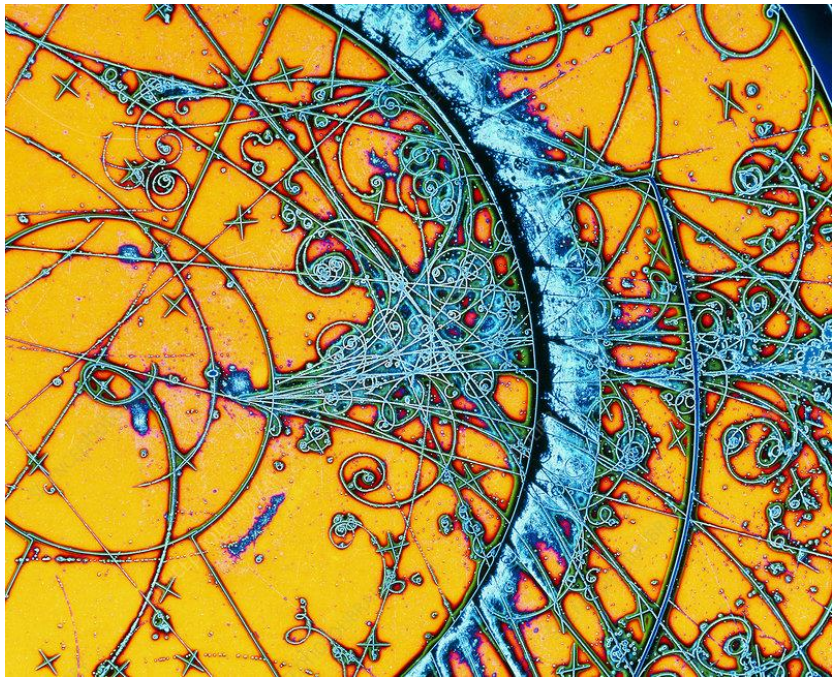

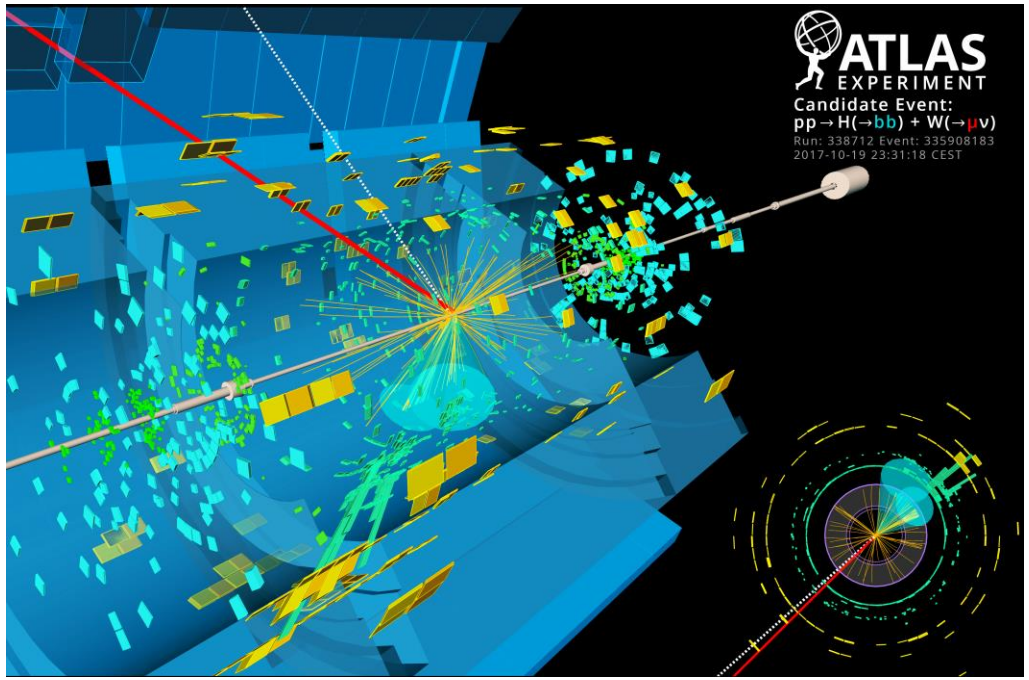
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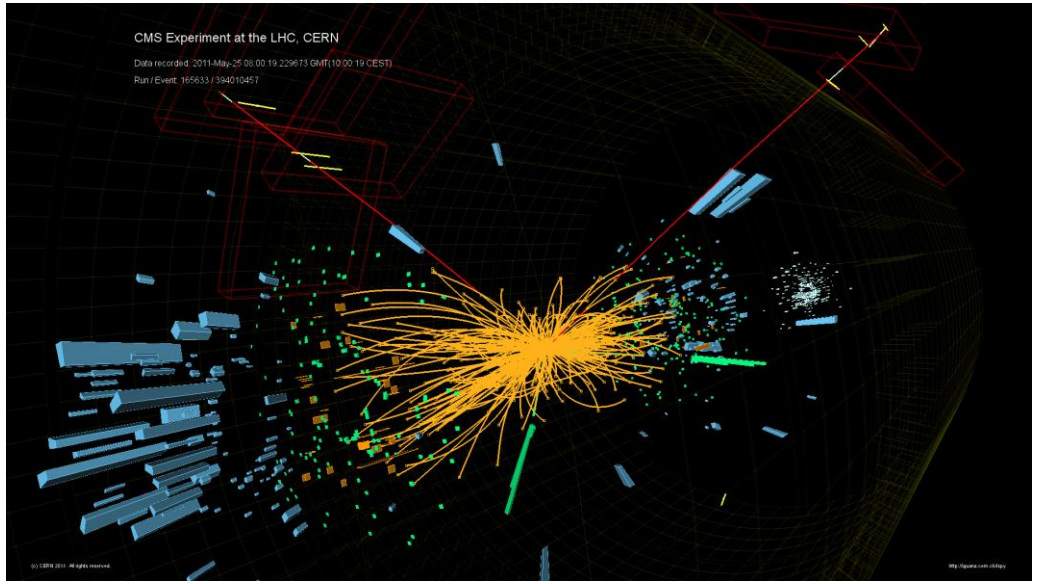

CMS Experiment at the LHC, CERN

Signals in Particle Detectors 3/4

Academic Training Lectures

Werner Riegler, CERN, werner.riegler@cern.ch

Signals in particle detectors

Main Auditorium, Mon. 2 Dec.

Lecture 1:

- Electrostatics
- Principles
- Reciprocity
- Induced currents
- Induced voltages
- Ramo-Shockley theorem
- Mean value theorem
- Capacitance matrix
- Equivalent circuits

Council Chamber, Tue. 3 Dec.

Lecture 2:

Signals in

- Ionization chambers
- Liquid argon calorimeters
- Diamond detectors
- Silicon detectors
- GEMs (Gas Electron Multiplier)
- Micromegas (Micromesh gas detector)
- APDs (Avalanche Photo Diodes)
- LGADs (Low Gain Avalanche Diodes)
- SiPMs (Silicon Photo Multipliers)
- Strip detectors
- Pixel detectors
- Wire Chambers
- Liquid Argon TPCs

TH conference room (4/3-006), Wed. 4 Dec.

Lecture 3:

- Media with conductivity
- Quasi-static approximations
- Signal theorem extensions
- Time dependent weighting fields
- Resistive plate chambers (RPCs)
- Un-depleted silicon sensors
- Monolithic pixel sensors

Filtration Plant (222/R-001), Thu. 5.Dec.

Lecture 4:

- Signal propagation
- Transmission lines
- Termination
- Linear signal processing
- Noise
- Optimum filters

Main Auditorium, Fri. 6 Dec.

Lecture 5:

- Possible overflow, wrap-up and Q&A session

Extensions of the Ramo-Shockley theorem

In the first two lectures we assumed that

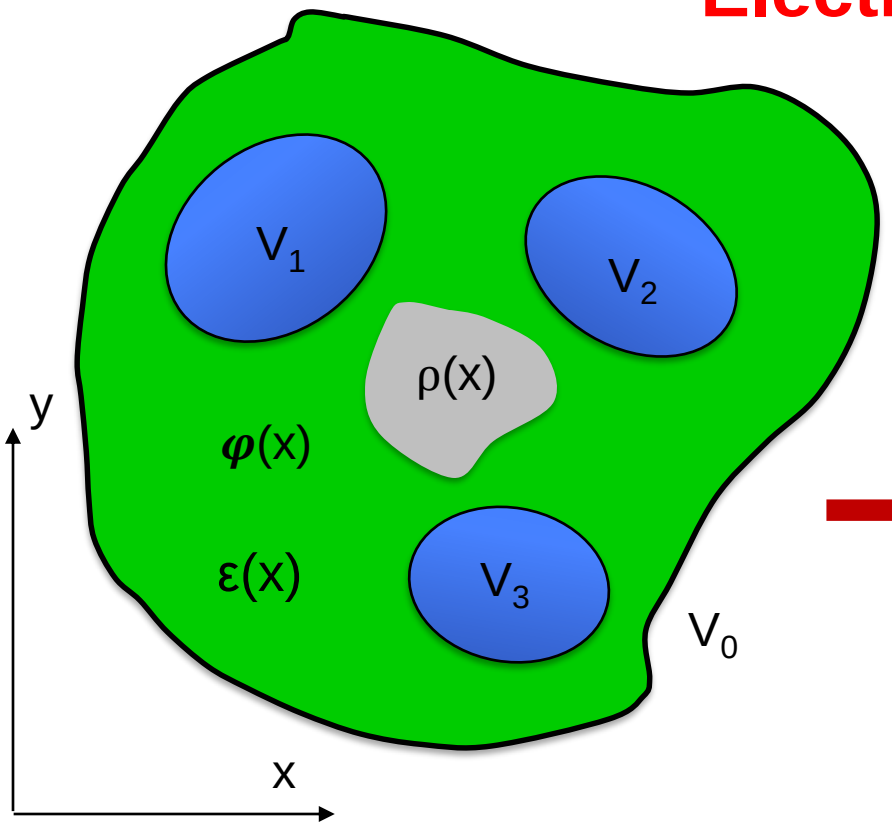
- the electrodes are perfectly conducting electrodes
- the electrodes are grounded (measuring induced current) or insulated (measuring induced voltage)
- the detector materials are perfect insulators

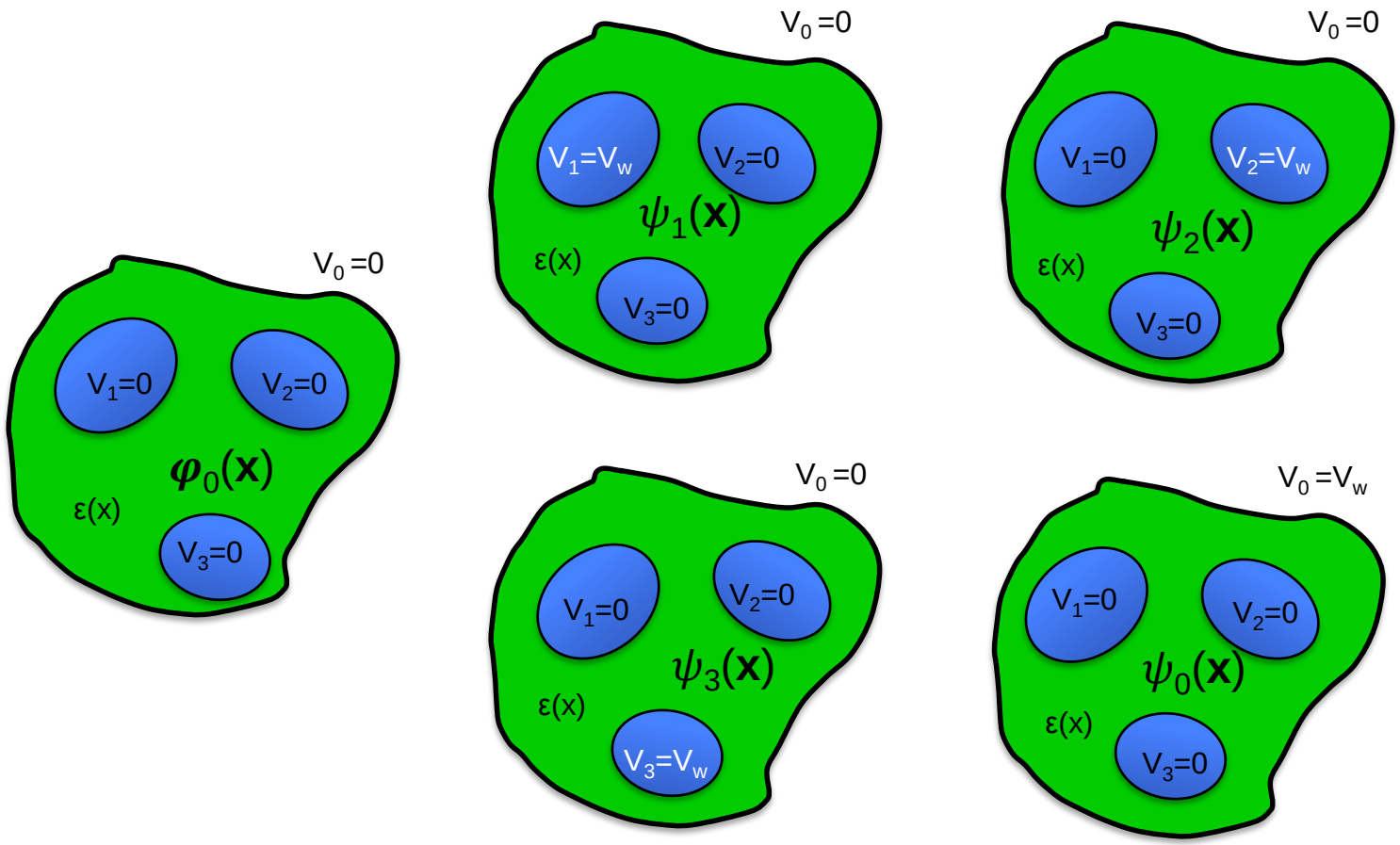
In a realistic detector, the electrodes are however neither grounded nor insulated, but they are connected to ground and among each other by impedance elements e.g. amplifiers, load resistors etc.

In addition the detector materials can have finite conductivity, like it is the case in Resistive Plate Chambers (RPCs), un-depleted silicon sensors, and detectors with resistive layers for application of High Voltage, spreading of charge or discharge protection.

For these situations we have to extend the Ramo-Schockley theorem.

Electrostatic in dielectric media





$$\nabla\left[\varepsilon(\mathbf{x})\nabla\varphi(\mathbf{x})\right] = -\rho(\mathbf{x}) \qquad \varphi(\mathbf{x})|_{\mathbf{x}=\mathbf{A}_n} = V_n$$

A solution that satisfies the boundary conditions (and is therefore unique):

$$\nabla\left[\varepsilon(\mathbf{x})\nabla\varphi_0(\mathbf{x})\right] = -\rho_0(\mathbf{x}) \qquad \varphi(\mathbf{x})|_{\mathbf{x}=\mathbf{A}_n} = 0$$

$$\nabla\left[\varepsilon(\mathbf{x})\nabla\psi_n(\mathbf{x})\right] = 0 \qquad \psi_n(\mathbf{x})|_{\mathbf{x}=\mathbf{A}_n} = V_w\delta_{mn}$$

$$\varphi(\mathbf{x}) = \varphi_0(\mathbf{x}) + \sum_{n=0}^{N}\frac{V_n}{V_w}\psi_n(\mathbf{x})$$

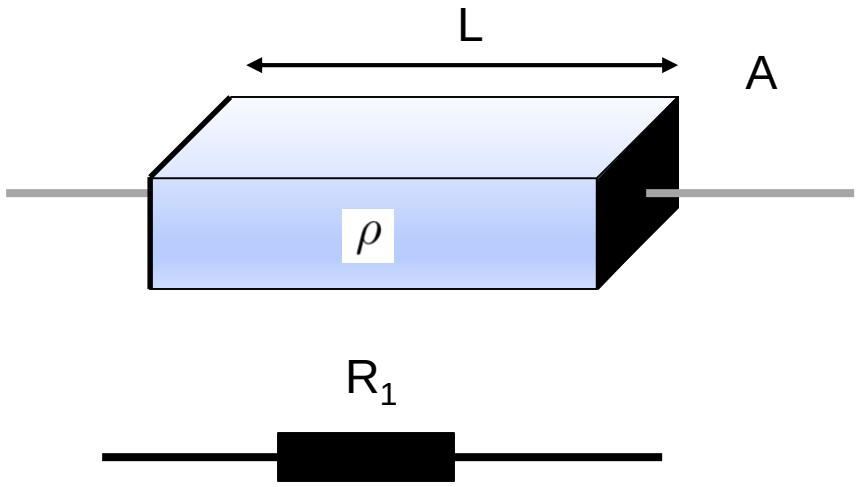
$$Q_n = \oint_{\mathbf{A}_n} \varepsilon(\mathbf{x})\mathbf{E}(\mathbf{x})d\mathbf{A}$$

$$c_{mn} = \frac{1}{V_w}\oint_{\mathbf{A}_n} \varepsilon(\mathbf{x})\nabla\psi_m(\mathbf{x})d\mathbf{A}$$

Ramo-Shockley theorem holds also for dielectric media !

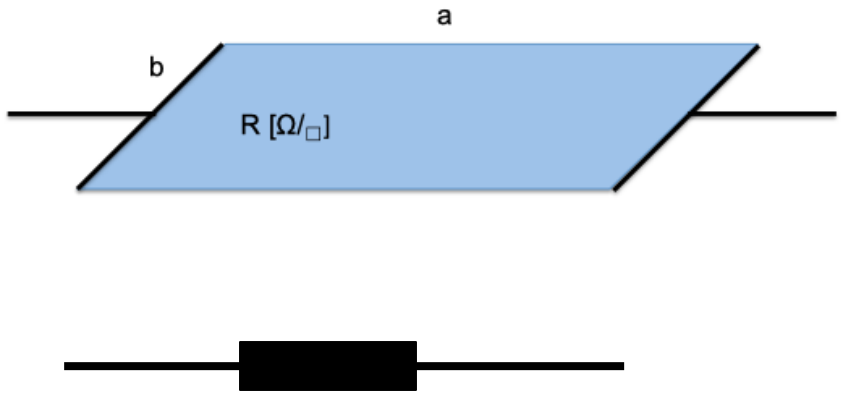
Conductivity, volume resistivity

Volume resistivity ρ [Ωm] – typically expressed as Ωcm
Conductivity $\sigma = 1/\rho$ [Siemens]



$$R_1 = \rho \frac{L}{A} \qquad I = \frac{U}{R_1}$$

Surface resistivity R [Ω/square]



$$R_1 = R\frac{a}{b}$$

Nonuniform conductivity (volume resistivity) relates the local current density to the local electric field:

$$\mathbf{j}(\mathbf{x}) = \sigma(\mathbf{x})\mathbf{E}(\mathbf{x}) = -\sigma(\mathbf{x})\nabla\varphi(\mathbf{x})$$

Quasi-static Approximation

In a medium with conductivity sigma there will be a current flowing according to

$$\mathbf{j}(\mathbf{x},t) = \sigma(\mathbf{x})\mathbf{E}(\mathbf{x},t) = -\sigma(\mathbf{x})\nabla\varphi(\mathbf{x})$$

In addition to this current we can have and externally impressed current $j_e(x, t)$, so the total current is

$$\mathbf{j}(\mathbf{x},t) = \sigma(\mathbf{x})\mathbf{E}(\mathbf{x},t) + \mathbf{j}_e(\mathbf{x},t) = -\sigma(\mathbf{x})\nabla\varphi(\mathbf{x},t) + \mathbf{j}_e(\mathbf{x},t)$$

Assuming the variation of the electric field to be slow, we can use the Poisson equation for a medium given by

$$\nabla[\varepsilon(\mathbf{x})\nabla\varphi(\mathbf{x},t)] = -\rho(\mathbf{x},t)$$

Performing the time derivative we have

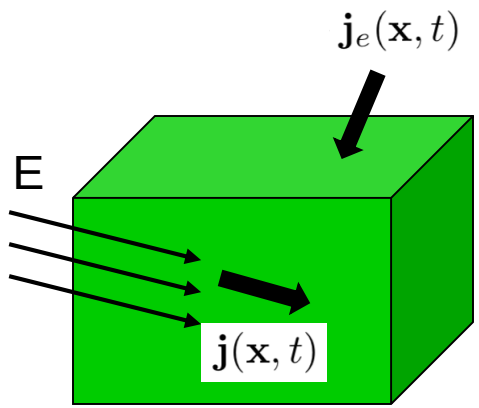
$$\nabla[\varepsilon(\mathbf{x})\nabla\frac{\partial\varphi(\mathbf{x},t)}{\partial t}] = -\frac{\partial\rho(\mathbf{x},t)}{\partial t}$$

And using

$$\nabla\mathbf{j}(\mathbf{x},t) = -\frac{\partial\rho(\mathbf{x},t)}{\partial t}$$

we have

$$\nabla\left[\varepsilon(\mathbf{x})\nabla\frac{\partial\varphi(\mathbf{x},t)}{\partial t} + \sigma(\mathbf{x})\nabla\varphi(\mathbf{x},t)\right] = -\frac{\partial\rho_e(\mathbf{x},t)}{\partial t}$$



Where ρ_e is the ‘externally impressed’ charge density.

Quasi-static Approximation of Maxwell's equations

Assuming a conductivity sigma of the material we have a current according to

$$\mathbf{j}(\mathbf{x},t) = \sigma(\mathbf{x})\mathbf{E}(\mathbf{x},t)$$

Maxwell's equations for this situation

$$\begin{aligned}
\nabla\mathbf{D}(\mathbf{x},t) &= \rho(\mathbf{x},t) \qquad \mathbf{D}(\mathbf{x},t) = \varepsilon(\mathbf{x})\mathbf{E}(\mathbf{x},t) \\
\nabla\mathbf{B}(\mathbf{x},t) &= 0 \qquad \mathbf{B}(\mathbf{x},t) = \mu(\mathbf{x})\mathbf{H}(\mathbf{x},t) \\
\nabla\times\mathbf{E}(\mathbf{x},t) &= -\frac{\partial\mathbf{B}(\mathbf{x},t)}{\partial t} \\
\nabla\times\mathbf{H}(\mathbf{x},t) &= \frac{\partial\mathbf{D}(\mathbf{x},t)}{\partial t} + \mathbf{j}_e(\mathbf{x},t) + \sigma(\mathbf{x})\mathbf{E}(\mathbf{x},t)
\end{aligned}$$

The current $j_e(x, t)$ is an 'externally impressed' current, which is related to the 'externally impressed' charge density ρ_e by

$$\nabla\mathbf{j}_e(\mathbf{x},t) = -\frac{\partial\rho_e(\mathbf{x},t)}{\partial t}$$

If we assume that this impressed current is only changing slowly we can neglect Faraday's law and approximate

$$\nabla\times\mathbf{E}(\mathbf{x},t) \approx 0 \qquad \mathbf{E}(\mathbf{x},t) = -\nabla\varphi(\mathbf{x},t)$$

and we can then write the electric field as the gradient of a potential, an by taking the divergence of the last equation …

$$\nabla(\nabla\times\mathbf{H}(\mathbf{x},t)) = \frac{\partial\nabla\mathbf{D}(\mathbf{x},t)}{\partial t} + \nabla\mathbf{j}_e(\mathbf{x},t) + \nabla[\sigma(\mathbf{x})\mathbf{E}(\mathbf{x},t)] = 0$$

$$\nabla\left[\varepsilon(\mathbf{x})\nabla\frac{\partial\varphi(\mathbf{x},t)}{\partial t} + \sigma(\mathbf{x})\nabla\varphi(\mathbf{x},t)\right] = -\frac{\partial\rho_e(\mathbf{x},t)}{\partial t}$$

→ the same equation as on the previous slide

Laplace Transform, Fourier Transform

Bilateral Laplace Transform

$$\mathscr{L}[f(t)] = F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt,$$

Fourier Transform

$$\mathscr{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = F(i\omega) \qquad \omega = 2\pi f,$$

Inverse Laplace and Fourier Transforms

$$f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s)\, e^{st}\, ds \qquad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(i\omega)\, e^{i\omega t}\, d\omega.$$

Relations that hold for Laplace and Fourier Transforms

(a) Addition
$\mathscr{L}[af(t) + bg(t)] = aF(s) + bG(s)$

(b) Convolution
$\mathscr{L}\left[\int_{-\infty}^{\infty} f(t-t')g(t')dt'\right] = F(s)\,G(s)$

(c) Time differentiation
$\mathscr{L}\left[f^{(n)}(t)\right] = s^n F(s)$

(d) Time integration
$\mathscr{L}\left[\int_{-\infty}^{t} f(t')dt'\right] = \frac{1}{s} F(s)$

(e) Time shift
$\mathscr{L}[f(t-t_0)] = F(s)e^{-st_0}$

(f) Time scaling
$\mathscr{L}[f(at)] = \frac{1}{a} F(\frac{s}{a})$

(g) Damping
$\mathscr{L}[e^{-s_0 t} f(t)] = F(s+s_0)$

(h) Multiplication
$\mathscr{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$

(i) Initial value if $f(t) = 0$ for $t < 0$
$f(0^+) = \lim_{s\to\infty} sF(s)$

(j) Final value
$f(\infty) = \lim_{s\to 0} sF(s)$

(k) Parseval's theorem
$\int_{-\infty}^{\infty} f(t)^2 dt = \int_{-\infty}^{\infty} |F(i2\pi f)|^2 df \qquad = 2\int_0^{\infty} |F(i2\pi f)|^2 df$

Quasi-static Approximation of Maxwell's equations

Performing the Laplace Transform of the quasi-static equation

$$\nabla\left[\varepsilon(\mathbf{x})\nabla\frac{\partial\varphi(\mathbf{x},t)}{\partial t}+\sigma(\mathbf{x})\nabla\varphi(\mathbf{x},t)\right]=-\frac{\partial\rho_e(\mathbf{x},t)}{\partial t}$$

we find

$$\nabla\left[\varepsilon(\mathbf{x})\nabla s\varphi(\mathbf{x},s)+\sigma(\mathbf{x})\nabla\varphi(\mathbf{x},s)\right]=-s\rho_e(\mathbf{x},s)$$

$$\nabla\left[(\varepsilon(\mathbf{x})+\sigma(\mathbf{x})/s)\nabla\varphi(\mathbf{x},s)\right]=-\rho_e(\mathbf{x},s)$$

[1] H.A. Haus, J.R. Melcher, Electromagentic Fields and Energy, Prentice-Hall, Englewood Cliffs, NJ, 1989.

