \qquad (4)$$

$$\downarrow$$

$$\chi_\alpha = q_\alpha - p_\alpha \qquad c_1 = c_2 = 0$$

The AVV triangle (b) does not contribute at LO of the derivative expansion to the amplitude $\pi^0 \to \gamma\gamma$.

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Relate diagram (b) to Landau-Yang theorem:

a massive unit spin particle cannot decay into two on shell massless photons, Landau (Dokl. Akad. Nauk '48), Yang (PR '56)

$a_1 \to \gamma\gamma$ decay is forbidden.

$$\downarrow$$

The axial-vector channel $\pi^0 \to a_1 \to \gamma\gamma$ induced by the πa_1-mixing is also forbidden.

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Generalization to NLO in powers of q and p of $\Gamma^{\sigma\mu\nu}(q,p)$

$\downarrow$

Effective Lagrangian for the hadronic $a_1\omega\rho$ vertex

$$\mathcal{L}_{a_1\omega\rho} = \frac{N_c g_\rho^3}{32\pi^2} e^{\sigma\mu\nu\alpha} \left\{ a^i_{1\sigma} \left(c_1 \omega_\mu \rho^i_{\alpha\nu} + c_2 \rho^i_\nu \omega_{\alpha\mu} \right) \right. \tag{5}$$
$$\left. - \frac{1}{2m^2} \left[\rho^i_{\alpha\beta} \left(\omega_{\sigma\nu} a^i_{1\beta\mu} + \omega_{\beta\mu} a^i_{1\sigma\nu} \right) + 2\rho^i_{\sigma\nu} a^i_{1\mu} \partial_\beta \omega_{\beta\alpha} \right] \right\}.$$

$b_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu, \qquad b = \omega, \rho^i, a^i_1$; isospin index i.

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- c_1, c_2 are not intrinsic to the triangle graph, but depend on the context in which they arise:
- When both vector ω and ρ mesons couple to photons the gauge symmetry is conserved if and only if $c_1 = c_2 = 0$.
- For $a_1 \to \gamma\rho$ decay: preserve transversality of the $\omega \to \gamma$ index and may abandon transversality related to the ρ field, i.e. the choice is $c_1 = 0, c_2 \neq 0$.
- Similarly $c_1 \neq 0, c_2 = 0$ for $a_1 \to \gamma\omega$ decay.

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Correspondence with bibliography, examples:

$c_1 = c_2 = 0$

- Volkov (Annals Phys '84): The three-derivative part alone was used to estimate widths $\Gamma(a_1 \to \gamma\rho) = 34\,\text{keV}$ and $\Gamma(a_1 \to \gamma\omega) = 300\,\text{keV}$.

$c_1 = c_2 \neq 0$,

- Kaiser, Meissner (NPA '90): Conservation of the axial-vector current in the AVV-triangle $\to$ the contribution of the diagram (b) vanishes due an accidental antisymmetry under the exchange of fields $\omega_\mu \leftrightarrow \rho^0_\mu$.

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Summary: Use the hadron vertex $a_1\omega\rho$ in the form (5), where parameters c_1, c_2 should be subsequently specified.

There is no a priori physical process associated with these three particles from which one could extract the values of c_1 and c_2.

Fix c_1 ,c_2 on theoretical or/and phenomenological grounds when (5) is an element of the Feynman diagram corresponding to a real physical process.

Jackiw (IJMP B, '00): "When radiative corrections are finite but undetermined"; our works on *Implicit Regularization*, e.g. Baeta-Scarpelli, Sampaio, Hiller, Nemes, PRD '00; - Batista, Hiller, Cherchiglia, Sampaio, PRD '18

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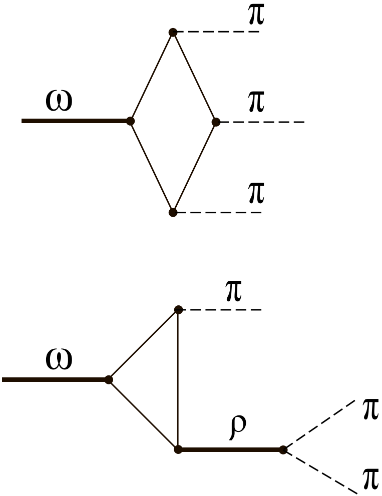
How does this work for the $\omega \to 3\pi$ amplitude?

