# Vector space of Feynman integrals and Intersection theory

Luca Mattiazzi

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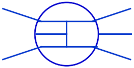
Institute of Nuclear Physics,
Polish Academy of Sciences, Kraków

Based on:

- **Pierpaolo Mastrolia** and **Sebastian Mizera**
  *Feynman integrals and Intersection theory*
  JHEP 1902 (2019) 139
- **H.Frellesvig, F.Gasparotto, S.Laporta, M.Mandal, P. Mastrolia, L.M., S.Mizera**
  *Decomposition of Feynman Integrals on the Maximal Cut by Intersection Numbers*
  JHEP 1905 (2019) 153
- **H.Frellesvig, F.Gasparotto, M.Mandal, P. Mastrolia, L.M., S.Mizera**
  *Vector Space of Feynman Integrals and Multivariate Intersection Numbers*
  Phys.Rev.Lett. 123 (2019) no.20, 201602

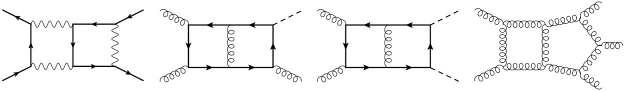
# Introduction

Scattering amplitudes are built out of many Feynman integrals:



$$I_{a_1,\dots,a_N} = \int \prod_{i=1}^{L} d^d k_i \frac{1}{D_1^{a_1}\dots D_N^{a_N}}$$

State of the art calculations at 2 loop, such as



,

has $\mathcal{O}(10000)$ integrals. Needs to evaluate them all? No!
Linear relation among integrals: **Integration By Parts Identities - IBPs**

[Chetyrkin, Tkachov (1981)]
[Laporta (2001)]...

$$\int \prod_{i=1}^{L} d^d k_i \frac{\partial}{\partial k_j^\mu}\left(\frac{v^\mu}{D_1^{a_1}\dots D_N^{a_N}}\right) = 0 \Rightarrow c_1 I_{a_1+1,\dots,a_N} + \dots + c_N I_{a_1,\dots,a_N+1} = 0$$

**Integrals** related by a total derivative

# Introduction

**Linear System** $\Rightarrow$ **Gauss Elimination** $\Rightarrow$ **Master Integrals** $\{J_i\}$ - **MIs**

Decomposition of an Integral in terms of **MIs**

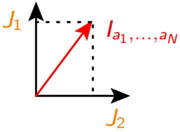
$$I_{a_1,\ldots,a_N} = \sum_{i=1}^{\nu} c_i J_i$$

**Drawbacks:**

- $\#$ equation grows dramatically
- manipulation large expressions

Possible bottleneck of multiloop calculation

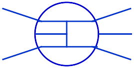
Can we directly project integrals on MI directly?



# Intersection theory

In **Baikov** representation

[Aomoto, Kita, Matsumoto, Mizera, ...]
[Mastrolia, Mizera (2018)]



$$I_{a_1,\ldots,a_N} = \int_{\mathcal{C}} u(z)\,\varphi(z)$$

$$u(z) = \prod_i P_i(z)^{\gamma_i} \;\Rightarrow\; u(z) \text{ multivalued function s.t. } P_i\,(\partial\mathcal{C}) = 0$$

$$\varphi(z) = \hat{\varphi}(z)\,\mathrm{d}z \Rightarrow \varphi(z) \text{ single valued form}$$

**Total derivative** translates to

$$\int_{\mathcal{C}} \mathrm{d}(u\,\varphi) = \int_{\mathcal{C}} \mathrm{d}u\,\varphi + u\,\mathrm{d}\varphi = \int_{\mathcal{C}} u\left(\frac{\mathrm{d}u}{u} + \mathrm{d}\right)\varphi$$

$$= \int_{\mathcal{C}} u\,(\omega + \mathrm{d})\,\varphi = \int_{\mathcal{C}} u\,\nabla_\omega\varphi = 0 \quad \mathcal{P}_\omega = \{z\,|\,z \text{ is a pole of } \omega\}$$

rewriting **Integration by Parts Identities** as

$$\int_{\mathcal{C}} u\,(\varphi + \nabla_\omega\xi) = \int_{\mathcal{C}} u\,\varphi \quad\Rightarrow\quad \varphi \sim \varphi + \nabla_\omega\xi$$

