Correction of Hamiltonian driving terms with wires

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With many thanks for the discussions: *Yannis Papaphilippou, Stephane Fartoukh, Kyriacos and Nikos*

Fermilab 2019 meeting on Wire Compensation

Wire correction using Hamiltonian Fourier Coefficients C_{mk} (Hamiltonian dr. terms)

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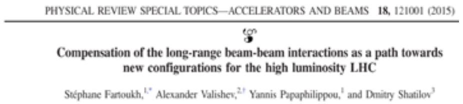

PHYSICAL REVIEW SPECIAL TOPICS—ACCELERATORS AND BEAMS 18, 121001 (2015)

Compensation of the long-range beam-beam interactions as a path towards new configurations for the high luminosity LHC

Stéphane Fartoukh,[1,*] Alexander Valishev,[2,†] Yannis Papaphilippou,[1] and Dmitry Shatilov[3]

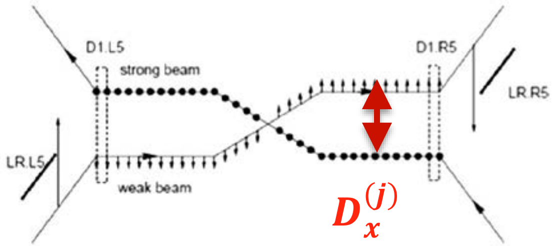
different dr terms c_{mn}
(expansion of the kick, not H)

Consider 1D correction X-plane in IR5

Left-right independent optimization of two wires in the plane of separation (x). So k=0.
Wish to install a wire than cancels this sum for some m.

Collision points (CP)

18 CPs left 18 CPs right



$D_x^{(j)}$

$\sigma_{x,y}^{(j),strong}$

$$C_m^{\mathrm{CP}} = \sum_{j=1}^{18} C_m^{(j)}$$

(diff in betatr. phases ignored)

$$-C_m^{\mathrm{CP}} = N^{\mathrm{W}} C_m^{\mathrm{W}}$$

Know real sep. and aspect ratios: $D_x^{(j)}, r^{(j)}$

Want these: D_x^W , $r^W = ?,\ N^W = ?$

"sigma" aspect ratio: $r = \frac{\sigma_y^{strong}}{\sigma_x^{strong}}$

Comparison of approaches: C_{mk} and c_{mk}

$$\sigma_{x,y} \equiv \sigma^{\rm str}_{x,y}, \quad \text{omit "strong" from now on}$$

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- **CP participates with beta aspect ratio at its location:**

$$r_{\rm W} = \frac{\beta_x^{\rm weak}}{\beta_y^{\rm weak}}$$

- **Exact anti-symmetry:**

$$\beta_{x,y}^{\rm weak} = \beta_{y,x}^{\rm strong}.$$

Optimization in 2D produces these

$$\begin{cases} c_{pq}^{w.{\rm L}} \equiv N_{w.{\rm L}} \times \dfrac{(\beta_x^{w.{\rm L}})^{p/2} (\beta_y^{w.{\rm L}})^{q/2}}{(d_{w.{\rm L}})^{p+q}} \\ c_{pq}^{w.{\rm R}} \equiv N_{w.{\rm R}} \times \dfrac{(\beta_x^{w.{\rm R}})^{p/2} (\beta_y^{w.{\rm R}})^{q/2}}{(d_{w.{\rm R}})^{p+q}} \end{cases}$$

$$c_{pq}^{\rm LR} \equiv \sum_{k \in {\rm LR}} \frac{\beta_x^{p/2}(s_k)\beta_y^{q/2}(s_k)}{d_{bb}^{p+q}(s_k)}$$

Here everything depends on strong beam (Ham. approach):

- **CP participates with sigma aspect ratio:**

$$r \equiv \frac{\sigma_y}{\sigma_x} \qquad r = \sqrt{r^{\rm W}}$$

- **Exact anti-symmetry:**

$$\sigma_{x,y}^{\rm weak} = \sigma_{y,x}$$

$$\sigma_x^{\rm weak} = \sigma_y = r\sigma_x; \quad \sigma_y^{\rm weak} = \sigma_x = \frac{1}{r}\sigma_y$$

- **For now Left-Right independent.**

Comparison of approaches: C_{mk} *and* c_{mk}

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- **No dependence on amplitude**

Here:

$a_{x,y}$ **n-sigma amplitudes**

$$x = \sqrt{2\beta_x^{\text{weak}} J_x} \sin\phi_x = a_x \sigma_x^{\text{weak}} \sin\phi_x = r a_x \sigma_x \sin\phi_x$$
$$y = \sqrt{2\beta_y^{\text{weak}} J_y} \sin\phi_y = a_y \sigma_y^{\text{weak}} \sin\phi_y = a_y \sigma_x \sin\phi_y$$

- **Define amplitude region: well between orbits**

$$r a_x > d_x \quad \text{outside strong beam orbit } (|x| > D_x)$$
$$r a_x < d_x \quad \text{between orbits } (|x| < D_x)$$
$$r a_x < d_x + \Delta \text{ and } a_x > \Delta \quad \leftarrow \text{“\textbf{well between orbits}”}$$

$\Delta \approx 2 \div 2.5$ in σ_x units.