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The quasi-static electromagnetic approximation for weakly conducting media ☆

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So we can write this equation as

$$\nabla\left[\varepsilon_{eff}(\mathbf{x})\nabla\varphi(\mathbf{x},s)\right]=-\rho_e(\mathbf{x},s) \qquad \varepsilon_{eff}(\mathbf{x})=\varepsilon(\mathbf{x})+\sigma(\mathbf{x})/s \qquad \rho(\mathbf{x},s)=-\nabla\left[\varepsilon(\mathbf{x})\nabla\varphi(\mathbf{x},s)\right] \qquad \rho_e(\mathbf{x},s)=-\nabla\left[\varepsilon_{eff}(\mathbf{x})\nabla\varphi(\mathbf{x},s)\right]$$

This is the Poisson equation with an effective permittivity !!

→ We can therefore find the time dependent solutions for a medium with a given conductivity by solving the electrostatic Poisson equation in the Laplace domain !

→ Knowing the electrostatic solution for a given permittivity ε(x) we just have to replace ε(x) by ε(x)+σ(x)/s and perform the inverse Laplace transform !

Many examples

Jinst

Published by IOP Publishing for Sissa Medialab

Received: *February 26, 2016*
Accepted: *October 19, 2016*
Published: *November 7, 2016*

Electric fields, weighting fields, signals and charge diffusion in detectors including resistive materials

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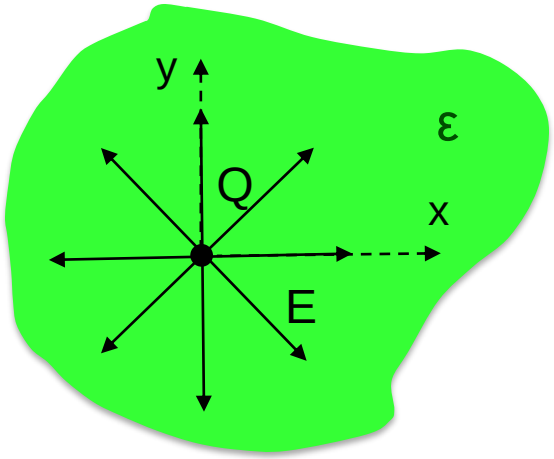
Abstract: In this report we discuss static and time dependent electric fields in detector geometries with an arbitrary number of parallel layers of a given permittivity and weak conductivity. We derive the Green's functions i.e. the field of a point charge, as well as the weighting fields for readout pads and readout strips in these geometries. The effect of 'bulk' resistivity on electric fields and signals is investigated. The spreading of charge on thin resistive layers is also discussed in detail, and the conditions for allowing the effect to be described by the diffusion equation is discussed. We apply the results to derive fields and induced signals in Resistive Plate Chambers, MICROMEGAS detectors including resistive layers for charge spreading and discharge protection as well as detectors using resistive charge division readout like the MicroCAT detector. We also discuss in detail how resistive layers affect signal shapes and increase crosstalk between readout electrodes.

Point charge Q in a uniform conducting medium

A point charge Q at x=0 inside a medium of constant permittivity ε

$$\rho(\mathbf{x}) = Q\delta(\mathbf{x}) \qquad \varphi(\mathbf{x}) = \frac{Q}{4\pi\varepsilon}\frac{1}{|\mathbf{x}|} \qquad \varepsilon = \varepsilon_0\varepsilon_r$$

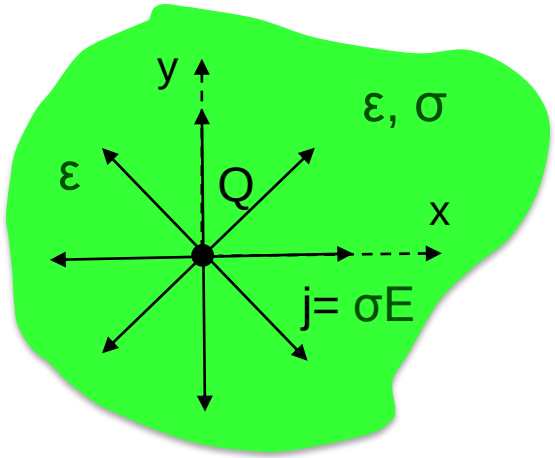


A point charge Q placed at x=0 into a medium of constant permittivity ε and constant conductivity σ

$$\rho_e(\mathbf{x},t) = Q\delta(\mathbf{x})\,\Theta(t) \qquad \rho_e(\mathbf{x},s) = \mathbf{L}\left[\rho_e(\mathbf{x},t)\right] = \frac{Q}{s}\,\delta(\mathbf{x})$$

$$\varphi(\mathbf{x},s) = \frac{Q/s}{\varepsilon+\sigma/s}\frac{1}{|\mathbf{x}|} = \frac{Q}{(s+1/\tau)}\frac{1}{4\pi\varepsilon|\mathbf{x}|} \qquad \tau = \frac{\varepsilon}{\sigma} = \frac{\varepsilon_r\varepsilon_0}{\sigma}$$

$$\varphi(\mathbf{x},t=0) = \lim_{s\to\infty} s\varphi(\mathbf{x},s) = \frac{Q}{4\pi\varepsilon|\mathbf{x}|} \qquad \varphi(\mathbf{x},t=\infty) = \lim_{s\to 0} s\varphi(\mathbf{x},s) = 0$$



At t=0 when the charge is placed at x=0, the potential is equal to the static potential in absence of conductivity, for long times the potential is zero.

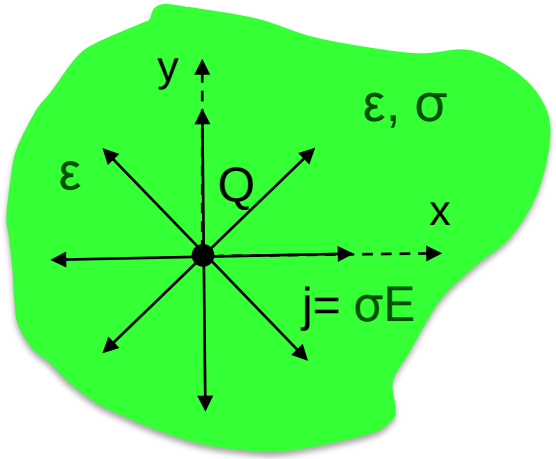
The time dependent potential is:

$$\varphi(\mathbf{x},t) = \mathbf{L}^{-1}\left[\varphi(\mathbf{x},s)\right] = \frac{Q}{4\pi\varepsilon|\mathbf{x}|}\,e^{-t/\tau}$$

Point charge Q in a uniform conducting medium

The charge density is given by

$$\rho(\mathbf{x},s) = -\nabla\left[\varepsilon(\mathbf{x})\nabla\varphi(\mathbf{x},s)\right] = -\varepsilon\Delta\varphi(\mathbf{x},s) = \frac{Q\delta(\mathbf{x})}{s+1/\tau}$$



In the time domain we have

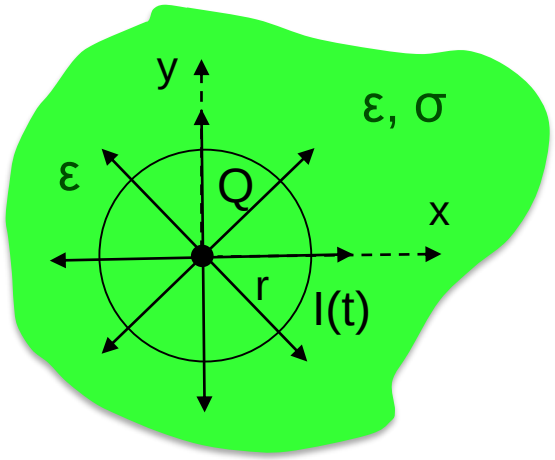
$$\rho(\mathbf{x},t) = Q\delta(\mathbf{x})e^{-t/\tau}$$

The situation therefore corresponds to a and exponentially decaying point charge at x=0. The radial current are given by

$$\mathbf{j}(\mathbf{x},s) = \sigma\mathbf{E}(\mathbf{x},s) = -\sigma\nabla\varphi(\mathbf{x},s) = -\frac{1}{\tau}\frac{Q}{(s+1/\tau)}\nabla\frac{1}{4\pi|\mathbf{x}|}$$

The total current flowing through the surface of a sphere of radius r is

$$I(s,r) = \oint_{S(r)}\mathbf{j}(\mathbf{x},s)d\mathbf{A} = \oint_V \nabla\mathbf{j}dV = -\frac{1}{\tau}\frac{Q}{(s+1/\tau)}\oint_V \Delta\frac{1}{4\pi|\mathbf{x}|}dV = \frac{1}{\tau}\frac{Q}{(s+1/\tau)}$$



In the time domain this reads as

$$I(t,r) = \mathbf{L}^{-1}\left[I(s,r)\right] = \frac{Q}{\tau}e^{-t/\tau}$$

Integrating this current over time gives

$$Q_{tot} = \int_0^\infty I(t,r)dt = \lim_{s\to 0} s\frac{1}{s}I(s,t) = Q$$

This reflects the fact that the entire charge that was place at x=0 at t=0 disappears.

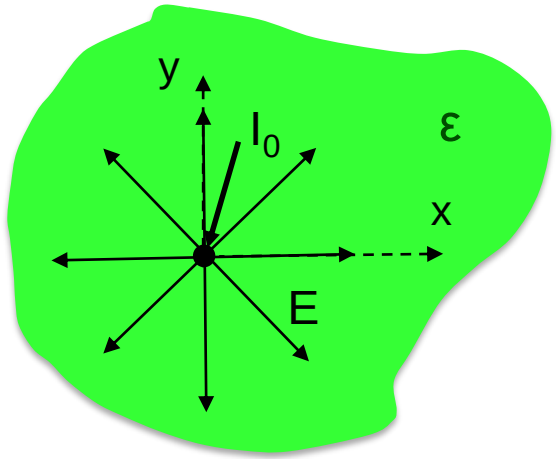
Point ‘current’ I_0 in a uniform conducting medium

Placing a constant current I_0 at x=0 at t=0 we have the charge

$$Q(t) = I_0 t \qquad Q(s) = \mathbf{L}\left[Q(t)\right] = I_0/s^2$$

and the potential is then

$$\varphi(\mathbf{x}, s) = \frac{I_0/s^2}{\varepsilon + \sigma/s}\frac{1}{4\pi|\mathbf{x}|} = \frac{I_0}{s(s+1/\tau)}\frac{1}{4\pi\varepsilon|\mathbf{x}|} = I_0\tau\left(\frac{1}{s} - \frac{1}{s+1/\tau}\right)\frac{1}{4\pi\varepsilon|\mathbf{x}|} \qquad \tau = \frac{\varepsilon}{\sigma} = \frac{\varepsilon_r\varepsilon_0}{\sigma}$$



For long times we then have

$$\varphi(\mathbf{x}, t=\infty) = \lim_{s\to 0} s\varphi(\mathbf{x}, s) = I_0\tau\frac{1}{4\pi\varepsilon|\mathbf{x}|}$$

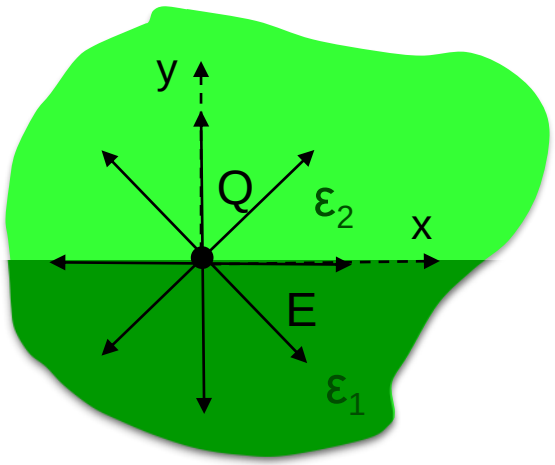
So the potential is equal to having a point charge $I_0\tau$ at x=0. The time dependent potential is

$$\varphi(\mathbf{x}, t) = I_0\tau(1 - e^{-t/\tau})\frac{1}{4\pi\varepsilon|\mathbf{x}|}$$

Point charge Q on the boundary of two media

Point charge Q at x=0 on the boundary between two infinite half-spaces of different permittivity

$$\varphi(\mathbf{x}) = \frac{Q}{2\pi(\varepsilon_1+\varepsilon_2)}\frac{1}{|\mathbf{x}|}$$

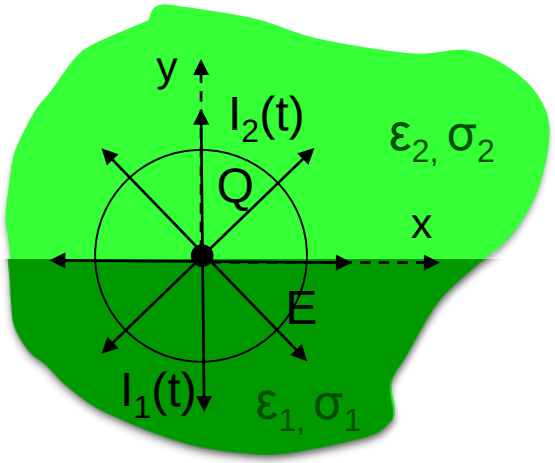


Placing a point charge Q placed at t=0 at x=0 on the boundary between two hal-spaces of different permittivity and conductivity, Laplace domain

$$\varphi(\mathbf{x},s) = \frac{Q}{2\pi(\varepsilon_1+\varepsilon_2)(s+1/\tau)}\frac{1}{|\mathbf{x}|} \qquad \tau = \frac{\varepsilon_1+\varepsilon_2}{\sigma_1+\sigma_2}$$

Time domain

$$\varphi(\mathbf{x},t) = \frac{Q}{2\pi(\varepsilon_1+\varepsilon_2)}\frac{1}{|\mathbf{x}|}e^{-t/\tau}$$



The current flowing through a half-spheres in the two layers are given by

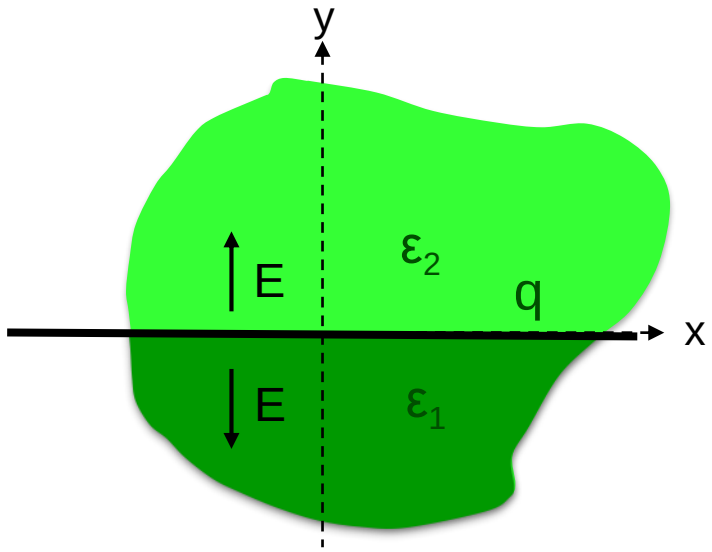
$$I_1(t) = \sigma_1|\mathbf{E}(r,t)|2r^2\pi = \frac{\sigma_1 Q}{(\varepsilon_1+\varepsilon_2)}e^{-t/\tau} \qquad I_2(t) = \sigma_2|\mathbf{E}(r,t)|2r^2\pi = \frac{\sigma_2 Q}{(\varepsilon_1+\varepsilon_2)}e^{-t/\tau}$$

$$Q_1 = \int_0^\infty I_1(t)dt = \frac{\sigma_1}{\sigma_1+\sigma_2}Q \qquad Q_2 = \int_0^\infty I_2(t)dt = \frac{\sigma_2}{\sigma_1+\sigma_2}Q \qquad Q_1+Q_2 = Q$$

Thin layer of charge on the boundary of two media

An infinite thin layer of charge at z=0 with charge density q [C/cm^2] on the boundary of two infinite dielectric half-spaces

$$E = \frac{q}{\varepsilon_1 + \varepsilon_2}$$



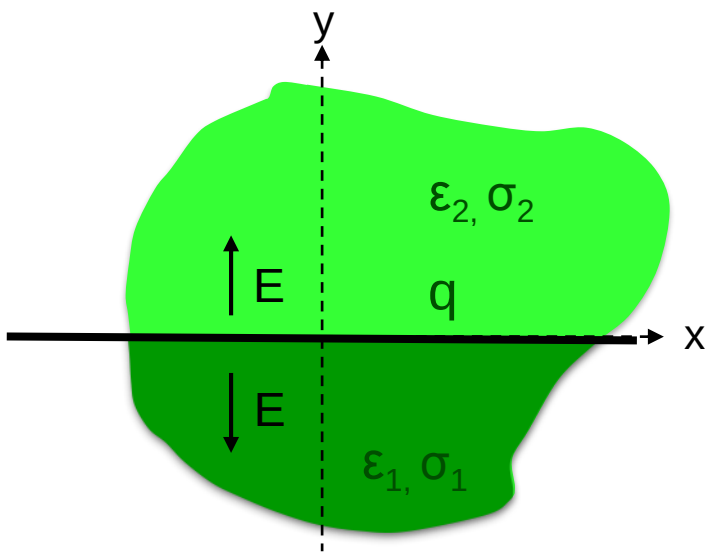
Placing the charge density q at t=0 with the two infinite half-spaces having a conductivities σ_1 and σ_2

$$E(s) = \frac{q}{(\varepsilon_1 + \varepsilon_2)(s + 1/\tau)} \qquad \tau = \frac{\varepsilon_1 + \varepsilon_2}{\sigma_1 + \sigma_2}$$

$$E(t) = \frac{q}{(\varepsilon_1 + \varepsilon_2)} e^{-t/\tau}$$

The time dependent charge on the interface is

$$\varepsilon_1 E(t) + \varepsilon_2 E(t) = \overline{q}(t) \qquad \rightarrow \qquad \overline{q}(t) = q\, e^{-t/\tau}$$



In case the half-space z<0 has conductivity σ and the half-space z>0 is insulating the time constant is

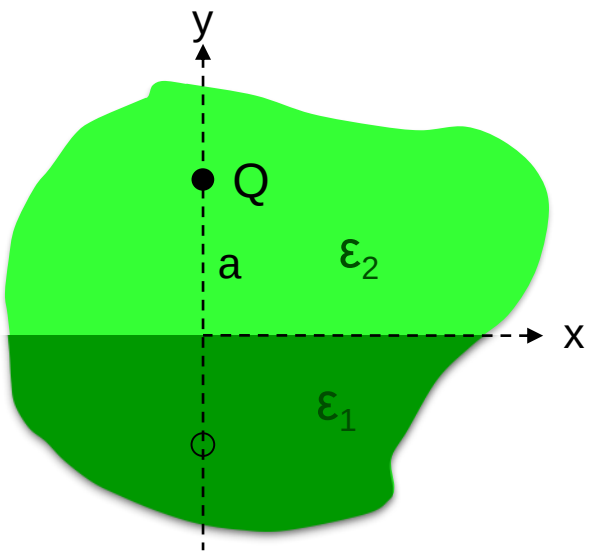
$$\tau = \frac{\varepsilon_0}{\sigma} (\varepsilon_r + 1)$$

Point charge Q in presence of a conducting half-space

Point charge Q at position 0,0,a in a geometry with two infinite half-spaces of different dielectric permittivities

$$\varphi(\mathbf{x}) = \frac{Q}{4\pi\varepsilon_2}\left[\frac{1}{\sqrt{x^2+y^2+(z-a)^2}} - \frac{\varepsilon_1-\varepsilon_2}{\varepsilon_2+\varepsilon_3}\frac{1}{\sqrt{x^2+y^2+(z+a)^2}}\right] \qquad z>0$$

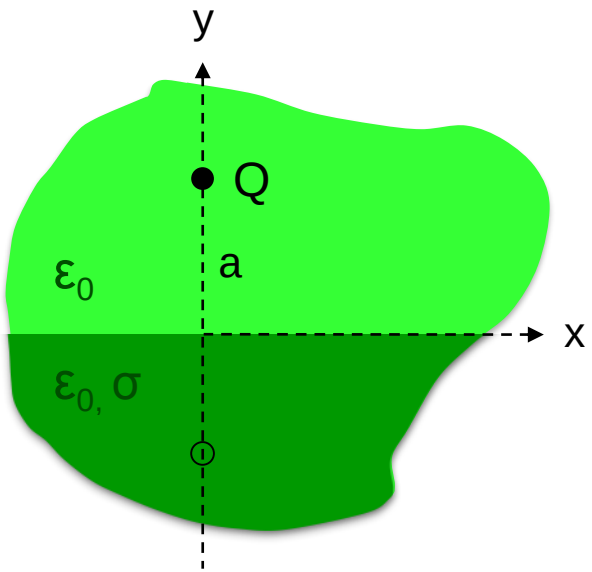
$$= \frac{Q}{4\pi\varepsilon_2}\frac{2\varepsilon_1}{\varepsilon_1+\varepsilon_2}\frac{1}{\sqrt{x^2+y^2+(z-a)^2}} \qquad z<0$$



Point charge Q at position 0, 0, a in a geometry with an infinite half-space of conductivity sigma. Replace ε_2 by ε_0, ε_1 by $\varepsilon_0 + \sigma/s$ and Q by Q/s

$$\varphi(\mathbf{x},t) = \frac{Q}{4\pi\varepsilon_0\sqrt{x^2+y^2+(z-a)^2}} - \frac{Q(1-e^{-t/\tau})}{4\pi\varepsilon_0\sqrt{x^2+y^2+(z+a)^2}} \qquad z>0$$

$$= \frac{Qe^{-t/\tau}}{4\pi\varepsilon_0\sqrt{x^2+y^2+(z-a)^2}} \qquad z<0 \qquad \tau = 2\varepsilon_0/\sigma$$



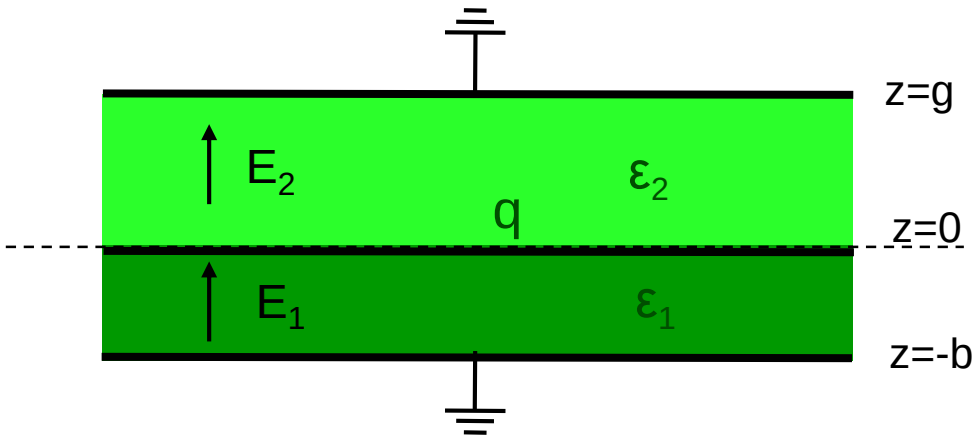
At t=0 the field is equal to a single point charge Q at 0,0,a. At t=infinite there is a point charge Q at 0,0,a and a mirror charge at 0,0,-a.

