Transverse Dynamics (part II)

JAI lectures - Michaelmas Term 2019

Hector Garcia-Morales

University of Oxford and CERN

hector.garcia.morales@cern.ch

Table of contents of part II of the course

General matrix formalism (linear systems)

The transformation between $x(s_0)$ and $x(s)$ can be expressed in a general way:

$$\mathbf{x}(s) = M(s|s_0)\mathbf{x}(s_0) \tag{1}$$

where the application $M(s|s_0)$ can be expressed in matrix formalism:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} C(s|s_0) & S(s|s_0) \\ C'(s|s_0) & S'(s|s_0) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} \tag{2}$$

where C and S are the cosine-like and sine-like functions and its derivatives C' and S' with respecto to s.

Element concatenation

The transfer matrices for different elements of the lattice can be concatenated to find the full transfer matrix between two points s_0 and s_n.

$$x(s_n) = M_n(s_n|s_{n-1})\dots M_3(s_3|s_2)M_2(s_2|s_1)M_1(s_1|s_0)x(s_0) \tag{3}$$

Remember to multiply matrices in reverse order.

Lattice design lecture

We will see more of how lattices are design in practice using MADX.

Thin lens approximation

When the focal length f of the quadrupole is much bigger than the length of the magnet itself L_q the transfer matrices can be rewriten:

$$M_{\text{foc}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \tag{4}$$

$$M_{\text{def}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \tag{5}$$

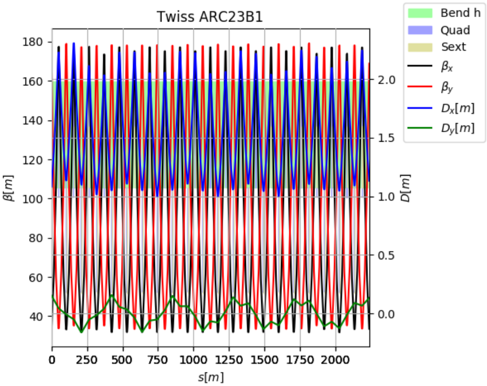
Take-home exercise

Derive the limits for the thin lens approximation and find the new matrices for quadrupoles in thin lens approximation.

Twiss parameters

$$u = \sqrt{2J_u\beta_u(s)}\sin(\phi_u(s) - \phi_{u0}) \quad (6)$$

$\beta_u(s)$ is a periodic function given by the focusing properties of the lattice.



$$\phi(s_0|s) = \int_{s_0}^{s} \frac{ds'}{\beta(s')} \quad (7)$$

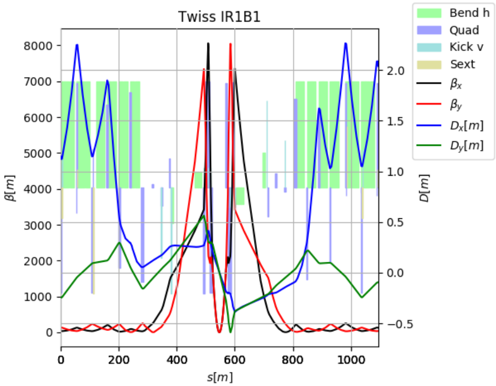
$$\alpha_u(s) = -\frac{1}{2}\beta'_u(s) \quad (8)$$

$$\gamma_u(s) = \frac{1+\alpha_u^2(s)}{\beta_u(s)} \quad (9)$$

Twiss parameters

$$u = \sqrt{2J_u\beta_u(s)}\,\sin(\phi_u(s) - \phi_{u0}) \qquad (10)$$

$\beta_u(s)$ is a periodic function given by the focusing properties of the lattice:



$$\phi(s_0|s) = \int_{s_0}^{s} \frac{ds'}{\beta(s')} \qquad (11)$$

$$\alpha_u(s) = -\frac{1}{2}\beta_u'(s) \qquad (12)$$

$$\gamma_u(s) = \frac{1 + \alpha_u^2(s)}{\beta_u(s)} \qquad (13)$$

Transfer matrix in terms of Twiss parameters

Express M in terms of initial and final Twiss parameters (instead of magnetic properties).