$d_{x,y} \equiv D_{x,y}/\sigma_{x,y}$ – normalized separations

Comparison of results

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- If for a pair m,k canceled, then all are approximately canceled
- above true for locations with

  $r^w \sim 2$ or $1/2$
- Verified with:
  - tune-shifts
  - MadX, SixTrack
  - FMA
  - DA

Here:

- Confirmed that all can be canceled in single plane (of collision)

  and working on 2D
- For single plane any location works

  and working on 2D
- Verified with
  - MadX tracking and
  - Effective Hamiltonian (beam-beam invariant)

HiLumi HL-LHC PROJECT

2D Fourier coefficient (modulo) as an integral along parametrized curve. It depends only on aspect ratio r and normalized separation to weak-beam sigma:

for wire, also observed in

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$$\frac{d_x}{r} = \frac{D_x}{\sigma_y} = \frac{D_x}{\sigma_x^{\mathrm{weak}}}$$

CP lattice location participates only via r and norm. sep:

$$C_{mk}\,(s, x, y, D_x) \;\rightarrow\; C_{mk}^{(r)}\,(a_x, d_x, a_y) \;\rightarrow$$

$$|C_{mk}^{(r)}(a_x, \frac{D_x}{\sigma_x^{\mathrm{weak}}}, a_y)| = \int_0^{a_x} d\xi \frac{2\,\mathbf{Q}_{mk}(r\xi, \frac{D_x}{a_x\sigma_x^{\mathrm{weak}}} r\xi, \frac{a_y}{a_x g_r(\xi)}\xi; 1)}{\xi\, g_r(\xi)}$$

$$g_r(\xi) = \sqrt{\frac{(r^2-1)\,\xi^2}{{a_x}^2}+1}$$

prefer to work with the modulo D=|C|

$$\mathbf{Q}_{mk}(\hat{a}_x, \hat{d}_x, \hat{a}_y; 1) = e^{-\frac{1}{2}(\hat{a_x}-\hat{d_x})^2-\frac{1}{2}\hat{a_y}^2}\mathbf{\Lambda}_m(\hat{a}_x\hat{d}_x, -\hat{a}_x^2/4)\;\mathbf{\Lambda}_k(0, -\hat{a}_y^2/4)$$

“well-between beams” parametric curves become parametric lines

Using novel basis: Generalized 2D Bessel functions

The **Q** above are the **W** in this paper

Beam-beam effects at the Fermilab Tevatron: Theory

T. Sen,* B. Erdelyi, M. Xiao, and V. Boocha

For several infinitesimal beam-beam kicks, the integral reduces to a sum over the kicks, and the resonance driving terms become

$$U^{++++}_{m_x,m_y,p} = \frac{1}{16}\frac{r_p}{2\pi\gamma_p}(-1)^{m_x+m_y-1}\sum_n N_{b,n}\int_0^1 \frac{dv}{v[v(r^2-1)+1]^{1/2}}\exp(-t_{x,n}-t_{y,n})W_{x,n}W_{y,n}\exp[i(m_x\alpha_{x,n}+m_y\alpha_{y,n}+p\theta_n)], \quad (48)$$

$$W_x = \sum_{l_x}(-1)^{l_x}[\exp(-s_x)I_{m_x-2l_x}(s_x)]\left[\exp\left(-\frac{r_x}{2}\right)I_{l_x}\left(\frac{r_x}{2}\right)\right], \quad (50)$$

$$W_y = \sum_{l_y}(-1)^{l_y}[\exp(-s_y)I_{m_y-2l_y}(s_y)]\left[\exp\left(-\frac{r_y}{2}\right)I_{l_y}\left(\frac{r_y}{2}\right)\right].$$

it turns out these W are known objects called 2D Bessels

[11] Clemente Cesarano and Claudio Fornaro, *Generalized Bessel functions in terms of generalized Hermite polynomials*, International Journal of Pure and Applied Mathematics Volume 112 No. 3 2017, 613-629 (see e.g. Eqn 27)

[12] H. J. Korsch, A. Klumpp , D. Witthaut, *On two-dimensional Bessel functions*, Journal of Physics A, V39, 48 (2006)

$$e^{-\frac{1}{2}(\hat{a}_x-\hat{d}_x)^2-\frac{1}{2}\hat{a_y}^2}\,\boldsymbol{\Lambda}_m(\hat{a}_x\hat{d}_x, -\hat{a}_x^2/4)\,\boldsymbol{\Lambda}_k(0,-\hat{a}_y^2/4)$$

Gaussian factor (near strong beam core)

