The $\mathcal{N} = 4$ Sudakov form factor and cusp anomalous dimension at four loops

Tobias Huber
Universität Siegen

UNIVERSITÄT SIEGEN

TRR 257 - Particle Physics Phenomenology after the Higgs Discovery

Boels, Yang, TH	1705.03444 (PRL)
Boels, Yang, TH	1711.08449 (JHEP)
v. Manteuffel, Panzer, Schabinger, Yang, TH	1912.13459

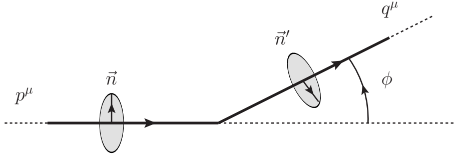
ParticleFace Meeting, Krakow, February 11th – 13th, 2020

Wilson loops and Γ_{cusp}

- Consider the Wilson loop [Korchemsky,Radyushkin'87]

$$W = \frac{1}{N_R} \, \text{Tr}_R \langle 0 | P \exp \left(ig \oint dx_\mu \, A_a^\mu(x) \, T_R^a \right) | 0 \rangle$$

- Path: Two segments that form a cusp of Euclidean angle ϕ, closed at infinity



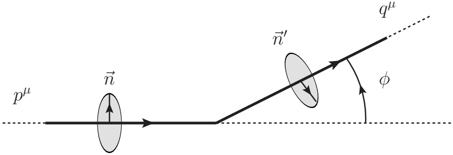
- Have $\log W = \log Z + \mathcal{O}(\epsilon^0)$
  - $Z = \overline{\text{MS}}$ renormalization constant (UV divergence)

Wilson loops and $\Gamma_{\rm cusp}$

- Consider the Wilson loop [Korchemsky,Radyushkin'87]

$$W = \frac{1}{N_R}\,\mathrm{Tr}_R\langle 0|P\exp\left(ig\oint dx_\mu\, A_a^\mu(x)\, T_R^a\right)|0\rangle$$

- Path: Two segments that form a cusp of Euclidean angle ϕ, closed at infinity



- Have $\quad \log W = \log Z + \mathcal{O}(\epsilon^0)$
  - $Z = \overline{\mathrm{MS}}$ renormalization constant (UV divergence)
- Angle-dependent cusp anomalous dimension defined via RGE of Z

$$\Gamma_{\text{cusp}}(\phi,\alpha_s) = \frac{d\log Z}{d\log\mu}$$

- Light-like limit $\gamma_{\rm cusp}$ of Γ_{cusp} via $\phi \to +i\infty$

Light-like cusp anomalous dimension

- γ_{cusp} is a universal quantity
- Governs structure of IR divergences in many physical quantities

  $\longrightarrow$ many applications

  - Collider physics: resummation, ...
  - Governs running of matching coefficients from QCD onto SCET

[Bell,Beneke,Li,TH'10]

$$\frac{d}{d\ln\mu}C_i(\mu) = \left[\gamma_{\text{cusp}}\ \ln\frac{q^2}{\mu^2} + \gamma' + \gamma_i\right]C_i(\mu)$$

- γ_{cusp} has a loop expansion, e.g. in QCD

$$\gamma_{\text{cusp}} = \frac{\alpha_s}{4\pi}\,\gamma_{\text{cusp}}^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2\gamma_{\text{cusp}}^{(2)} + \dots$$

- Light-like γ_{cusp} mostly extracted from splitting functions, form factors, ...

Light-like cusp anomalous dimension

- In QCD, γ_{cusp} is known analytically to three loops [Moch,Vermaseren,Vogt'05]
  - Color structure to three loops

$$\gamma_{\text{cusp}} = \frac{\alpha_s}{4\pi}\, C_R\, \gamma_{\text{cusp}}^{R} + \Big(\frac{\alpha_s}{4\pi}\Big)^2 C_R \left[C_A\, \gamma_{\text{cusp}}^{RA} + (n_f T_F)\, \gamma_{\text{cusp}}^{Rf}\right]$$

$$+ \Big(\frac{\alpha_s}{4\pi}\Big)^3 C_R \left[C_A^2\, \gamma_{\text{cusp}}^{RAA} + (n_f T_F)^2\, \gamma_{\text{cusp}}^{Rff} + C_F\, (n_f T_F)\, \gamma_{\text{cusp}}^{RFf} + C_A\, (n_f T_F)\, \gamma_{\text{cusp}}^{RAf}\right] + \mathcal{O}(\alpha_s^4)$$

  - Observe quadratic Casimir scaling up to three loops

[for generalisation at four loops, see: Sven's talk; Moch,Ruijl,Ueda,Vermaseren,Vogt'18; Becher,Neubert'19]

Light-like cusp anomalous dimension

- In QCD, γ_{cusp} is known analytically to three loops [Moch,Vermaseren,Vogt'05]
  - Color structure to three loops

$$\gamma_{\text{cusp}} = \frac{\alpha_s}{4\pi} C_R\, \gamma^{R}_{\text{cusp}} + \left(\frac{\alpha_s}{4\pi}\right)^2 C_R \left[C_A\, \gamma^{RA}_{\text{cusp}} + (n_f T_F)\, \gamma^{Rf}_{\text{cusp}}\right]$$

$$+ \left(\frac{\alpha_s}{4\pi}\right)^3 C_R \left[C_A^2\, \gamma^{RAA}_{\text{cusp}} + (n_f T_F)^2\, \gamma^{Rff}_{\text{cusp}} + C_F\, (n_f T_F)\, \gamma^{RFf}_{\text{cusp}} + C_A\, (n_f T_F)\, \gamma^{RAf}_{\text{cusp}}\right] + \mathcal{O}(\alpha_s^4)$$

  - Observe quadratic Casimir scaling up to three loops

[for generalisation at four loops, see: Sven's talk; Moch,Ruijl,Ueda,Vermaseren,Vogt'18; Becher,Neubert'19]

- Color structure at four loops

$$\gamma^{(4)}_{\text{cusp}} = C_R\, C_A^3\, \gamma^{RAAA}_{\text{cusp}} + \frac{d_R d_A}{N_R}\, \gamma^{dRA}_{\text{cusp}} + (n_f T_F)^3\, C_R\, \gamma^{Rfff}_{\text{cusp}} + C_R\, C_F\, (n_f T_F)^2\, \gamma^{RFff}_{\text{cusp}} + C_R\, C_A\, (n_f T_F)^2\, \gamma^{RAff}_{\text{cusp}}$$

$$+ C_R\, C_F^2\, (n_f T_F)\, \gamma^{RFFf}_{\text{cusp}} + C_R\, C_F\, C_A\, (n_f T_F)\, \gamma^{RFAf}_{\text{cusp}} + C_R\, C_A^2\, (n_f T_F)\, \gamma^{RFAAf}_{\text{cusp}} + \frac{d_R d_F}{N_R}\, (n_f T_F)\, \gamma^{dRFf}_{\text{cusp}}$$

  - New color structure: Quartic Casimir invariant build from

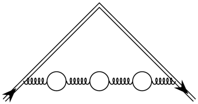
$$d_R^{abcd} = \frac{1}{6} \text{Tr}[T_R^a T_R^b T_R^c T_R^d + \text{perms.}(b,c,d)]$$

Light-like cusp anomalous dimension

[figures from 1902.05076]