**Equivalence class** between forms defines the Twisted cohomology group.

$$_{\omega}\langle\varphi| \equiv \{\varphi|\nabla_\omega\varphi = 0\}/\{\nabla_\omega\xi\} = H^n_\omega$$

**Key** relation between IBPs and Twisted Cohomology

[Aomoto (1975)]
[Lee, Pomeransky (2013)]

$$\begin{aligned}\nu &= dim\,(H^n_\omega) \\ &= \chi(X) = (-1)^n\,(n+1-\chi(\mathcal{P}_\omega)\,) \\ &= \{\#\text{ of solutions of } \omega = 0\}\end{aligned}$$

**Contours** have similar structure

$$\int_{\mathcal{C}} u\varphi = \int_{\mathcal{C}+\partial_\omega g} u\varphi \quad \Rightarrow \quad |\mathcal{C}] = H^\omega_n \text{Twisted Homology group}$$

Feynman integrals are **pairing**

$$\langle\varphi|\mathcal{C}] = \underbrace{\int_{\mathcal{C}} u(z)}_{cycle}\overbrace{\varphi(z)}^{cocycle}$$

Dual integrals

$$[\mathcal{C}|\varphi\rangle = \int_{\mathcal{C}} u^{-1}(z)\,\varphi(z) = \int_{\mathcal{C}} u^{-1}\,(\varphi + \nabla_{-\omega}\xi)$$

$$\Rightarrow |\varphi\rangle = H^n_{-\omega} \quad , \quad [C| = H_n^{-\omega}$$

# Twisted intersection number

Seeking relations between integrals (forms) $\Rightarrow$ **Intersection number**

[Mastrolia, Mizera (2018)]
[Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, **L.M.**, Mizera (2019)]

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{(2\pi i)^n} \int \iota(\varphi_L) \wedge \varphi_R$$

IBP built in naturally within such formalism.

$$\langle \varphi_L + \nabla_\omega \xi | \varphi_R \rangle = \langle \varphi_L | \varphi_R + \nabla_{-\omega} \xi \rangle = \langle \varphi_L | \varphi_R \rangle$$

Define basis of independent form and dual form

$$\langle e_1|, \quad \langle e_2|, \quad \cdots, \quad \langle e_\nu| \quad , \quad |h_1\rangle, \quad |h_2\rangle, \quad \cdots, \quad |h_\nu\rangle$$

Build the matrix

$$\mathbf{M} = \begin{pmatrix} \langle \varphi_L | \varphi_R \rangle & \langle \varphi_L | h_1 \rangle & \langle \varphi_L | h_2 \rangle & \dots & \langle \varphi_L | h_\nu \rangle \\ \langle e_1 | \varphi_R \rangle & \langle e_1 | h_1 \rangle & \langle e_1 | h_2 \rangle & \dots & \langle e_1 | h_\nu \rangle \\ \langle e_2 | \varphi_R \rangle & \langle e_2 | h_1 \rangle & \langle e_2 | h_2 \rangle & \dots & \langle e_2 | h_\nu \rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle e_\nu | \varphi_R \rangle & \langle e_\nu | h_1 \rangle & \langle e_\nu | h_2 \rangle & \dots & \langle e_\nu | h_\nu \rangle \end{pmatrix} \equiv \begin{pmatrix} \langle \varphi_L | \varphi_R \rangle & \mathbf{A}^\intercal \\ \mathbf{B} & \mathbf{C} \end{pmatrix}$$

# Master Decomposition Formula

$\langle \varphi_L |$ **depends** on the basis element

$$\det \mathbf{M} = \det \mathbf{C} \left( \langle \varphi_L | \varphi_R \rangle - \mathbf{A}^\intercal \, \mathbf{C}^{-1} \, \mathbf{B} \right) = 0$$

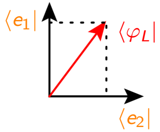
$$\Downarrow$$

$$\langle \varphi_L | \varphi_R \rangle = \mathbf{A}^\intercal \, \mathbf{C}^{-1} \, \mathbf{B} = \sum_{i,j=1}^{\nu} \langle \varphi_L | h_j \rangle \, (\mathbf{C}^{-1})_{ji} \, \langle e_i | \varphi_R \rangle$$