Quasi-static Approximation of Maxwell's equations

Layer of charge on the boundary between two dielectric layers and two grounded plates at z=-b and z=g.

$$E_1 b + E_2 g = 0 \qquad -\varepsilon_1 E_1 + \varepsilon_2 E_2 = q$$

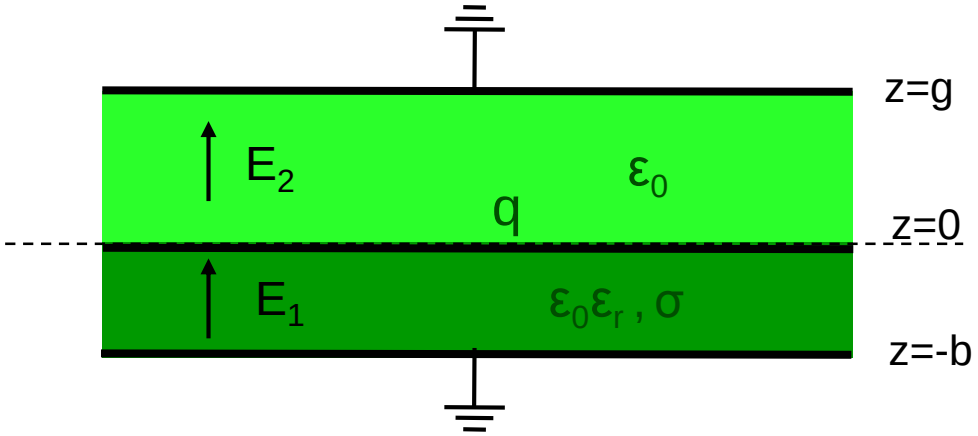
$$E_1 = -\frac{g\,q}{\varepsilon_1 g + \varepsilon_2 b} \qquad E_2 = \frac{b\,q}{\varepsilon_1 g + \varepsilon_2 b}$$



Replace ε_1 by $\varepsilon_0\varepsilon_r + \sigma/s$, ε_2 by ε_0 and q by q/s

$$E_1(t) = -\frac{g\,q}{\varepsilon_1 g + \varepsilon_2 b}\,e^{-t/\tau} \qquad E_2(t) = \frac{b\,q}{\varepsilon_1 g + \varepsilon_2 b}\,e^{-t/\tau} \qquad \tau = \frac{\varepsilon_0}{\sigma}\left(\varepsilon_r + \frac{b}{g}\right)$$

In case b=g the time constant becomes equal to the case of two infinite half-spaces.



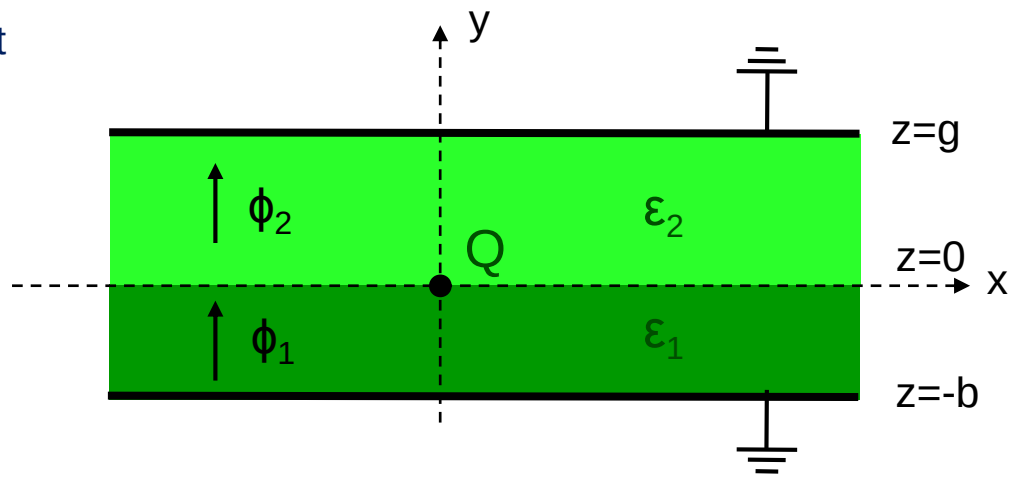
Quasi-static Approximation of Maxwell's equations

Point charge on the boundary of two dielectric media with grounded planes at z=-b and z=g

$$\phi_1(r,z) = \frac{Q}{2\pi}\int_0^\infty J_0(kr)\frac{4\sinh(gk)\sinh(k(b+z))}{D(k)}dk \qquad -b<z<0$$

$$\phi_2(r,z) = \frac{Q}{2\pi}\int_0^\infty J_0(kr)\frac{4\sinh(bk)\sinh(k(g-z))}{D(k)}dk \qquad 0<z<g$$

$$D(k) = 4[\varepsilon_1\cosh(bk)\sinh(gk)+\varepsilon_2\sinh(bk)\cosh(gk)]$$

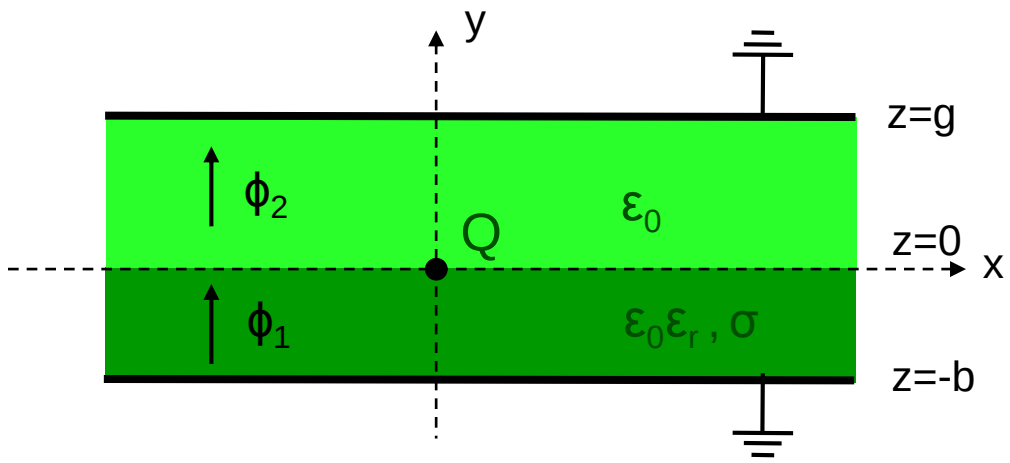


Assuming layer 1 to have conductivity σ we have

$$\varepsilon_1 = \varepsilon_0\varepsilon_r+\sigma/s \quad \varepsilon_2=\varepsilon_0 \quad Q_1 = Q/s \tag{4.14}$$

$$E_1(r,z,s) = -\frac{Q}{2\pi s}\int_0^\infty k\,J_0(kr)\frac{\sinh(gk)\cosh(k(b+z))}{\varepsilon_0[\sinh(bk)\cosh(gk)+(\varepsilon_r+\sigma/(\varepsilon_0 s))\cosh(bk)\sinh(gk)]}dk \tag{4.15}$$

$$E_2(r,z,s) = \frac{Q}{2\pi s}\int_0^\infty k\,J_0(kr)\frac{\sinh(bk)\cosh(k(g-z))}{\varepsilon_0[\sinh(bk)\cosh(gk)+(\varepsilon_r+\sigma/(\varepsilon_0 s))\cosh(bk)\sinh(gk)]}dk$$



We find the time dependent fields by performing the inverse Laplace transforms and have

$$E_1(r,z,t) = -\frac{Q}{2\pi}\int_0^\infty k\,J_0(kr)\frac{\sinh(gk)\cosh(k(b+z))}{\varepsilon_0 D(k)}e^{-t/\tau(k)}dk \tag{4.16}$$

$$E_2(r,z,t) = \frac{Q}{2\pi}\int_0^\infty k\,J_0(kr)\frac{\sinh(bk)\cosh(k(g-z))}{\varepsilon_0 D(k)}e^{-t/\tau(k)}dk$$

with

$$\tau(k) = \frac{\varepsilon_0}{\sigma}\left(\varepsilon_r+\frac{\tanh(bk)}{\tanh(gk)}\right)$$

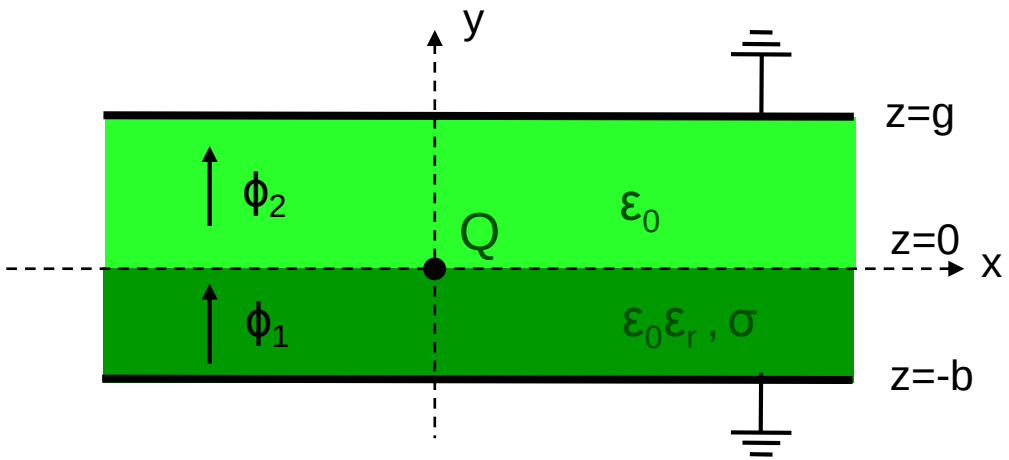
Quasi-static Approximation of Maxwell's equations

$$E_1(r,z,t) = -\frac{Q}{2\pi}\int_0^\infty k\,J_0(kr)\frac{\sinh(gk)\cosh(k(b+z))}{\varepsilon_0 D(k)}e^{-t/\tau(k)}dk$$

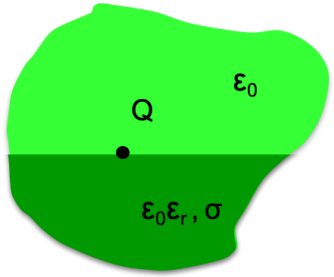
$$E_2(r,z,t) = \frac{Q}{2\pi}\int_0^\infty k\,J_0(kr)\frac{\sinh(bk)\cosh(k(g-z))}{\varepsilon_0 D(k)}e^{-t/\tau(k)}dk$$

with

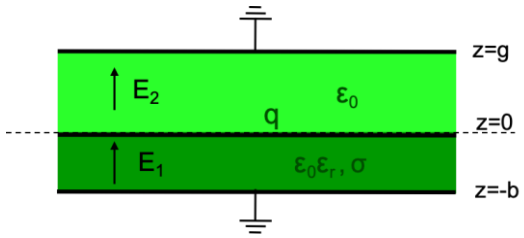
$$\tau(k) = \frac{\varepsilon_0}{\sigma}\left(\varepsilon_r + \frac{\tanh(bk)}{\tanh(gk)}\right)$$



$$\tau(k=\infty) = \frac{\varepsilon_0}{\sigma}(\varepsilon_r + 1) = \tau_1$$

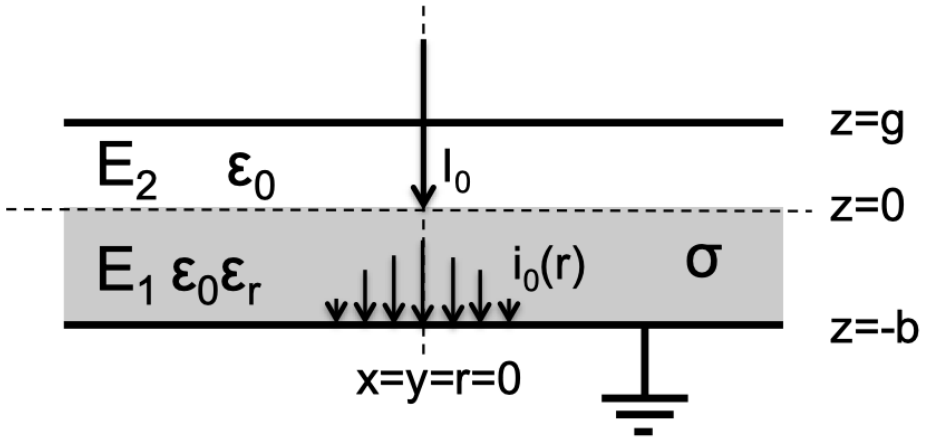


$$\tau(k=0) = \frac{\varepsilon_0}{\sigma}\left(\varepsilon_r + \frac{b}{g}\right) = \tau_2$$



The charge disappears with a continuous distribution of time constants of τ_1 and τ_1.
This situation would correspond to a charge deposit in a Resistive Plate Chamber (RPC).

Point current on a resistive layers



$$r_{50\%} \approx b \quad r_{90\%} \approx 2.3b \quad r_{99\%} \approx 3.9b$$

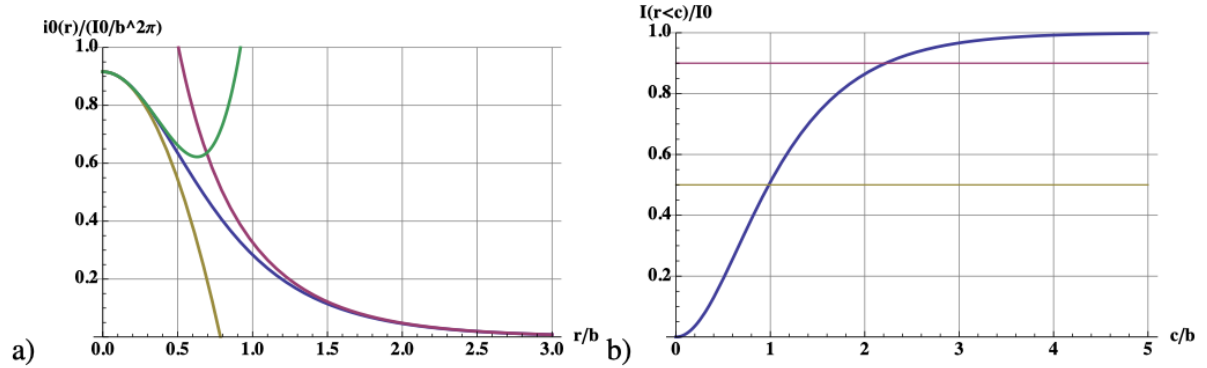


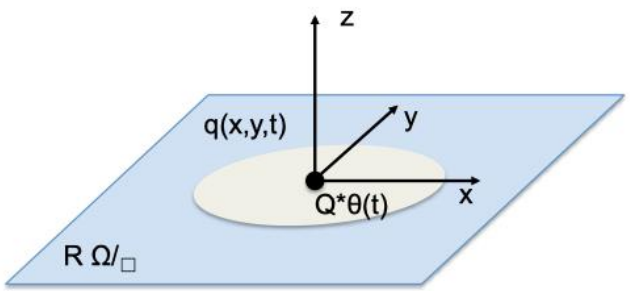
Figure 12. a) Current density $i_0(r)$ at $z = -b$. The exact curve together with the 2nd order and 4th order approximation from eq. (4.20) and the exponential approximation from eq. (4.22). b) Total current at $z = -b$ flowing inside a radius r from eq. (4.23).

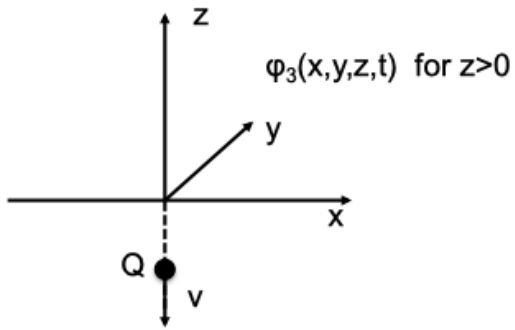
$$i_0(r) = -\sigma E_1(r, z = -b) = \frac{I_0}{b^2\pi} \int_0^\infty \frac{1}{2} J_0\left(y\frac{r}{b}\right) \frac{y}{\cosh(y)} dy$$

$$\int_0^\infty \frac{1}{2} J_0\left(y\frac{r}{b}\right) \frac{y}{\cosh(y)} dy = \frac{\pi}{2} \sum_{n=0}^{\infty} (-1)^n (2n+1) K_0\left(\frac{(2n+1)\pi}{2}\frac{r}{b}\right) \approx \frac{\pi}{2\sqrt{r/b}} e^{-\pi r/(2b)} \quad \text{for} \quad \frac{r}{b} \gg 1$$

$$I(r) = \int_0^r 2r\pi i_0(r')dr' = I_0\left[1 - 2\sum_{n=0}^{\infty}(-1)^n \frac{r}{b} K_1\left(\frac{(2n+1)\pi}{2}\frac{r}{b}\right)\right]$$

Point charge on an infinitely extended resistive layer





A point charge Q is placed on an infinitely extended resistive layer at r=0, t=0. The potential for z>0 and z<0 is given by

$$\phi_1(r,z,t) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2+(-z+vt)^2}} \qquad \phi_3(r,z,t) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2+(z+vt)^2}} \qquad v = \frac{1}{2\varepsilon_0 R}$$

The potential is equivalent to a point charge Q moving along the z-axis at a velocity v

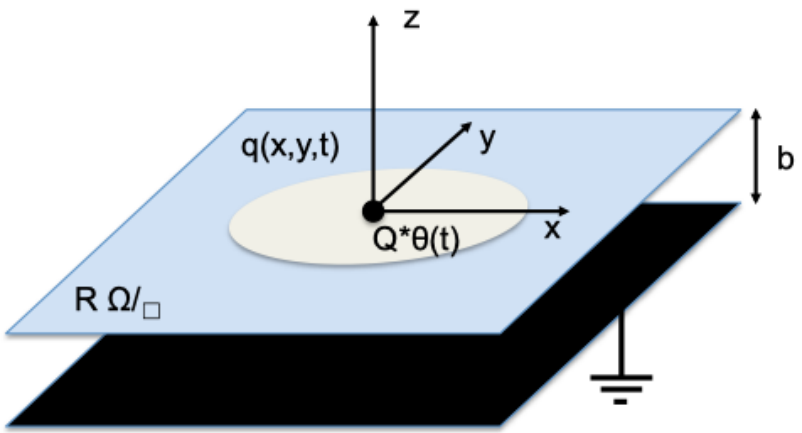
The charge distribution on the resistive layer as a function of time is

$$q(r,t) = \varepsilon_0\frac{\partial\phi_1}{\partial z}|_{z=0} - \varepsilon_0\frac{\partial\phi_3}{\partial z}|_{z=0} = \frac{Q}{2\pi}\frac{vt}{\sqrt{(r^2+v^2t^2)^3}}$$

At no time this charge distribution assumes a Gaussian shape.

Point charge on an infinitely extended resistive layer

A resistive layer in presence of a grounded layer. A point charge is placed on the layer at r=0, t=0



$$q(r,t) = \frac{Q}{b^2\pi}\frac{1}{2}\int_0^\infty \kappa J_0\left(\kappa\frac{r}{b}\right)\exp\left[-\kappa(1-e^{-2\kappa})\frac{t}{T}\right]d\kappa \qquad T = \frac{b}{v} = 2b\varepsilon_0 R$$

For long times we can make the approximation

$$-\kappa(1-e^{-2\kappa})\frac{t}{T} \approx -2\kappa^2\frac{t}{T}$$

And the charge distribution becomes

$$q(r,t) = \frac{Q}{b^2\pi}\frac{1}{8t/T}e^{-\frac{r^2}{8b^2t/T}}$$

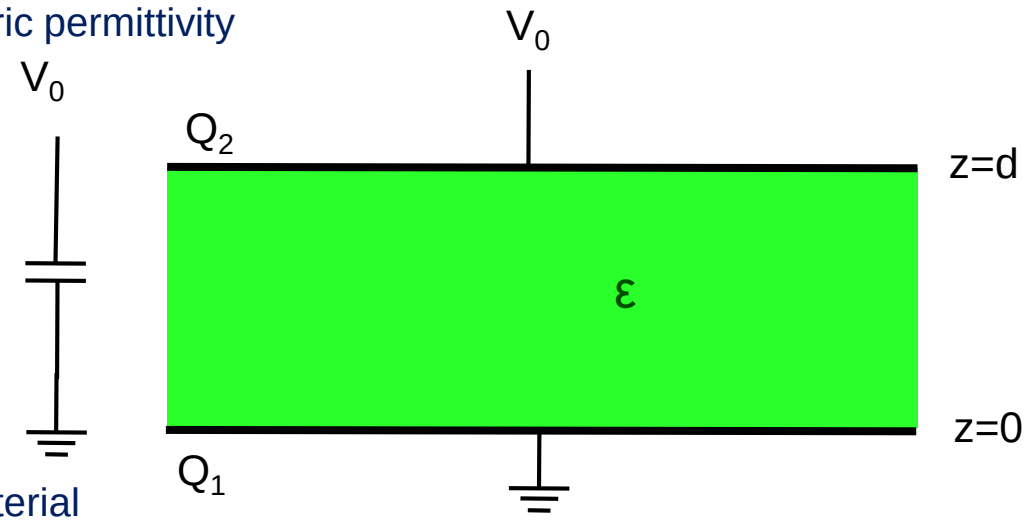
The charge distribution does indeed assume Gaussian shape …

Capacitance, Impedance

A constant voltage applied to a capacitor with homogeneous dielectric permittivity

$$\varphi(z) = \frac{V}{d}\, z$$

$$Q_1 = -\varepsilon \frac{V_0}{d} A \qquad Q_2 = \varepsilon \frac{V_0}{d} A$$



A time dependent voltage applied to a capacitor with conductive material

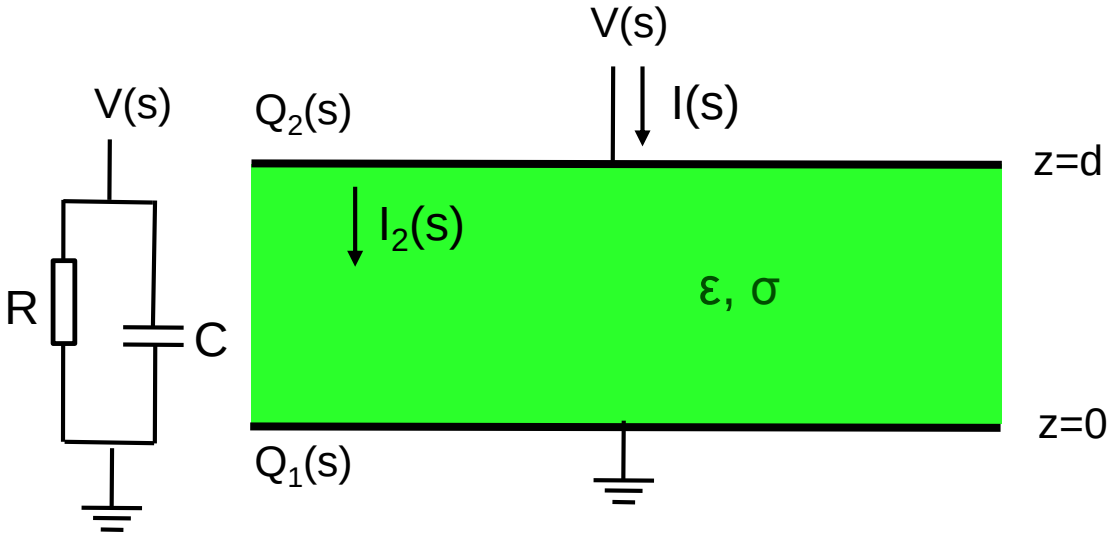
$$Q_1(s) = -\varepsilon \frac{V(s)}{d} A \qquad Q_2(s) = \varepsilon \frac{V(s)}{d} A$$

$$I_2(s) = \sigma \frac{V(s)}{d} A$$

$$\frac{dQ_2(t)}{dt} = -I_2(t) + I(t)$$

$$I(s) = sQ_2(s) + I_2(s) = \left(s\varepsilon \frac{A}{d} + \sigma \frac{A}{d}\right) V(s) = \left(sC + \frac{1}{R}\right) V(s) = \frac{V(s)}{Z(s)}$$

$$Z(s) = \frac{\frac{1}{sC} R}{\frac{1}{sC} + R}$$



Quasi-static Approximation of Maxwell's equations

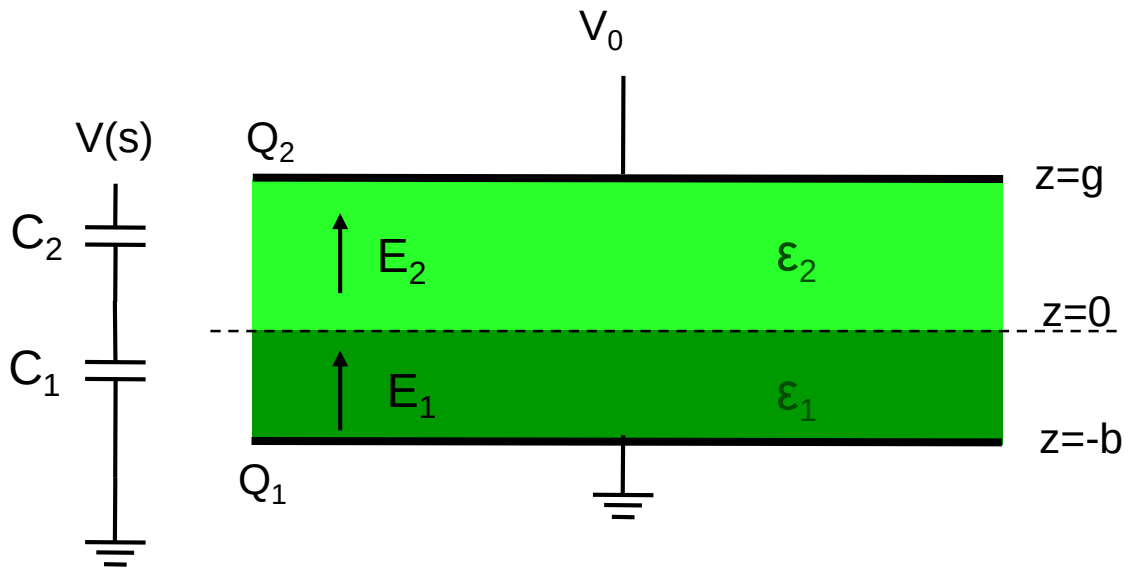
A constant voltage applied to a geometry with two insulating layers

$$-E_1 b - E_2 g = V_0 \qquad -\varepsilon_1 E_1 + \varepsilon_2 E_2 = 0$$

$$E_1 = -\frac{\varepsilon_2 V_0}{\varepsilon_1 g + \varepsilon_2 b} \qquad E_2 = -\frac{\varepsilon_1 V_0}{\varepsilon_1 g + \varepsilon_2 b}$$

$$Q_1 = \varepsilon_1 E_1 A = -\frac{\varepsilon_1 \varepsilon_2 A V_0}{\varepsilon_1 g + \varepsilon_2 b} \qquad Q_2 = -\varepsilon_2 E_2 A = \frac{\varepsilon_1 \varepsilon_2 A V_0}{\varepsilon_1 g + \varepsilon_2 b}$$

$$Q_1 = -CV_0 \qquad Q_2 = CV_0 \qquad C = \frac{C_1 C_2}{C_1 + C_2} \qquad C_1 = \varepsilon_1 \frac{A}{b} \qquad C_1 = \varepsilon_2 \frac{A}{g}$$