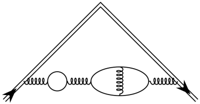
- Status of calculation of $\gamma_{\text{cusp}}^{(4)}$

$(T_F n_f)^3 C_R$



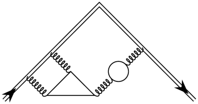
[Beneke,Braun'95]
[v. Manteuffel,Schabinger'16]

$(T_F n_f)^2 C_R C_F$



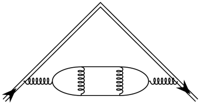
[Grozin,Henn,Korchemsky,Marquard'15]
[Grozin'16; v. Manteuffel,Schabinger'19]

$(T_F n_f)^2 C_R C_A$



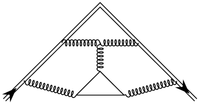
[Henn,Smirnov,Smirnov,Steinhauser'16]
[Davies,Vogt,Ruijl,Ueda,Vermaseren'16; v. Manteuffel,Schabinger'19]

$(T_F n_f) C_R C_F^2$



[Grozin'18]

$(T_F n_f) C_R C_F C_A$



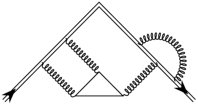
[Moch,Ruijl,Ueda,Vermaseren,Vogt'17 (num.)]
[Brüser,Grozin,Henn,Stahlhofen'19 (conj.)]

Light-like cusp anomalous dimension

[figures from 1902.05076]

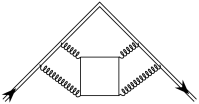
- Status of calculation of $\gamma_{\text{cusp}}^{(4)}$

$(T_F n_f) C_R C_A^2$



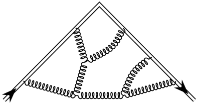
[Moch,Ruijl,Ueda,Vermaseren,Vogt'17 (num.)]
[Brüser,Grozin,Henn,Stahlhofen'19 (conj.)]

$n_f \frac{d_R d_F}{N_R}$



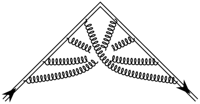
[Moch,Ruijl,Ueda,Vermaseren,Vogt'17'18 (num.)]
[Lee,Smirnov,Smirnov,Steinhauser'19; Henn,Peraro,Stahlhofen,Wasser'19]

$C_R C_A^3$



[Moch,Ruijl,Ueda,Vermaseren,Vogt'17'18 (num.)]

$\frac{d_R d_A}{N_R}$



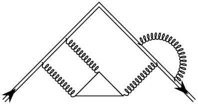
[Moch,Ruijl,Ueda,Vermaseren,Vogt'17'18 (num.)]

Light-like cusp anomalous dimension

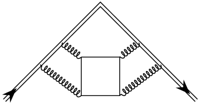
[figures from 1902.05076]

- Status of calculation of $\gamma_{\text{cusp}}^{(4)}$

$(T_F n_f) C_R C_A^2$



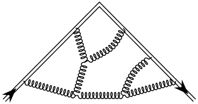
$n_f \frac{d_R d_F}{N_R}$



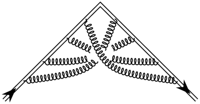
[Moch,Ruijl,Ueda,Vermaseren,Vogt'17 (num.)]
[Brüser,Grozin,Henn,Stahlhofen'19 (conj.)]

[Moch,Ruijl,Ueda,Vermaseren,Vogt'17'18 (num.)]
[Lee,Smirnov,Smirnov,Steinhauser'19; Henn,Peraro,Stahlhofen,Wasser'19]

$C_R C_A^3$



$\frac{d_R d_A}{N_R}$



[Moch,Ruijl,Ueda,Vermaseren,Vogt'17'18 (num.)]

[Moch,Ruijl,Ueda,Vermaseren,Vogt'17'18 (num.)]

- Contributions in lower line not known analytically until recently
  - Sufficient to calculate one, since complete large N_c contribution known
    [Henn,Lee,Smirnov,Smirnov,Steinhauser'16]
  - Missing piece has leading transcendentality $\Rightarrow$ compute in $\mathcal{N}=4$ SYM

Wilson loop with a Lagrangian insertion
[Henn,Korchemsky,Mistlberger'19]

Sudakov form factor
[v. Manteuffel,Panzer,Schabinger,Yang,TH'19]

$\mathcal{N} = 4$ SYM

- Unique maximal supersymmetric gauge theory in $D = 4$
- $\mathcal{N} = 4$ SYM has vanishing β-function
- Related to QCD via maximal transcendentality principle

[Kotikov,Lipatov,Onishchenko,Velizhanin'04]

  - $\mathcal{N} = 4$ SYM is 'hardest part' of QCD

$$\gamma_{\text{cusp}}^{\text{QCD}} = 4\,\left(\frac{\alpha_s}{4\pi}\right) + \left[C_A\left(-8\zeta_2 - \frac{268}{9}\right) - \frac{40}{9}\,n_f\right]\,\left(\frac{\alpha_s}{4\pi}\right)^2 + \ldots$$

$$\gamma_{\text{cusp}}^{\mathcal{N}=4} = 4\;2\,g^2 - 8\zeta_2\;2\,g^4 + \ldots$$

  - BES equation predicts leading-colour ($\propto N_c^L$) $\gamma_{\text{cusp}}^{\mathcal{N}=4}$ to all orders

[Beisert,Eden,Staudacher'04]

    - Verified to four loops [Bern,Dixon,Smirnov'05; Henn,TH'13]

- Couplings: $g^2 = \frac{g_{YM}^2\,N_c}{(4\pi)^2}\,(4\pi)^\epsilon\,e^{-\epsilon\gamma_E}$ 't Hooft coupling: $a = 2g^2$

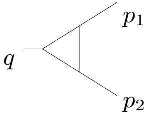
The Sudakov form factor in $\mathcal{N} = 4$ SYM

- Introduce bilinear UV protected operator: $\mathcal{O} = \mathrm{Tr}(\phi_{12}\phi_{12})$
- Definition of the form factor

$$\mathcal{F}_\mathcal{S} = \langle \phi^a_{34}(p_1)\phi^b_{34}(p_2)\,\mathcal{O}\rangle = \mathrm{Tr}(T^a T^b) F_\mathcal{S}$$

  - $\phi^a_{34}(p_i)$: on-shell states in the adjoint representation

- Kinematics: $p_1^2 = p_2^2 = 0, \quad q^2 = (p_1 + p_2)^2$
- Use dim. reg. with $D = 4 - 2\epsilon$

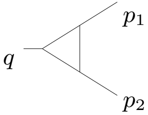


The Sudakov form factor in $\mathcal{N} = 4$ SYM

- Introduce bilinear UV protected operator: $\mathcal{O} = \mathrm{Tr}(\phi_{12}\phi_{12})$
- Definition of the form factor

$$\mathcal{F}_S = \langle \phi_{34}^a(p_1)\phi_{34}^b(p_2)\,\mathcal{O}\rangle = \mathrm{Tr}(T^a T^b) F_S$$

  - $\phi_{34}^a(p_i)$: on-shell states in the adjoint representation
- Kinematics: $p_1^2 = p_2^2 = 0, \quad q^2 = (p_1 + p_2)^2$
- Use dim. reg. with $D = 4 - 2\epsilon$