Since $| \varphi_R \rangle$ is arbitrary

$$\langle \varphi_L | = \sum_{i,j=1}^{\nu} \langle \varphi_L | h_j \rangle \, \left( \mathbf{C}^{-1} \right)_{ji} \langle e_i | \overset{\mathbf{C}_{ij} = \delta_{ij}}{=} \sum_{i=1}^{\nu} \langle \varphi_L | h_i \rangle \, \langle e_i | = \sum_{i=1}^{\nu} c_i \, J_i$$

**Direct** projection



# Univariate computation

$$\langle\varphi_L|\varphi_R\rangle = \frac{1}{(2\pi i)}\int \iota(\varphi_L)\wedge\varphi_R = \sum_{p\in\mathcal{P}_\omega} \underset{z=p}{Res}\left(\left(\nabla_\omega^{-1}\varphi_L\right)\varphi_R\right) = -\sum_{p\in\mathcal{P}_\omega} \underset{z=p}{Res}\left(\varphi_L\nabla_{-\omega}^{-1}\varphi_R\right)$$

$$= \sum_{p\in\mathcal{P}_\omega} \underset{z=p}{Res}\left(\psi_p\,\varphi_R\right) \overset{dLog}{=} \sum_{p\in\mathcal{P}_\omega} \frac{Res_{z=p}\left(\varphi_L\right)\,Res_{z=p}\left(\varphi_R\right)}{Res_{z=p}\left(\omega\right)}$$

$$\psi_p = \nabla_\omega^{-1}\varphi_L \quad\Rightarrow\quad (\mathrm{d}+\omega)\,\psi_p = \varphi_L$$

only **local** solution to $\psi_p$ needed $\Rightarrow$ power series ansatz

$$\psi_p = \sum_{j=\min}^{\max} \psi_p^{(j)}\tau^j + \mathcal{O}\left(\tau^{\max+1}\right)$$

$\psi_p$ obtained by **pattern mathcing**

Sanity check:

$$\langle\nabla_\omega\xi|\varphi_R\rangle = \sum_{p\in\mathcal{P}_\omega} \underset{z=p}{Res}\left(\xi\varphi_R\right)$$

# Univariate computation

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{(2\pi i)} \int \iota(\varphi_L) \wedge \varphi_R = \sum_{p \in \mathcal{P}_\omega} \underset{z=p}{Res} \left( \left( \nabla_\omega^{-1} \varphi_L \right) \varphi_R \right) = - \sum_{p \in \mathcal{P}_\omega} \underset{z=p}{Res} \left( \varphi_L \nabla_{-\omega}^{-1} \varphi_R \right)$$

$$= \sum_{p \in \mathcal{P}_\omega} \underset{z=p}{Res} \left( \psi_p \, \varphi_R \right) \overset{dLog}{=} \sum_{p \in \mathcal{P}_\omega} \frac{Res_{z=p} \left( \varphi_L \right) \; Res_{z=p} \left( \varphi_R \right)}{Res_{z=p} \left( \omega \right)}$$

$$\psi_p = \nabla_\omega^{-1} \varphi_L \quad \Rightarrow \quad (\mathrm{d} + \omega) \, \psi_p = \varphi_L$$

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$$\psi_p = \sum_{j=\min}^{\max} \psi_p^{(j)} \tau^j + \mathcal{O} \left( \tau^{\max+1} \right)$$

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Sanity check:

$$\langle \nabla_\omega \xi | \varphi_R \rangle = \sum_{p \in \mathcal{P}_{\xi \varphi_R}} \underset{z=p}{Res} \left( \xi \varphi_R \right)$$

# Univariate computation

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{(2\pi i)} \int \iota(\varphi_L) \wedge \varphi_R = \sum_{p \in \mathcal{P}_\omega} \underset{z=p}{Res} \left( \left( \nabla_\omega^{-1} \varphi_L \right) \varphi_R \right) = - \sum_{p \in \mathcal{P}_\omega} \underset{z=p}{Res} \left( \varphi_L \nabla_{-\omega}^{-1} \varphi_R \right)$$

$$= \sum_{p \in \mathcal{P}_\omega} \underset{z=p}{Res} \left( \psi_p \, \varphi_R \right) \overset{dLog}{=} \sum_{p \in \mathcal{P}_\omega} \frac{Res_{z=p} \left( \varphi_L \right) \; Res_{z=p} \left( \varphi_R \right)}{Res_{z=p} \left( \omega \right)}$$

$$\psi_p = \nabla_\omega^{-1} \varphi_L \quad \Rightarrow \quad (\mathrm{d} + \omega) \, \psi_p = \varphi_L$$

only **local** solution to $\psi_p$ needed $\Rightarrow$ power series ansatz

$$\psi_p = \sum_{j=\min}^{\max} \psi_p^{(j)} \tau^j + \mathcal{O}\left( \tau^{\max+1} \right)$$

