Using Loop-Tree Duality to understand physical and non-physical thresholds

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Introduction.

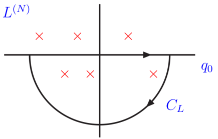
Propagator:

$$G_F(q) \propto \frac{1}{q^2 - m^2 + \imath 0}. \qquad (1)$$

Cauchy's residue theorem:

$$\text{Res}[f(z), z_0] = \frac{1}{2\pi\imath} \oint_\Gamma f(z) dz.$$

$$\boxed{\text{Amplitude} \longrightarrow \text{Residue theorem} \longrightarrow \text{LTD}}$$



Our duality approach.

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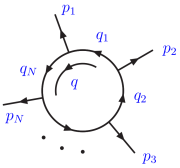
Singular behaviour @1-loop.

Loop-Tree Duality @1-loop:

$$\mathcal{A} = \int \mathcal{N}(\ell, \{p_j\}_N) \prod_i G_F(q_i)$$
$$= -\int \mathcal{N}(\ell, \{p_j\}_N) \otimes \sum_i \tilde{\delta}(q_i) \prod_{j \neq i} G_D(q_j; q_i), \qquad (2)$$

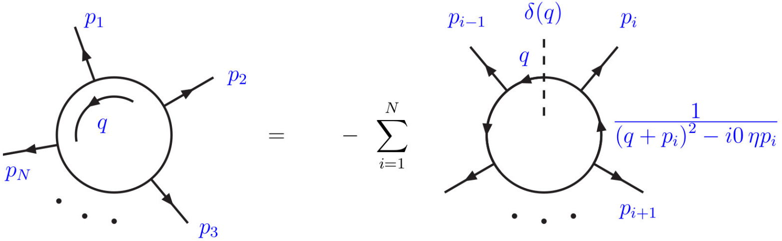
$$G_D(q_i; q_j) = \frac{1}{q_j^2 - m_j^2 - \imath 0 \eta k_{ji}}, \qquad (3)$$

with $k_{ji} = q_j - q_i$ and η a future-like vector ($\eta_0 > 0$, $\eta^2 \geq 0$).



Singular behaviour @1-loop.

LTD@1-loop:



Defining $q_{j,0}^{(+)} = \sqrt{\boldsymbol{q}_j^2 + m_j^2}$,

$$\mathcal{A} = -\sum_i \int_\ell \mathcal{N}(\ell, \{p_j\}_N)\tilde{\delta}(q_i) \prod_{j \neq i} \frac{1}{q_{j,0}^2 - \left(q_{j,0}^{(+)}\right)^2 - \imath 0 \eta k_{ji}} \qquad (4)$$

Singular behaviour @1-loop.

Dual amplitudes singularities: As $q_j = q_i + k_{ji}$, then,

$$q_{i,0}^{(+)} + q_{j,0}^{(+)} + k_{ji,0} = 0. \quad (5)$$

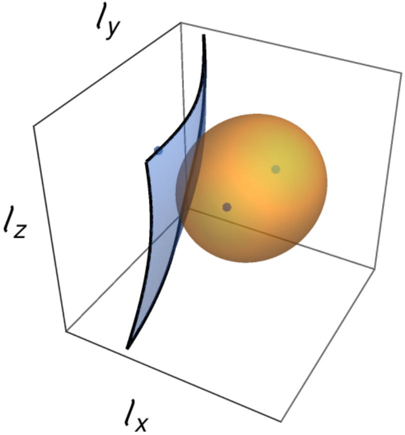
$$q_{i,0}^{(+)} - q_{j,0}^{(+)} + k_{ji,0} = 0. \quad (6)$$

$$\text{Equation } (5) \longrightarrow \text{Ellipsoid in the } 3 - \text{momentum space.}$$

$$\text{Equation } (6) \longrightarrow \text{Hyperboloid in the } 3 - \text{momentum space.}$$

Defining $\lambda_{ij}^{\pm\pm} = \pm q_{i,0}^{(+)} \pm q_{j,0}^{(+)} + k_{ji,0}$, the conditions of the divergence of the dual amplitude are given by $\lambda_{ij}^{\pm\pm} \to 0$.

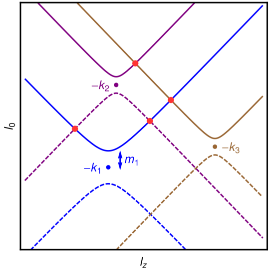
Singular behaviour @1-loop.



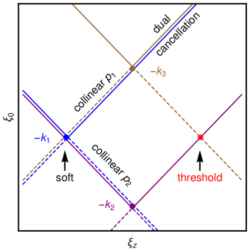
$\lambda_{ij}^{++} \to 0$: *orange*; $\lambda_{ij}^{+-} \to 0$: *blue*.

Singular behaviour @1-loop.