(a)

(b)



(a) and (b) representative of full set of possible diagrams without and with 1, 2, and 3 πa_1-mixing effects on the pion line.

$$A_{\omega\to 3\pi} = -\frac{N_c g_\rho}{4\pi^2 f_\pi^3} \epsilon_{\mu\nu\alpha\beta} \epsilon^\mu(q) p_0^\nu p_+^\alpha p_-^\beta F_{\omega\to 3\pi}, \qquad (6)$$

p_0, p_+, p_-: pion momenta; $\epsilon^\mu(q)$: ω polarization; In color: new.

$a = \frac{m_\rho^2}{g_\rho^2 f_\pi^2}$

$$F_{\omega\to 3\pi} = \left(1 - \frac{3}{a} + \frac{3}{2a^2} + \frac{1}{8a^3}\right) + \left(1 - \frac{c}{2a}\right) \sum_{k=0,+,-} \frac{g_\rho^2 f_\pi^2}{m_\rho^2 - (q-p_k)^2}. \quad (7)$$

1st parentheses: box diagrams with 0, 1, 2, and 3 πa_1-transitions.

Last term: ρ-exchange graphs, where $c = c_1 - c_2$ controls the magnitude of an arbitrary local part of the AVV-quark-triangle.

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Low-energy limit in (7),

$$\sum_{k=0,+,-} \frac{g_\rho^2 f_\pi^2}{m_\rho^2-(q-p_k)^2} \to \frac{3}{a}$$

$$\downarrow$$

• Full cancellation at order $1/a$, Wakamatsu (Ann Phys '89).

•The surface term contributes at order of $1/a^2$. Without it ($c_1 = c_2$) we reproduce the πa_1-mixing effect found in Wakamatsu ('89) at that order.

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- Can one use c to cancel all πa_1-mixing effects?

$\rightarrow c = 1 + 1/(12a)$

- Not supported phenomenologically:

$\Gamma(\omega \rightarrow \pi^+\pi^0\pi^-) = 3.2\,\mathrm{MeV}$ too low compared to experimental value
$\Gamma(\omega \rightarrow \pi^+\pi^0\pi^-) = 7.57 \pm 0.13\,\mathrm{MeV}$.

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- Cohen (PLB '89): The chiral WI for $\gamma \to 3\pi$ process require that the chiral triangle and the box anomaly contribute to the total amplitude with the weights

$$A^{tot}_{\gamma\to 3\pi} = \frac{3}{2}A^{AVV} - \frac{1}{2}A^{VAAA}, \tag{8}$$

where A^{VAAA} is the point $\gamma \to \omega \to \pi\pi\pi$ amplitude and A^{AVV} is the amplitude for the $\gamma \to \omega \to \pi\rho \to \pi\pi\pi$ process.

- This result is consistent both with the WIs and the KSFR relation, which arises in NJL at $a = 2$.

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That's what one obtains from eq. (7) if:

1- one neglects the terms of order $1/a^2$ and higher in the box contribution

2- puts $c = 0$ in the ρ-exchange term.

- If c is chosen to cancel πa_1-mixing effects, these amplitudes contribute with a relative weight of $-7/64$ and $71/64$, respectively.

• Observation I: the surface term c cannot be used to resolve the πa_1-mixing puzzle. Its value is unambiguously fixed by the chiral WI, which require that $c = 0$.

• Observation II: This pattern has been considered in Schechter (PRD '84), Kaiser (NPA '90), Wakamatsu (Ann Phys '89), and reproduces well the phenomenological value of the width.

• Observation III: If VMD is a valid theoretical hypothesis, $\gamma \to \omega \to 3\pi$ contains contributions from πa_1-mixing which violate the LET (1)

$$A_{\gamma\to 3\pi} = -F^{3\pi} e_{\mu\nu\alpha\beta}\epsilon^{\mu}(q)p_0^{\nu}p_+^{\alpha}p_-^{\beta}, \qquad (9)$$

$$F^{3\pi} = \frac{N_c e}{12\pi^2 f_\pi^3}\left(1+\frac{3}{2a^2}+\frac{1}{8a^3}\right) \neq \frac{N_c e}{12\pi^2 f_\pi^3}. \qquad (10)$$

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In the following we will show that it is possible to combine the phenomenologically successful value $c = 0$ with a full cancellation of πa_1-mixing effects within the NJL approach.