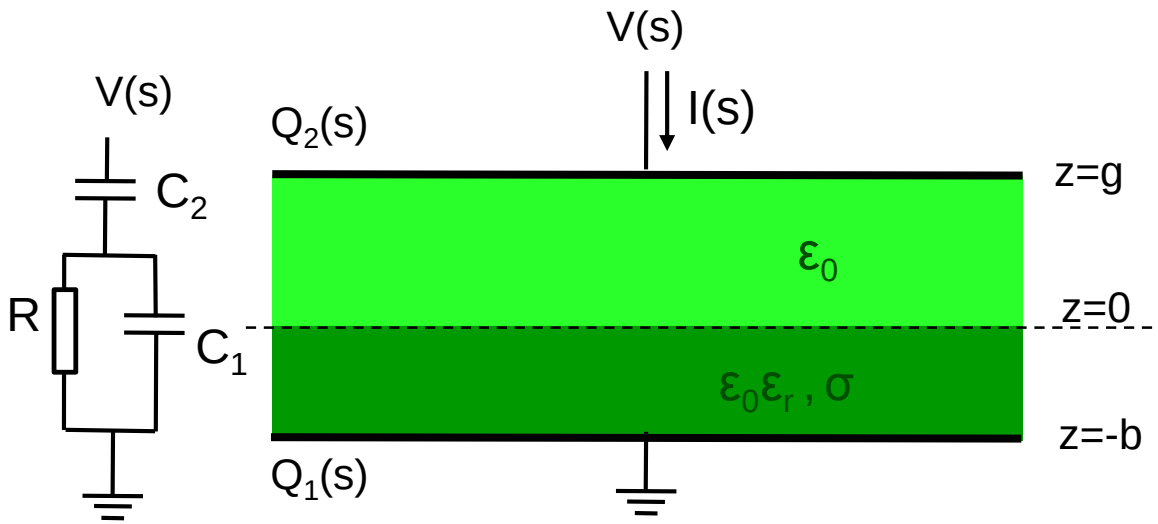


A time dependent voltage applied to a geometry with an insulating and a conductive layer

$$E_2(s) = -\frac{(\varepsilon_0 \varepsilon_r s + \sigma) V(s)}{(b + \varepsilon_r g)\varepsilon_0 s + g\sigma} \qquad Q_2 = -\varepsilon_0 E_2(s) A$$

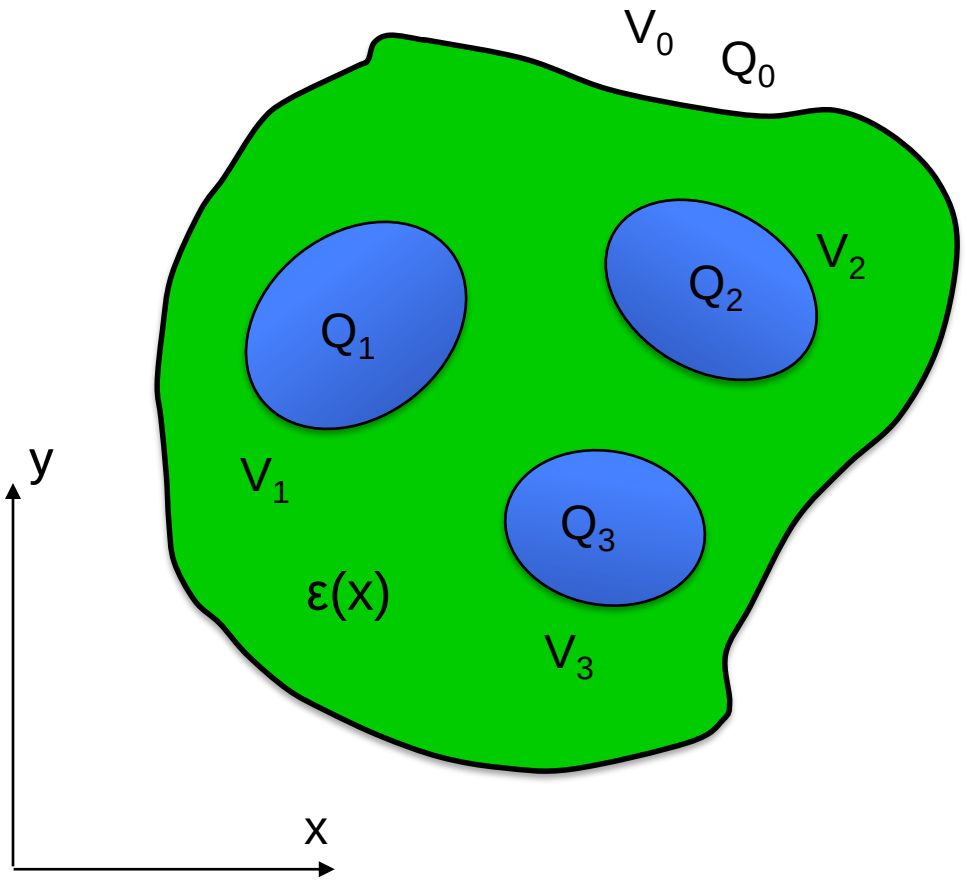
$$I(s) = sQ_2 = \frac{V(s)}{Z(s)} \qquad Z(s) = \frac{R\frac{1}{sC_1}}{R + \frac{1}{sC_1}} + \frac{1}{sC_2}$$

$$C_1 = \varepsilon_0 \varepsilon_r \frac{A}{b} \qquad C_2 = \varepsilon_0 \frac{A}{g} \qquad R = \frac{1}{\sigma}\frac{b}{A}$$



Capacitance matrix for insulating media

N+1 metal electrodes at potentials V_n will result in charges Q_n on the electrodes.



$$\nabla\left[\varepsilon(\mathbf{x})\nabla\varphi(\mathbf{x})\right] = -\rho(\mathbf{x}) \qquad \varphi(\mathbf{x})|_{\mathbf{x}=\mathbf{A}_n} = V_n$$

$$\mathbf{E}(\mathbf{x}) = -\nabla\varphi(\mathbf{x})$$

$$Q_n = \oint_{\mathbf{A}_n} \varepsilon(\mathbf{x})\mathbf{E}(\mathbf{x})d\mathbf{A}$$

Using the weighting potential for each electrode

$$\nabla\left[\varepsilon(\mathbf{x})\nabla\psi_n(\mathbf{x})\right] = 0 \qquad \psi_n(\mathbf{x})|_{\mathbf{x}=\mathbf{A}_n} = V_w\delta_{mn}$$

we can construct the solution to the problem

$$\varphi(\mathbf{x}) = \sum_m \frac{V_m}{V_w}\psi_m(\mathbf{x})$$

Inserting this in the above relation results in

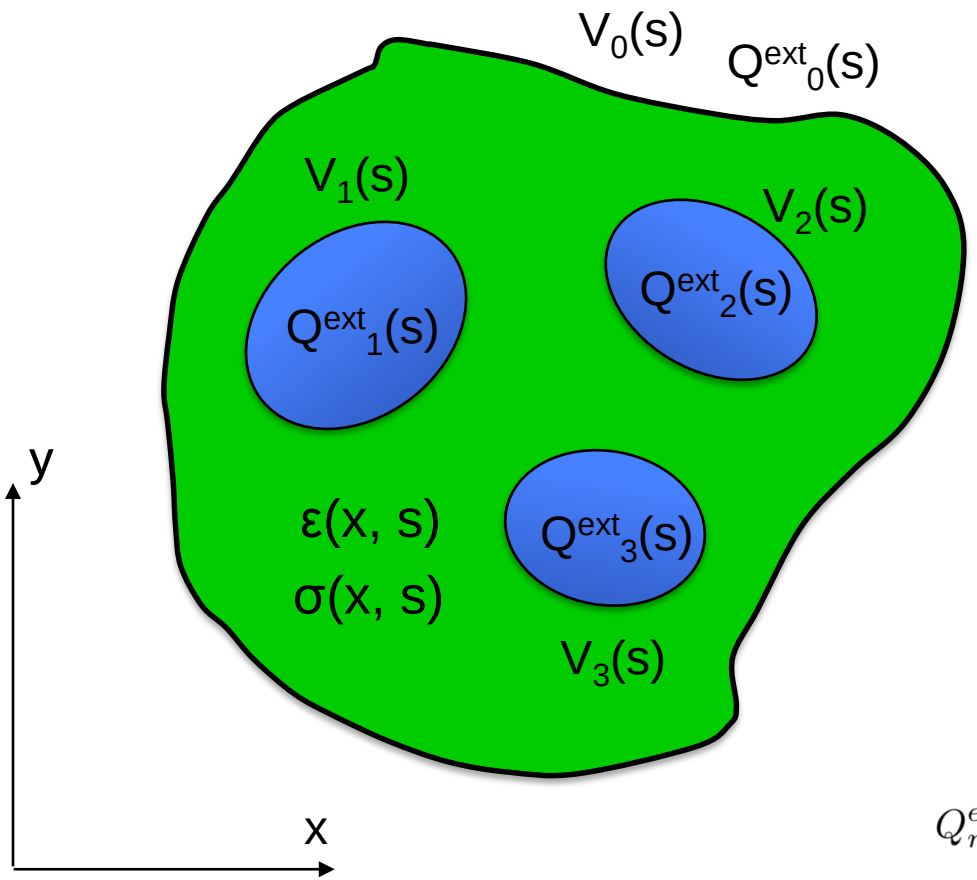
$$Q_n = \sum_m \frac{V_m}{V_w}\oint_{\mathbf{A}_n} \varepsilon(\mathbf{x})\nabla\psi_m(\mathbf{x})d\mathbf{A} = c_{nm}V_m$$

This defines the capacitance matrix of the system

$$c_{mn} = \frac{1}{V_w}\oint_{\mathbf{A}_n} \varepsilon(\mathbf{x})\nabla\psi_m(\mathbf{x})d\mathbf{A}$$

Admittance matrix for conductive media

N+1 metal electrodes at potentials V_n will result in charges Q_n on the electrodes.



$$\nabla\left[\varepsilon_{eff}(\mathbf{x})\nabla\varphi(\mathbf{x},s)\right]=0 \qquad \varphi(\mathbf{x},s)|_{\mathbf{x}=\mathbf{A}_n}=V_n(s)$$

$$\varepsilon_{eff}(\mathbf{x},s)=\varepsilon(\mathbf{x},s)+\frac{\sigma(\mathbf{x},s)}{s}$$

$$\mathbf{E}(\mathbf{x},s)=-\nabla\varphi(\mathbf{x},s)$$

$$Q_n^{ext}(s)=\oint_{\mathbf{A}_n}\varepsilon_{eff}(\mathbf{x},s)\mathbf{E}(\mathbf{x},s)d\mathbf{A}$$

Using the weighting potential for each electrode

$$\nabla\left[\varepsilon_{eff}(\mathbf{x},s)\nabla\psi_n(\mathbf{x},s)\right]=0 \qquad \psi_n(\mathbf{x},s)|_{\mathbf{x}=\mathbf{A}_n}=V_w\delta_{mn}$$

we can construct the solution to the problem

$$\varphi(\mathbf{x},s)=\sum_m\frac{V_m}{V_w}\psi_m(\mathbf{x},s)$$

Inserting this in the above relation results in

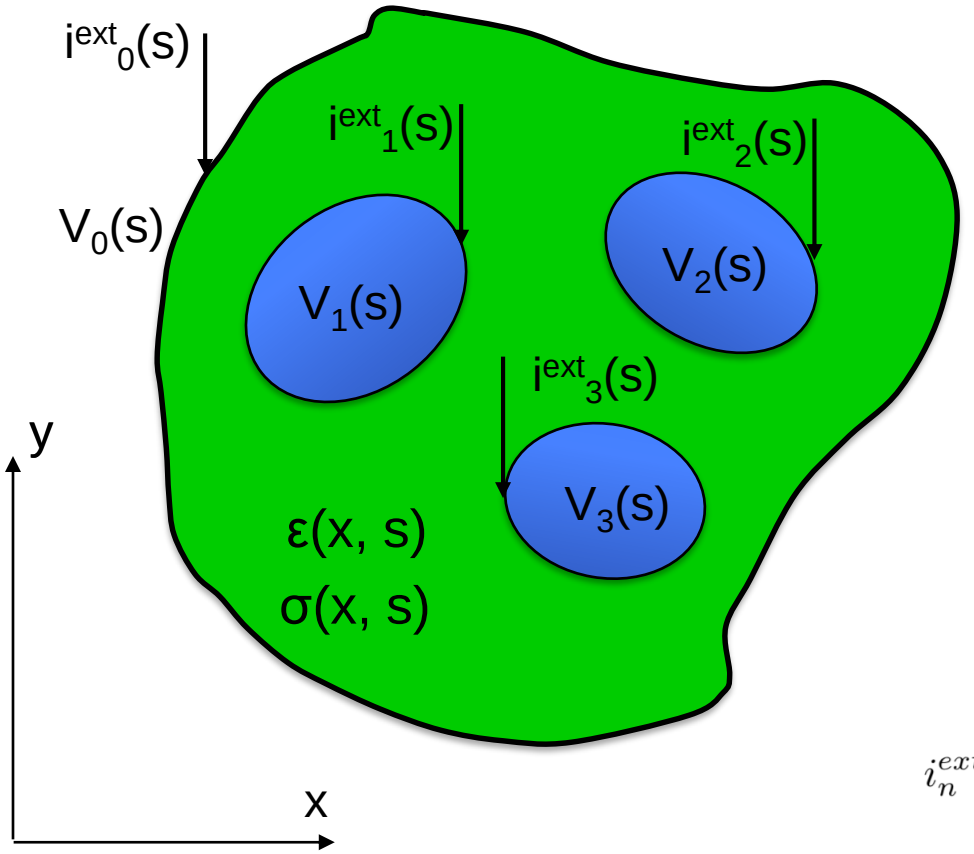
$$Q_n^{ext}(s)=\sum_m\frac{V_m(s)}{V_w}\oint_{\mathbf{A}_n}\varepsilon_{eff}(\mathbf{x},s)\nabla\psi_m(\mathbf{x},s)d\mathbf{A}=\sum_m c_{nm}(s)V_m$$

$$i_n^{ext}(s)=sQ_n^{ext}(s)=\sum_m\frac{sV_m(s)}{V_w}\oint_{\mathbf{A}_n}\varepsilon_{eff}(\mathbf{x},s)\nabla\psi_m(\mathbf{x},s)d\mathbf{A}=\sum_m y_{nm}(s)V_m$$

This defines the 'admittance' matrix of the system:

$$y_{mn}(s)=\frac{s}{V_w}\oint_{\mathbf{A}_n}\varepsilon_{eff}(\mathbf{x},s)\nabla\psi_m(\mathbf{x},s)d\mathbf{A}$$

Impedance Matrix



As in the case of the capacitance matrix we define one electrode as the reference electrode in order to have a unique relation between currents and voltages

$$U_n(s) = V_n(s) - V_0(s)$$

$$y_{mn}(s) = \begin{pmatrix} \cancel{y_{00}(s)} & \cancel{y_{01}(s)} & \cancel{\dots} & \cancel{y_{0N}(s)} \\ y_{10}(s) & y_{11}(s) & \dots & y_{1N}(s) \\ \vdots & \vdots & \ddots & \vdots \\ y_{N0}(s) & y_{N1}(s) & \dots & y_{NN}(s) \end{pmatrix}$$

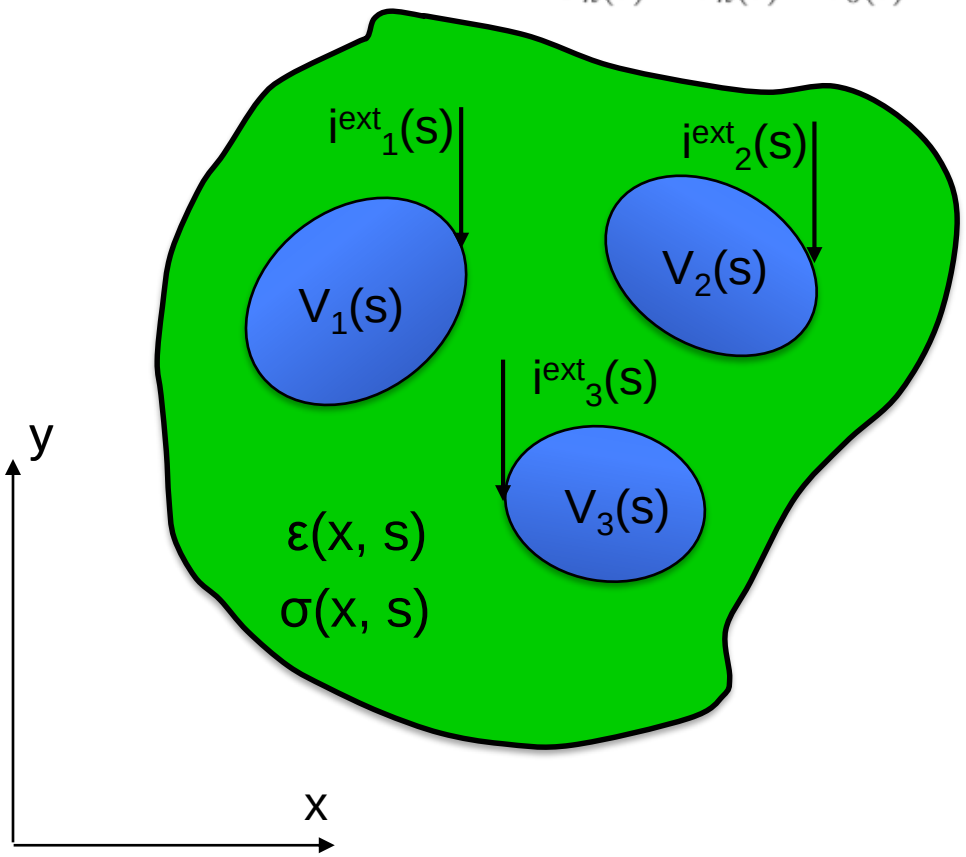
$$i_n^{ext}(s) = \sum_{m=1}^{N} y_{nm}(s) U_m(s) \qquad U_n(s) = \sum_{m=1}^{N} z_{nm}(s) i_m^{ext}(s) \qquad z_{mn} = y_{mn}^{-1}$$

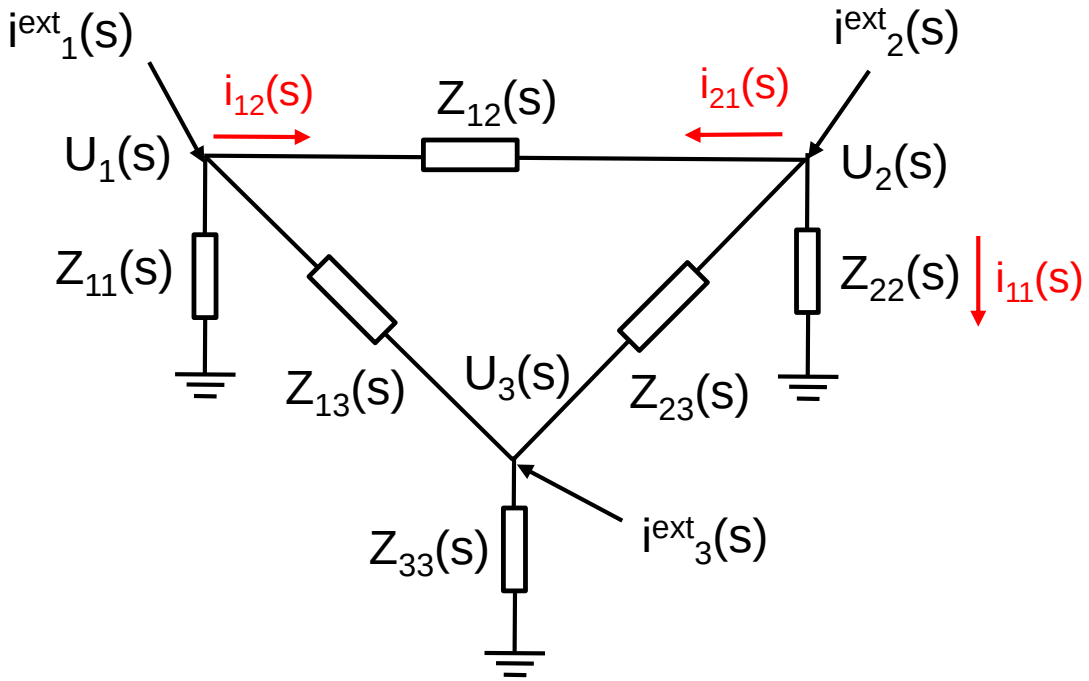
$$y_{mn}(s) = \frac{s}{V_w} \oint_{\mathbf{A}_n} \varepsilon_{eff}(\mathbf{x}, s) \nabla \psi_m(\mathbf{x}, s) d\mathbf{A}$$

The matrix y_{mn} the Admittance Matrix of the system
The matrix z_{mn} is the Impedance Matrix of the system

Equivalent circuit, Impedance elements

$$U_n(s) = V_n(s) - V_0(s)$$





$$y_{mn}(s) = \frac{s}{V_w} \oint_{\mathbf{A}_n} \varepsilon_{eff}(\mathbf{x}, s) \nabla \psi_m(\mathbf{x}, s) d\mathbf{A}$$

$$i_n^{ext}(s) = \sum_{m=1}^{N} y_{nm}(s) U_m(s)$$

The impedance elements of the equivalent circuit are defined by

$$i_{nn}(s) = \frac{U_n(s)}{Z_{nn}(s)} \qquad i_{nm}(s) = \frac{U_n(s) - U_m(s)}{Z_{mn}(s)}$$

Using the fact that the sum of these current at each node is zero

$$i_n^{ext}(s) = \sum_{m=1}^{N} i_{nm}(s)$$

we can relate the impedance elements Z_{mn} to the Admittance Matrix y_{mn}

$$Z_{nm}(s) = -\frac{1}{y_{nm}(s)} \quad n \neq m \qquad Z_{nn}(s) = \frac{1}{\sum_{m=1}^{N} y_{nm}(s)} = -\frac{1}{y_{0n}} \quad n = m$$

Impedance elements

The weighting field of electrode 1

$$E_1^w(s) = -\frac{s\varepsilon_0 V_w}{\varepsilon_0 s(b+\varepsilon_r g)+g\sigma}$$