Potential singularities: $\lambda_{ij}^{\pm\pm} = \pm q_{i,0}^{(+)} \pm q_{j,0}^{(+)} + k_{ji,0} \to 0.$



Location of the singularities in the massive case.



Location of the singularities in the massless case.

Causal singularities @1-loop.

First potential singularity: $\lambda_{ij}^{++} = q_{i,0}^{(+)} + q_{j,0}^{(+)} + k_{ji,0} \to 0.$

- Only possible for $k_{ji,0} < 0.$
- Restricted by the masses: $(m_j + m_i)^2 \leq k_{ji}^2$

Defining $\mathcal{S}_{ij}^{(1)} = (2\pi\imath)^{-1}\left[G_D(q_i;q_j)\tilde{\delta}(q_i) + G_D(q_j;q_i)\tilde{\delta}(q_j)\right]$ it is given that

$$\lim_{\lambda_{ij}^{++}\to 0} \mathcal{S}_{ij}^{(1)} = \frac{\theta(-k_{ji,0})\theta(k_{ji}^2 - (m_i+m_j)^2)}{4q_{i,0}^{(+)}q_{j,0}^{(+)}(-\lambda_{ij}^{++} - \imath 0 k_{ji,0})} + \mathcal{O}\left((\lambda_{ij}^{++})^0\right) \qquad (7)$$

These singularities are unitary thresholds, and their position do not depend on the representation.

Second potential singularity: $\lambda_{ij}^{+-} = q_{i,0}^{(+)} - q_{j,0}^{(+)} + k_{ji,0} \to 0.$

- Possible for $k_{ji,0} \in \mathbb{R}$.
- Restricted by the masses: $k_{ji}^2 \leq (m_j - m_i)^2$

Each of the singularities is unphysical and the sum of the two involved dual contributions is finite:

$$\lim_{\lambda_{ij}^{+-} \to 0} \mathcal{S}_{ij}^{(1)} = \mathcal{O}\left((\lambda_{ij}^{+-})^0 \right) \tag{8}$$

As $k_{kj,0} = k_{ki,0} - k_{ji,0}$, for non-singular propagators $\lim_{\lambda_{ij}^{+-}} G_D(q_j; q_k) = \lim_{\lambda_{ij}^{+-}} G_D(q_i; q_k)$.

Unphysical singularities @1-loop.

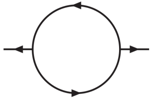
Second potential singularity: $\lambda_{ij}^{+-} = q_{i,0}^{(+)} - q_{j,0}^{(+)} + k_{ji,0} \to 0$.
The cancellation of this integrand singularity is fully local,

$$q_{j,0}^{(+)} G_D(q_i; q_j)\Big|_{\lambda_{ij}^{+-}\to 0} = -q_{i,0}^{(+)} G_D(q_j; q_i)\Big|_{\lambda_{ij}^{+-}\to 0} . \tag{9}$$

The position of these singularities does depend on the momentum flow. *E.g.*

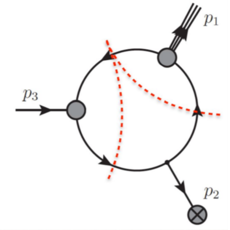
$$\int_{\ell_1} \frac{1}{((\ell_1 - p)^2 - m_1^2 + \imath 0)(\ell_1^2 - m_2^2 + \imath 0)}, \tag{10}$$

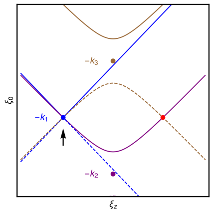
with $p = (p_0, \mathbf{0})$. If $0 < m_2 - m_1 < p_0 < m_1 + m_2$, the two point function is free of singularities.



Anomalous thresholds @1-loop.

Anomalous thresholds: More than two internal particles go on-shell.





Example of an anomalous threshold.

Anomalous thresholds @1-loop.

Defining $\mathcal{S}^{(1)}_{ijk} = (2\pi\imath)^{-1} G_D(q_i; q_j) G_D(q_i; q_k) \tilde{\delta}(q_i) + perm.$ and if $x_{ijk} = 8 q^{(+)}_{i,0} q^{(+)}_{j,0} q^{(+)}_{k,0}$, it is given that

$$\lim_{\lambda^{++}_{ij}, \lambda^{++}_{ik} \to 0} \mathcal{S}^{(1)}_{ijk} = \frac{1}{x_{ijk}} \prod_{r=j,k} \frac{\theta(-k_{ri,0}) \theta(k^2_{ri} - (m_i + m_r)^2)}{-\lambda^{++}_{ir} - \imath 0 k_{ri,0}} + \mathcal{O}\left((\lambda^{++}_{ij})^{-1}, (\lambda^{++}_{ik})^{-1} \right) \tag{11}$$

Although $\lambda^{-+}_{jk} = \lambda^{++}_{ik} - \lambda^{++}_{ij}$, the series expansion of $\mathcal{S}^{(+)}_{ijk}$ around λ^{-+}_{jk} is free of singularities.