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The πa_1-mixing and $\gamma \to 3\pi$ amplitude

- Recall: the πa_1 diagonalization is usually performed by a linearized transformation of the axial vector field.

In the NJL:

$$a_\mu \to a_\mu + \frac{\partial_\mu \pi}{a g_\rho f_\pi}, \tag{11}$$

where $\pi = \tau_i \pi^i$, $a_\mu = \tau_i a^i_\mu$ and τ_i are the $SU(2)$ Pauli matrices.

- This replacement that has been used in the calculations above.

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- Osipov (JTEP '18), Osipov, Kahlifa (PRD '18):

- The gauge noncovariant replacement (11) violates gauge symmetry, e.g. the anomalous low energy amplitude $a_1 \to \gamma\pi^+\pi^-$ decay is not transverse.

- Gauge symmetry of the $a_1 \to \gamma\pi^+\pi^-$ amplitude can be restored if one uses the covariant derivative $\mathcal{D}_\mu \pi$

$$a_\mu \to a_\mu + \frac{\mathcal{D}_\mu \pi}{a g_\rho f_\pi}, \quad \mathcal{D}_\mu \pi = \partial_\mu \pi - ieA_\mu [Q, \pi]. \quad (12)$$

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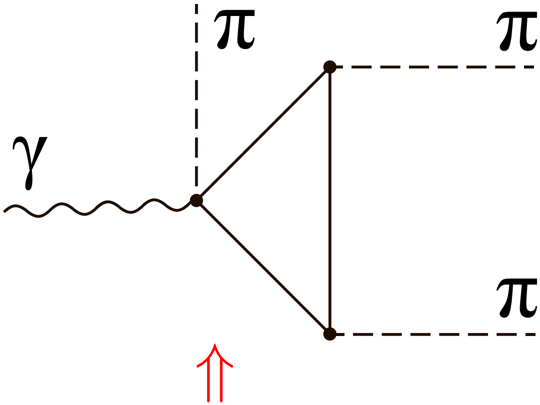
- Osipov, Hiller, Zhang (PRD '18, MPLA'19):

- Generalization of $\mathcal{D}_\mu \pi$ to electroweak sector

- Gauge covariant derivative is important for processes with breaking of the intrinsic parity $\sim \epsilon_{\mu\nu\alpha\beta}$

- It does not affect current-algebra theorems related to the non-anomalous part of the action.

- Osipov, Khalifa, Hiller (PRD ’20):
- $\mathcal{D}_\mu \pi$ contributes with additional diagram to $A_{\gamma\to 3\pi}$ (with 3 πa_1-transitions):



The vertex $\bar{q}q\gamma\pi$ induces a deviation from the complete VMD.

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It is an anomalous AAA amplitude to $\gamma \to 3\pi$

Shift ambiguity of formal linear divergence of integral $\longrightarrow$ undetermined 4-vector υ_ρ,

$$A = -\frac{N_c e}{4\pi^2 f_\pi^3} \epsilon_{\mu\nu\sigma\rho} \epsilon^\mu(q) p_0^\nu (p_+ + p_-)^\sigma \left(\frac{\upsilon^\rho}{4a^3} \right) \quad (13)$$

.

Parametrize

$$v_\mu = b_1 q_\mu + b_2 (p_+ - p_-)_\mu + b_3 (p_+ + p_-)_\mu$$

where only term $\sim b_2$ survives in A and yields an extra contribution to $F^{3\pi}$

b_2 is dimensionless and as yet undetermined. Fix it using LET (1), requiring that the unwanted terms ($\sim 1/a^2, \sim 1/a^3$) in (10) vanish

$$b_2 = a + \frac{1}{12} = 1.92. \tag{14}$$

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Conclusions

The solution to the breaking of low energy theorem (LET) by πa_1 mixing terms in NJL proceeds via:

- Gauge covariant diagonalization of the mixing
- $\longrightarrow$ New vertex $\gamma\pi\bar{q}q$, beyound VMD.
- It contributes in an AAA triangle diagram as pure surface term (ST).
- Careful analysis of all ST shows that this ST is the crucial element needed to restore the LET.