Using the general formula for the impedance matrix

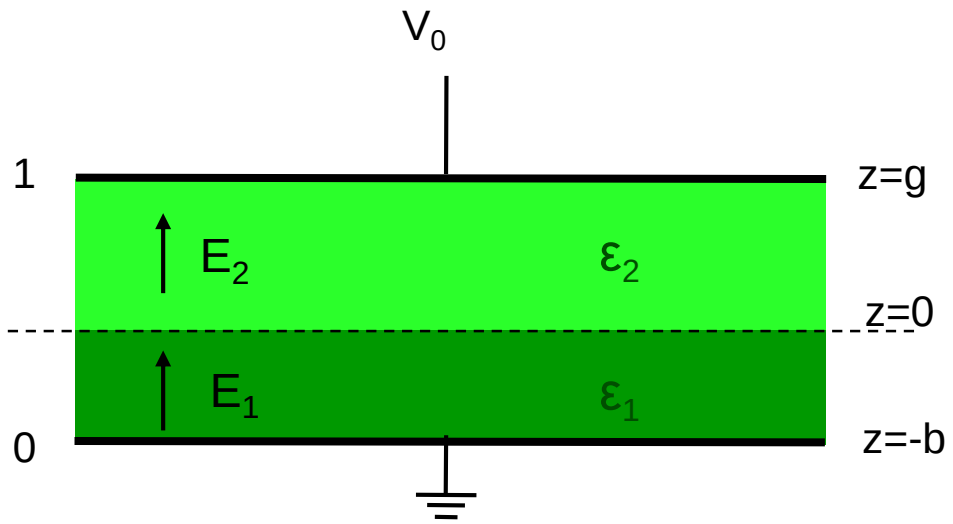
$$y_{mn}(s) = \frac{s}{V_w}\oint_{\mathbf{A}_n} \varepsilon_{eff}(\mathbf{x},s)\nabla\psi_m(\mathbf{x},s)d\mathbf{A}$$

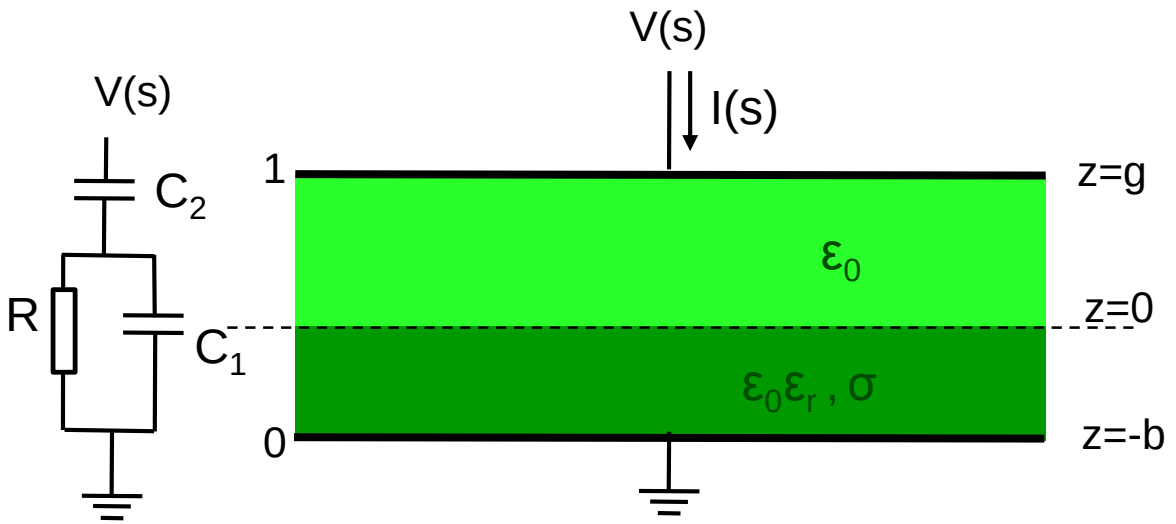
we have

$$y_{10} = \frac{s}{V_w}A\left(\varepsilon_0\varepsilon_r + \frac{\sigma}{s}\right)E_1^w(s) = \frac{1}{Z_{11}(s)}$$

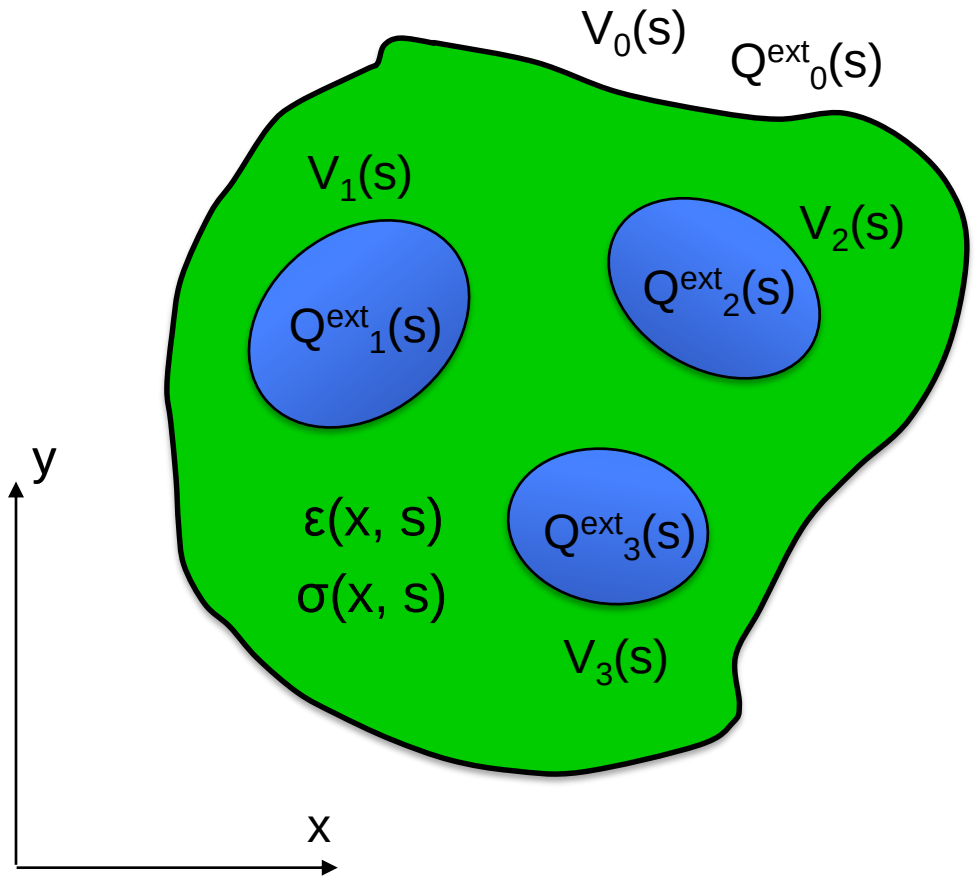
$$Z_{11}(s) = \frac{R\frac{1}{sC_1}}{R+\frac{1}{sC_1}} + \frac{1}{sC_2}$$

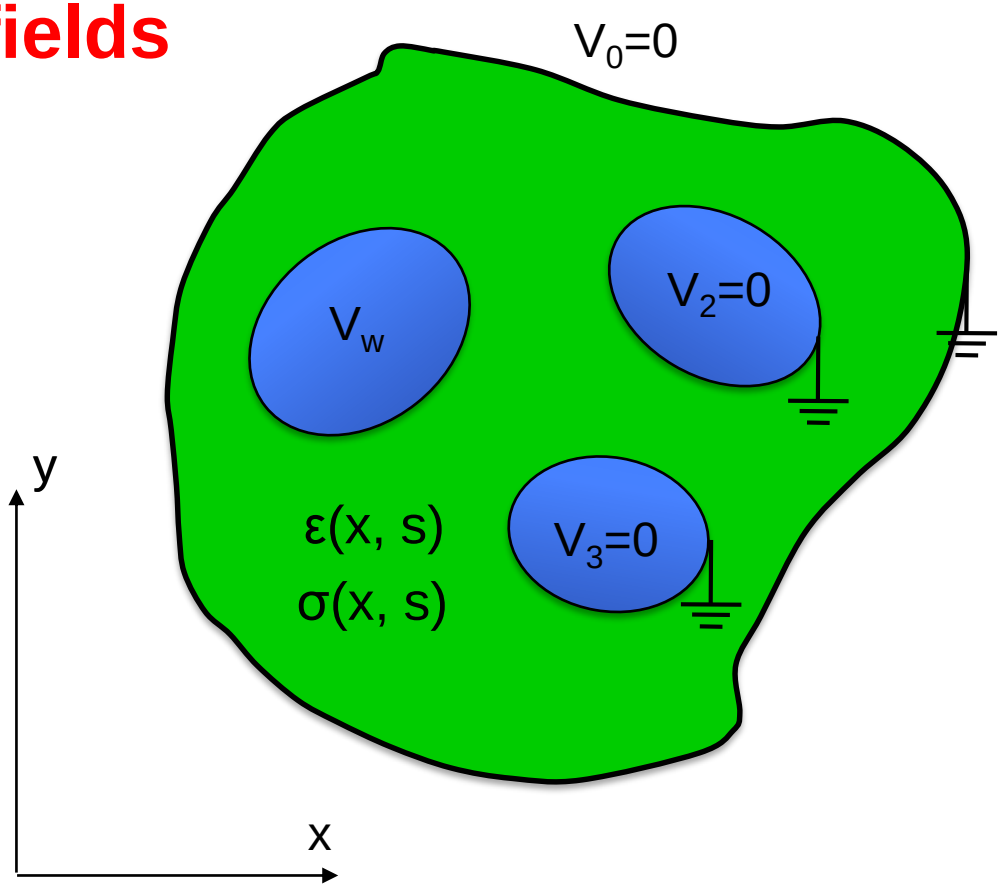
$$C_1 = \varepsilon_0\varepsilon_r\frac{A}{b} \qquad C_2 = \varepsilon_0\frac{A}{g} \qquad R = \frac{1}{\sigma}\frac{b}{A}$$





Weighting fields





$$\nabla\left[\varepsilon_{eff}(\mathbf{x},s)\nabla\psi_n(\mathbf{x},s)\right]=0 \qquad \psi_n(\mathbf{x},s)|_{\mathbf{x}=\mathbf{A}_n}=V_w\delta_{mn}$$

$$\varphi(\mathbf{x},s)=\sum_m \frac{V_m}{V_w}\psi_m(\mathbf{x},s)$$

$$\varphi(\mathbf{x},t)=\frac{1}{V_w}\sum_m\int_0^t \psi_m(\mathbf{x},t-t')V_m(t')dt'$$

The weighting potentials in the Laplace domain are due to application of V_w to the electrode in question and grounding all the others.

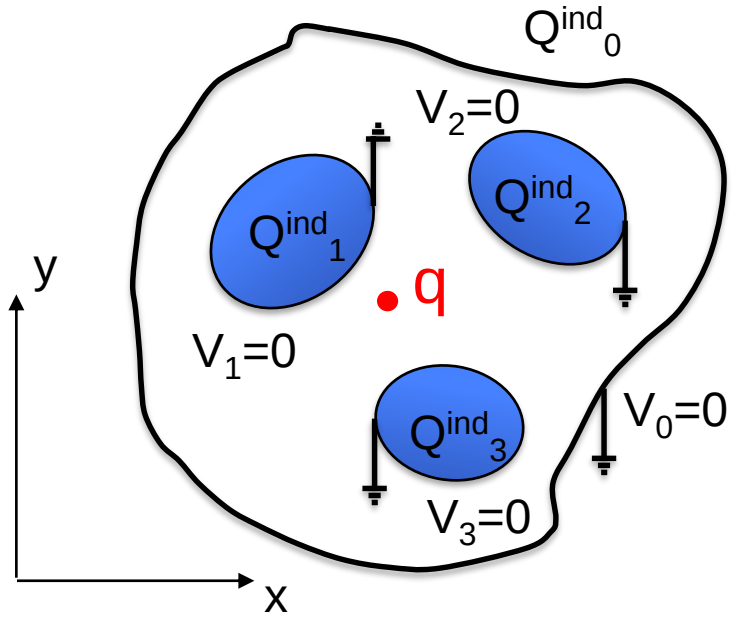
In the time domain this refers to a delta function $V_w\,\delta(t)$ applied to the electrode in question.

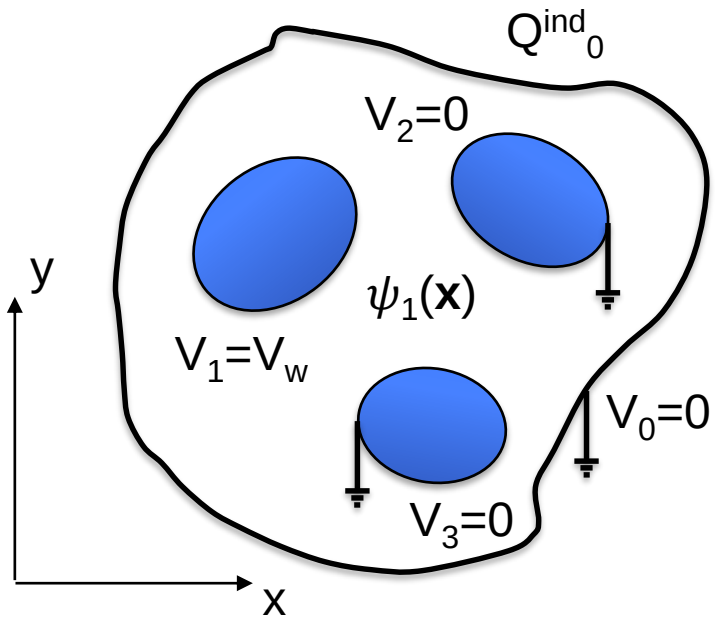
Extension of the Ramo Shockley theorem

[4] E. Gatti, G. Padovini, V. Radeka, Signal evaluation in multielectrode radiation detectors by means of a time dependent weighting vector, Nucl. Instrum. Methods A 193 (1982) 651–653.

[9] W. Riegler, Extended theorems for signal induction in particle detectors, Nucl. Instrum. Methods A 535 (2004) 287–293.

Theorem, induced charge





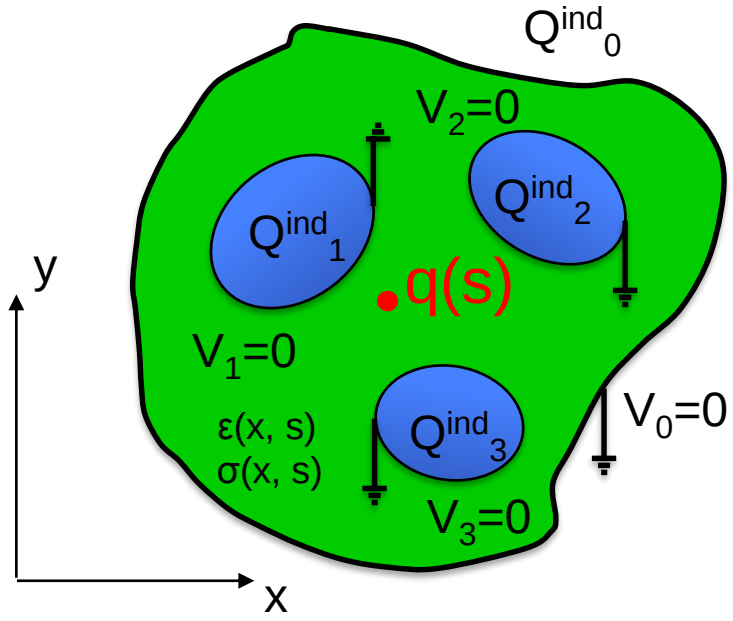
The charge induced on a grounded conducting electrode by a point charge q at position **x** can be calculated the following way:

Remove the point charge, put the electrode in question to potential V_w while keeping all other electrodes at ground potential.

This defines the potential $\psi_n(\mathbf{x})$ and the induced charge is

$$Q_n^{ind} = -\frac{q}{V_w}\psi_n(\mathbf{x})$$

Theorem, induced charge

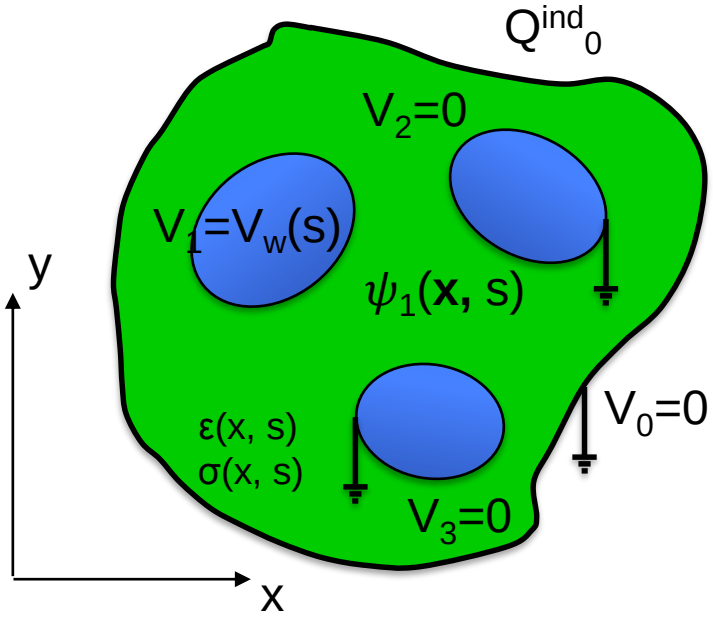


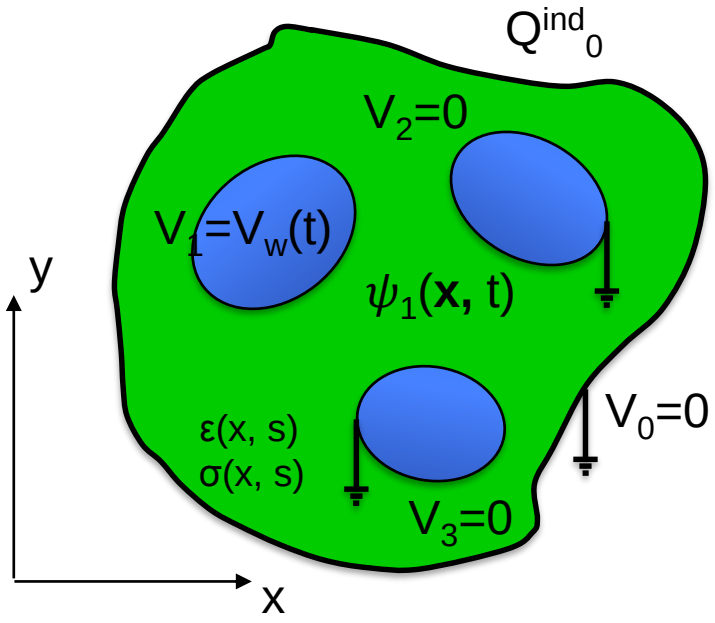
The charge induced on a grounded conducting electrode by a point charge q(s) at position **x** can be calculated the following way:

Remove the point charge, apply a voltage $V_w(s)$ to the electrode in question while keeping all other electrodes at ground potential

$$Q_n^{ind}(s) = Q^{ext}(s) = -\frac{q(s)}{V_w(s)}\psi_n(\mathbf{x}, s)$$

$$Q^{ind}(t) = \mathbf{L}^{-1}\left[Q^{ind}(s)\right]$$

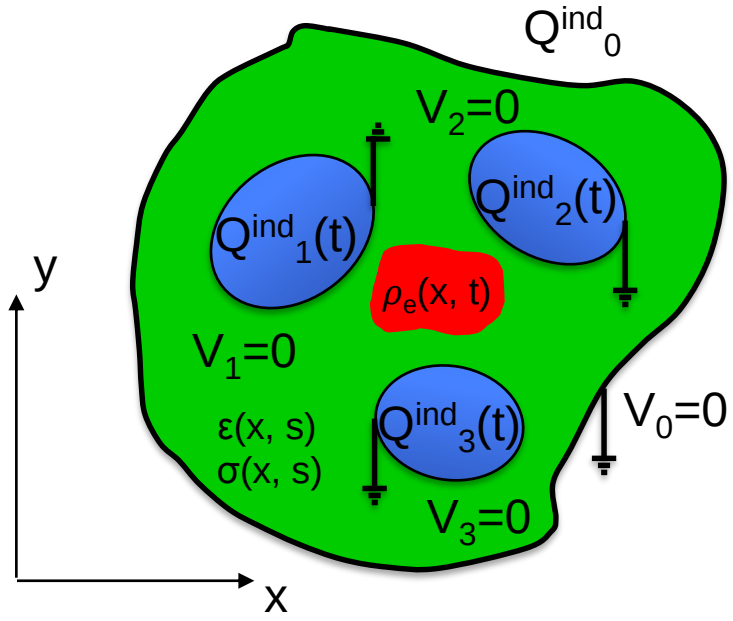


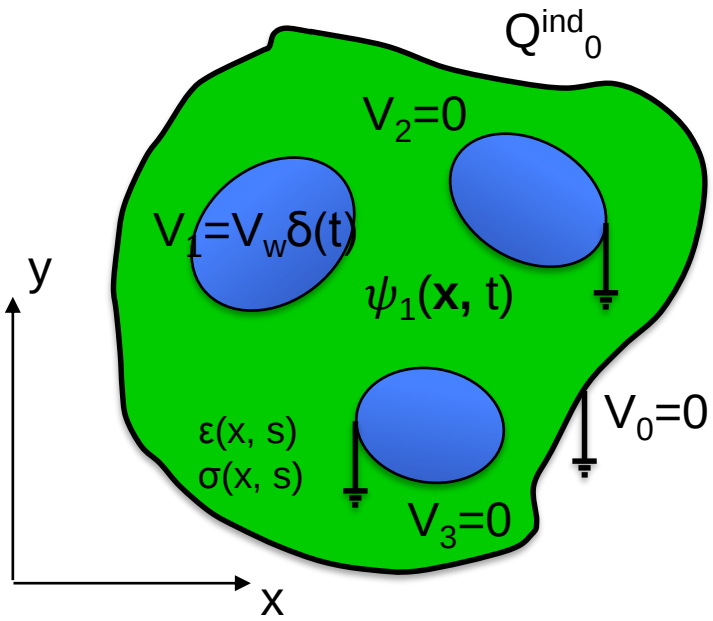


Note that this charge does not refer to the charge that is sitting on the electrode but to the charge that is brought into the system i.e. the charge that has moved between ground an the electrode.

That's exactly that charge one measures if one connects an amplifier to the electrode.

Theorem, induced charge





In case we have a time varying external charge density in between the electrodes we have

$$Q_n^{ext}(s) = -\frac{1}{V_w(s)} \int_V \psi_n(\mathbf{x}, s)\rho_e(\mathbf{x}, s) d^3x \qquad Q^{ind}(t) = \mathbf{L}^{-1}\left[Q^{ind}(s)\right]$$

In case we chose to apply a delta function for finding the weighting field we have

$$Q_n^{ext}(s) = -\frac{1}{V_w} \int_V \psi_n(\mathbf{x}, s)\rho_e(\mathbf{x}, s) d^3x \qquad Q^{ind}(t) = \mathbf{L}^{-1}\left[Q^{ind}(s)\right]$$

$$Q_n^{ext}(t) = \frac{1}{V_w} \int_0^t \int \psi_n(\mathbf{x}, t-t')\rho_e(\mathbf{x}, t') d^3x dt$$

If there is a charge moving along a trajectory $x_1(t)$ the charge density amounts to

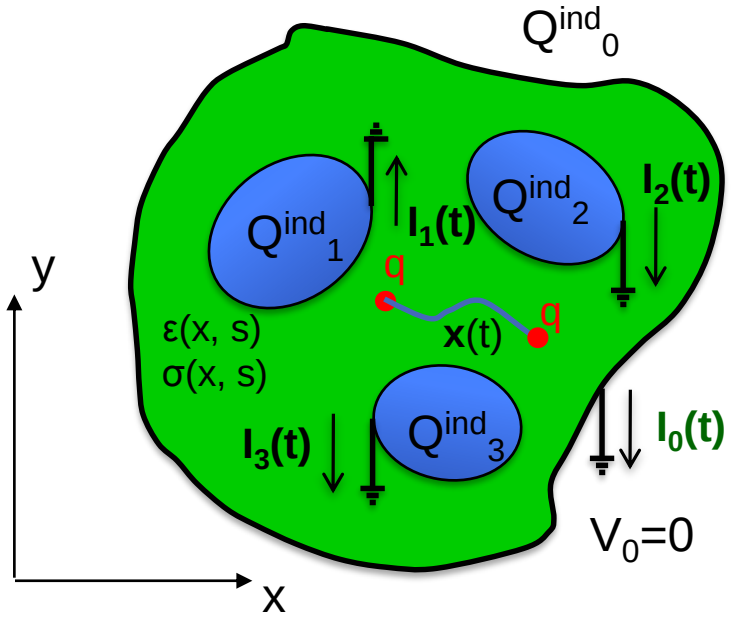
$$\rho_e(\mathbf{x}, t) = q\delta(\mathbf{x} - \mathbf{x}_1(t))$$

And the induced charge is

$$Q_n^{ext}(t) = -\frac{q}{V_w} \int_0^t \psi_n(\mathbf{x}_1(t'), t-t') d^3x dt$$

Note that ψ_n is not a physical potential, since the delta function gives it a dimension of V/s.