$\psi_p$ obtained by **pattern mathcing**

Sanity check:

$$\langle \nabla_\omega \xi | \varphi_R \rangle = \sum_{p \in \mathcal{P}_{\xi \varphi_R}} \underset{z=p}{Res} \left( \xi \varphi_R \right) = 0$$

# Reduction on the Maximal Cut

**Maximal Cut** $\Rightarrow$ univariate integral representation


$$u = \left(\frac{1}{4} z^2 (s-2z-1)(s-2z+3)\right)^{\frac{d-5}{2}}$$

$\omega = d\log u = 0$ 2 sols. $\Rightarrow$ 2 MIs $\nu = 2$

The MIs chosen as

$$J_1 = I_{1,1,1,1,1,1,1,1,1;0} = \langle e_1|\mathcal{C}] = \langle 1|\mathcal{C}] \quad \& \quad J_2 = I_{1,1,1,1,1,1,1,1,1;-1} = \langle e_2|\mathcal{C}] = \langle z|\mathcal{C}]$$

**Decompose**

$$I_{1,1,1,1,1,1,1,1,1;-2} = \langle \varphi|\mathcal{C}] = \langle z^2|\mathcal{C}]$$

**Compute**

$$\langle \varphi|e_i\rangle \quad i = 1,2 \quad \& \quad \mathbf{C}_{ij} = \langle e_i|e_j\rangle \quad i,j = 1,2$$

Plug in the **Master Decomposition Formula**

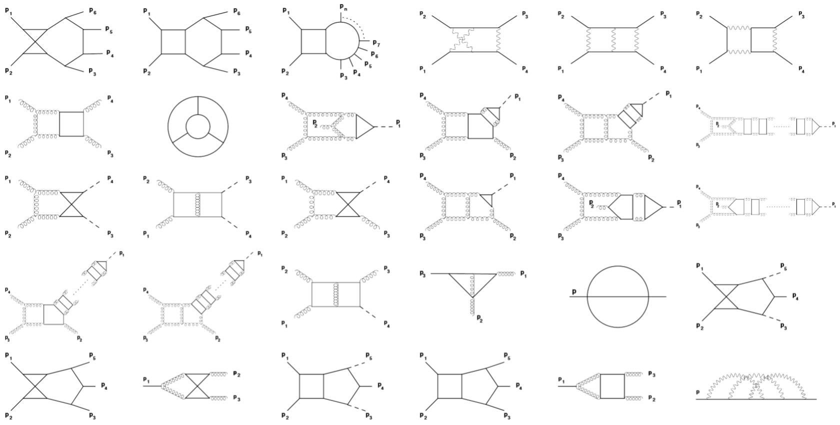
$$c_i = \sum_{j=1}^{2} \langle \varphi|e_j\rangle \left(\mathbf{C}^{-1}\right)_{ji} \quad c_1 = -\frac{(d-4)(s-1)(s+3)}{4(2d-7)} \quad c_2 = \frac{(3d-11)(s+1)}{2(2d-7)}$$

agreement with SYS

# Examples

$\mathcal{O}(30)$ examples checked on the maximal cut

[Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, **L.M.**, Mizera (2019)]



# Multivariate :: Fibration

Univariate: **known**; Multivariate?

