Transverse Dynamics (part I)

JAI lectures - Michaelmas Term 2019

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Table of contents of part I of the course

Goal of this course

- Introduce you to one of the core topics in accelerator physics.
- Explain you the basics of the formalism.
- Give you and idea of the related phenomenology.
- Full derivations are not included in main lectures.
- Most important things: learn something and enjoy!

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Some references

Books

1. Wilson, An introduction to particle accelerators.
2. Lee, Accelerator Physics.
3. Wiedemann, Particle Accelerator Phsyics.
4. A. Wolski, Beam Dynamics in High Energy Particle Accelerators.
5. E. Forest, Beam Dynamics: A new Attitude Framework.
6. A. Chao, Handbook of Accelerator Physics and Engineering.

Other courses

1. A. Latina, JUAS lectures on Transverse Dynamics (2019) (main reference).
2. CAS lectures (recent Introductory Course).
3. USPAS lectures.

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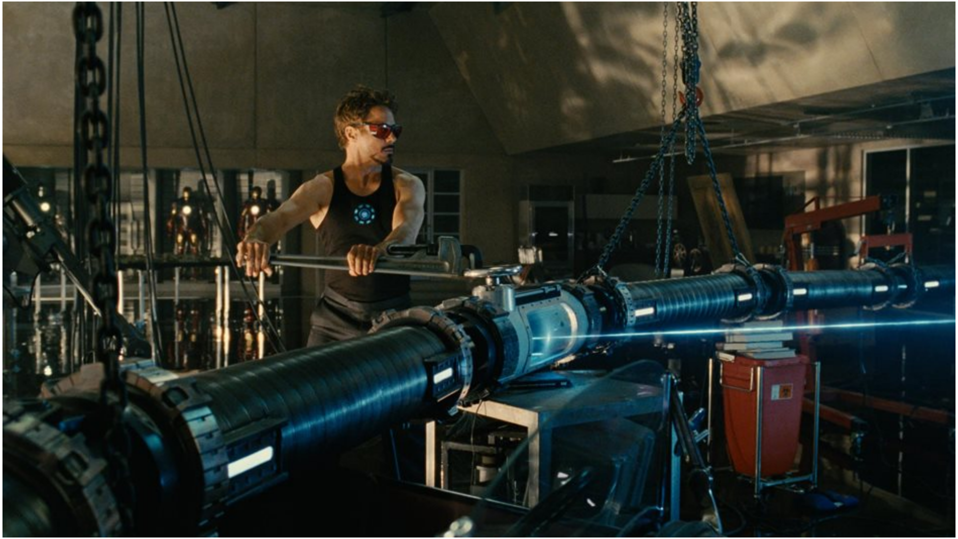
I did not know how complex an accelerator was...



Why these lectures?

What do we want to study?

High energy particles travelling through intense magnetic fields (usually periodic).

Why transverse dynamics?

- It covers $\sim 2/3$ of the phase space (4 out of 6 dimensions).
- Magnets act primarily on the transverse plane.
- Main accelerator parameters are determined (at first order) by transverse properties:
  - Luminosity, emittance, brilliance, beam losses, instabilities, tune...

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Special Relativity recap.

We need to study the motion of charged particles at (very) high energies.

$$E = \sqrt{m^2c^4 + p^2c^2} \tag{1}$$

where m is the mass of the particle and p the particle momentum.

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2}$$

Ultrarelativsitic approximation ($\gamma \gg 1$):

$$E \approx pc \tag{3}$$

Question: what is faster?

a) An electron/positron at LEP ($E \approx 100$ GeV)
b) A proton at the LHC ($E \approx 7$ TeV)

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Lorentz Force

The force experienced by a charge q and speed v under an electric field E and magnetic field B.

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{4}$$

- Electric field for increasing (decreasing) speed.
- Magnetic field for bending and focusing the beam trajectory.

Question: Why do we use magnets for bending the trajectory of the beam?

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Beam rigidity

Lorentz force:

$$F_L = qvB \quad (5)$$

Centripetal force:

$$F_c = m\frac{v^2}{\rho} \quad (6)$$

Null force condition ($\sum F = 0$):

$$F_L = F_c \Rightarrow \frac{p}{q} = B\rho \quad (7)$$

Beam rigidity:

$$B\rho \approx 3.33p[\mathrm{GeV/c}] \quad (8)$$

Applications

- Given size and magnet technology determines physics reach.
- Given magnet technology and physics goal determine required size.
- Given size and physics goal determines technology needed.

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Given current technology ($B_{\mathbf{max}} \sim 10$ T):

- What is the maximum energy of a particle accelerator around the Earth equator?
- and of an accelerator around the Solar System?