Theorem, induced current



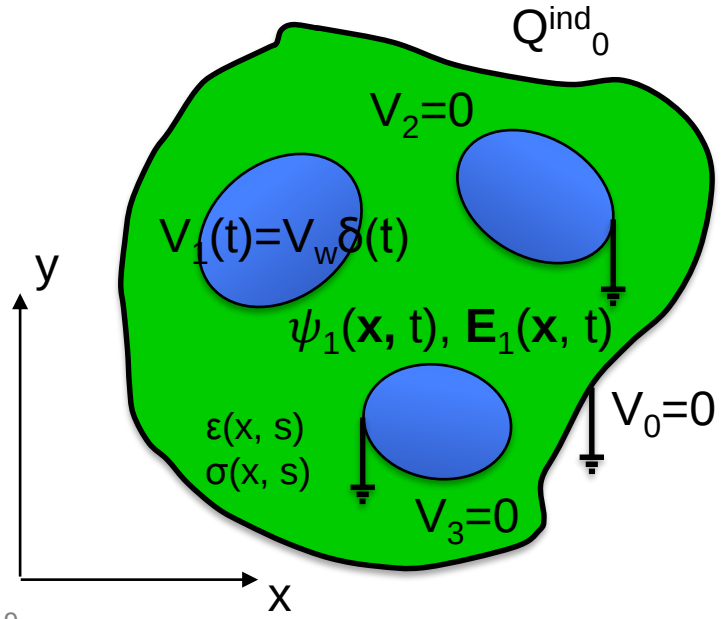
Applying the delta voltage pulse to the electrode in question we find the potential $\psi_n(x, t)$ and the field $E_n(x, t)$ from which the induced current can be calculated the following way:

$$I_n^{ext}(s) = -sQ_n^{ext}(s) = \frac{s}{V_w}\int_V \psi_n(\mathbf{x}, s)\rho_e(\mathbf{x}, s)d^3x$$

$$\rho_e(\mathbf{x}, t) = q\delta(\mathbf{x} - \mathbf{x}_1(t))$$

$$I_n^{ext}(t) = -\frac{q}{V_w}\int_0^t \mathbf{E}_n(\mathbf{x}_1(t'), t - t')\dot{\mathbf{x}}_1(t')dt'$$

→ Ramo-Shockley theorem extension for conducting media



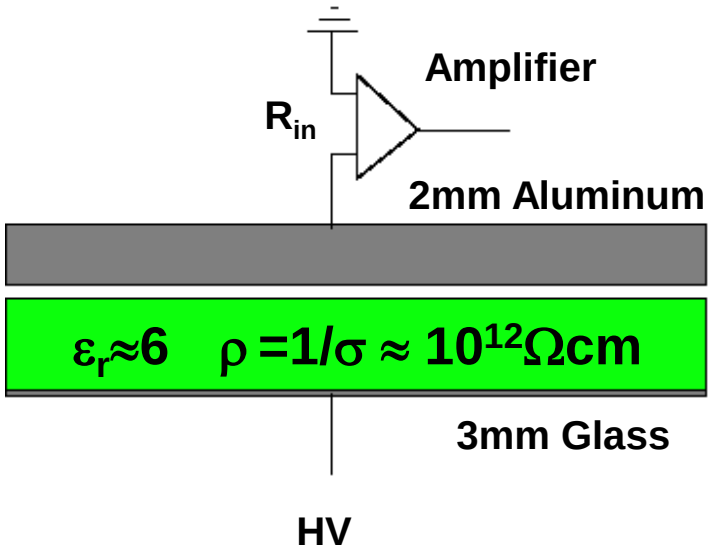
Note that E_n is not physical potential, since the delta function gives it a dimension of V/cm s.

In case the material is an insulator there is no time dependence of the weighting field and we recuperate Ramo's theorem.

$$\mathbf{E}_n(\mathbf{x}, t) = \mathbf{E}_{n0}(\mathbf{x})\delta(t - t') \qquad I_n^{ext}(t) = -\frac{q}{V_w}\mathbf{E}_{n0}(\mathbf{x}_1(t))\dot{\mathbf{x}}_1(t)dt$$

Example

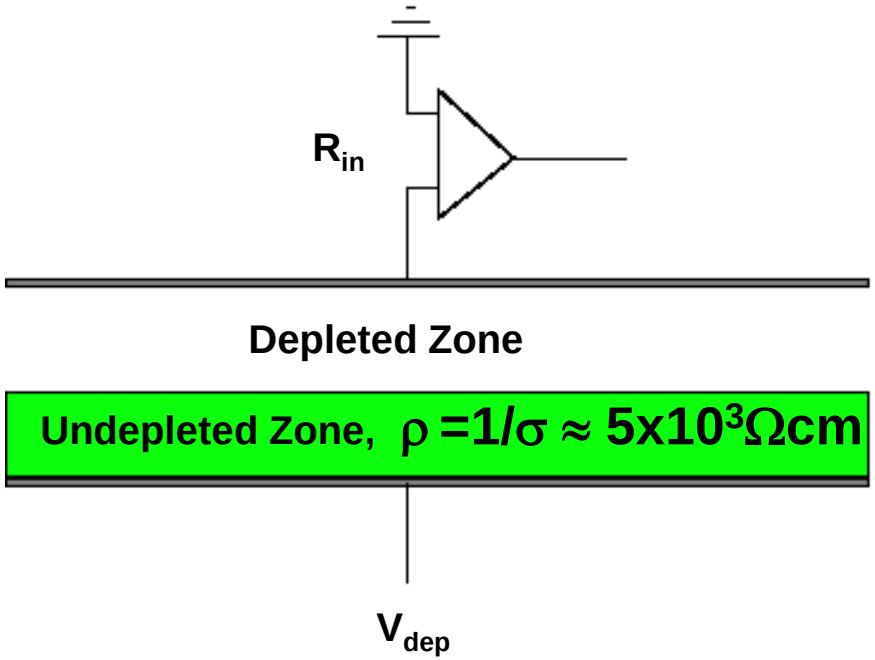
RPC



300μm Gas Gap

$\tau \approx \varepsilon_0 / \sigma \approx$ 100msec

Silicon Detector



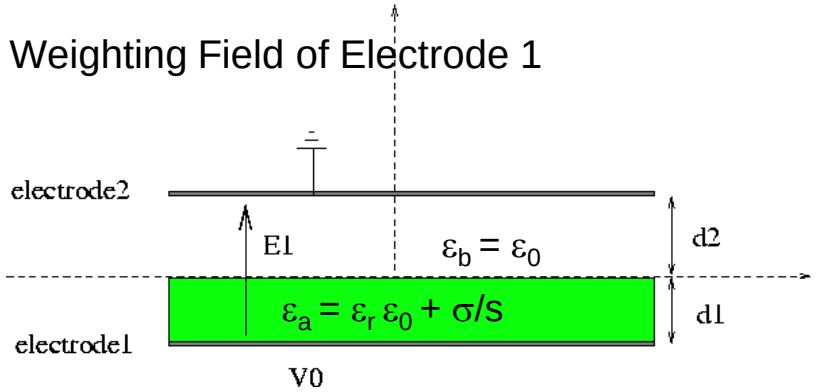
$\tau \approx \varepsilon_0 / \sigma \approx$ 1ns

heavily irradiated silicon has larger resistivity

that can give time constants of a few hundreds of ns

Example

Weighting Field of Electrode 1

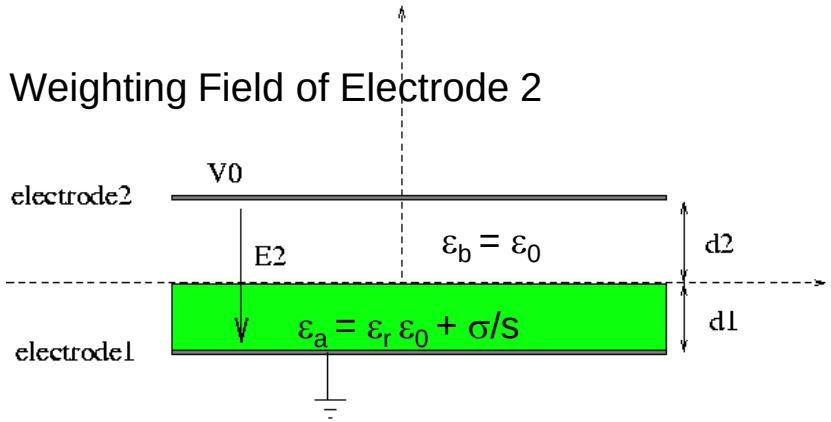


$$E_{1z}(s) = \frac{\varepsilon_a V_0}{\varepsilon_a d_2 + \varepsilon_b d_1} = \frac{V_0 \varepsilon_r}{(d_1 + d_2 \varepsilon_r)} \frac{s + \frac{1}{\tau_1}}{s + \frac{1}{\tau_2}} \quad z > 0$$

$$= \frac{\varepsilon_b V_0}{\varepsilon_a d_2 + \varepsilon_b d_1} = \frac{V_0}{(d_1 + d_2 \varepsilon_r)} \frac{s}{s + \frac{1}{\tau_2}} \quad z < 0$$

$$\tau_1 = \frac{\varepsilon_r \varepsilon_0}{\sigma} \qquad \tau_2 = \frac{\varepsilon_0}{\sigma} \left(\frac{d_1 + d_2 \varepsilon_r}{d_2} \right)$$

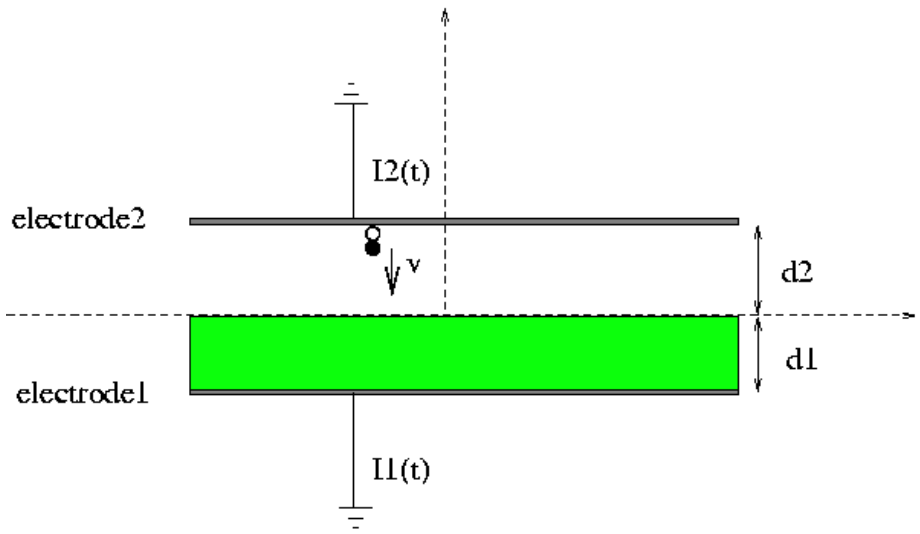
Weighting Field of Electrode 2



$$E_{2z}(s) = -E_{1z}(s)$$

Example

At t=0 a pair of charges q, -q is created at z=d_2.
One charge is moving with velocity v to z=0
Until it hits the resistive layer at T=d_2/v.



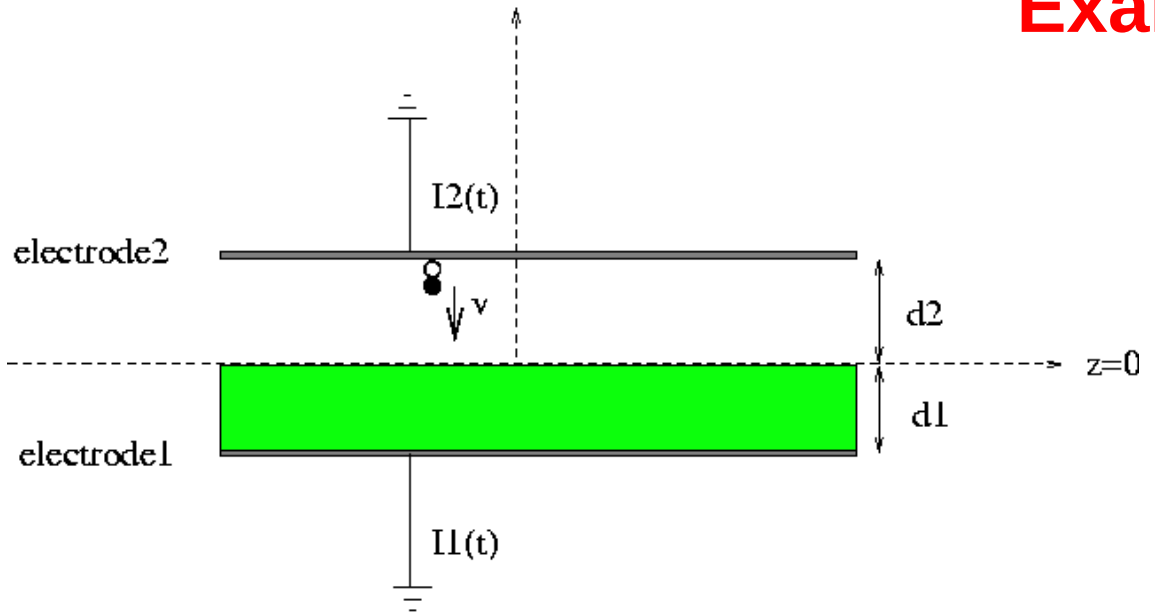
$$x_0(t) = d_2 - vt \qquad t < T$$
$$= 0 \qquad t > T$$

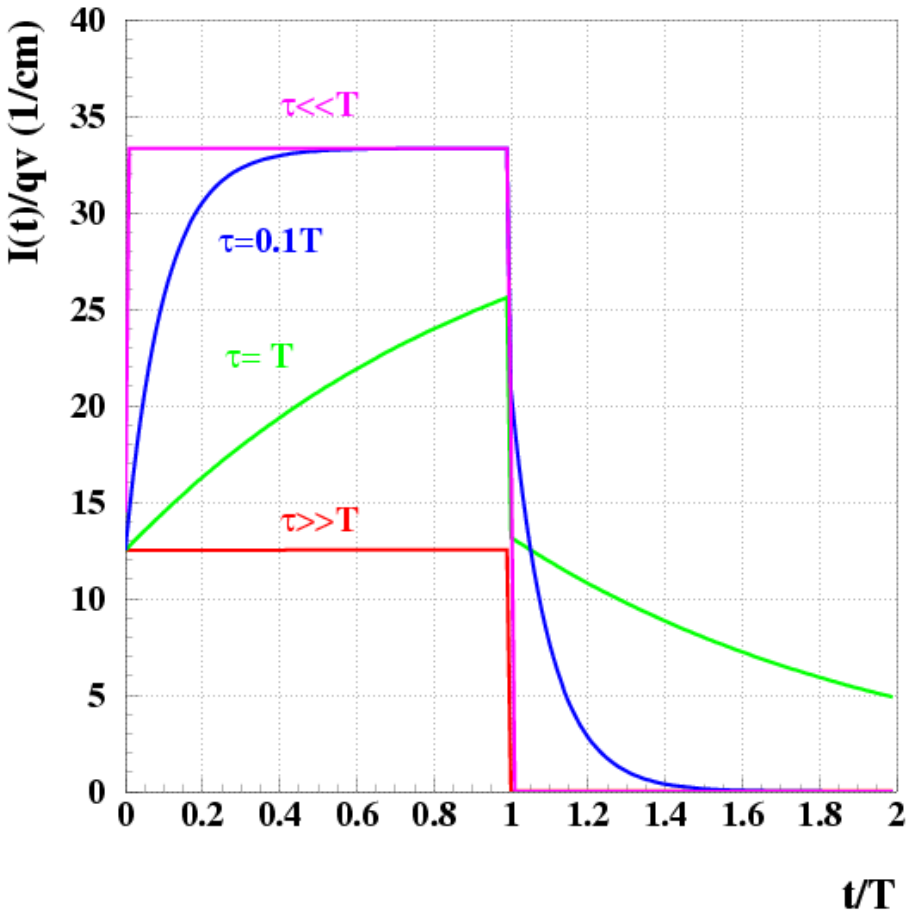
$$\dot{x}_0(t) = -v \qquad t < T$$
$$= 0 \qquad t > T$$

$$E_{1z}(\vec{x},t) = \frac{\varepsilon_r V_0}{d_1+\varepsilon_r d_2}\left[\delta(t) + \frac{\tau_2-\tau_1}{\tau_1\tau_2}\, e^{-\frac{t}{\tau_2}}\right] \qquad z > 0$$

$$I_1(t) = qv\frac{\varepsilon_r}{d_1+\varepsilon_r d_2}\left[1 + \frac{d_1}{d_2\varepsilon_r}(1 - e^{-\frac{t}{\tau_2}})\right] \qquad t < T$$
$$= qv\frac{1}{d_1+\varepsilon_r d_2}\frac{d_1}{d_2}\left(e^{\frac{T}{\tau_2}} - 1\right) e^{-\frac{t}{\tau_2}} \qquad t > T$$

Example





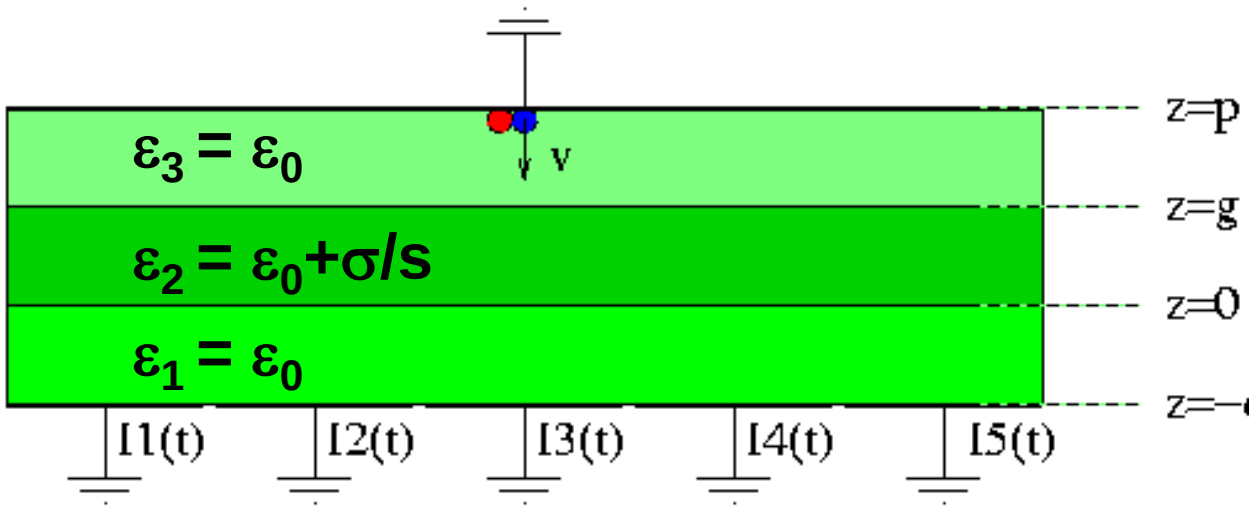
In case of high resistivity (τ>>T, RPCs, irradiated silicon) the layer is an insulator.

In case of very low resistivity (τ <<T, silicon) the layer acts like a metal plate and the scenario is equal to a parallel plate geometry with plate separation d_2.

The total induced charge is always equal to q !

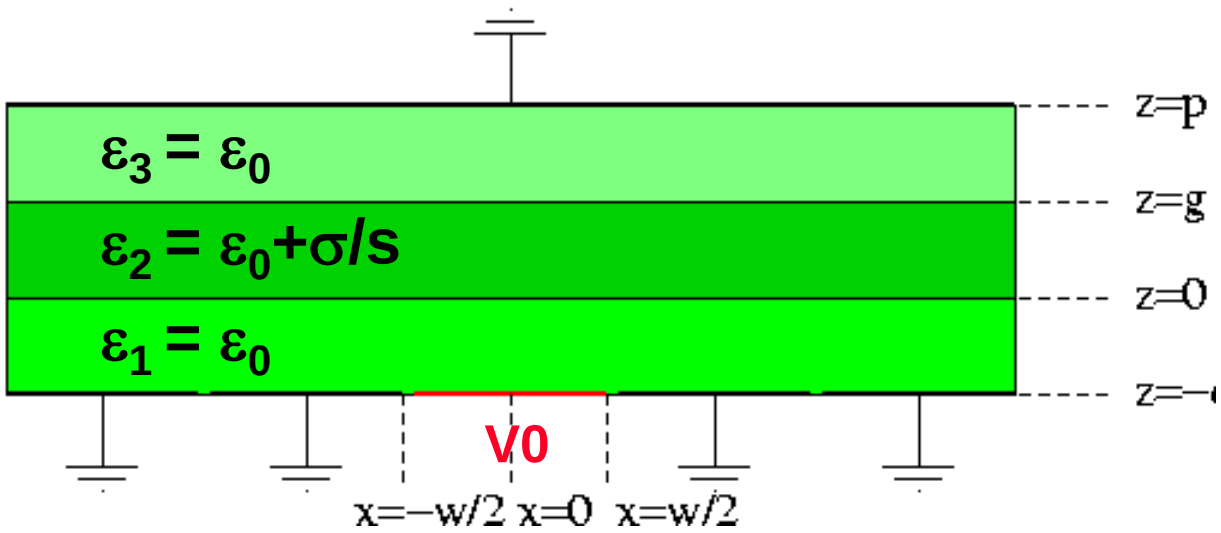
$$\int_0^\infty I_1(t)dt = q$$

Strip Example



What is the effect of a conductive layer between the readout strips and the place where a charge is moving ?

Strip Example



Electrostatic Weighting field (derived from B. Schnizer et. al, CERN-OPEN-2001-074):

$$E_z(x,z) = \frac{4V_0}{\pi}\int_0^\infty d\kappa \cos(\kappa x)\sin(\kappa\frac{w}{2})\frac{2\varepsilon_1\varepsilon_2\cosh[\kappa(p-z)]}{(\varepsilon_1+\varepsilon_2)(\varepsilon_2+\varepsilon_3)\ \sinh[\kappa(p+q)] - (\varepsilon_1-\varepsilon_2)(\varepsilon_2+\varepsilon_3)\ \sinh[\kappa(q-p)] - (\varepsilon_1+\varepsilon_2)(\varepsilon_2-\varepsilon_3)\ \sinh[\kappa(2g+q-p)] + (\varepsilon_1-\varepsilon_2)(\varepsilon_2-\varepsilon_3)\ \sinh[\kappa(p+q-2g)]}$$

Replace $\varepsilon_1 \rightarrow \varepsilon_0$, $\varepsilon_2 \rightarrow \varepsilon_0+\sigma/s$, $\varepsilon_3 \rightarrow \varepsilon_0$ and perform inverse Laplace Transform $\rightarrow E_z(x,z,t)$. Evaluation with MATHEMATICA:

Strip Example

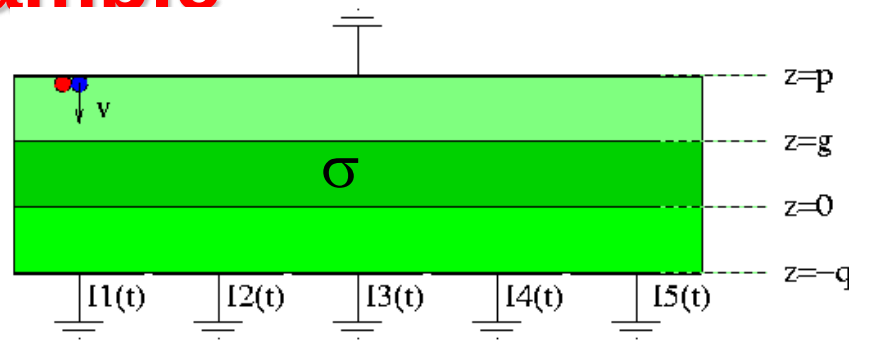
$T \ll \tau$

$T = \tau$

$T = 10\tau$

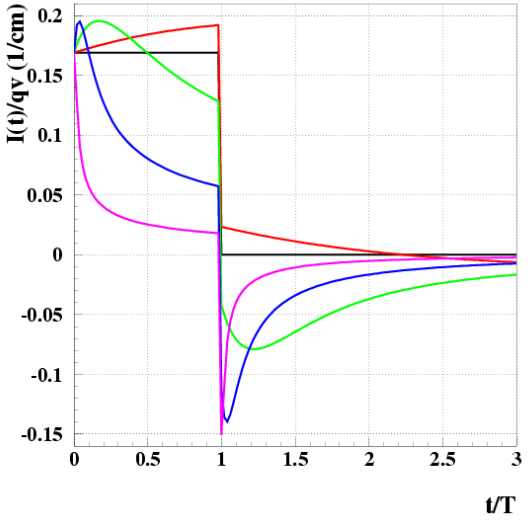
$T = 50\tau$

$T = 500\tau$

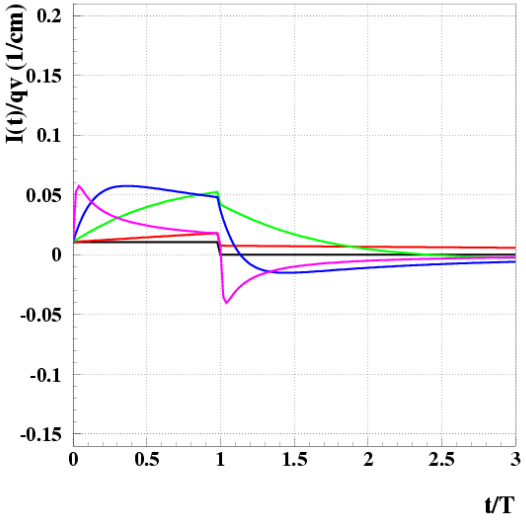


$\tau = \varepsilon_0/\sigma$

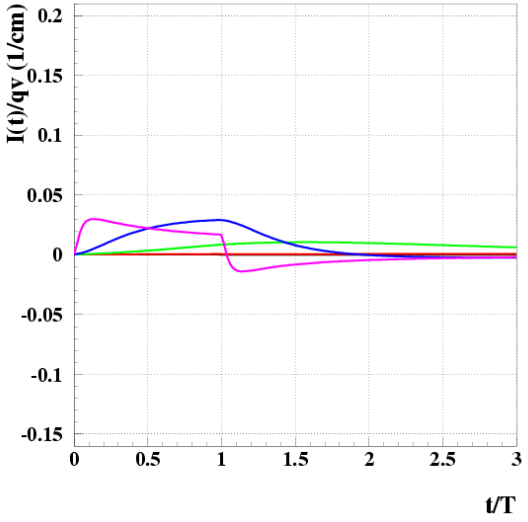
$I_1(t)$



$I_3(t)$

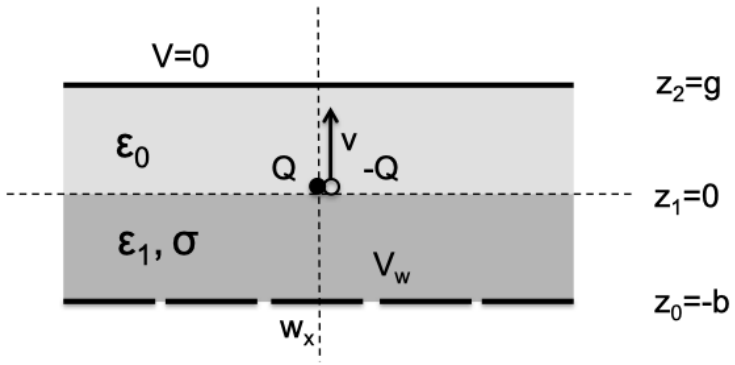


$I_5(t)$



The conductive layer ‘spreads’ the signals across the strips.