Wire Fourier coefficient

The wire corrector potential can be described just as another long-range CP, whose strong-beam sigmas $\sigma_{x,y}$ are both decreased by a large factor denoted here with f_w [2]. Hence, the Fourier coefficients for a wire corrector can be obtained by taking

$$\sigma_{x,y} \to \sigma_{x,y}/f_w, \tag{6.1}$$

$$f_w \text{ is large, say} \sim 100$$

Expectaton: C_m^W ***should not depend on the value of*** f_W

$$\lim_{f_w \to \infty}$$

X-plane. CP and Wire Fourier coef. modulo D=|C| as an integral along parametrized line

Because of the Gaussian factor, for a_x well between orbits

$$\mathbf{Q}_m \to 0 \quad \text{hence} \quad D_m \to const,$$

and D^{CP} differs from D^{W} only because of the presence of g_r under the integral. This is valid for any m.

$$\text{line slope } \psi = \frac{\tilde{d}_x}{a_x} = \frac{D_x}{\sigma_x^{\mathrm{weak}} a_x}$$

$$D_m^{\mathrm{CP},(\mathrm{r})}\left(a_x, \frac{D_x}{\sigma_x^{\mathrm{weak}}}\right) = \int_0^{a_x} d\xi \, \frac{2\,\mathbf{Q}_m(r\xi, \psi\, r\xi; 1)}{g_r(\xi)\,\xi}$$

$$g_r(\xi) = \sqrt{\frac{(r^2-1)\,\xi^2}{a_x{}^2} + 1}$$

$$D_m^{\mathrm{W}}(\psi) = \lim_{f_w \to \infty} \int_0^{f_w a_x} d\xi \, \frac{2\,\mathbf{Q}_m(r\xi, \psi\, r\xi; 1)}{\xi} =$$

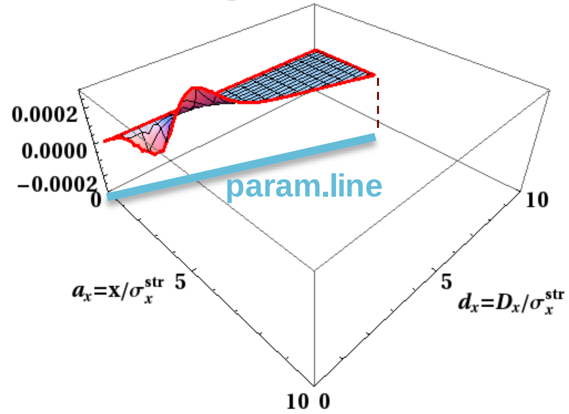
$$= \frac{1}{2\pi} \int_0^{2\pi} e^{-i\, m\, \phi_x} \ln P \, d\phi_x$$

$$P = \frac{1}{2}(d_x + r a_x \sin\phi_x)^2$$

Q_m surface

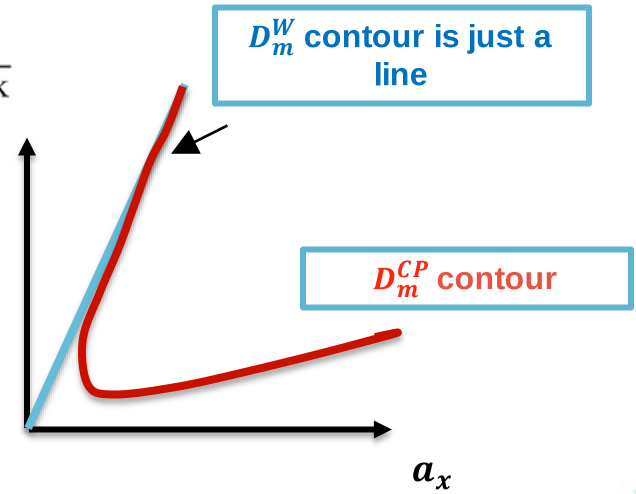
function under integral for m=4

$\sim \mathrm{Exp}[-1/2\,(a_x - d_x)^2]\,\Lambda_4$



Wire coef D_m^W is just the asymptotic of D_m^{CP}

$$\tilde{d}_x \equiv \frac{D_x}{\sigma_x^{\text{weak}}}$$



$$\text{line slope } \psi = \frac{\tilde{d}_x}{a_x} = \frac{D_x}{\sigma_x^{\text{weak}} a_x}$$

$$D_m^{\text{CP},(\text{r})}\left(a_x, \frac{D_x}{\sigma_x^{\text{weak}}}\right) = \int_0^{a_x} d\xi \, \frac{2\,\mathbf{Q}_m(r\xi, \psi\, r\xi; 1)}{g_r(\xi)\,\xi}$$