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Harmonic oscillator returns

Restoring force:

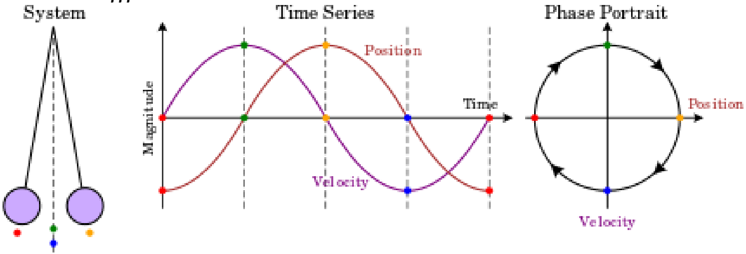
$$F = -kx \tag{9}$$

Equation of motion:

$$u'' = -\frac{k}{m} u \tag{10}$$

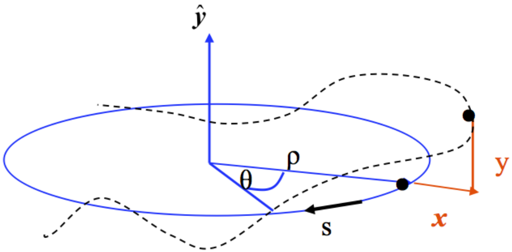
Solution:

$$u = a\cos(\omega t + \phi) \tag{11}$$



Frenet-Serret reference system

6D phase space: $(x, x', y, y', z, \delta)$



The coordinates are relative to the reference particle/trajectory.

Reference trajectory: $(0, 0, 0, 0, 0, 0)$

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{P_x}{P_z} \approx \frac{P_y}{P_0} \tag{12}$$

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$$\delta = \frac{\Delta P}{P_0} \tag{14}$$

Pay attention! This is not the set of canonical variables used in Hamilton's equation.

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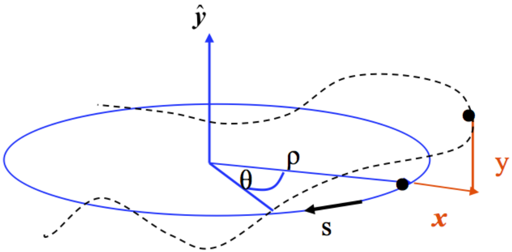

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Multipolar expansion

Any magnetic field can be descomposed by:

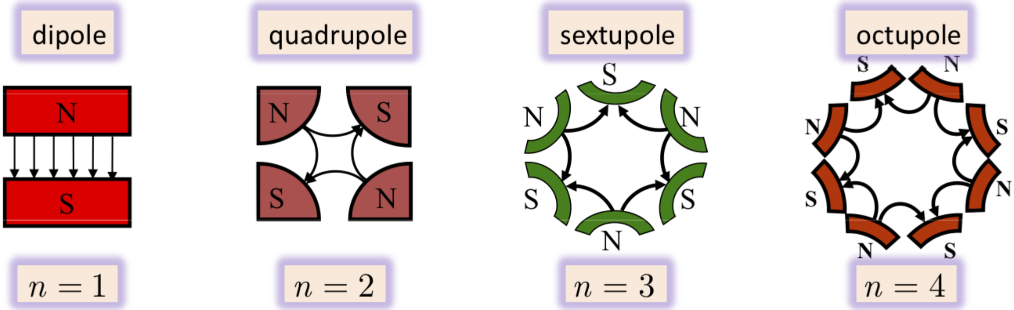
$$B_y + iB_x = \sum_{n=1}^{\infty} c_n (x + iy)^{n-1} \tag{15}$$

where

$$c_n = b_n + ia_n \tag{16}$$

- b_n are the normal coefficients.
- a_n are the skew coefficients.

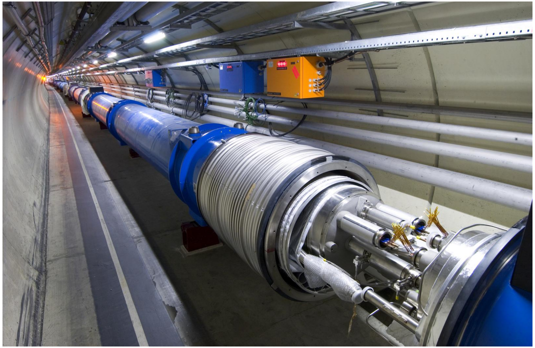
Magnet types[1]



[1] LHC magnet types (link)

Magnet types: Dipoles

- Two magnetic poles.
- Bend particle trajectory.
- Provide weak focusing.
- Not required in linear colliders.



Take home exercice: LHC dipoles

The LHC contains 1232 dipole magnets.
Each is 15 m long.