Taking $s(0) = s_0$ and $\phi(0) = 0$ we can take Eq. (**??**) to obtain,

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(\cos\phi_s + \alpha_0 \sin\phi_0) & \sqrt{\beta_s\beta_0}\sin\phi_s \\ \frac{(\alpha_0-\alpha_s)\cos\phi_s-(1+\alpha_0\alpha_s)\sin\phi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}(\cos\phi_s - \alpha_s\sin\phi_s) \end{pmatrix} \tag{14}$$

This expression is very useful when Twiss parameters are known at two different locations.

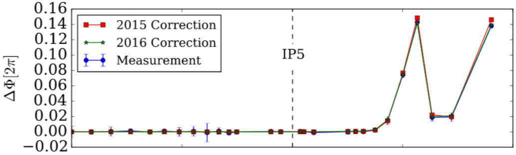
How do we mesure β and ϕ?

Phase ϕ

- Harmonic analysis of orbit oscillations.

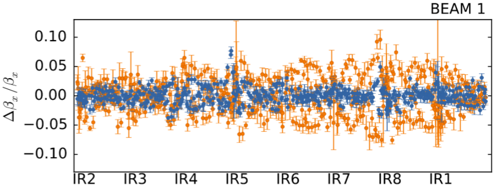
Betatron tune Q

- FFT of transverse beam position over many turns.



β-function

- β from phase.
- β from amplitude.
- K-modulation.



One matrix to rule them all: the one-turn matrix

If we take previous matrix (Eq. [14]) and consider the case of one full turn (i.e. $\beta_s = \beta_0$, $\alpha_s = \alpha_0$) the matrix simplifies

$$\mathcal{M} = \begin{pmatrix} \cos\phi_L + \alpha_s \sin\phi_L & \beta_s \sin\phi_L \\ \gamma_s \sin\phi_L & \cos\phi_L - \alpha_s \sin\phi_L \end{pmatrix} \tag{15}$$

remember that the tune is the phase advance in units of 2π:

$$Q = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)} = \frac{\phi}{2\pi} \tag{16}$$

then, the one turn matrix in terms of tune is,

$$\mathcal{M} = \begin{pmatrix} \cos(2\pi Q) + \alpha_s \sin(2\pi Q) & \beta_s \sin(2\pi Q) \\ \gamma_s \sin(2\pi Q) & \cos(2\pi Q) - \alpha_s \sin(2\pi Q) \end{pmatrix} \tag{17}$$

Properties of transfer matrices

1. Phase space area preservation:

$$\det(M) = 1 \tag{18}$$

2. Motion is stable over $N \to \infty$ turns if:

$$\text{trace}(M) \leq 2 \tag{19}$$

Stability condition (derivation)

Let's consider a general transfer matrix M for a periodic system:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we want the motion to be stable after $N \to \infty$ turns:

$$x_N = M^N x_0$$

How can we compute M^N?

Stability condition (derivation)

$$x_N = M^N x_0$$

- $\det(M) = ad - bc = 1$
- $\text{trace}(M) = a + d$

If we diagonalise M, we can rewrite it as:

$$M = U \cdot \begin{pmatrix} \lambda_1 & 0 \\ & \lambda_2 \end{pmatrix} \cdot U^T$$

where U is some unitary matrix and λ_1 and λ_2 are the eigenvalues.

Stability condition (derivation)

After N turns:

$$M^N = U \cdot \begin{pmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{pmatrix} \cdot U^T$$

Given that $\det(M) = 1$:

$$\lambda_1 \lambda_2 = 1 \rightarrow \lambda_{1,2} = \exp^{\pm ix}$$

To have stable motion $x \in \mathbb{R}$. To find the eigenvalues we use the characteristic equation:

$$\det(M - \lambda \mathbb{I}) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = 0$$

Solve it:

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

$$\lambda^2 - \text{trace}(M)\lambda + 1 = 0$$

$$\text{trace}(M) = \lambda + 1/\lambda = \exp^{ix} + \exp^{-ix} =$$

$$= 2\cos(x)$$

Since $x \in \mathbb{R}$:

$$|\text{trace}(M)| \leq 2$$

Twiss transport matrix and Twiss parameters evolution

Instead of transporting coordinates x and x' we can transport Twiss parameters (β, α, γ).