$$g_r(\xi) = \sqrt{\frac{(r^2-1)\,\xi^2}{a_x{}^2} + 1}$$

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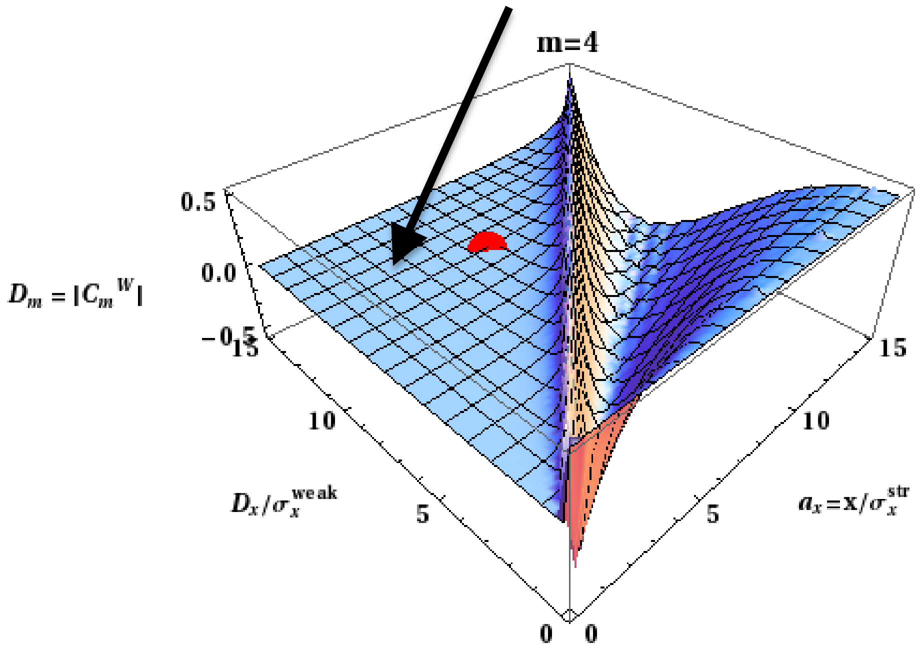
$$= \frac{1}{2\pi} \int_0^{2\pi} e^{-i\,m\,\phi_x} \ln P \, d\phi_x$$

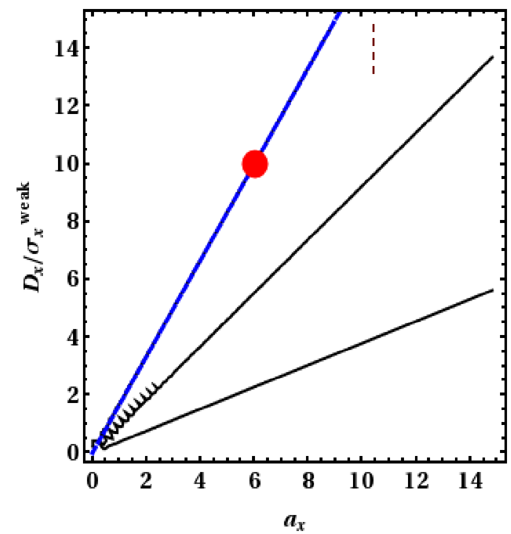
$$P = \frac{1}{2}(d_x + r a_x \sin\phi_x)^2$$

For ax "well between orbits" the only difference is in form factor g under int.

Well between orbits wire coef. depends only on param line

D_m^W surface "well between orbits" (above diagonal). Any contour of const. value of D_m^W coincides with its param line.

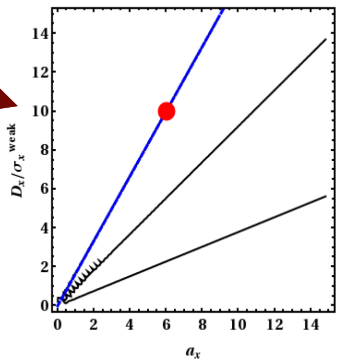


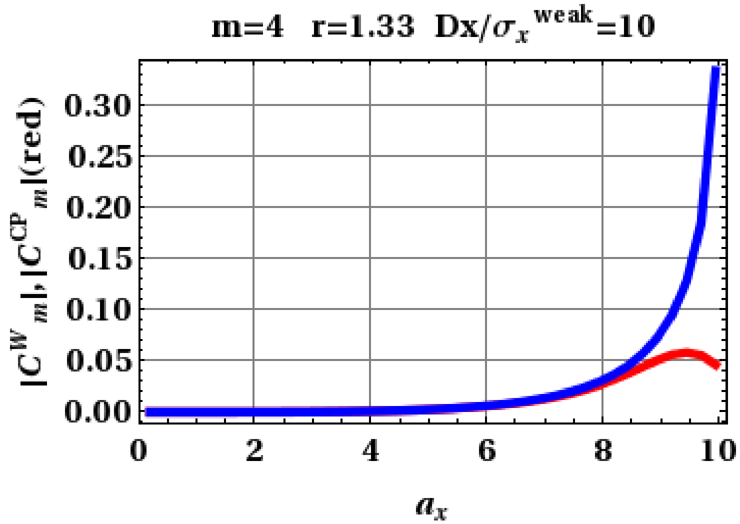


for large $f_w \to \infty$,
it depends only on ψ

Detour: Single CP and Wire both at same norm sep

Showing sections here of the two surfaces: CP (red) and wire (blue)





ax "well between orbits"