- **What is the length of the full circumference?**

Magnet types: Quadrupoles

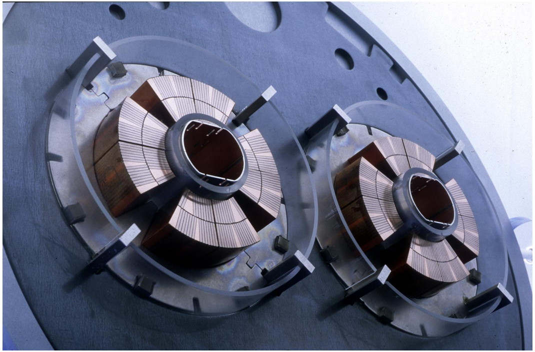
- Four poles.
- Focus the beam (horizontally or vertically).

Normalized focusing strength:

$$k = \frac{G}{P/q}[\mathrm{m}^{-2}] \tag{17}$$

$$k[\mathrm{m}^{-2}] \approx 0.3\frac{G[\mathrm{T/m}]}{P[\mathrm{GeV/c}]/q[\mathrm{e}]} \tag{18}$$

k is also known as k_1.



Magnet types: Quadrupoles

The focal lenght of a quadrupole is:

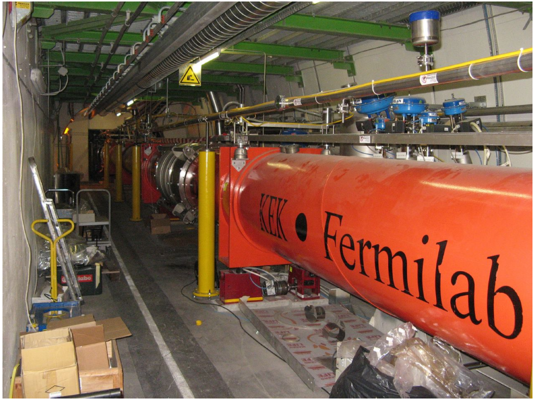
$$f = \frac{1}{k \cdot L}[\mathrm{m}] \qquad (19)$$

where L is the length of the quadrupole.

Example: Q1 LHC

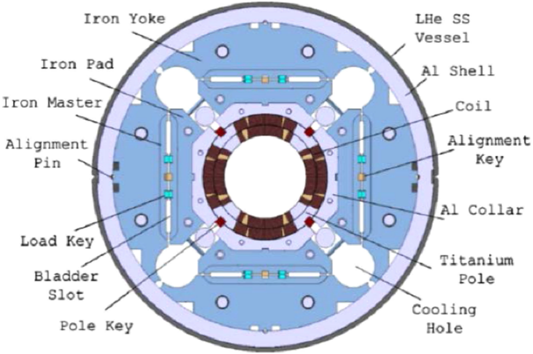
$$L = 6.37 \text{ m}$$

$$kL = -5.54 \cdot 10^{-2} \mathrm{m}^{-1}$$



New HL-LHC quadrupoles

- The LHC upgrade will require stronger focusing at IP1 and IP5.
- New quadrupole magnets with stronger grandients are required.
- Successful tests on short models.

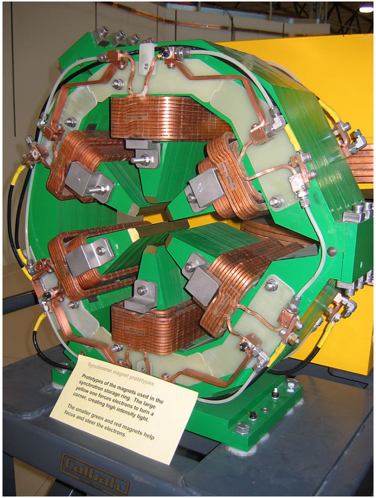


Magnet types: Sextupoles

- Six poles.
- Correct chromatic effects.
- Usually distributed in the arcs.
- Essential for accelerator performance.

Other multipoles

- Octupoles.
- Decapoles.
- Dodecapoles.



Hamiltonian approach[2]

Hamiltonian of a particle with mass m, charge q and momentum $\mathbf{p}$ in presence of an electromagnetic field $(\phi, \mathbf{A})$:

$$H = c\sqrt{(\mathbf{p} - q\mathbf{A})^2 + m^2c^2} + q\phi \qquad (20)$$

Hamilton equations:

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q} \qquad (21)$$

Equation (20) will be explained in future lectures and the derivation of the dynamics.