- Perturbative expansion of the FF

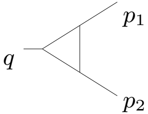
$$F_S = 1 + g^2\,(-q^2)^{-\epsilon}\,F_S^{(1)} + g^4\,(-q^2)^{-2\epsilon}\,F_S^{(2)} + g^6\,(-q^2)^{-3\epsilon}\,F_S^{(3)}$$

The Sudakov form factor in $\mathcal{N}=4$ SYM

- Introduce bilinear UV protected operator: $\mathcal{O} = \mathrm{Tr}(\phi_{12}\phi_{12})$
- Definition of the form factor

$$\mathcal{F}_S = \langle \phi_{34}^a(p_1)\phi_{34}^b(p_2)\,\mathcal{O}\rangle = \mathrm{Tr}(T^aT^b)F_S$$

  - $\phi_{34}^a(p_i)$: on-shell states in the adjoint representation
- Kinematics: $p_1^2 = p_2^2 = 0, \quad q^2 = (p_1+p_2)^2$
- Use dim. reg. with $D = 4-2\epsilon$



- Perturbative expansion of the FF

$$F_S = 1 + g^2\,(-q^2)^{-\epsilon}\,F_S^{(1)} + g^4\,(-q^2)^{-2\epsilon}\,F_S^{(2)} + g^6\,(-q^2)^{-3\epsilon}\,F_S^{(3)}$$

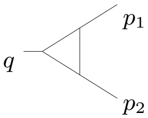
$$+\, g^8\,(-q^2)^{-4\epsilon}\left[F_{S,P}^{(4)} + \frac{1}{N_c^2}\,F_{S,NP}^{(4)}\right] + \mathcal{O}(g^{10})$$

The Sudakov form factor in $\mathcal{N}=4$ SYM

- Introduce bilinear UV protected operator: $\mathcal{O} = \mathrm{Tr}(\phi_{12}\phi_{12})$
- Definition of the form factor

$$\mathcal{F}_S = \langle \phi_{34}^a(p_1)\phi_{34}^b(p_2)\, \mathcal{O}\rangle = \mathrm{Tr}(T^aT^b)F_S$$

  - $\phi_{34}^a(p_i)$: on-shell states in the adjoint representation
  - Kinematics: $p_1^2 = p_2^2 = 0, \quad q^2 = (p_1+p_2)^2$
  - Use dim. reg. with $D = 4-2\epsilon$



- Perturbative expansion of the FF

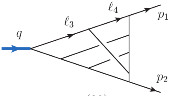
$$F_S = 1 + g^2\,(-q^2)^{-\epsilon}\,F_S^{(1)} + g^4\,(-q^2)^{-2\epsilon}\,F_S^{(2)} + g^6\,(-q^2)^{-3\epsilon}\,F_S^{(3)}$$

$$+\, g^8\,(-q^2)^{-4\epsilon}\left[F_{S,P}^{(4)} + \frac{1}{N_c^2}\,F_{S,NP}^{(4)}\right] + \mathcal{O}(g^{10})$$

  - $F_S^{(L)}$ has leading divergence $\propto 1/\epsilon^{2L}$, but $F_{S,NP}^{(4)}$ must be $\propto 1/\epsilon^2$ only!

The Sudakov form factor in $\mathcal{N} = 4$ SYM

- Non-planar correction at four loops appears through quartic Casimir invariant

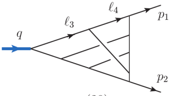


(28)

$$\mathrm{Tr}[T_A^a T_A^b T_A^c T_A^d T_A^a T_A^b T_A^c T_A^d]/N_A = \frac{d_A^{abcd} d_A^{abcd}}{N_A} - \frac{1}{24} C_A^4 = \frac{3}{2} N_c^2$$

The Sudakov form factor in $\mathcal{N} = 4$ SYM

- Non-planar correction at four loops appears through quartic Casimir invariant



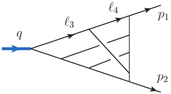
(28)

$$\mathrm{Tr}[T_A^a T_A^b T_A^c T_A^d T_A^a T_A^b T_A^c T_A^d]/N_A = \frac{d_A^{abcd} d_A^{abcd}}{N_A} - \frac{1}{24}\, C_A^4 = \frac{3}{2}\, N_c^2$$

- Relation to light-like cusp AD

$$\log F_S = \sum_{L=1}^{\infty} g^{2L} \left(-q^2\right)^{-L\epsilon} \left[-\frac{\gamma_{\text{cusp}}^{(L)}}{(2L\epsilon)^2} - \frac{\mathcal{G}_{\text{coll}}^{(L)}}{2L\epsilon} \right] + \mathcal{O}\left(\epsilon^0\right)$$

- Non-planar correction at four loops appears through quartic Casimir invariant



(28)

$$\mathrm{Tr}[T_A^a T_A^b T_A^c T_A^d T_A^a T_A^b T_A^c T_A^d]/N_A = \frac{d_A^{abcd} d_A^{abcd}}{N_A} - \frac{1}{24} C_A^4 = \frac{3}{2} N_c^2$$

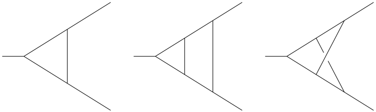
- Relation to light-like cusp AD

$$\log F_S = \sum_{L=1}^{\infty} g^{2L} \left(-q^2\right)^{-L\epsilon} \left[- \frac{\gamma_{\text{cusp}}^{(L)}}{(2L\epsilon)^2} - \frac{\mathcal{G}_{\text{coll}}^{(L)}}{2L\epsilon} \right] + \mathcal{O}\left(\epsilon^0\right)$$

- Form factor to two loops

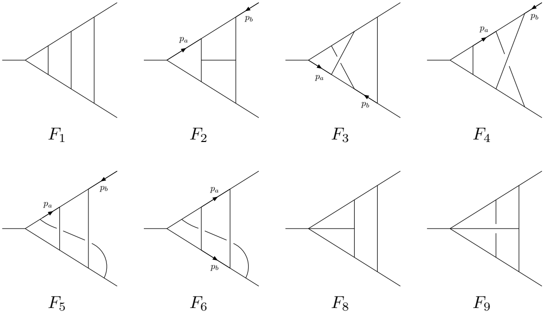
[van Neerven'86]

$$F_S = 1 + a\, x^{\epsilon}\, R_\epsilon \cdot 2\, D_1 + a^2\, x^{2\epsilon}\, R_\epsilon^2 \cdot [4\, E_1 + E_2]$$



D_1 E_1 E_2

Form factor at three loops



$$F_S = 1 + a\, x^{\epsilon}\, R_{\epsilon} \cdot 2\, D_1 + a^2\, x^{2\epsilon}\, R_{\epsilon}^2 \cdot [4\, E_1 + E_2]$$

$$+ a^3\, x^{3\epsilon}\, R_{\epsilon}^3 \cdot [8\, F_1 - 2\, F_2 + 4\, F_3 + 4\, F_4 - 4\, F_5 - 4\, F_6 - 4\, F_8 + 2\, F_9]$$

[Gehrmann,Henn,TH'11]

- Through to three loops the result is remarkably simple.
- Single irreducible numerators $(p_a + p_b)^2$
- So far all integrals are of uniform transcendentality (UT) in ϵ-expansion