Charge spreading



$$\tau_0 = \varepsilon_0/\sigma = \varepsilon_0\rho \quad T = g/v$$

In some detectors, a resistive layer is applied on top of the readout strips to spread out the charge and therefore 'increase' the pad response function.

This example shows a resistive 'bulk' layer on top of the readout strips. The layer is in contact with the strips, so charge can move from the strips into the resistive layer.

- The solid blue line shows the situation for the given time constant.
- The dashed blue line shows the situation for zero conductivity
- The dashed magenta line shows the situation for infinite conductivity

τ_0=10T

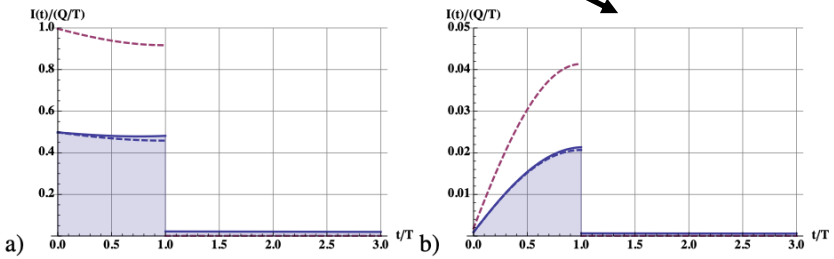


Figure 31. Uniform charge movement from $z = 0$ to $z = g$, with $\varepsilon_r = 1, w_x = 4g, b = g, \tau_0 = 10T$ for a)$x = 0$ and b) $x = 4g$.

τ_0=T

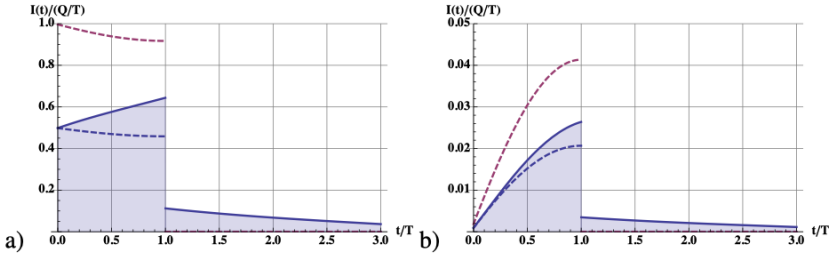


Figure 32. Uniform charge movement from $z = 0$ to $z = g$, with $\varepsilon_r = 1, w_x = 4g, b = g, \tau_0 = T$ for a)$x = 0$ and b) $x = 4g$.

τ_0=0.1T

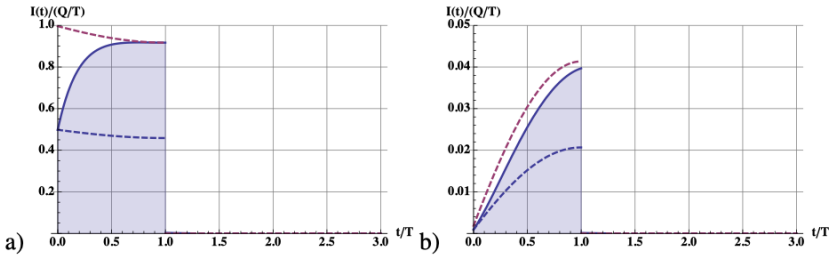
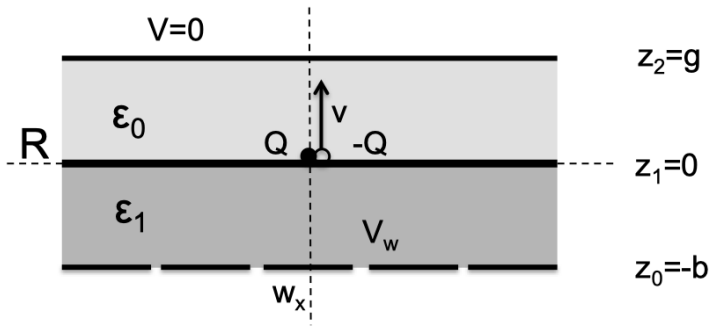


Figure 33. Uniform charge movement from $z = 0$ to $z = g$, with $\varepsilon_r = 1, w_x = 4g, b = g, \tau_0 = 0.1T$ for a)$x = 0$ and b) $x = 4g$.

Charge spreading

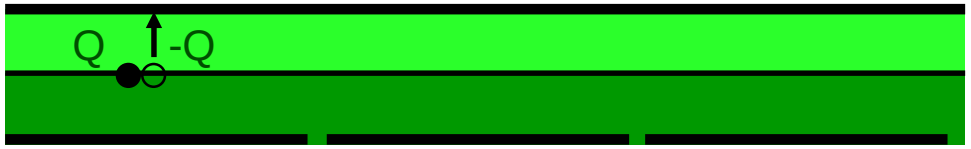


$$T_0 = \varepsilon_0 R g \qquad T = g/v$$

In some detectors, a resistive layer is applied on top of the readout strips to spread out the charge and therefore 'increase' the pad response function.

This example shows a thin resistive layer on top of the readout strips. The layer is insulated from the strips.

- The solid line shows the situation for different time constants
- The dashed line shows the situation for infinite resistivity



T_0=10T

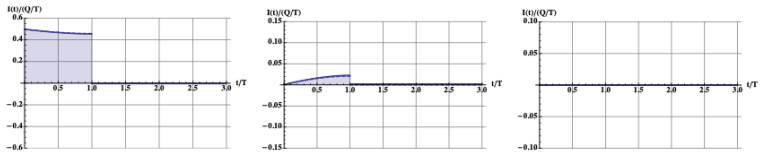


Figure 34. $\varepsilon_r = 1, w_x = 4g, b = g, T_0 = 10T$ for $x = 0, x = 4g, x = 8g$.

T_0=T

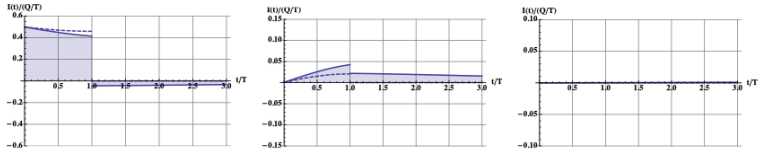


Figure 35. $\varepsilon_r = 1, w_x = 4g, b = g, T_0 = T$ for $x = 0, x = 4g, x = 8g$.

T_0=0.1T

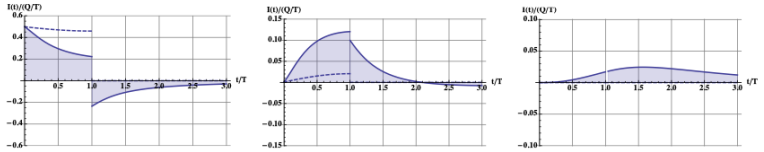


Figure 36. $\varepsilon_r = 1, w_x = 4g, b = g, T_0 = 0.1T$ for $x = 0, x = 4g, x = 8g$.

T_0=0.01T

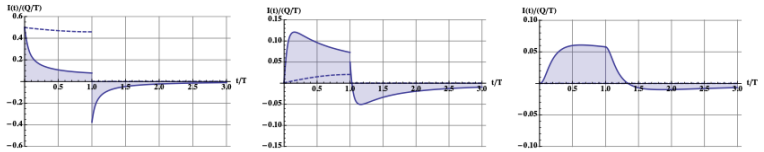


Figure 37. $\varepsilon_r = 1, w_x = 4g, b = g, T_0 = 0.01T$ for $x = 0, x = 4g, x = 8g$.

T_0=0.001T

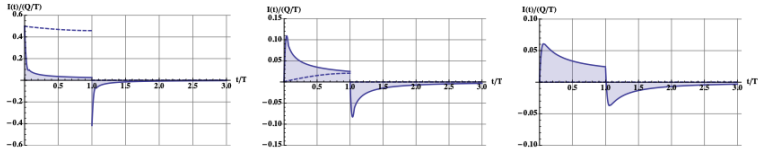
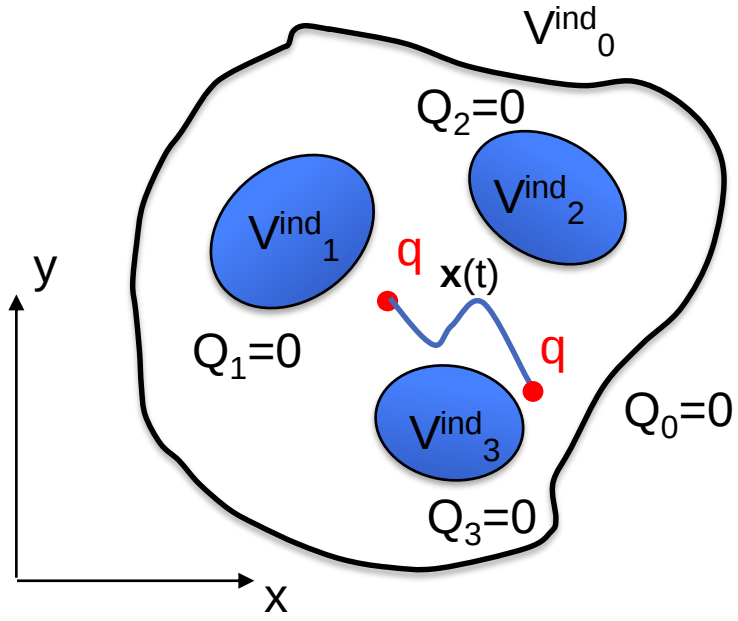
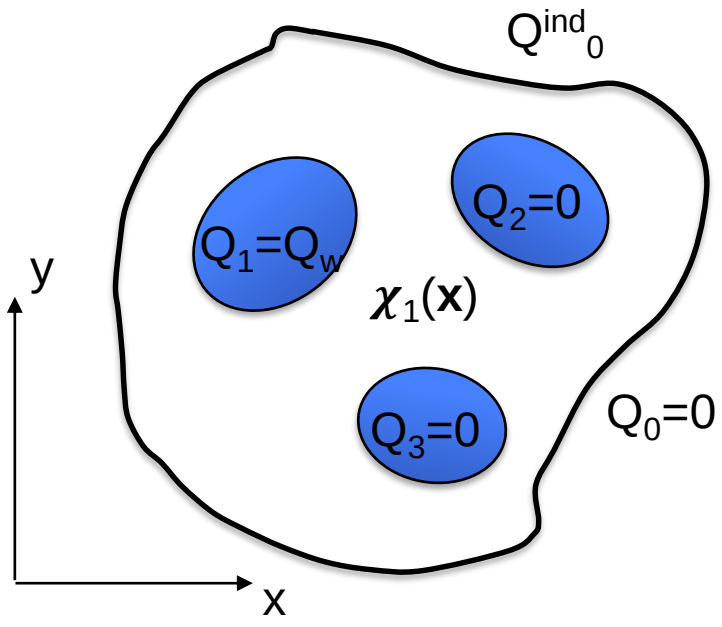


Figure 38. $\varepsilon_r = 1, w_x = 4g, b = g, T_0 = 0.001T$ for $x = 0, x = 4g, x = 8g$.

Induced voltage, equivalent circuit

Theorem, induced voltage, static





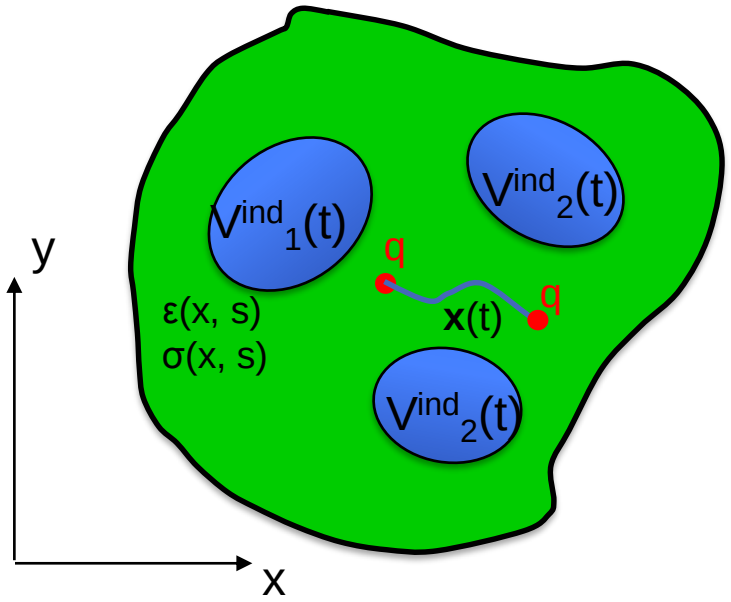
The voltage induced on an uncharged and insulated conducting electrode by a point charge q at position **x** can be calculated the following way:

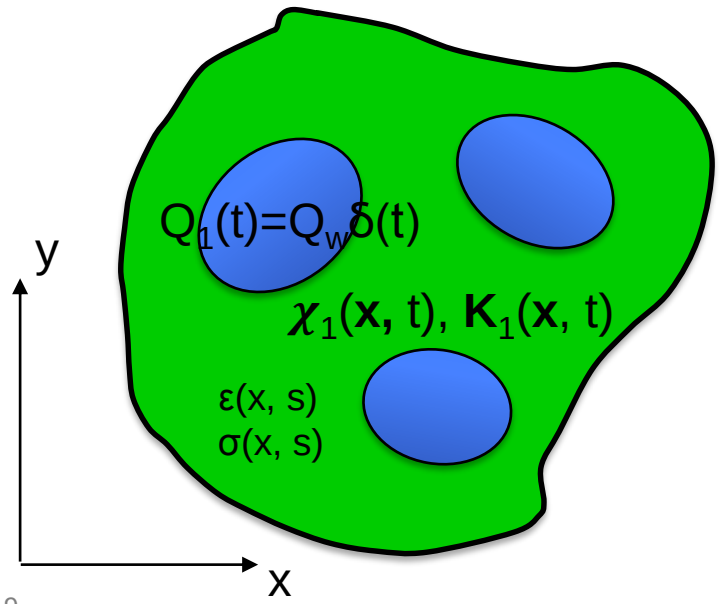
Remove the point charge, put a charge Q_w on the electrode in question while keeping all other electrodes insulated and uncharged.

This defines the potential $\chi_n(\mathbf{x})$ and the induced voltage is

$$V_n^{ind}(t) = \frac{q}{Q_0}\chi_n(\mathbf{x}(t))$$

Theorem, induced voltage, dynamic





Applying the delta voltage pulse to the electrode in question we find the potential $\psi_n(x, t)$ and the field $E_n(x, t)$ from which the induced current can be calculated the following way:

$$V_n^{ind}(s) = \frac{s}{Q_w}\int_V \chi_n(\mathbf{x}, s)\rho_e(\mathbf{x}, t)d^3x$$

$$\rho_e(\mathbf{x}, t) = q\delta(\mathbf{x} - \mathbf{x}_1(t))$$

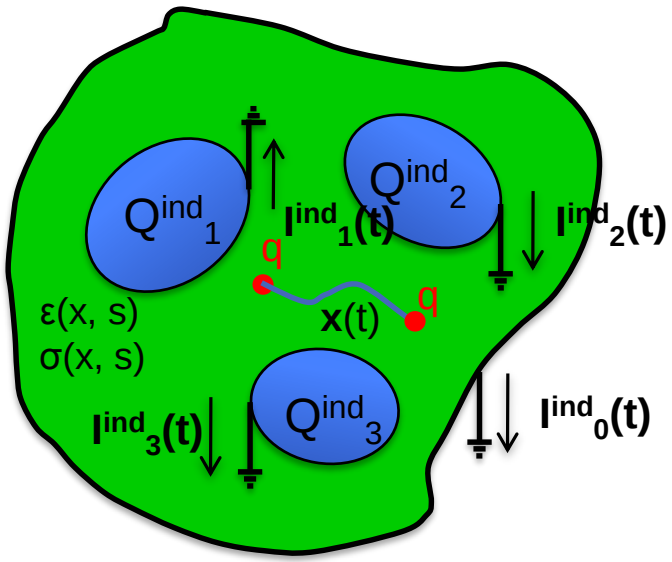
$$V_n^{ind}(s) = -\frac{q}{Q_w}\int_0^t \mathbf{K}(\mathbf{x}_1(t'), t - t')\dot{\mathbf{x}}_1(t')dt'$$

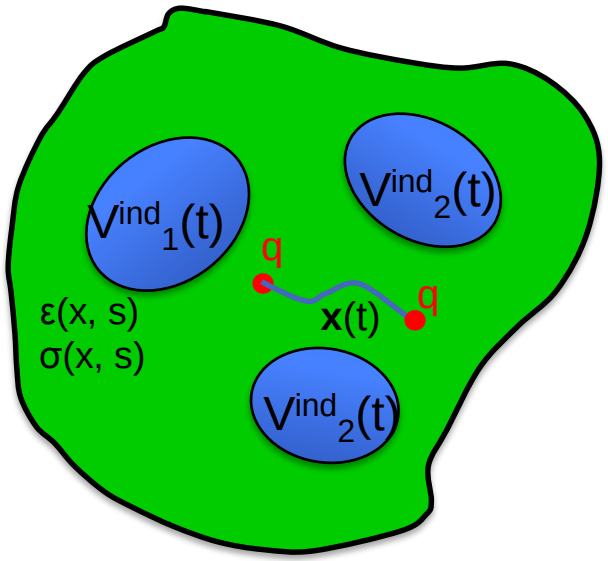
Since the admittance matrix relates currents and voltages on the electrodes in absence of charge, the admittance matrix relates the weighting fields $\mathbf{E_n}$ and $\mathbf{K_n}$ and therefore related the currents induced on grounded electrodes and the voltages induced on insulated electrodes.

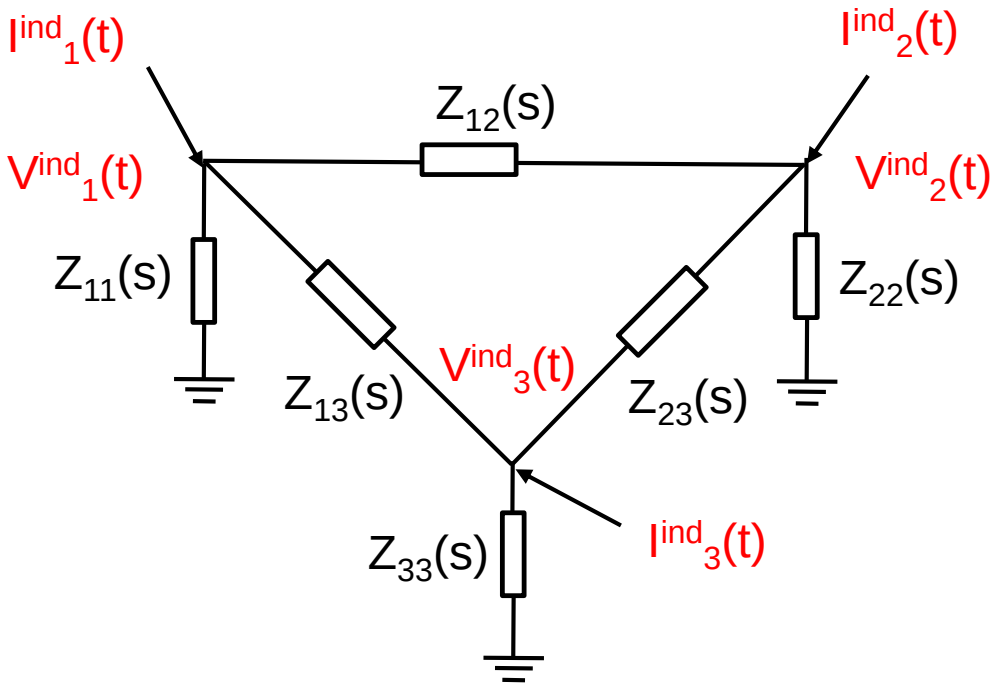
$$I_n^{ind}(s) = \sum_{n=1}^{N} y_{nm}(s)V_m^{ind}(s)$$

This means in turn that we can first calculate the current induced on grounded electrodes and then place these currents as ideal current sources on the equivalent circuit of the medium.

Equivalent circuit, Impedance elements



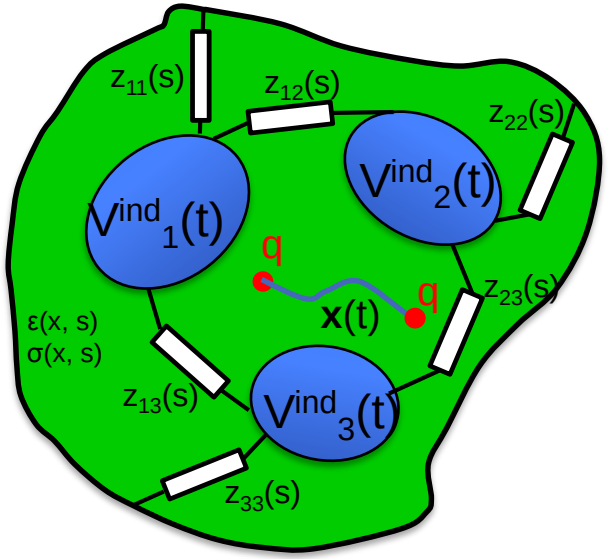




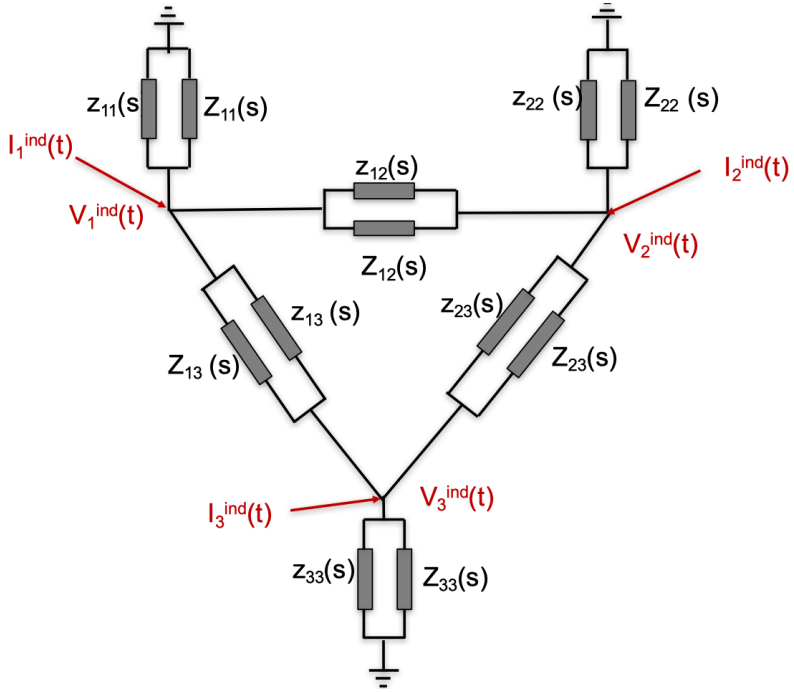
$$Z_{nm}(s) = -\frac{1}{y_{nm}(s)} \quad n \neq m \qquad Z_{nn}(s) = \frac{1}{\sum_{m=1}^{N} y_{nm}(s)} = -\frac{1}{y_{0n}} \quad n = m$$

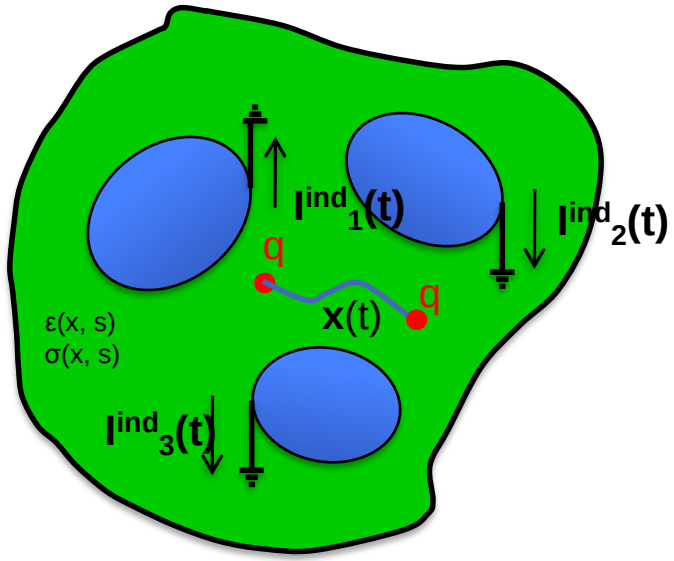
$$y_{mn}(s) = \frac{s}{V_w} \oint_{\mathbf{A}_n} \varepsilon_{eff}(\mathbf{x}, s) \nabla \psi_m(\mathbf{x}, s) d\mathbf{A}$$

Equivalent circuit, Impedance elements



In case the electrodes are not insulated but connected with discrete linear impedance components we can consider them as part of the medium and we therefore just have to add these elements in the equivalent circuit.





$$Z_{nm}(s) = -\frac{1}{y_{nm}(s)} \quad n \neq m \qquad Z_{nn}(s) = \frac{1}{\sum_{m=1}^{N} y_{nm}(s)} = -\frac{1}{y_{0n}} \quad n = m$$

$$y_{mn}(s) = \frac{s}{V_w} \oint_{\mathbf{A}_n} \varepsilon_{eff}(\mathbf{x}, s) \nabla \psi_m(\mathbf{x}, s) d\mathbf{A}$$

Nonlinear media

[6] L.A. Hamel, M. Julien, Comments on Ramo's theorem, in: Ralph B. James (Ed.), Hard X-ray and Gamma-Ray Detector Physics III, in: Proceedings of SPIE, 4507, 2001, pp. 255–263.