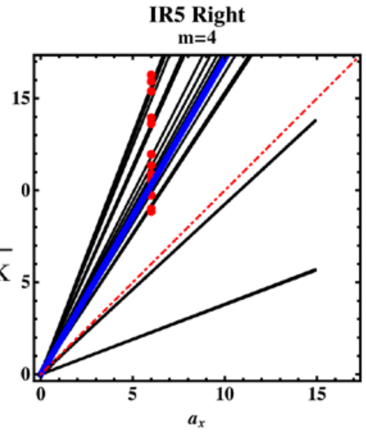
ax near strong beam core

Need to solve **this** Equation for IR5 Left or Right. (intensity 2.2 E11, 250 mkrad)

$$D_m^{CP} = \sum_{j=1}^{18} D_m^{(r_j)}(a_x, \tilde{d}_x^{(j)})$$

(diff in betatr. phases ignored)

$$\tilde{d}_x \equiv \frac{D_x}{\sigma_x^{\mathrm{weak}}}$$



$$-D_m^{CP} = N^W D_m^W(\psi)$$

$$\psi = \frac{\tilde{d}_x^{W}}{a_x}$$

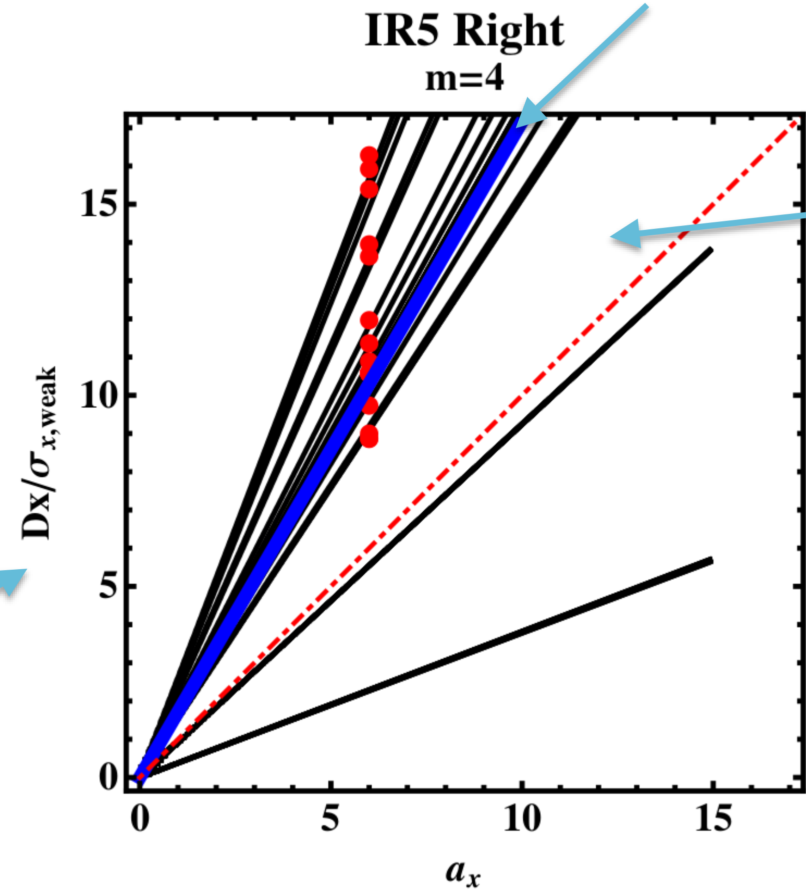
two unknowns: ψ and and the wire charge N^W in units of CP charge.

Illustration for m=4, ax=6

Red points represent 18 CP (stay same on plot for any m)

Black lines repr. wire contours for red-point values. They deviate slightly from red pts because of factors g.