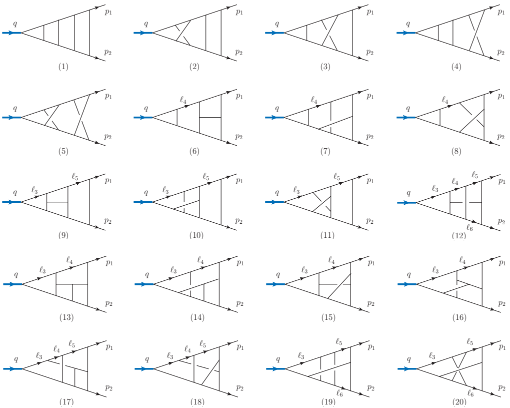
Four-loop integrand

- Four-loop integrand was derived from colour-kinematic duality

[Boels,Kniehl,Tarasov,Yang'12]

  - Checks via unitarity cuts

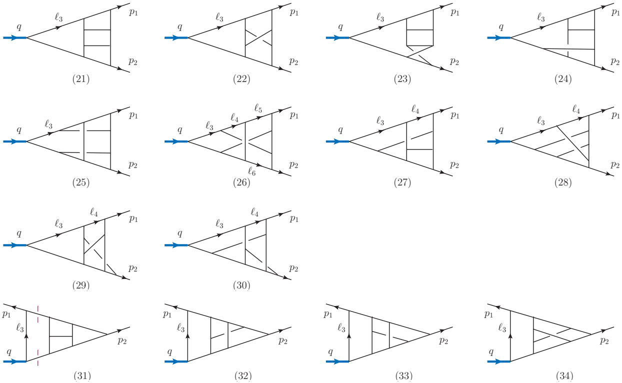
- Purely planar diagrams, six (8,11,15,16,18,20) have vanishing colour-factor



- 12 propagators and 18 independent scalar products / topology

Four-loop integrand

- Diagrams with non-planar contributions [Boels,Kniehl,Tarasov,Yang'12]
  - Four diagrams (21, 25, 30, 31) have also planar contributions



- All but one topology (26) have internal box(es) or triangle(s)
- Many have one or more graph symmetries

Four-loop integrand, integration strategies

- Complicated integrals with up to double irreducible numerators

Graph	Numerator factor	Color factor	Symmetry factor
(27)	$-(\ell_3\cdot p_1)^2-(\ell_3\cdot p_2)^2-6(\ell_3\cdot p_1)(\ell_3\cdot p_2)$ $+(\ell_3\cdot p_1)(\ell_4\cdot(p_1+5p_2))$ $-(\ell_3\cdot p_2)(\ell_4\cdot(3p_1-p_2))$ $+(p_1\cdot p_2)\big[2\ell_3\cdot(p_{12}+\ell_3-\ell_4)$ $+\ell_4\cdot(p_1-p_2)-p_1\cdot p_2\big]$ $+(\alpha_1+1)\big[(\ell_3\cdot p_{12}-p_1\cdot p_2)^2$ $-\frac{1}{7}(\ell_3\cdot p_{12}-p_1\cdot p_2)(\ell_4\cdot(7p_1-p_2))$ $-\frac{2}{7}(\ell_3\cdot(\ell_3-p_{12})+p_1\cdot p_2)(p_1\cdot p_2)\big]$	$24\,N_c^2\,\delta_{a_1a_2}$	1

- IBP & Laporta reduction to ~ 280 master integrals achieved with Reduze

[Boels,Kniehl,Yang'15]

  - Free parameter α_1 drops out of result
  - Subset of 'rational IBP relations' will prove useful

Four-loop integrand, integration strategies

- Complicated integrals with up to double irreducible numerators

Graph	Numerator factor	Color factor	Symmetry factor
(27)	$-(\ell_3 \cdot p_1)^2 - (\ell_3 \cdot p_2)^2 - 6(\ell_3 \cdot p_1)(\ell_3 \cdot p_2)$ $+(\ell_3 \cdot p_1)(\ell_4 \cdot (p_1 + 5p_2))$ $-(\ell_3 \cdot p_2)(\ell_4 \cdot (3p_1 - p_2))$ $+(p_1 \cdot p_2)\big[2\ell_3 \cdot (p_{12} + \ell_3 - \ell_4)$ $+\ell_4 \cdot (p_1 - p_2) - p_1 \cdot p_2\big]$ $+(\alpha_1 + 1)\big[(\ell_3 \cdot p_{12} - p_1 \cdot p_2)^2$ $-\frac{1}{7}(\ell_3 \cdot p_{12} - p_1 \cdot p_2)(\ell_4 \cdot (7p_1 - p_2))$ $-\frac{2}{7}(\ell_3 \cdot (\ell_3 - p_{12}) + p_1 \cdot p_2)(p_1 \cdot p_2)\big]$	$24\, N_c^2\, \delta_{a_1 a_2}$	1

- IBP & Laporta reduction to ~ 280 master integrals achieved with Reduze

[Boels,Kniehl,Yang'15]

  - Free parameter α_1 drops out of result
  - Subset of 'rational IBP relations' will prove useful

- Our integration strategies
  - Integrate non-planar FF numerically in UT basis of masters [Boels,Yang,TH'17]
  - Integrate complete FF analytically in ϵ-finite basis

[v. Manteuffel,Panzer,Schabinger,Yang,TH'19]

UT basis and dlog forms

- How to find a UT basis in single-scale problems?
- How to know an integral is UT without explicit calculation?

UT basis and dlog forms

- How to find a UT basis in single-scale problems?
- How to know an integral is UT without explicit calculation?
- dLog form [Arkani-Hamed,Bourjaily,Cachazo,Trnka'14; Bern,Herrmann,Litsey,Stankowicz,Trnka'14'15]
  - Change variables to scalar parameters a_i, b_i, c_i, d_i in four dimensions

$$
\begin{aligned}
l_6 &= a_1 p_1 + a_2 p_2 + a_3 q_1 + a_4 q_2 \,, & l_3 &= d_1 p_1 + d_2 p_2 + d_3 q_1 + d_4 q_2 \,,\\
l_5 &= b_1 p_1 + b_2 p_2 + b_3 q_1 + b_4 q_2 \,, & q_i^2 &= q_i \cdot p_j = 0 \quad \forall i,j\\
l_4 &= c_1 p_1 + c_2 p_2 + c_3 q_1 + c_4 q_2 \,, & q_1 \cdot q_2 &= -p_1 \cdot p_2
\end{aligned}
$$

  - In terms of spinor-helicity variables

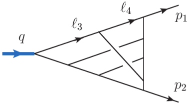
$$
\begin{aligned}
p_1 &= \lambda_1\,\tilde{\lambda}_1 \,, & q_1 &= \lambda_1\,\tilde{\lambda}_2 \,,\\
p_2 &= \lambda_2\,\tilde{\lambda}_2 \,, & q_2 &= \lambda_2\,\tilde{\lambda}_1
\end{aligned}
$$

$$
\int d^4l_3\, d^4l_4\, d^4l_5\, d^4l_6\,(\ldots) = (p_1\cdot p_2)^8 \int \prod_{i=1}^{4} da_i\, db_i\, dc_i\, dd_i\,(\ldots) = (p_1\cdot p_2)^8 \int \prod_{j=1}^{16} dlog\left[f_j(\vec{a},\vec{b},\vec{c},\vec{d})\right]
$$