[7] L. Hamel, M. Julien, Generalized demonstration of Ramo's theorem with space charge and polarization effects, Nucl. Instrum. Methods A 597 (2008) 207–211.

Nuclear Inst. and Methods in Physics Research, A 940 (2019) 453–461

An application of extensions of the Ramo–Shockley theorem to signals in silicon sensors

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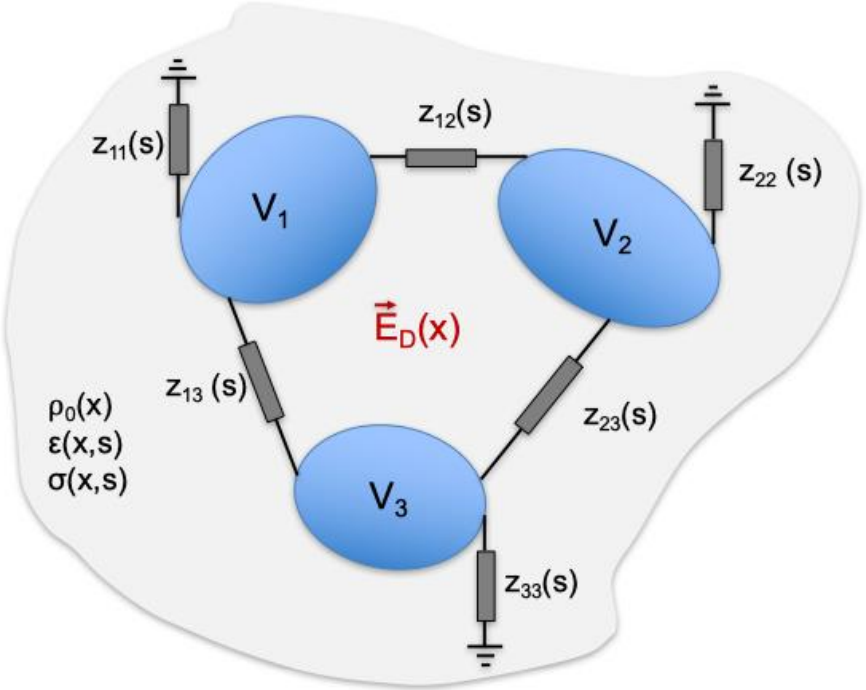
Static electric field in a biased silicon sensor



Metal electrodes embedded in a medium with

- Static space-charge $\rho_0(x)$
- Position and frequency dependent permittivity $\varepsilon(x, s)$
- Position and frequency dependent conductivity $\sigma(x, s) = 1/\rho(x,s)$ (ρ … volume resistivity)
- Connection of the electrodes with discrete impedance elements $z_{mn}(s)$
- Nonlinear material i.e. $\rho_0(x)$, $\varepsilon(x, s)$, $\sigma(x, s)$ can depend on the voltages V_n applied to the electrodes.

Laplace parameter $s=i\omega$
ω … frequency

→ Silicon sensor !

Induce voltage

2.1. Induced voltage

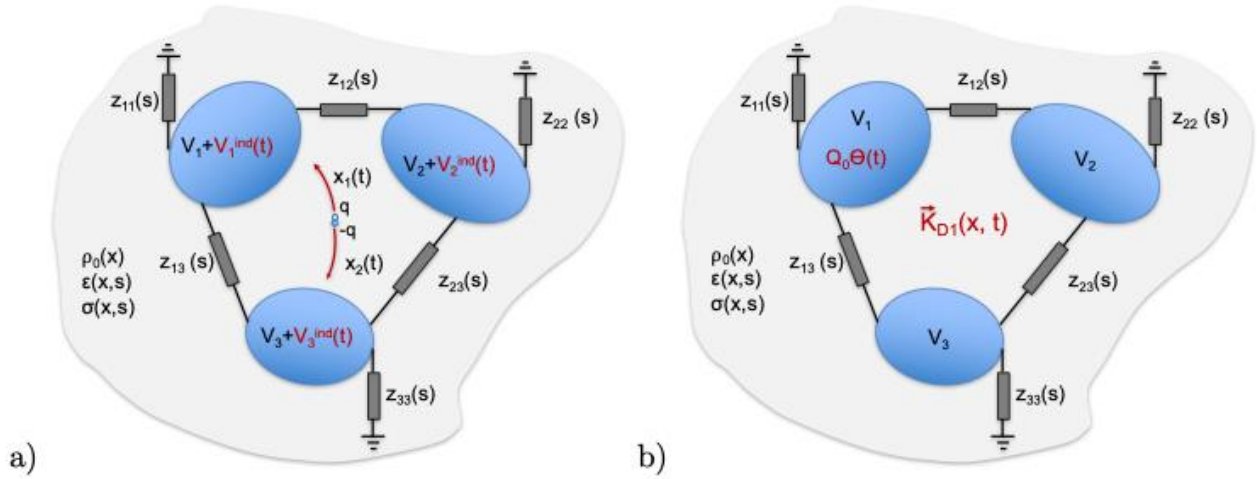


Figure 2: a) Two point charges $q, -q$ placed in the detector at $t = 0$. Their movement induces voltages $V_n(t)$ on the electrodes. b) The electric field $\vec{K}_{D1}(\vec{x}, t)$ due to placing a charge $Q(t) = Q_0\Theta(t)$ on electrode 1. This is equivalent to placing a current $I(t) = Q_0\delta(t)$ on the electrode.

A pair of charges $q, -q$ is created in the detector at t=0 and these charges are moving in the electric field $\vec{E}_D(\vec{x})$ along trajectories $\vec{x}_1(t)$ and $\vec{x}_2(t)$. The voltage $V_n^{ind}(t)$ induced on electrode n (Fig. 2a) can be calculated by

$$V_n^{ind}(t) = -\frac{q}{Q_0}\int_0^t \vec{K}_n\left[\vec{x}_1(t'), t-t'\right]\dot{\vec{x}}_1(t')dt' + \frac{q}{Q_0}\int_0^t \vec{K}_n\left[\vec{x}_2(t'), t-t'\right]\dot{\vec{x}}_2(t')dt' \quad (5)$$

where the weighting field $\vec{K}_n(\vec{x}, t)$ is defined the following way (Fig. 2b): the charges $q, -q$ are removed, an 'infinitesimal' charge $Q(t) = Q_0\Theta(t)$ is added to electrode n, which results in a field $\vec{K}_{Dn}(\vec{x}, t)$, from which the weighting field $\vec{K}_n(\vec{x}, t)$ is derived as

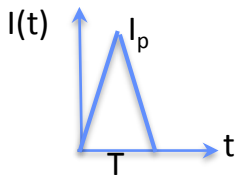
$$\vec{K}_n(\vec{x}, t) = \vec{K}_{Dn}(\vec{x}, t) - \vec{E}_D(\vec{x}) \quad (6)$$

This weighting field is defined by placing a 'step charge' or 'delta current' on the electrode in question and calculating the resulting electric field.

This theorem is very well suited for calculation of signals with TCAD simulation programs.

One can add the entire discrete circuitry like biasing network, amplifier etc. to the TCAD model and directly find the voltages induced on the nodes.

In TCAD one can e.g. use a triangular current pulse with duration T and peak value I_p and then use $Q_0 = I_p \times T/2$, where T must be chosen much smaller than the reaction time of the medium.



In general any current I(t) with $Q_0 = \int I(t)dt$ can be used, as long as the duration is much smaller than the reaction time of the medium.

Induce charge on grounded electrodes

This weighting field is defined by placing a step voltage on the electrode in question and calculating the resulting electric field.

This theorem is well suited for calculation of signals with TCAD simulation programs when the input impedance of the amplifier that connects to the electrode is negligible with respect to the other impedances in the circuit.

In that case $I_n(t)=-dQ_n(t)/dt$ directly gives the input current to the amplifier.

2.2. Induced charge on grounded electrode

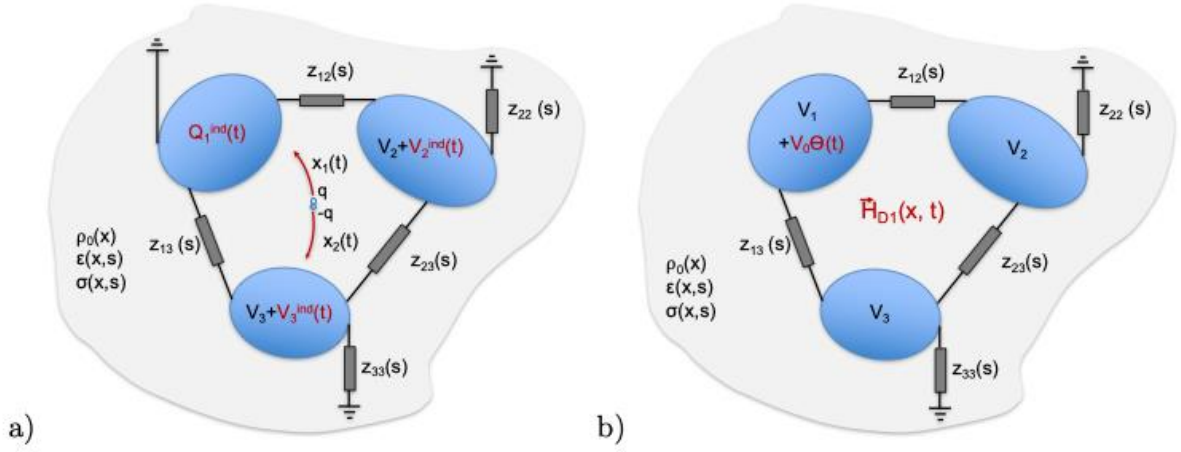


Figure 3: a) Two point charges $q, -q$ placed in the detector at $t = 0$. Their movement induces a charge $Q_n(t)$ on a grounded electrode. b) The electric field $\vec{H}_{D1}(\vec{x}, t)$ due to adding a voltage $V(t) = V_0\Theta(t)$ on electrode 1.

A pair of charges $q, -q$ is created in the detector at t=0 and these charges are moving in the electric field $\vec{E}_D(\vec{x})$ along trajectories $\vec{x}_1(t)$ and $\vec{x}_2(t)$. The charge $Q_n^{ind}(t)$ induced on the grounded electrode n (Fig. 3a) can be calculated by

$$Q_n^{ind}(t) = \frac{q}{V_0}\int_0^t \vec{H}_n\left[\vec{x}_1(t'), t-t'\right]\dot{\vec{x}}_1(t')dt' - \frac{q}{V_0}\int_0^t \vec{H}_n\left[\vec{x}_2(t'), t-t'\right]\dot{\vec{x}}_2(t')dt' \quad (7)$$

where the weighting field $\vec{H}_n(\vec{x}, t)$ is defined the following way (Fig. 3b): the charges $q, -q$ are removed, an 'infinitesimal' voltage $V(t) = V_0\Theta(t)$ is added to electrode n, which results in a field $\vec{H}_{Dn}(\vec{x}, t)$, from which the weighting field $\vec{H}_n(\vec{x}, t)$ is derived as

$$\vec{H}_n(\vec{x}, t) = \vec{H}_{Dn}(\vec{x}, t) - \vec{E}_D(\vec{x}) \quad (8)$$

2.3. Induced current on grounded electrode

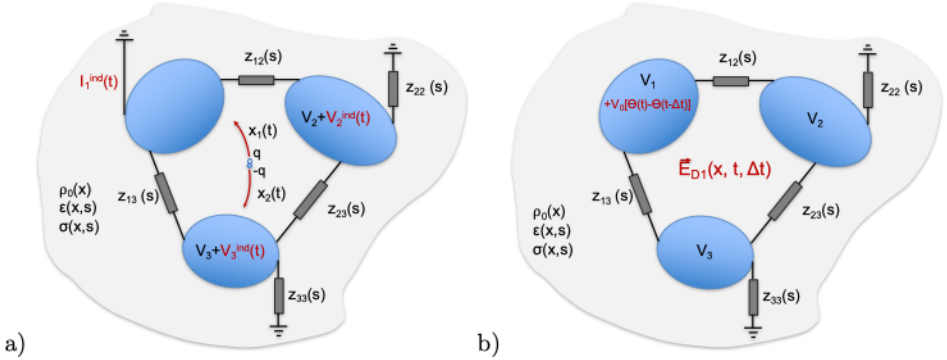


Figure 4: a) Two point charges $q, -q$ placed in the detector at $t = 0$. Their movement induces a current $I_n(t)$ on grounded electrodes b) The electric field $\vec{H}_{D1}(\vec{x}, t, \Delta t)$ due to putting a 'square voltage pulse' $V_0[\Theta(t) - \Theta(t - \Delta t)]$ on electrode 1.

The current induced on a grounded electrode is simply defined as the derivative of the induced charge, which gives

$$\begin{aligned} I_n^{ind}(t) &= -\frac{dQ_n^{ind}(t)}{dt} \qquad (10) \\ &= -\frac{q}{V_0}\left[\vec{H}_n\left[\vec{x}_1(t), 0\right]\dot{\vec{x}}_1(t) + \int_0^t \vec{H}_n^{(t)}\left[\vec{x}_1(t'), t - t'\right]\dot{\vec{x}}_1(t')dt'\right] \\ &\quad + \frac{q}{V_0}\left[\vec{H}_n\left[\vec{x}_2(t), 0\right]\dot{\vec{x}}_2(t) + \int_0^t \vec{H}_n^{(t)}\left[\vec{x}_2(t'), t - t'\right]\dot{\vec{x}}_2(t')dt'\right] \end{aligned}$$

A pair of charges $q, -q$ is created in the detector at t=0 and these charges are moving in the electric field $\vec{E}_D(\vec{x})$ along trajectories $\vec{x}_1(t)$ and $\vec{x}_2(t)$. The current $I_n^{ind}(t)$ induced on a grounded electrode (Fig. 3a) can be calculated by

$$\begin{aligned} I_n^{ind}(t) &= -\frac{q}{V_0}\int_0^t \vec{W}_n\left[\vec{x}_1(t'), t - t'\right]\dot{\vec{x}}_1(t')dt' \qquad (15) \\ &\quad + \frac{q}{V_0}\int_0^t \vec{W}_n\left[\vec{x}_2(t'), t - t'\right]\dot{\vec{x}}_2(t')dt' \end{aligned}$$

where the weighting vector $\vec{W}_n(\vec{x}, t)$ is defined the following way (Fig. 4b): the charges $q, -q$ are removed, an 'infinitesimal' voltage pulse $V(t) = V_0[\Theta(t) - \Theta(t - \Delta t)]$ is added to electrode n, which results in a field $\vec{E}_{Dn}(\vec{x}, t, \Delta t)$, from which the weighting vector $\vec{W}_n(\vec{x}, t)$ is derived as

$$\vec{W}_n(\vec{x}, t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\vec{E}_{Dn}(\vec{x}, t, \Delta t) - \vec{E}_D(\vec{x})\right] \qquad (16)$$

In contrast to $\vec{K}(\vec{x}, t)$ and $\vec{H}(\vec{x}, t)$, the vector $\vec{W}(\vec{x}, t)$ does not represent an electric field but a vector with units V/cm s. For the case where $E_D(\vec{x}) = 0$ the weighting vector $W_n(\vec{x}, t)$ then becomes the response to a delta pulse $V_0\delta(t)$ on the electrode n.

Induced Current on grounded electrodes

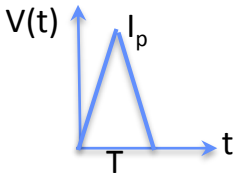
The induced current can also be calculated by a 'weighting field' or 'weighting vector' W_n that is causes by a small voltage pulse on the electrode in question.

W(x, t) has units of V/cm*s and therefore does not represent an electric field.

Using this weighting vector the induced current can be calculated directly.

This weighting vector will always have a 'prompt' component that follows the short pulse and a 'delayed' component that includes the reaction of the medium.

When using this weighting field for numerical simulations it is useful to use these two components separately to avoid numerical issues.



Induced Currents and Voltages

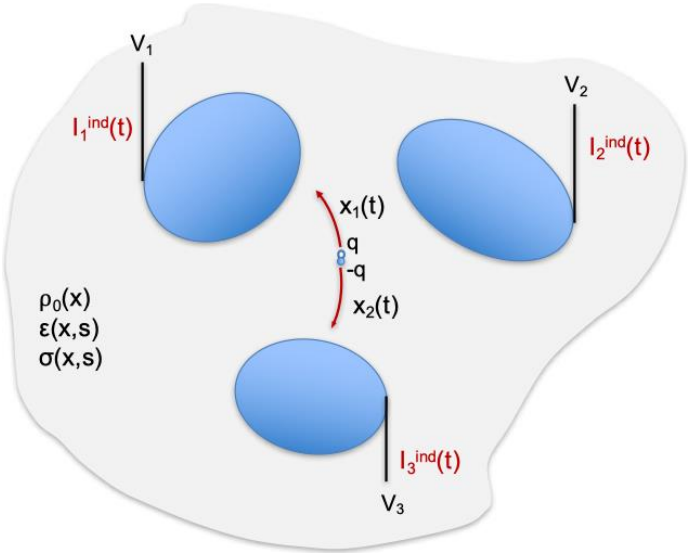
$$\vec{W}_n(\vec{x},t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\vec{E}_{Dn}(\vec{x},t,\Delta t) - \vec{E}_D(\vec{x})]$$

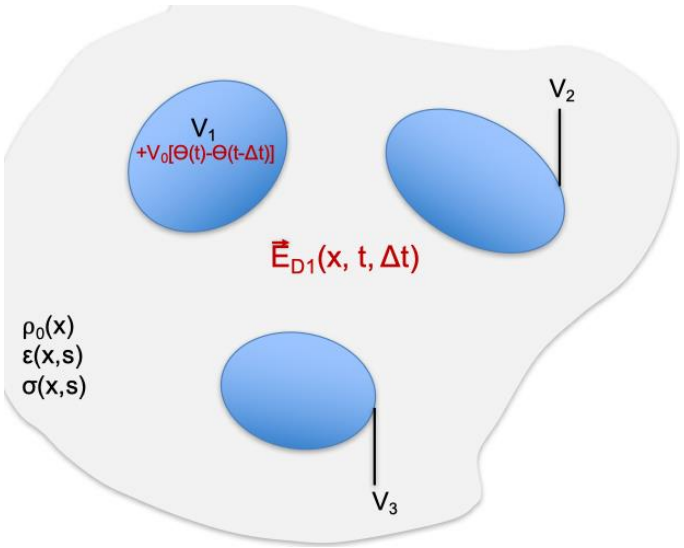
The charges $Q^{ind}_n(t)$ and current $I^{ind}_n(t) = -dQ_n(t)/dt$ are related to the voltages $V^{ind}_n(t)$ induced on the electrodes that are connected by the discrete impedance elements $z_{mn}(s)$ throught the admittance matrix $y_{mn}(s)$.

Let us assume we have calculated the weighting field H(x, t) or the weighting vector W(x, t) for the induced charge or the induced current, for the case where the other electrodes are held at fixed potentials i.e. the interconnecting impedance elements $z_{mn}(s)$ do not play a role.

We perform the Laplace transform and have $H_n(x, s)$ and $W_n(x, s)$.

We define the admittance matrix $y_{mn}(s)$ by integrating these weighting fields over the electrode surfaces.

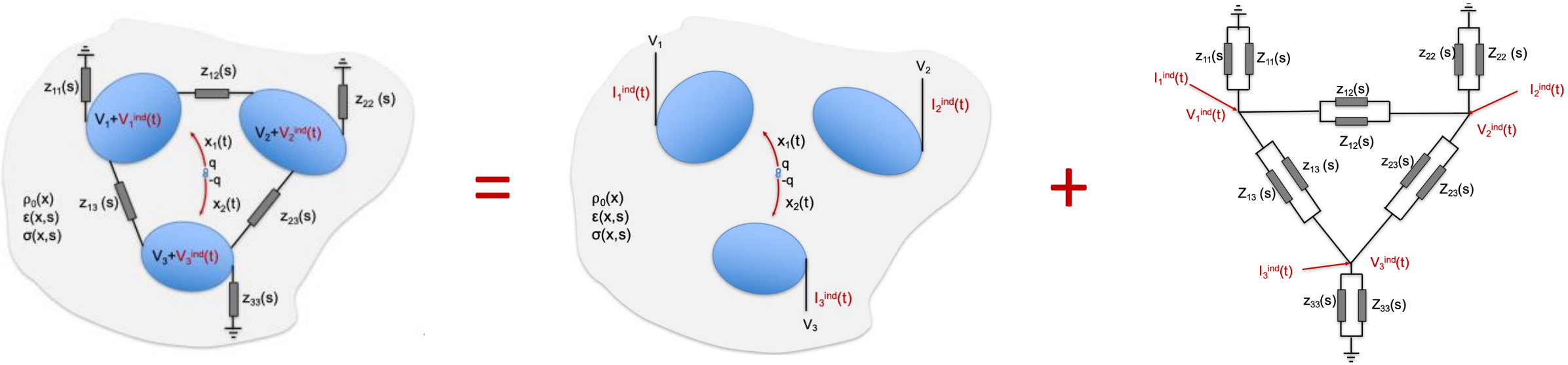




$$\begin{aligned} I_n^{ind}(t) &= -\frac{q}{V_0}\int_0^t \vec{W}_n\left[\vec{x}_1(t'), t-t'\right]\dot{\vec{x}}_1(t')dt' \\ &\quad + \frac{q}{V_0}\int_0^t \vec{W}_n\left[\vec{x}_2(t'), t-t'\right]\dot{\vec{x}}_2(t')dt' \end{aligned}$$

$$\begin{aligned} y_{mn}(s) &= \frac{s}{V_0}\oint_{A_m}\left[\varepsilon(\vec{x},s) + \sigma(\vec{x},s)/s\right]\vec{W}_n(\vec{x},s)d\vec{A} \\ Z_{nm}(s) &= -\frac{1}{y_{nm}(s)} \quad m \neq n \\ Z_{nn}(s) &= \frac{1}{\sum_{m=1}^{N} y_{mn}(s)} \end{aligned}$$

Equivalent Circuit



The electrodes and the medium can be represented by nodes that are connected by impedance elements.

The induced voltage signals for the case where the electrodes are connected by arbitrary impedance elements can then be calculated by the induced currents on grounded electrodes together with the equivalent circuit diagram.

In case the medium has no conductivity, i.e. σ=0, these impedance elements are $Z_{nm}=1/sC_{nm}$ with C_{nm} being the mutual electrode capacitances.

This second method for calculating the induced voltages has the advantage that one calculates the currents $I^{ind}_n(t)$ and the impedance elements $Z_{mn}(s)$ once and can then perform all further calculations for different readout and biasing circuits in a separate SPICE simulation.