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Hill's equation

- We expect a solution in the form of a quasi harmonic oscillation: amplitude and phase will depend on hte position s in the ring.
- The linear motion (dipoles and quadrupoles) of particles can be described by:

$$u'' + K(s)u = 0 \tag{22}$$

where $K(s) = \left(\frac{1}{\rho^2} + k\right)$ is composed by linear fields (dipoles and quadrupoles).

Hill's equation

$$u'' + K(s)u = 0 \tag{23}$$

Some remarks

- $K(s)$ is a non-constant (s-dependent) restoring force.
- $K(s)$ is a periodic function with period $L \Rightarrow K(s+L) = K(s)$
- Usually in the vertical plane $1/\rho = 0$, therefore $K_y = k_y$.
- In a quadrupole $1/\rho = 0$ and $K_x = -K_y$ i.e. a horizontal focusing quadrupole defocuses in the vertical plane (and vice versa).
- In a bending magnet $k = 0$ so $K = 1/\rho^2$.

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Hill's equation: general solution

General solution

For $K(s) = K(s+L)$:

$$u = \sqrt{2J_u\beta_u(s)}\sin(\phi_u(s) - \phi_{u0}) \tag{24}$$

$$u' = -\frac{\sqrt{2J_u}}{\sqrt{\beta_u(s)}}\left[\cos(\phi_u(s) - \phi_{u0}) + \sin(\phi_u(s) - \phi_{u0})\right] \tag{25}$$

where $u = x, y$

Integration constants

- Action: J is a constant (related to emittance).
- Phase constant: ϕ_0.

- Beta-function: $\beta(s)$, periodic function.

$$\beta(s+L) = \beta(s) \tag{26}$$

- Phase advance: $\phi(s_0|s) = \int_{s_0}^{s} \frac{ds'}{\beta(s')}$

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Weak focusing and cyclotrons ($K = 0$)

In cyclotrons only dipole magnets are used.
But still, there is some focusing effect:

$$u'' + \left(\frac{1}{\rho^2} + k\right) u = 0 \xrightarrow[\boxed{k=0}]{} u'' + \frac{1}{\rho^2} u = 0 \tag{27}$$

- Small and low energy accelerators.
- Example: mass spectrometer.

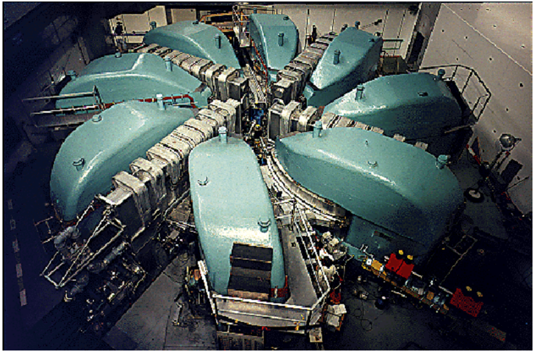

Figure: PSI cyclotron (250 MeV protons)

Strong focusing ($K > 0$)

Initial conditions: $x = x_0\,, \quad x' = x_0'$

Solution:

$$x(s) = x_0 \cos(\sqrt{K}s) + x_0' \frac{1}{\sqrt{K}} \sin(\sqrt{K}s) \tag{28}$$

$$x'(s) = -x_0 \sqrt{K} \sin(\sqrt{K}s) + x_0' \cos(\sqrt{K}s) \tag{29}$$

Matrix formalism:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}} \sin(\sqrt{K}L) \\ -\sqrt{K} \sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} \tag{30}$$

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Strong focusing ($K < 0$)

Initial conditions: $x = x_0\,, \quad x' = x'_0$

Solution:

$$x(s) = x_0 \cosh(\sqrt{|K|}s) + x'_0 \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|}s) \tag{31}$$

$$x'(s) = x_0 \sqrt{|K|} \sinh(\sqrt{|K|}s) + x'_0 \cosh(\sqrt{|K|}s) \tag{32}$$

Matrix formalism:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cosh(\sqrt{|K|}L) & \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|}L) \\ \sqrt{|K|} \sinh(\sqrt{|K|}L) & \cosh(\sqrt{|K|}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} \tag{33}$$

Strong focusing ($K < 0$)

Initial conditions: $x = x_0 , \quad x' = x'_0$
Solution:

$$x(s) = x_0 \cosh(\sqrt{|K|}s) + x'_0 \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|}s) \tag{31}$$

$$x'(s) = x_0 \sqrt{|K|} \sinh(\sqrt{|K|}s) + x'_0 \cosh(\sqrt{|K|}s) \tag{32}$$

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- Special relativity and magnetic properties.
- Reference system and Hill's equation (without derivation).
- Solution of linear homogeneus Hill's equations.
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- Matrix formulation for dipoles and quadrupoles.

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- Example: FODO.
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