Blue line repr. wire solution Ψ , N^W for this m



"well betw orbits" means above diagonal

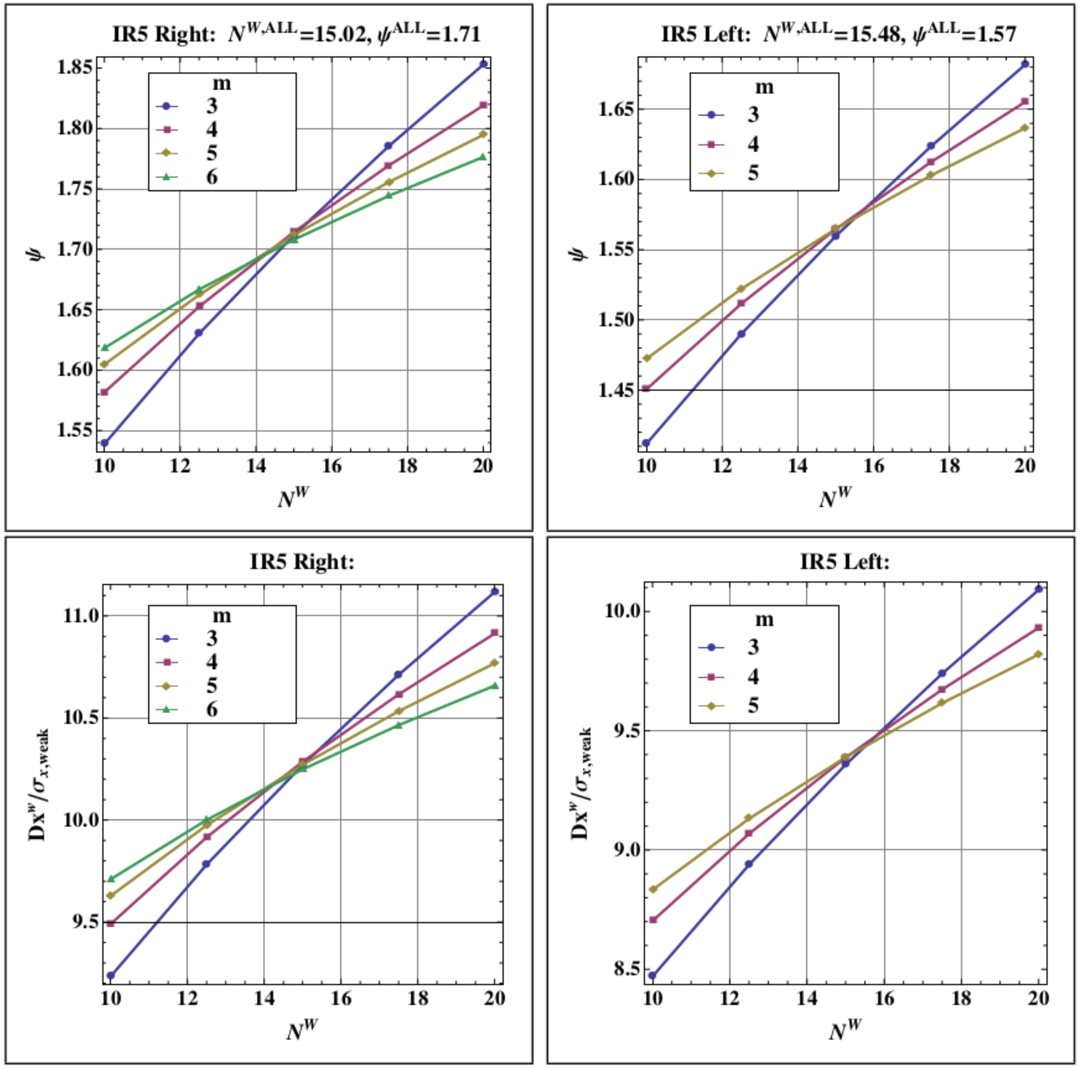
$$\widetilde{d}_x \equiv \frac{D_x}{\sigma_x^{\text{weak}}}$$

Any ax "well between orb" will give the same

Solving numerically for several values of m and N^{W} shows that there exist ψ^{ALL} and $N^{W\ ALL}$

m=1, m=2 excluded (lin. and quad. terms in H)

Wire of N^{W} installed here cancels all m



Recursion property of wire Fourier coefficient.
It explains existence of Ψ^{ALL} and $N^{W\,ALL}$

The $D_m^{\mathrm{w}}(\psi)$ obey homogeneous difference equation for the index m with this analytical solution:

$$D_m^{\mathrm{w}}(\psi) = R_m(D_3^{\mathrm{w}}, D_4^{\mathrm{w}}; \psi) =$$

$$= \frac{B^{m-3}\,(3\,A\,D_3^{\mathrm{w}} - 4\,D_4^{\mathrm{w}}) - A^{m-3}\,(3\,B\,D_3^{\mathrm{w}} - 4\,D_4^{\mathrm{w}})}{m\,(A-B)}$$

$$A = \psi + \sqrt{\psi^2 - 1}, \quad B = \psi - \sqrt{\psi^2 - 1}$$

By knowing ψ and D_3^{w} and D_4^{w} one can find D_m for any m.

Stephane: driving terms generated by wire corrector for different indices m, k may not be independent. We see that, at least in the single plane case this is indeed the case, but for Hamiltonian driving terms.

Solving (numerically) this equation to find ψ^{ALL} (arbitrary m_1 and m_2)

$$\frac{D^{CP}_{m_1}}{R_{m_1}(\psi, D_3(\psi), D_4(\psi))} = \frac{D^{CP}_{m_2}}{R_{m_2}(\psi, D_3(\psi), D_4(\psi))}$$

Table 1: Values of wire charge and normalized separation at the wire corrector that cancel all Hamiltonian driving terms (any index *m*). Single-plane (of separation) assumed and left-right independent optimization.