$$\int u(z)\varphi_L(z) \quad \overset{?}{\Leftarrow} \quad \int \mathrm{d}z_n \int \mathrm{d}z_{n-1} \cdots \int \mathrm{d}z_1\, u(z_1,\ldots,z_n)\,\varphi_L(z_1,\ldots,z_n)$$

$$\int \mathrm{d}z_n\, \varphi_L^{(n)}(z_n)\, u^{(n)}(z_n)$$

**Integration** connects them. How does it translates to our formalism?

$$\int u(\mathbf{z})\varphi_L(\mathbf{z}) = \int \mathrm{d}z_n \cdots \int \mathrm{d}z_2 \underbrace{\int \mathrm{d}z_1\, u(z_1,\ldots,z_n)\,\hat{\varphi}_L(z_1,\ldots,z_n)}_{\exists\,\nu_1 \text{ MI in } z_1}$$

$$= \int \mathrm{d}z_n \cdots \int \mathrm{d}z_3 \underbrace{\int \mathrm{d}z_2 \sum_i c_i^{(1)}(z_n,\ldots,z_2)\, J_i^{(1)}(z_n,\ldots,z_2)}_{\exists\,\nu_2 \text{ MI in } z_2}$$

$$\vdots$$

$$= \int \mathrm{d}z_n \sum_i c_i^{(n)}(z_n) J_i^{(n)}(z_n)$$

# Multivariate :: Fibration

$$\int u(\mathbf{z})\varphi_L(\mathbf{z}) = \int \mathrm{d}z_n \sum_i c_i^{(n)}(z_n) J_i^{(n)}(z_n) = \int \mathrm{d}z_n \sum_i \varphi_{L,i}^{(n)}(z_n) \int u(\mathbf{z})\, e_i^{(\mathbf{n-1})}(\mathbf{z})$$

$$\Downarrow$$

[Mizera (2019)]
[Frellesvig, Gasparotto, Mandal, Mastrolia, **L.M.**, Mizera (2019)]

$$\langle\varphi_L| = \sum_{i=1}^{\nu_{\mathbf{n-1}}} \langle\varphi_{L,i}^{(n)}| \wedge \langle e_i^{(\mathbf{n-1})}|$$

$\varphi_{L,i}^{(n)}$ **coefficient** of the reduction

$$\langle\varphi_{L,i}^{(n)}| = \sum_j \langle\varphi_L|h_j^{(\mathbf{n-1})}\rangle \left(\mathbf{C}_{(\mathbf{n-1})}^{-1}\right)_{ji}$$

Same decomposition on the **dual** basis

$$|\varphi_R\rangle = \sum_{i=1}^{\nu_{\mathbf{n-1}}} |\varphi_{R,i}^{(n)}\rangle \wedge |h_i^{(\mathbf{n-1})}\rangle \quad \Rightarrow \quad |\varphi_{R,i}^{(n)}\rangle = \sum_j \left(\mathbf{C}_{(\mathbf{n-1})}^{-1}\right)_{ij} \langle e_j^{(\mathbf{n-1})}|\varphi_R\rangle$$

# Multivariate :: Computation

[Mizera (2019)]
[Frellesvig, Gasparotto, Mandal, Mastrolia, **L.M.**, Mizera (2019)]

We are interested in

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{(2\pi i)^2} \int_X \iota(\varphi_L) \wedge \varphi_R$$

$$= \frac{1}{(2\pi i)} \int_{X_2} \left( \iota(\varphi^{(2)}_{L,i}) \wedge \varphi^{(2)}_{R,j} \overbrace{\frac{1}{(2\pi i)} \int_{X_1} \iota(e^{(1)}_i) \wedge h^{(1)}_j}^{\langle e^{(1)}_i | h^{(1)}_j \rangle = \mathbf{C}^{(1)}_{ij}} \right)$$

$$= \frac{1}{(2\pi i)} \int_{X_2} \left( \iota(\varphi^{(2)}_{L,i}) \wedge \varphi^{(2)}_{R,j} \; \mathbf{C}^{(1)}_{ij} \right)$$

$$= \sum_{p \in \mathcal{P}_{\Omega^{(2)}}} \underset{z_2 = p}{Res} \left( \psi^{(2)}_{p,i} \; \hat{\varphi}^{(2)}_{R,j} \; \mathbf{C}^{(1)}_{ij} \right)$$

$$\Rightarrow \quad \frac{1}{(2\pi i)^n} \int \iota(\varphi_L) \wedge \varphi_R = \sum_{p \in \mathcal{P}_{\Omega^{(n)}}} \underset{z_n = p}{Res} \left( \psi^{(n)}_{p,i} \; \varphi^{(n)}_{R,j} \; \mathbf{C}^{(\mathbf{n-1})}_{ij} \right)$$

$\psi^{(n)}_i$ **generalization** of univariate case

$$\partial_{z_n} \psi^{(n)}_i + \psi^{(n)}_j \hat{\Omega}^{(n)}_{ji} = \hat{\varphi}^{(n)}_{L,i}$$