Four-loop example with two boxes:

Topology 28 with numerator

$$(l_3 - l_4 - p_2)^2 \times (l_3 - p_1)^2$$



(28)

$$\frac{da_1 \ldots dd_4\ (d_1 d_2 - d_3 d_4)(-c_1 + c_1 c_2 - c_3 c_4 + d_1 - c_2 d_1 - c_1 d_2 + d_1 d_2 + c_4 d_3 + c_3 d_4 - d_3 d_4)}{(a_1 a_2 - a_3 a_4)(a_1 a_2 - a_3 a_4 - a_2 b_1 - a_1 b_2 + b_1 b_2 + a_4 b_3 + a_3 b_4 - b_3 b_4)(c_1 c_2 - c_3 c_4)(-c_2 + c_1 c_2 - c_3 c_4)(d_1 + d_1 d_2 - d_3 d_4)(-d_2 + d_1 d_2 - d_3 d_4)}$$

$$\times \frac{1}{(a_1 a_2 - a_3 a_4 - a_2 c_1 - a_1 c_2 + c_1 c_2 + a_4 c_3 + a_3 c_4 - c_3 c_4)(b_1 b_2 - b_3 b_4 - b_2 c_1 - b_1 c_2 + c_1 c_2 + b_4 c_3 + b_3 c_4 - c_3 c_4)}$$

$$\times \frac{1}{(-a_1 + a_1 a_2 - a_3 a_4 + d_1 - a_2 d_1 - a_1 d_2 + d_1 d_2 + a_4 d_3 + a_3 d_4 - d_3 d_4)(b_1 b_2 - b_3 b_4 - b_2 d_1 - b_1 d_2 + d_1 d_2 + b_4 d_3 + b_3 d_4 - d_3 d_4)}$$

$$\times \frac{1}{(-b_1 + b_1 b_2 - b_3 b_4 + d_1 - b_2 d_1 - b_1 d_2 + d_1 d_2 + b_4 d_3 + b_3 d_4 - d_3 d_4)(c_1 c_2 - c_3 c_4 - c_2 d_1 - c_1 d_2 + d_1 d_2 + c_4 d_3 + c_3 d_4 - d_3 d_4)}$$

$$= -\, dLog[\frac{(c_1 c_2 - c_3 c_4)}{(c_1 c_2 - c_3 c_4 - c_2 d_1 + c_4 d_3)}] dLog[\frac{(-c_2 + c_1 c_2 - c_3 c_4)}{(c_1 c_2 - c_3 c_4 - c_2 d_1 + c_4 d_3)}] dLog[\frac{d_3}{(-1 + d_1 - d_2 + d_1 d_2 - d_3 d_4)}] dLog[\frac{d_4}{(-1 + d_1 - d_2 + d_1 d_2 - d_3 d_4)}]$$

$$\times\, dLog[\frac{((a_1 a_2 - a_3 a_4)(c_3 d_1 - c_1 d_3))}{(-(a_3 c_1 c_2 d_1) + a_1 a_2 c_3 d_1 - a_3 a_4 c_3 d_1 + a_3 c_3 c_4 d_1 + a_3 c_2 d_1^2 - a_2 c_3 d_1^2 - a_1 a_2 c_1 d_3 + a_3 a_4 c_1 d_3 + a_1 c_1 c_2 d_3 - a_1 c_3 c_4 d_3 + a_2 c_1 d_1 d_3 - a_1 c_2 d_1 d_3 + a_4 c_3 d_1 d_3 - a_3 c_4 d_1 d_3 - a_4 c_1 d_3^2 + a_1 c_4 d_3^2)}]$$

$$\times\, dLog[\frac{((a_1 a_2 - a_3 a_4 - a_2 c_1 - a_1 c_2 + c_1 c_2 + a_4 c_3 + a_3 c_4 - c_3 c_4)(c_3 d_1 - c_1 d_3))}{(-(a_3 c_1 c_2 d_1) + a_1 a_2 c_3 d_1 - a_3 a_4 c_3 d_1 + a_3 c_3 c_4 d_1 + a_3 c_2 d_1^2 - a_2 c_3 d_1^2 - a_1 a_2 c_1 d_3 + a_3 a_4 c_1 d_3 + a_1 c_1 c_2 d_3 - a_1 c_3 c_4 d_3 + a_2 c_1 d_1 d_3 - a_1 c_2 d_1 d_3 + a_4 c_3 d_1 d_3 - a_3 c_4 d_1 d_3 - a_4 c_1 d_3^2 + a_1 c_4 d_3^2)}]$$

$$\times\, dLog[\frac{((c_3 d_1 - c_1 d_3)(-a_1 + a_1 a_2 - a_3 a_4 + d_1 - a_2 d_1 - a_1 d_2 + d_1 d_2 + a_4 d_3 + a_3 d_4 - d_3 d_4))}{(-(a_3 c_1 c_2 d_1) + a_1 a_2 c_3 d_1 - a_3 a_4 c_3 d_1 + a_3 c_3 c_4 d_1 + a_3 c_2 d_1^2 - a_2 c_3 d_1^2 - a_1 a_2 c_1 d_3 + a_3 a_4 c_1 d_3 + a_1 c_1 c_2 d_3 - a_1 c_3 c_4 d_3 + a_2 c_1 d_1 d_3 - a_1 c_2 d_1 d_3 + a_4 c_3 d_1 d_3 - a_3 c_4 d_1 d_3 - a_4 c_1 d_3^2 + a_1 c_4 d_3^2)}]$$

$$\times\, dLog[\frac{((c_3 d_1 - c_1 d_3)(a_1 a_2 c_1 - a_3 a_4 c_1 - a_1 c_1 c_2 + a_1 c_3 c_4 - a_1 a_2 d_1 + a_3 a_4 d_1 - a_2 c_1 d_1 + a_1 c_2 d_1 + c_1 c_2 d_1 - c_3 c_4 d_1 + a_2 d_1^2 - c_2 d_1^2 + a_4 c_1 d_3 - a_1 c_4 d_3 - a_4 d_1 d_3 + c_4 d_1 d_3 + a_3 c_1 d_4 - a_1 c_3 d_4 - a_3 d_1 d_4 + c_3 d_1 d_4 + a_1 d_3 d_4 - c_1 d_3 d_4)}{(-(a_3 c_1 c_2 d_1) + a_1 a_2 c_3 d_1 - a_3 a_4 c_3 d_1 + a_3 c_3 c_4 d_1 + a_3 c_2 d_1^2 - a_2 c_3 d_1^2 - a_1 a_2 c_1 d_3 + a_3 a_4 c_1 d_3 + a_1 c_1 c_2 d_3 - a_1 c_3 c_4 d_3 + a_2 c_1 d_1 d_3 - a_1 c_2 d_1 d_3 + a_4 c_3 d_1 d_3 - a_3 c_4 d_1 d_3 - a_4 c_1 d_3^2 + a_1 c_4 d_3^2)}$$