CP setup	$N^{W,ALL}(I^{W})$	$D^{W}_{x}/\sigma^{weak,W}_{x}$	ψ^{ALL}
18, IR5 Right	15 (158 A)	10.3	1.71
18, IR5 Left	15.5 (164 A)	9.4	1.57
the result in [3] for $r_w = 1$	$\sim$140 A	10.3	

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Testing with MadX tracking. Wires installed at ψ^{ALL} *and* $N^{W\ ALL}$ as in Table 1 (x-plane)

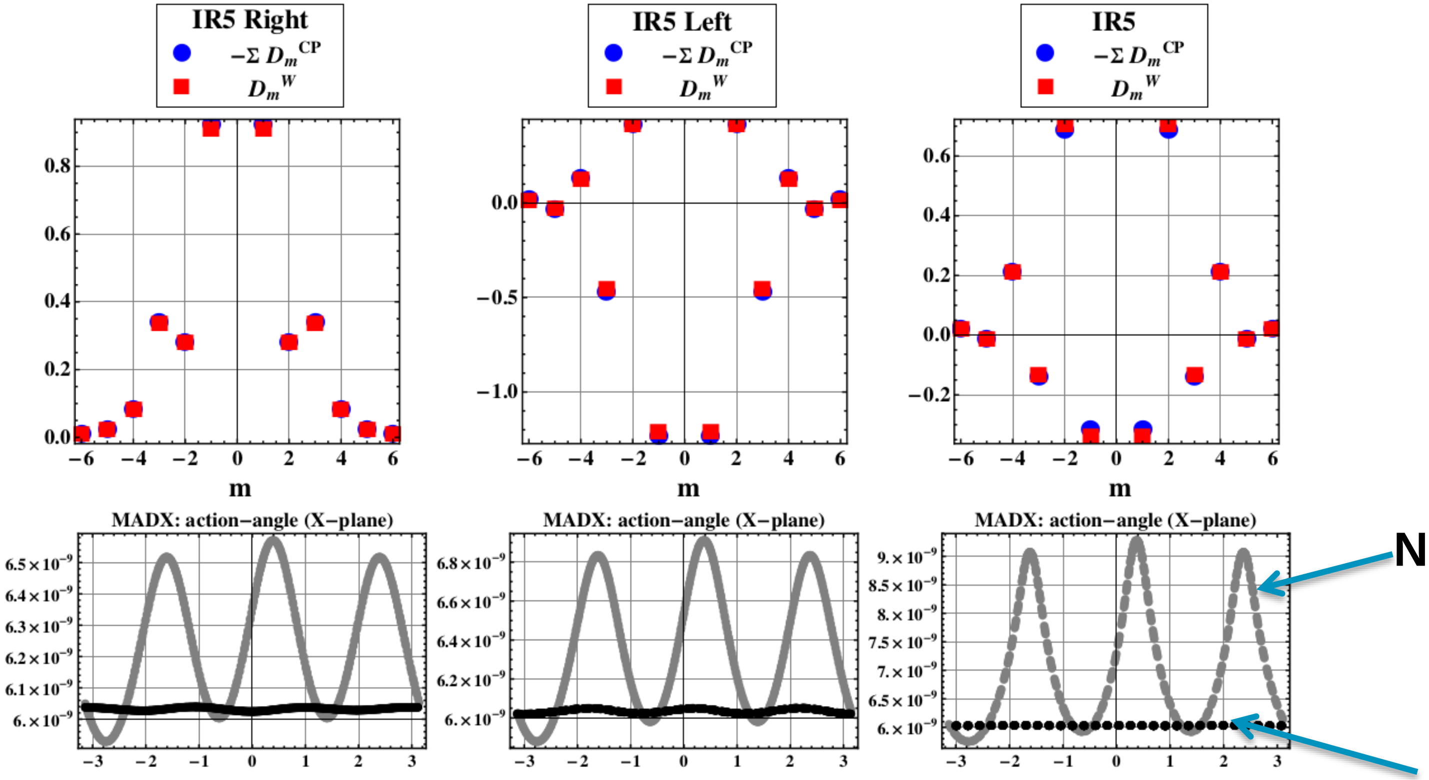
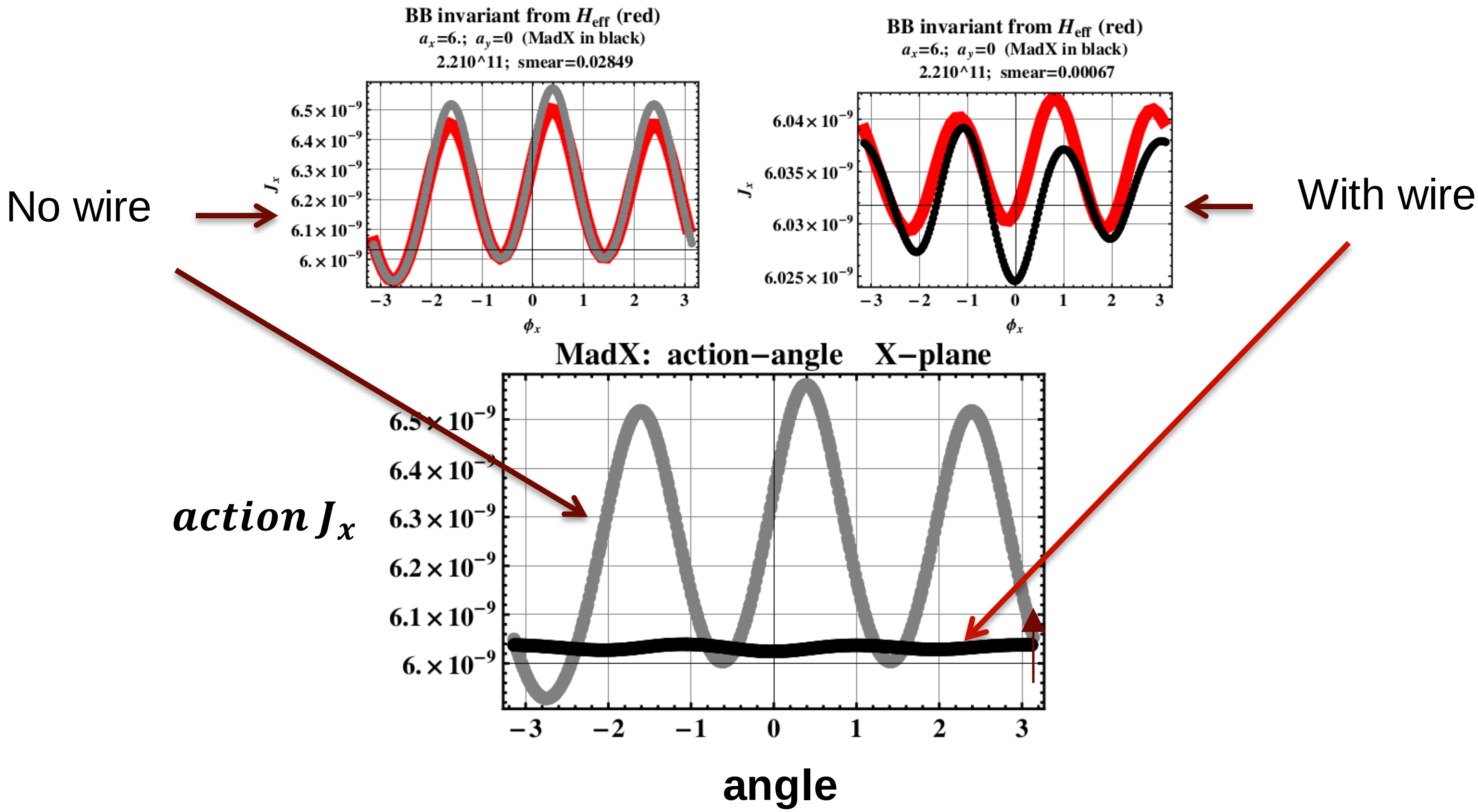