# Multivariate :: Connection

New connection arise: $\Omega_{ij}^{(n)}$. In the 2 variables case:

$$\int \varphi(z_1, z_2)\, u(z_1, z_2) = \sum_i^{\nu_1} \int \mathrm{d}z_2 \varphi_i^{(2)}(z_2) \int \mathrm{d}z_1\, e_i^{(1)}(z_1, z_2)\, u(z_1, z_2)$$
$$= \sum_i^{\nu_1} \int \mathrm{d}z_2 \varphi_i^{(2)}(z_2) u_i(z_2)$$

**total derivative** is

$$0 = \sum_i^{\nu_1} \int \mathrm{d}\left(\varphi_i^{(2)}(z_2) u_i(z_2)\right) = \sum_i^{\nu_1} \int \left(\mathrm{d}\varphi_i^{(2)}(z_2) u_i(z_2) + \varphi_i^{(2)}(z_2)\, \mathrm{d}u_i(z_2)\right)$$
$$= \sum_i^{\nu_1} \int \left(\mathrm{d}\varphi_i^{(2)}(z_2)\delta_{ij} + \varphi_i^{(2)}(z_2)\, \Omega_{ij}^{(2)}(z_2)\right) u_j(z_2)$$

# Multivariate :: Connection

$$
\begin{aligned}
\mathrm{d}u_i(z_2) &= \mathrm{d}_{z_2} \int e_i^{(1)}(z_1,z_2)u(z_1,z_2) \\
&= \int \left( \mathrm{d}_{z_2} e_i^{(1)}(z_1,z_2) + \frac{\mathrm{d}_{z_2} u(z_1,z_2)}{u(z_1,z_2)} \wedge e_i^{(1)}(z_1,z_2) \right) u(z_1,z_2) \\
&= \int u(z_1,z_2)(\mathrm{d}_{z_2} + \omega_2\wedge)e_i^{(1)}(z_1,z_2) \Rightarrow \Omega_{ij}^{(2)}(z_2) \int e_i^{(1)}(z_1,z_2)u(z_1,z_2)
\end{aligned}
$$

$\Omega_{ij}^{(2)}(z_2)$ **projection** of the outer connection

$$
\Omega_{ij}^{(2)}(z_2) = \sum_k \langle (\mathrm{d}_{z_2} + \omega_2\wedge)\, e_i^{(1)} | h_k^{(1)} \rangle \left( \mathbf{C}_{(1)}^{-1} \right)_{kj}
$$

$$\Downarrow$$

$$
\hat{\Omega}_{ij}^{(n)}(z_2) = \sum_k \langle (\mathrm{d}_{z_n} + \omega_n\wedge)\, e_i^{(\mathbf{n-1})} | h_k^{(\mathbf{n-1})} \rangle \left( \mathbf{C}_{(\mathbf{n-1})}^{-1} \right)_{kj}
$$

# Multivariate :: The algorithm

**Goal**: $\langle \varphi_L | \varphi_R \rangle$

**Inputs**

$\langle \varphi_L | \,,\, | \varphi_R \rangle$ n-forms,

$\omega = \sum_i^n \omega_i$ connection,

$\nu_{n-1}$ number master (n-1)-forms

$\langle e_i^{(\mathbf{n-1})} | \,,\, | h_j^{(\mathbf{n-1})} \rangle$ inner basis

$\Rightarrow$

**Get**

$\mathbf{C}_{ij}^{(\mathbf{n-1})} = \langle e_i^{(\mathbf{n-1})} | h_j^{(\mathbf{n-1})} \rangle$ metric matrix,

$\langle \varphi_{L,i}^{(n)} | \,,\, | \varphi_{R,i}^{(n)} \rangle$ projected form,

$\Omega_{ij}^{(n)}$ projected connection

**Compute**

see also [Matsumoto (1998)]
[Matsubara et al. (2019)]
[Weinzierl (2020)]

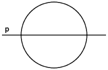
$$\langle \varphi_L | \varphi_R \rangle = \sum_{p \in \mathcal{P}_{\Omega^{(n)}}} \underset{z_n = p}{Res} \left( \psi_{p,i}^{(n)} \, \varphi_{R,j}^{(n)} \, \mathbf{C}_{ij}^{(\mathbf{n-1})} \right)$$

$$\partial_{z_n} \psi_i^{(n)} + \hat{\Omega}_{ij}^{(n)} \psi_j^{(n)} = \hat{\varphi}_{L,i}^{(n)}$$