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and

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The Model: Chiral U(2)xU(2) NJL

$$L = \bar{q}\left(i\gamma^{\mu} D_{\mu} - \hat{m}\right)q + L_S + L_V + L_{EW}$$

$$D_{\mu} = \partial_{\mu} - ieQA_{\mu} - \frac{igZ_{\mu}}{\cos\theta_W}\left(T_3 P_L - Q\sin^2\theta_W\right) - igP_L\left(T_+ W_{\mu}^{+} + T_- W_{\mu}^{-}\right)$$

$$L_S = \frac{G_S}{2}\left[\left(\bar{q}\tau_a q\right)^2 + \left(\bar{q}i\gamma_5\tau_a q\right)^2\right]$$

$$L_V = -\frac{G_V}{2}\left[\left(\bar{q}\gamma_{\mu}\tau_a q\right)^2 + \left(\bar{q}\gamma_{\mu}\gamma_5\tau_a q\right)^2\right]$$

Chiral U(2)xU(2) and Gauge SU(2)xU(1) Transformations of Quark Fields

$$\delta_c q = i(\alpha_c + \gamma_5 \beta_c) q$$

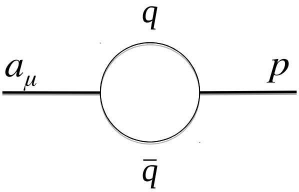
$$\alpha_c = \frac{1}{2}\alpha_c^a \tau_a, \quad \beta_c = \frac{1}{2}\beta_c^a \tau_a$$

$$\delta_g q_R = i e \alpha_g Q q_R$$

$$\delta_g q_L = i(\omega_g + e \alpha_g Y_L) q_L$$

$$\alpha_g = \alpha_g(x) \qquad \omega_g = \frac{1}{2}\vec{\omega}_g(x)\vec{\tau}$$

The Essence of the Problem



$$\longrightarrow \quad m a^{\mu} \partial_{\mu} p$$

$$a_{\mu} \rightarrow a_{\mu} + \kappa m \partial_{\mu} p$$

$$D_m = i\gamma^{\mu} D_{\mu} - m + s + i\gamma_5 p + \gamma^{\mu} v_{\mu} + \gamma^{\mu}\gamma_5 \left(a_{\mu} + \kappa m \partial_{\mu} p\right)$$

This breaks the gauge symmetry in the presence of electroweak interactions and changes chiral transformations of spin-1 fields .

Chiral-Covariant Form of the pi-a1 Diagonalization

$$a_\mu \to a_\mu + \frac{\kappa}{2}\left(\{p, \partial_\mu \bar{s}\} - \{\bar{s}, \partial_\mu p\}\right)$$

$$v_\mu \to v_\mu + i\,\frac{\kappa}{2}\left([p, \partial_\mu p] + [\bar{s}, \partial_\mu \bar{s}]\right)$$

$$\delta_c a_\mu = i[\alpha_c, a_\mu] + i[\beta_c, v_\mu]$$

$$\delta_c v_\mu = i[\alpha_c, v_\mu] + i[\beta_c, a_\mu]$$

[1] A.A. Osipov, B. Hiller, Phys. Rev. D62, 114013 (2000).
[2] A.A. Osipov, M. Sampaio, B. Hiller, Nucl. Phys. A703, 378 (2002).

Gauge-Covariant Form of the pi-a1 Diagonalization

$$a_\mu \to a_\mu + \frac{\kappa}{2}\left(\{p, D_\mu \bar{s}\} - \{\bar{s}, D_\mu p\}\right) \equiv a_\mu + \frac{\kappa}{2} Y_\mu$$

$$v_\mu \to v_\mu + i\frac{\kappa}{2}\left([p, D_\mu p] + [\bar{s}, D_\mu \bar{s}]\right) \equiv v_\mu + \frac{\kappa}{2} X_\mu$$