Figure 6: **Top**: Equal compensation of the total Hamiltonian driving term for each m for the left-right independent compensation with two wires installed near $s = \pm 93.5$ m from IP5, parameters taken as in Table 1. **Bottom**: Action-angle coordinates $J_x(\phi_x)$ of a single particle with $a_x = 6$ tracked for 1000 turns: with wire (darker) and without.

Testing with BB Invariant (x plane)



Summary

We found that *any lattice location* (r_w) for the wire will provide the same uniform (all m) cancellation of Hamiltonian driving terms in this plane as long as two wires are installed between the beam orbits at distances 10.3 (right wire) or 9.4 (left wire) σ_x^{weak} from the weak beam axis, with currents $\sim$ 160 A (for intensity 2.210^{11} and crossing angle 250 μ rad, as in [3]).

Thus the optimum setups are very close for left and right wires and also agree with the $r_w = 1$ result in [3], see the bottom line in Table 1.

The result may be explained as follows. We have seen that for a fixed IP5 optics (lattice) and a fixed m, the sum of all m-th CP coefficients depends on the set of values of r and $\widetilde{d}_x$ at these CPs so it effectively has two degrees of freedom. The wire coefficient to be fitted also has two degrees of freedom: wire charge N^{W} and single other parameter, since being the limit of the round-beam CP case ($r = 1$), the wire coefficient depends on only one lattice parameter $\widetilde{d}_x$. In addition, wire coefficients for different m are related. Therefore, not surprisingly, the optimization procedure finds (for each left or right) region, a solution pair that cancel the sum for all m.

$$\widetilde{d}_x \equiv \frac{D_x}{\sigma_x^{\text{weak}}}$$

Many Thanks for your attention

Wire cancellation at ampl. a_x as a “jump” along characteristic line

- take a particle traversing at x round-beam CP of charge λ and separation D_x. The coef. depends on normalized ampl. and sep.:

$$C_m(a_x, d_x), \quad \text{where} \quad d_x = D_x/\sigma_x^{\text{str}}, \quad a_x = x/\sigma_x^{\text{str}}$$

- position “wire corrector CP" at the same location (back to back)

$$\sigma_x^{\text{str}} \to \sigma_x^{\text{str}}/f_w, \quad f_w = 100$$

- Cancellation (with $-\lambda$) requires

$$C_m(a_x, d_x) = C_m(100\, a_x, 100\, d_x)$$

over some range of a_x. It becomes violated when a_x approaches d_x.

- It is a f_w–times "jump" from red to blue point (same surface). Here is a 3 times jump (middle) and range of canceled a_x (right)