**Terminating conditions**: $\Omega^{(1)} = \omega_1, \; \mathbf{C}^{(0)} = \mathbf{1}, \; \varphi_{L,R}^{(1)} = \varphi_{L,R}$

# Multivariate :: Sunrise

**Maximal Cut** $\Rightarrow$ 2 variables integral representation




$$u = (z_1 z_2(1 - z_1 - z_2))^{\gamma}$$

**Decompose** $I_{1,1,1,0,-1} = \langle \varphi | \mathcal{C}] = \langle z_2 | \mathcal{C}]$ on MIs $J_1 = I_{1,1,1,0,0} = \langle e^{(12)} | \mathcal{C}] = \langle 1 | \mathcal{C}]$

**Compute** $\langle z_2 | 1 \rangle$ & $\mathbf{C} = \langle 1 | 1 \rangle$

**Inputs**

$$\langle \varphi_L | = \langle 1 | \ , \ | \varphi_R \rangle = | 1 \rangle \ ,$$
$$\omega = \sum_i^n \omega_i \ , \ \nu_1 = 1$$
$$\langle e^{(1)} | = \langle z_1 | \ , \ | h^{(1)} \rangle = | z_1 \rangle$$

$\Rightarrow$

**Get**

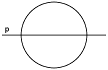
$$\mathbf{C}^{(1)} = \frac{\gamma (z_2 - 1)^4}{8(2\gamma - 1)(2\gamma + 1)},$$
$$\langle \varphi_L^{(2)} | = \langle -\frac{2}{z_2 - 1} | \ , \ | \varphi_R^{(2)} \rangle = | -\frac{2}{z_2 - 1} \rangle$$
$$\hat{\Omega}^{(2)} = \frac{(3\gamma + 2) z_2 - \gamma}{(z_2 - 1) z_2}$$

**Compute**

$$\langle 1 | 1 \rangle = \frac{\gamma^2}{3(3\gamma - 2)(3\gamma - 1)(3\gamma + 1)(3\gamma + 2)}$$

# Multivariate :: Sunrise

**Maximal Cut** $\Rightarrow$ 2 variables integral representation




$$u = (z_1 z_2 (1 - z_1 - z_2))^{\gamma}$$

**Decompose** $I_{1,1,1,0,-1} = \langle \varphi | \mathcal{C}] = \langle z_2 | \mathcal{C}]$ on MIs $J_1 = I_{1,1,1,0,0} = \langle e^{(12)} | \mathcal{C}] = \langle 1 | \mathcal{C}]$

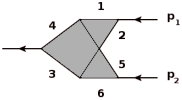
**Compute** $\langle z_2 | 1 \rangle$ & $\mathbf{C} = \langle 1 | 1 \rangle$

$$\langle z_2 | 1 \rangle = \frac{\gamma^2}{9(3\gamma - 2)(3\gamma - 1)(3\gamma + 1)(3\gamma + 2)} \quad \langle 1 | 1 \rangle = \frac{\gamma^2}{3(3\gamma - 2)(3\gamma - 1)(3\gamma + 1)(3\gamma + 2)}$$

Plug in the **Master Decomposition Formula**

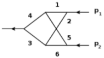
$$c_i = \sum_{i=1}^{\nu} \langle \varphi | e_j \rangle \left( \mathbf{C}^{-1} \right)_{ji} = \frac{\langle z_2 | 1 \rangle}{\langle 1 | 1 \rangle} \quad \Rightarrow \quad c = \frac{1}{3}$$

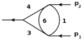
# Multivariate :: Non Planar Triangle

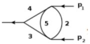


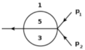

$$= \int \frac{u(\mathbf{z})\, f(\mathbf{z})}{z_1 z_2 z_3 z_4 z_5 z_6} \mathrm{d}^7 z$$

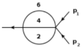
5 MIs
$$\begin{cases} \nu_{(1,2,3,4,5,6)} = 1 \\ \nu_{(1,3,4,6)} = 1 \\ \nu_{(2,3,4,6)} = 1 \\ \nu_{(2,4,6)} = 1 \\ \nu_{(1,3,5)} = 1 \end{cases}$$