$$D_\mu p = \partial_\mu p - i[N_\mu, p] - \{K_\mu, \bar{s}\}$$

$$D_\mu \bar{s} = \partial_\mu \bar{s} - i[N_\mu, \bar{s}] + \{K_\mu, p\}$$

$$N_\mu = \frac{1}{2}\left(g \bar{A}_\mu + g' B_\mu T_3\right)$$

$$K_\mu = \frac{1}{2}\left(g \bar{A}_\mu - g' B_\mu T_3\right)$$

$$\delta_g D_\mu p = i[\theta_g, D_\mu p] - \{\beta_g, D_\mu \bar{s}\}$$

$$\delta_g D_\mu \bar{s} = i[\theta_g, D_\mu \bar{s}] + \{\beta_g, D_\mu p\}$$

$$\theta_g \equiv \frac{1}{2}\left(\omega_g + e\alpha_g T_3\right)$$

$$\beta_g \equiv -\frac{1}{2}\left(\omega_g - e\alpha_g T_3\right)$$

[1] A.A. Osipov, B. Hiller, P.M. Zhang, Phys. Rev. D98, 113007 (2018).

Hidden Symmetries of the Non-Anomalous Part of the Effective Action

The real part of the mesonic effective action is not affected by the covariant form of the pi-a1 diagonalization. It follows from the Chisholm's theorem and an existence of the hidden symmetry under the gauge and chiral transformations

$$\delta_g a_\mu = i[\theta_g, a_\mu] + i[\beta_g, v_\mu] - i\kappa m[\partial_\mu \theta_g, p] + \kappa m \partial_\mu \{\beta_g, \bar{s}\}$$

$$\delta_g v_\mu = i[\theta_g, v_\mu] + i[\beta_g, a_\mu] + i\kappa m[\beta_g, \partial_\mu p]$$

[1] A.A. Osipov, B. Hiller, P.M. Zhang, arXiv:1811.02991 [hep-ph] (2018).

$$\delta_c a_\mu = i[\alpha_c, a_\mu] + i[\beta_c, v_\mu] + \kappa m\{\beta_c, \partial_\mu \bar{s}\}$$

$$\delta_c v_\mu = i[\alpha_c, v_\mu] + i[\beta_c, a_\mu] + i\kappa m[\beta_c, \partial_\mu p]$$

[2] A.A. Osipov, M.K. Volkov, Ann. Phys. 382, 50 (2017).

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It is an anomalous AAA amplitude to $\gamma \to 3\pi$

$$A = \frac{N_c e}{4a^3 f_\pi^3} \left\{ p_-^\sigma [J_{\mu\nu\sigma}(p_0, p_-) - J_{\mu\sigma\nu}(p_-, p_0)] + p_+^\sigma [J_{\mu\nu\sigma}(p_0, p_+) - J_{\mu\sigma\nu}(p_+, p_0)] \right\} \epsilon^\mu(q) p_0^\nu . \quad (15)$$

Low energy expansion of the quark loop integral $J_{\mu\nu\sigma}$ starts from a linear term

$$J_{\mu\nu\sigma}(p_0, p_-) = \frac{1}{24\pi^2} e_{\mu\nu\sigma\rho} (p_0 - p_- - 3\upsilon)^\rho + \dots \quad (16)$$

Shift ambiguity of formal linear divergence of integral $\longrightarrow$ undetermined 4-vector υ_ρ,

$$A = -\frac{N_c e}{4\pi^2 f_\pi^3} \epsilon_{\mu\nu\sigma\rho} \epsilon^\mu(q) p_0^\nu (p_+ + p_-)^\sigma \left(\frac{\upsilon^\rho}{4a^3} \right) \quad (17)$$

.

Parametrize

$$v_\mu = b_1 q_\mu + b_2(p_+ - p_-)_\mu + b_3(p_+ + p_-)_\mu$$

where only term $\sim b_2$ survives in A and yields an extra contribution to $F^{3\pi}$

$$\Delta F^{3\pi} = \frac{N_c e}{12\pi^2 f_\pi^3}\left(\frac{-3b_2}{2a^3}\right), \qquad (18)$$

b_2 is dimensionless and as yet undetermined. Fix it using LET (1), requiring that the unwanted terms ($\sim 1/a^2, \sim 1/a^3$) in (10) vanish

$$b_2 = a + \frac{1}{12} = 1.92. \qquad (19)$$