$$\times\, dLog[\frac{(c_1 c_2 - c_3 c_4 - c_2 d_1 - c_1 d_2 + d_1 d_2 + c_4 d_3 + c_3 d_4 - d_3 d_4)}{(c_1 c_2 - c_3 c_4 - c_2 d_1 + c_4 d_3)}] dLog[\frac{(c_1 c_2 d_1 - c_3 c_4 d_1 - c_2 d_1^2 + c_4 d_1 d_3 + c_3 d_1 d_4 - c_1 d_3 d_4)}{(c_1 c_2 - c_3 c_4 - c_2 d_1 + c_4 d_3)}] dLog[\frac{(d_1 + d_1 d_2 - d_3 d_4)}{(-1 + d_1 - d_2 + d_1 d_2 - d_3 d_4)}] dLog[\frac{(-d_2 + d_1 d_2 - d_3 d_4)}{(-1 + d_1 - d_2 + d_1 d_2 - d_3 d_4)}]$$

$$\times\, dLog[\frac{(a_1 a_2 - a_3 a_4 - a_2 b_1 - a_1 b_2 + b_1 b_2 + a_4 b_3 + a_3 b_4 - b_3 b_4)}{f[b_1, b_2, b_3, b_4]}] dLog[\frac{(b_1 b_2 - b_3 b_4 - b_2 c_1 - b_1 c_2 + c_1 c_2 + b_4 c_3 + b_3 c_4 - c_3 c_4)}{f[b_1, b_2, b_3, b_4]}]$$

$$\times\, dLog[\frac{(b_1 b_2 - b_3 b_4 - b_2 d_1 - b_1 d_2 + d_1 d_2 + b_4 d_3 + b_3 d_4 - d_3 d_4)}{f[b_1, b_2, b_3, b_4]}] dLog[\frac{(-b_1 + b_1 b_2 - b_3 b_4 + d_1 - b_2 d_1 - b_1 d_2 + d_1 d_2 + b_4 d_3 + b_3 d_4 - d_3 d_4)}{f[b_1, b_2, b_3, b_4]}]$$

✓

Four-loop Sudakov form factor in UT basis

- Use representation in terms of a_i, b_i, c_i, d_i to find more UT candidates
  - Procedure: Take subsequent residues in all 16 parameters $a_1, \ldots, d_4$
  - Rule: If other than simple pole appears for any of the remaining parameters: not UT

- Idea of algorithm: Make product-ansatz for numerator

$$\left[A_1\, l_6^2 + A_2\, l_5^2 + \ldots + A_{19} q^2\right] \times \left[B_1\, l_6^2 + B_2\, l_5^2 + \ldots + B_{19} q^2\right]$$

  - Appearance of other than simple pole gives constraint on A_i, B_j

  $\Rightarrow$ find set of UT candidates

Four-loop Sudakov form factor in UT basis

- Use representation in terms of a_i, b_i, c_i, d_i to find more UT candidates
  - Procedure: Take subsequent residues in all 16 parameters $a_1, \ldots, d_4$
  - Rule: If other than simple pole appears for any of the remaining parameters: not UT
- Idea of algorithm: Make product-ansatz for numerator

$$\left[A_1\, l_6^2 + A_2\, l_5^2 + \ldots + A_{19} q^2\right] \times \left[B_1\, l_6^2 + B_2\, l_5^2 + \ldots + B_{19} q^2\right]$$

  - Appearance of other than simple pole gives constraint on A_i, B_j

    $\Rightarrow$ find set of UT candidates
- Use IBP relations to write four-loop $\mathcal{N} = 4$ Sudakov form factor in UT basis
  - Leading color piece is rational linear combination of 32 integrals
  - Sub-leading color piece is rational linear combination of only 23 integrals
  - Topologies (31) – (34) drop out completely!
  - Most UT integrals come with full $1/\epsilon^8$ pole

Numerical integration of UT integrals

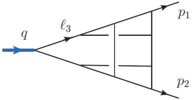
- Use two strategies: Sector decomposition and Mellin Barnes
- Sector decomposition: Mostly FIESTA, also SecDec [Smirnov'08+; Borowka,Heinrich et al.'12+]
  - UT integrals generate considerably fewer integration terms than non-UT siblings of comparable complexity
  - Pole resolution and preparation of integrand still takes $\sim$ two weeks for each integral
  - Subsequent numerical integration takes $\mathcal{O}$(days)
- Mellin Barnes: Use mostly `MB.m` [Czakon'05]

Numerical integration of UT integrals

- Use two strategies: Sector decomposition and Mellin Barnes
- Sector decomposition: Mostly FIESTA, also SecDec [Smirnov'08+; Borowka,Heinrich et al.'12+]
    - UT integrals generate considerably fewer integration terms than non-UT siblings of comparable complexity
    - Pole resolution and preparation of integrand still takes $\sim$ two weeks for each integral
    - Subsequent numerical integration takes $\mathcal{O}$(days)
- Mellin Barnes: Use mostly `MB.m` [Czakon'05]

12-line integral in topology 25 with numerator

$$\left[(p_1-\ell_5)^2+2(\ell_4-\ell_5)^2+(\ell_3-\ell_4)^2-(\ell_3-\ell_5)^2-(p_1-\ell_4)^2\right]^2$$
$$-\,4\,(\ell_4-\ell_5)^2\,(p_1-\ell_3+\ell_4-\ell_5)^2$$



(25)

$$=\frac{0.00347222}{\epsilon^8}+\frac{0.0114231(13)}{\epsilon^6}+\frac{1.163106(20)}{\epsilon^5}+\frac{14.04762(26)}{\epsilon^4}+\frac{109.8742(34)}{\epsilon^3}+\frac{647.669(44)}{\epsilon^2}$$
$$+\frac{3530.846\pm1.921}{\epsilon}$$

Numerical Results

- Non-planar part of four-loop Sudakov form factor in $\mathcal{N} = 4$ SYM

$$F^{(4)}_{S,NP} = 2 \times 24 \times \left[\frac{c_{-8}}{\epsilon^8} + \frac{c_{-7}}{\epsilon^7} + \ldots + \frac{c_{-1}}{\epsilon} + \mathcal{O}\left(\epsilon^0\right) \right]$$

ϵ order	−8	−7	−6	−5
result	-3.8×10^{-8}	$+4.4 \times 10^{-9}$	-1.2×10^{-6}	-1.2×10^{-5}
uncertainty	–	$\pm 5.7 \times 10^{-7}$	$\pm 1.0 \times 10^{-5}$	$\pm 1.2 \times 10^{-4}$

ϵ order	−4	−3	−2	−1
result	$+3.5 \times 10^{-6}$	+ 0.0007	+1.60	−17.98
uncertainty	$\pm 1.5 \times 10^{-3}$	± 0.0186	±0.19	± 3.25

Numerical Results

- Non-planar part of four-loop Sudakov form factor in $\mathcal{N}=4$ SYM

$$F^{(4)}_{S,NP} = 2 \times 24 \times \left[\frac{c_{-8}}{\epsilon^8} + \frac{c_{-7}}{\epsilon^7} + \ldots + \frac{c_{-1}}{\epsilon} + \mathcal{O}\left(\epsilon^0\right) \right]$$

ϵ order	-8	-7	-6	-5
result	-3.8×10^{-8}	$+4.4 \times 10^{-9}$	-1.2×10^{-6}	-1.2×10^{-5}
uncertainty	–	$\pm 5.7 \times 10^{-7}$	$\pm 1.0 \times 10^{-5}$	$\pm 1.2 \times 10^{-4}$