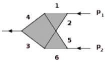
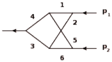
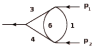
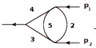
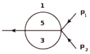
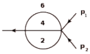
$J_1 =$ 

$J_2 =$ 

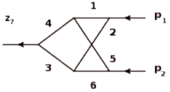
$J_3 =$ 

$J_4 =$ 

$J_5 =$ 

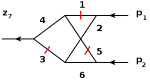
 $= c_1$  $+ c_2$  $+ c_3$  $+ c_4$  $+ c_5$ 

# Multivariate :: Non Planar Triangle




$$= \int \frac{u(\mathbf{z})\, z_7}{z_1 z_2 z_3 z_4 z_5 z_6} \mathrm{d}^7 z$$

**Cut**$_{\{1,3,5\}}$




$$= \int u_{135} \varphi_{135}$$

$$u_{135} = z_2^{\rho_2} z_4^{\rho_4} z_6^{\rho_6} u(0, z_2, 0, z_4, 0, z_6, z_7)$$

$$\hat{\varphi}_{135} = \frac{z_7}{z_2 z_4 z_6}$$

$$\varphi_{135} = \hat{\varphi}_{135} \mathrm{d}z_2 \wedge \mathrm{d}z_4 \wedge \mathrm{d}z_6 \wedge \mathrm{d}z_7$$

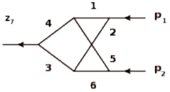
Number of MIs

$$\nu_{(2467)} = 2$$
$$\nu_{(246)} = 3$$
$$\nu_{(24)} = 2$$
$$\nu_{(2)} = 2$$

Inner basis

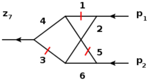
$$\hat{e}_1^{(2467)} = \frac{1}{z_2 z_4 z_6} \quad \hat{e}_2^{(2467)} = 1$$
$$\hat{e}_1^{(246)} = z_6 \quad \hat{e}_2^{(246)} = z_4 \quad \hat{e}_3^{(246)} = z_2$$
$$\hat{e}_1^{(24)} = z_4 \quad \hat{e}_2^{(24)} = z_2$$
$$\hat{e}_1^{(2)} = 1 \quad \hat{e}_2^{(2)} = z_2$$

# Multivariate :: Non Planar Triangle




$$= \int \frac{u(\mathbf{z})\, z_7}{z_1 z_2 z_3 z_4 z_5 z_6} \mathrm{d}^7 z$$

**Cut**$_{\{1,3,5\}}$

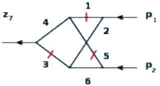



$$= \int u_{135} \varphi_{135}$$

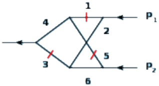
$$u_{135} = z_2^{\rho_2} z_4^{\rho_4} z_6^{\rho_6} u(0, z_2, 0, z_4, 0, z_6, z_7)$$

$$\hat{\varphi}_{135} = \frac{z_7}{z_2 z_4 z_6}$$

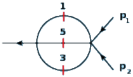
$$\varphi_{135} = \hat{\varphi}_{135} \mathrm{d}z_2 \wedge \mathrm{d}z_4 \wedge \mathrm{d}z_6 \wedge \mathrm{d}z_7$$




$= c_1$




$+ c_4$




$$c_i = \sum_{j=1}^{2} \langle \varphi_{135} | h_j^{(2467)} \rangle \left( \mathbf{C}^{-1}_{(2467)} \right)_{ji} \quad \Rightarrow \quad c_1 = -\frac{s}{2} \;, \quad c_2 = \frac{(d-3)(3d-10)(3d-8)}{2(d-4)^3 s^2}$$

# Outline

Giving a **new** persective

- Direct decomposition in integral basis and direct construction of system of differential equations
- Algebra of Feynman integrals controlled by intersection number
- Intersection number: Scalar product/Projection between Feynman integrals
- useful for both Physics and Mathematics

A **lot** of new possibilities

- study of **Differential Equations** for Feynman integrals
- application to different representations
- Combine with Finite Fields
- alternatives algorithm for the multivariate intersection number and for the MI reduction
- Quadratic relations $\Leftrightarrow$ **Riemann twisted Period Relation**

Thank you for your attention