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Singular behaviour @2-loop.

Definitions:

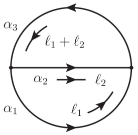
$$G_F(\alpha) = \prod_{i\in\alpha} G_F(q_i), \qquad G_D(\alpha) = \sum_{i\in\alpha} \tilde{\delta}(q_i) \prod_{\substack{j\in\alpha \\ j\neq i}} G_D(q_j, q_i). \tag{12}$$

LTD @2-loops:

$$\mathcal{A}(\{p_n\}_N) = \int_{\ell_1,\ell_2} \mathcal{N}(\ell_1, \ell_2, \{p_n\}_N)[G_D(\alpha_1)G_D(\alpha_2 \cup \alpha_3) + G_D(-\alpha_1 \cup \alpha_2)G_F(\alpha_3) - G_D(\alpha_1)G_F(\alpha_2)G_D(\alpha_3)]. \tag{13}$$

(Alternative representation, *Selomit's talk*.)

Singular behaviour @2-loop.



Two-loop Feynman diagram and momentum flow.

With the shorthand notation $i \in \alpha_1$, $j \in \alpha_2$, $k \in \alpha_3$ and $\tilde{\delta}(q_i, q_j) = \tilde{\delta}(q_i)\tilde{\delta}(q_j)$, we focus on the function

$$\mathcal{S}^{(2)}_{ijk} = (2\pi\imath)^{-2} \Big[G_D(q_j; q_k)\tilde{\delta}(q_i, q_j) + G_D(-q_j; q_i)\tilde{\delta}(q_j; q_k) + [G_D(q_k; q_j) + G_D(q_i; -q_j) - G_F(q_j)]\tilde{\delta}(q_i, q_k) \Big] . \quad (14)$$

Singular behaviour @2-loop.

The function $\mathcal{S}_{ijk}^{(2)}$ becomes singular whenever

$$\lambda_{ijk}^{\pm\pm\pm} = \pm q_{i,0}^{(+)} \pm q_{j,0}^{(+)} \pm q_{k,0}^{(+)} + k_{k(ij),0} \to 0. \tag{15}$$

With the choice of the momenta, $k_{k(ij)} = q_k - (q_i +_j)$ does not depend on the internal momenta. Also

$$\lim_{\lambda_{ijk}^{+++}\to 0} \mathcal{S} = \frac{\theta(-k_{k(ij),0})\theta(k_{k(ij)}^2 - (m_i + m_j + m_k)^2)}{x_{ijk}\left(-\lambda_{ijk}^{+++} - \imath 0 k_{kj,0}\right)} + \mathcal{O}\left((\lambda_{ijk}^{+++})^0\right) \tag{16}$$

However, $k_{kj,0}|_{\lambda_{ijk}^{+++}\to 0} = q_{i,0}^{(+)} + k_{k(ij),0} < 0$, so that $-\imath 0 k_{kj,0} = +\imath 0$. This happens whenever the momentum flow of the three internal lines go in the same direction.

Singular behaviour @2-loop.

For the divergences $\lambda_{ijk}^{++-} \to 0$ and $\lambda_{ijk}^{+--} \to 0$, the conditions are given by

$$\begin{aligned} \lambda_{ijk}^{++-} \to 0 &\Longrightarrow q_{k,0}^{(+)} - k_{k(ij),0} > 0, \\ \lambda_{ijk}^{+--} \to 0 &\Longrightarrow q_{i,0}^{(+)} + k_{k(ij),0} > 0. \end{aligned} \tag{17}$$

However these singularities vanish because of the momentum dependence of the $\imath 0$ prescription.
For instance, the sunrise diagram can be written as

$$L^{(2)} = \int_{\vec{\ell}_1, \vec{\ell}_2} \frac{1}{2 q_{3,0}^{(+)}} \left(\frac{1}{\lambda_{ijk}^{---}} - \frac{1}{\lambda_{ijk}^{+++}} \right). \tag{18}$$

Singular behaviour at more than two loops.

For a Feynman diagram with more than two loop, the arising singularities are the intersection of several on-shell manifolds, such as elipsoids and hyperboloids. The former are physical singularities and they remain in this duality representation, while the latter are spurious singularities that vanish when adding all the dual contributions **dual cancellations** (*Selomit's talk*).

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Conclusions.

- Loop-Tree Duality splits loop-level diagrams into connected tree-level diagrams (dual amplitudes).
- This Loop-Tree Duality approach introduces non-physical singularities in each of the dual amplitudes.
- Singularities in dual amplitudes are interpreted as ellipsoids and hyperboloids in the 3-momenta space.
- The spurious singularities vanish locally when the dual amplitudes are added.
- Physical singularities are contained in a compact region of the 3-momenta space.

THANK YOU!