ϵ order	-4	-3	-2	-1
result	$+3.5 \times 10^{-6}$	$+$ 0.0007	$+$1.60	$-$17.98
uncertainty	$\pm 1.5 \times 10^{-3}$	$\pm$ 0.0186	$\pm$0.19	$\pm$ 3.25

- Numerical value of c_{-2} confirmed by more precise numerical analysis $(1.528(2))$ [Henn,Peraro,Stahlhofen,Wasser'19] and analytic result (1.5274) [Henn,Korchemsky,Mistlberger'19; v. Manteuffel,Panzer,Schabinger,Yang,TH'19]

Numerical Results

- Non-planar part of four-loop Sudakov form factor in $\mathcal{N}=4$ SYM

$$F^{(4)}_{S,NP} = 2 \times 24 \times \left[\frac{c_{-8}}{\epsilon^8} + \frac{c_{-7}}{\epsilon^7} + \ldots + \frac{c_{-1}}{\epsilon} + \mathcal{O}\left(\epsilon^0\right) \right]$$

ϵ order	−8	−7	−6	−5
result	-3.8×10^{-8}	$+4.4 \times 10^{-9}$	-1.2×10^{-6}	-1.2×10^{-5}
uncertainty	–	$\pm 5.7 \times 10^{-7}$	$\pm 1.0 \times 10^{-5}$	$\pm 1.2 \times 10^{-4}$

ϵ order	−4	−3	−2	−1
result	$+3.5 \times 10^{-6}$	+ 0.0007	+1.60	−17.98
uncertainty	$\pm 1.5 \times 10^{-3}$	± 0.0186	±0.19	± 3.25

  - Numerical value of c_{-2} confirmed by more precise numerical analysis $(1.528(2))$ [Henn,Peraro,Stahlhofen,Wasser'19] and analytic result (1.5274) [Henn,Korchemsky,Mistlberger'19; v. Manteuffel,Panzer,Schabinger,Yang,TH'19]

- Non-planar light-like cusp AD

$$\gamma^{(4)}_{\text{cusp, NP}} = -\frac{2 \times 24 \times 64}{N_c^2} \times (1.60 \pm 0.19) \overset{N_c=3}{\approx} -546 \pm 65$$

Numerical Results

- Non-planar part of four-loop Sudakov form factor in $\mathcal{N}=4$ SYM

$$F^{(4)}_{S,NP} = 2 \times 24 \times \left[\frac{c_{-8}}{\epsilon^8} + \frac{c_{-7}}{\epsilon^7} + \ldots + \frac{c_{-1}}{\epsilon} + \mathcal{O}\left(\epsilon^0\right) \right]$$

ϵ order	−8	−7	−6	−5
result	-3.8×10^{-8}	$+4.4 \times 10^{-9}$	-1.2×10^{-6}	-1.2×10^{-5}
uncertainty	–	$\pm 5.7 \times 10^{-7}$	$\pm 1.0 \times 10^{-5}$	$\pm 1.2 \times 10^{-4}$

ϵ order	−4	−3	−2	−1
result	$+3.5 \times 10^{-6}$	+ 0.0007	+1.60	−17.98
uncertainty	$\pm 1.5 \times 10^{-3}$	± 0.0186	±0.19	± 3.25

  - Numerical value of c_{-2} confirmed by more precise numerical analysis $(1.528(2))$ [Henn,Peraro,Stahlhofen,Wasser'19] and analytic result (1.5274) [Henn,Korchemsky,Mistlberger'19; v. Manteuffel,Panzer,Schabinger,Yang,TH'19]

- Non-planar light-like cusp AD

$$\gamma^{(4)}_{\text{cusp, NP}} = -\frac{2 \times 24 \times 64}{N_c^2} \times (1.60 \pm 0.19) \overset{N_c=3}{\approx} -546 \pm 65$$

- Non-planar light-like collinear AD

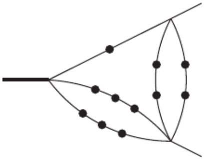
$$\mathcal{G}^{(4)}_{\text{coll, NP}} = -\frac{384}{N_c^2} \times (-17.98 \pm 3.25) \overset{N_c=3}{\approx} +767 \pm 139$$

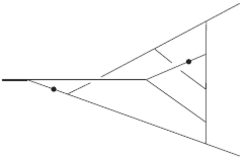
- Switch to a basis of finite master integrals. Requires

[Panzer'14; v. Manteuffel,Panzer,Schabinger'14'15; Schabinger'18]

  - finding of finite integrals
  - relation to uniformly transcendental basis
  - analytic integration of finite integrals through to weight six

- Use integral finder in `Reduze 2` to find finite integrals [v. Manteuffel,Studerus'12]
  - Need shifted dimension and many dots on propagators





Analytic calculation

- To relate uniform and finite basis need
  - IBP reduction of integrals with many dots
  - dimensional recurrence relations
- In reduction procedure, apply methods from
  - finite field artihmetic (finite fields, rational reconstruction)
    [v. Manteuffel,Schabinger'14'16; Maierhöfer,Usovitsch,Uwer'17; Peraro'16'19]
    - No intermediate expression swell, allows for massive parallelization
  - syzygies to avoid numerators in reductions of integrals with many dots
    [Gluza,Kajda,Kosower'10; Kosower'18; Lee'14; v. Manteuffel,Schabinger'19; Bitoun,Bogner,Klausen,Panzer'17]

Analytic calculation

- To relate uniform and finite basis need
  - IBP reduction of integrals with many dots
  - dimensional recurrence relations
- In reduction procedure, apply methods from
  - finite field artihmetic (finite fields, rational reconstruction)
    [v. Manteuffel,Schabinger'14'16; Maierhöfer,Usovitsch,Uwer'17; Peraro'16'19]
    - No intermediate expression swell, allows for massive parallelization
  - syzygies to avoid numerators in reductions of integrals with many dots
    [Gluza,Kajda,Kosower'10; Kosower'18; Lee'14; v. Manteuffel,Schabinger'19; Bitoun,Bogner,Klausen,Panzer'17]
- If finite integrals are linearly reducible, integrate their Feynman parameter representation with `HyperInt` [Panzer'14]
  - Otherwise, one can choose finite integrals in the relevant topology which first contribute at weight seven
  - Works for cusp, but not for full FF

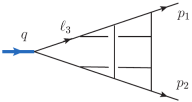
Analytic results for integrals

- Confirm uniform-weight expansion of all 55 integrals through to weight six

[partial results in Henn,Smirnov,Smirnov,Lee,Steinhauser'16'17'19; v. Manteuffel,Schabinger'16'19]

12-line integral in topology 25 with numerator

$$\left[(p_1-\ell_5)^2+2(\ell_4-\ell_5)^2+(\ell_3-\ell_4)^2-(\ell_3-\ell_5)^2-(p_1-\ell_4)^2\right]^2$$
$$-\,4\,(\ell_4-\ell_5)^2\,(p_1-\ell_3+\ell_4-\ell_5)^2$$



(25)

$$=\frac{0.00347222}{\epsilon^8}+\frac{0.0114231(13)}{\epsilon^6}+\frac{1.163106(20)}{\epsilon^5}+\frac{14.04762(26)}{\epsilon^4}+\frac{109.8742(34)}{\epsilon^3}+\frac{647.669(44)}{\epsilon^2}$$

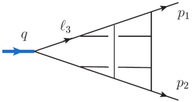
$$=\frac{1}{288\epsilon^8}+\frac{\zeta_2}{144\epsilon^6}+\frac{209\zeta_3}{216\epsilon^5}+\frac{623\zeta_4}{48\epsilon^4}+\frac{1}{\epsilon^3}\left(\frac{39449}{360}\zeta_5-\frac{205}{108}\zeta_3\zeta_2\right)+\frac{1}{\epsilon^2}\left(\frac{11621}{162}\zeta_3^2+\frac{38501}{315}\zeta_2^3\right)$$

Analytic results for integrals

- Confirm uniform-weight expansion of all 55 integrals through to weight six

[partial results in Henn,Smirnov,Smirnov,Lee,Steinhauser'16'17'19; v. Manteuffel,Schabinger'16'19]

12-line integral in topology 25 with numerator

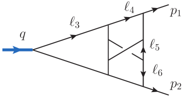


(25)

$$\left[(p_1-\ell_5)^2+2(\ell_4-\ell_5)^2+(\ell_3-\ell_4)^2-(\ell_3-\ell_5)^2-(p_1-\ell_4)^2\right]^2$$
$$-4\,(\ell_4-\ell_5)^2\,(p_1-\ell_3+\ell_4-\ell_5)^2$$

$$=\frac{0.00347222}{\epsilon^8}+\frac{0.0114231(13)}{\epsilon^6}+\frac{1.163106(20)}{\epsilon^5}+\frac{14.04762(26)}{\epsilon^4}+\frac{109.8742(34)}{\epsilon^3}+\frac{647.669(44)}{\epsilon^2}$$
$$=\frac{1}{288\epsilon^8}+\frac{\zeta_2}{144\epsilon^6}+\frac{209\zeta_3}{216\epsilon^5}+\frac{623\zeta_4}{48\epsilon^4}+\frac{1}{\epsilon^3}\left(\frac{39449}{360}\zeta_5-\frac{205}{108}\zeta_3\zeta_2\right)+\frac{1}{\epsilon^2}\left(\frac{11621}{162}\zeta_3^2+\frac{38501}{315}\zeta_2^3\right)$$

10-line integral in topology 22 with numerator



(22)

$$\ell_6^2\,(p_1-\ell_4)^2$$

$$=\frac{1.34678628(2)}{\epsilon^2}=\frac{1}{\epsilon^2}\left(\frac{1}{4}\zeta_3^2+\frac{31}{140}\zeta_2^3\right)$$

Analytic results for form factor and cusp

- Four-loop Sudakov FF in $\mathcal{N} = 4$SYM

$$F^{(4)} = \frac{1}{\epsilon^8}\left(\frac{2}{3}\right) + \frac{1}{\epsilon^6}\left(\frac{2}{3}\zeta_2\right) + \frac{1}{\epsilon^5}\left(-\frac{38}{9}\zeta_3\right) + \frac{1}{\epsilon^4}\left(\frac{5}{18}\zeta_2^2\right) + \frac{1}{\epsilon^3}\left(\frac{1082}{15}\zeta_5 + \frac{23}{3}\zeta_3\zeta_2\right)$$

$$+ \frac{1}{\epsilon^2}\left(\frac{10853}{54}\zeta_3^2 + \frac{95477}{945}\zeta_2^3\right) + \frac{1}{N_c^2}\left[\frac{1}{\epsilon^2}\left(18\zeta_3^2 + \frac{372}{35}\zeta_2^3\right)\right] + \mathcal{O}\left(\epsilon^{-1}\right).$$

  - Analytic result of uniform weight through to $\mathcal{O}(\epsilon^{-2})$
  - $\epsilon^{-8} - \epsilon^{-3}$ poles can be predicted from lower-loop results

Analytic results for form factor and cusp

- Four-loop Sudakov FF in $\mathcal{N} = 4$SYM

$$F^{(4)} = \frac{1}{\epsilon^8}\left(\frac{2}{3}\right) + \frac{1}{\epsilon^6}\left(\frac{2}{3}\zeta_2\right) + \frac{1}{\epsilon^5}\left(-\frac{38}{9}\zeta_3\right) + \frac{1}{\epsilon^4}\left(\frac{5}{18}\zeta_2^2\right) + \frac{1}{\epsilon^3}\left(\frac{1082}{15}\zeta_5 + \frac{23}{3}\zeta_3\zeta_2\right)$$

$$+ \frac{1}{\epsilon^2}\left(\frac{10853}{54}\zeta_3^2 + \frac{95477}{945}\zeta_2^3\right) + \frac{1}{N_c^2}\left[\frac{1}{\epsilon^2}\left(18\zeta_3^2 + \frac{372}{35}\zeta_2^3\right)\right] + \mathcal{O}\left(\epsilon^{-1}\right).$$

  - Analytic result of uniform weight through to $\mathcal{O}(\epsilon^{-2})$
  - $\epsilon^{-8} - \epsilon^{-3}$ poles can be predicted from lower-loop results

- Light-like cusp anomalous dimension at four loops

$$\frac{1}{2}\,\gamma_{\rm cusp}^{(4)} = -32\zeta_3^2 - \frac{7008}{35}\zeta_2^3 + \frac{1}{N_c^2}\left[-576\zeta_3^2 - \frac{11904}{35}\zeta_2^3\right].$$

  - Leading color piece known since long [Bern et al.06; Beisert,Eden,Staudacher'06; Henn,TH'13]
  - Analytic result of sub-leading color piece agrees with [Henn,Korchemsky,Mistlberger'19]
  - Complete four-loop light-like QCD $\gamma_{\rm cusp}$ now known analytically

Conclusion and Outlook

- Conclusion
  - We computed the Sudakov form factor to four loops in $\mathcal{N} = 4$ SYM
    - First calculation:
      Numerical integration of sub-leading colour piece in UT basis to $\mathcal{O}(1/\epsilon)$
    - Second calculation:
      Analytic integration of complete FF in ϵ-finite basis to $\mathcal{O}(1/\epsilon^2)$
  - Leading-transcendental part of the light-like cusp anomalous dimension at four loops in QCD can be extracted
    - Last missing ingredient to obtain a complete analytic expression for $\gamma_{\text{cusp}}^{\text{QCD}}$ at four loops [except for two “num.+conj.” colour structures]
    - Agrees with analytic result by [Henn,Korchemsky,Mistlberger’19]

Conclusion and Outlook

- Conclusion
  - We computed the Sudakov form factor to four loops in $\mathcal{N} = 4$ SYM
    - First calculation:
      Numerical integration of sub-leading colour piece in UT basis to $\mathcal{O}(1/\epsilon)$
    - Second calculation:
      Analytic integration of complete FF in ϵ-finite basis to $\mathcal{O}(1/\epsilon^2)$
  - Leading-transcendental part of the light-like cusp anomalous dimension at four loops in QCD can be extracted
    - Last missing ingredient to obtain a complete analytic expression for $\gamma_{\text{cusp}}^{\text{QCD}}$ at four loops [except for two "num.+conj." colour structures]
    - Agrees with analytic result by [Henn,Korchemsky,Mistlberger'19]
- Outlook
  - Five-loop calculation? $\mathcal{N} = 4$ SYM integrand known [Yang'16]