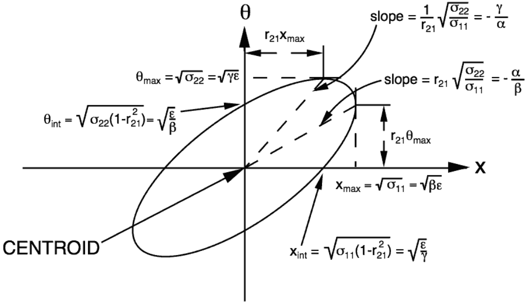
$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2CS & S^2 \\ -CC' & CS' + S'C & -SS' \\ C'^2 & -2C'S' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s_0} \tag{20}$$

- Given the twiss parameters at any point in the lattice we can transform them and compute their values at any other point in the ring.
- the transfer matrix is given by the focusing properties of the lattice elements, the same matrix elements to compute single particle trajectories.

Phase Space properties

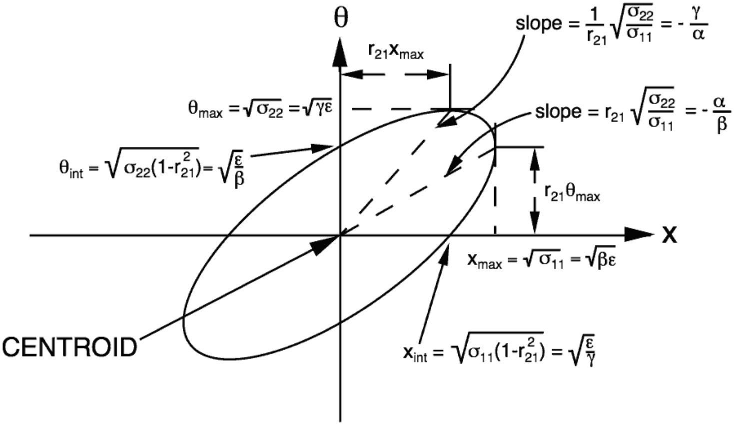
- Area is preserved.
- Beam size: $\sigma_u = \sqrt{J_u \beta_u}$.
- When σ_u is large σ'_u is small.
- In a β minima/maxima $\alpha = 0$ and the ellipse is not tilted.

Phase space ellipse.



$$J = \gamma_x(s)x(s)^2 + 2\alpha_x(s)x(s)x'(s) + \beta(s)x'(s)^2 \tag{21}$$

Phase Space properties

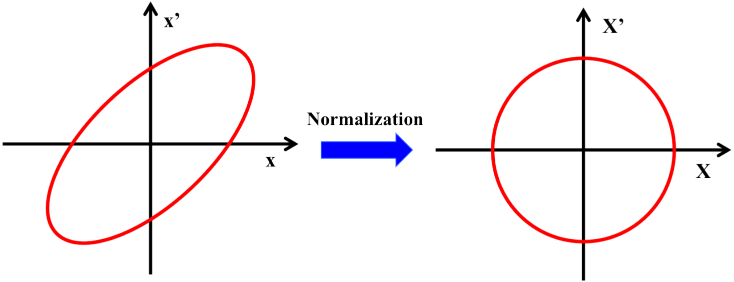


$$J = \gamma_x(s)x(s)^2 + 2\alpha_x(s)x(s)x'(s) + \beta(s)x'(s)^2$$

Normalized phase space

Can we use another reference frame so it is simpler to describe the system?

$$\mathcal{M} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\cos\phi & \sin\phi \end{pmatrix} \tag{22}$$



For linear systems is fine but it gets much more complex when non-linearities are included (we will see more details in the tutorial).

Beam emittance: single particle definition.

The geometric emittance is a constant of motion only if the beam energy is preserved (conservative system). This quantity is related to the action J that appeared in the solution of the Hill's equation.

Normalized emittance takes into account beam energy. It is a constant of motion even if energy is not constant:

$$\epsilon_n \equiv \beta_{\text{rel}} \gamma_{\text{rel}} \epsilon \tag{23}$$

The beam size at any location of the lattice is given by,

$$\sigma = \sqrt{\epsilon \beta} \tag{24}$$

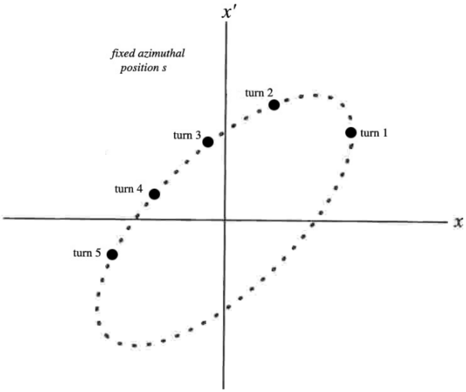
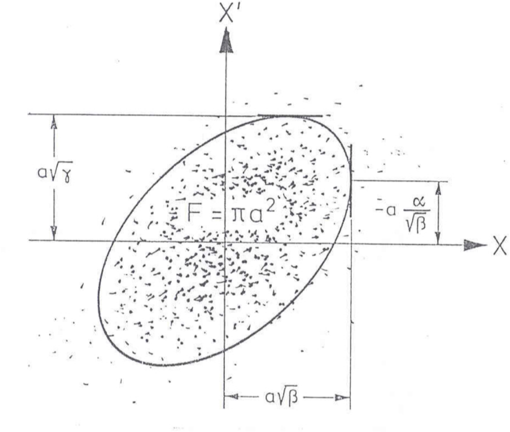


Figure: Single particle emittance

Beam emittance: statistical definition

The beam is composed of particles distributed in phase space.



Emittance is defined by

$$\epsilon_{\text{rms}} = \sqrt{\sigma_u^2 \sigma_{u'}^2 - \sigma_{uu'}^2} \qquad (25)$$

The rms emittance of a ring in phase space, i.e. particles uniformly distributed in phase ϕ coordinate at a fixed action J, is

$$\epsilon_{\text{rms}} = J.$$

If the accelerator is composed of linear elements and no dissipative forces act ϵ_{rms} is invariant.

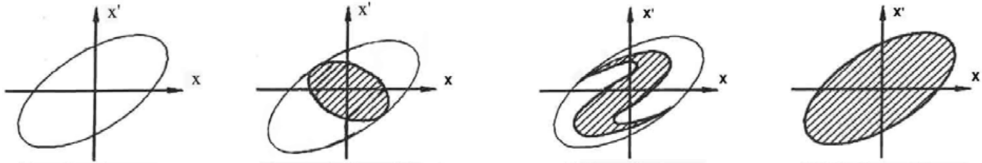
Beam emittance: phenomenology

What determines beam emittance?

- Amount of particles.
- Injectors manipulations.
- Beam transfer efficiency.

Sources of emittance growth

- Intrabeam scattering.
- Optics mismatch.
- Beam-gas scattering.
- Beam-beam interaction.
- Betatron resonances.
- Ground motion and PS ripples.



(a) machine phase space (b) unmatched beam injected (c) filamenting beam (d) fully filamented beam

Liouville's theorem and symplectic condition

The Liouville equation describes the time evolution of the phase space distribution function, ρ.

Liouville's theorem

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \sum_{i=1}^{N} \left(\frac{\partial \rho}{\partial q_i} \dot{q}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i \right) = 0 \tag{26}$$

where (q_i, p_i) are the canonical coordinates and the system is Hamiltonian.

Symplectic condition

Lioville's theorem $\Rightarrow$ invariant volume in phase space.

$$M^T J M = J \tag{27}$$

where J is the 6D symplectic matrix

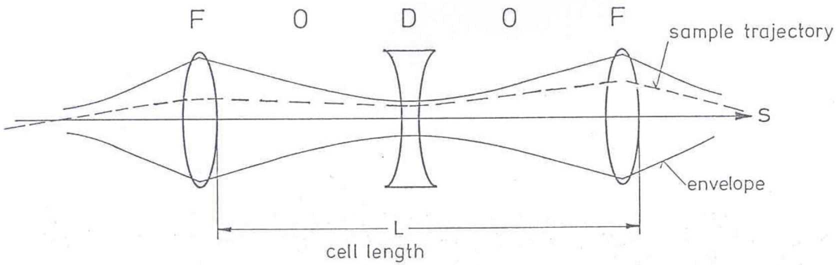
$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{28}$$

Take home exercice

Prove that Eq. (27) holds for the matrices described above.

FODO lattice

The FODO lattice is a sequence of a Focusing magnet (F), a Drift space (O), a Defocusing magnet (D) and a second drift space (0).



$$M_{\mathrm{FODO}} = M_0 M_{\mathrm{def}} M_0 M_{\mathrm{foc}} = \begin{pmatrix} 1 + \frac{L}{2f} & L + \frac{L^2}{4f} \\ -\frac{L}{2f^2} & 1 - \frac{L}{2f} - \frac{L^2}{4f^2} \end{pmatrix} \tag{29}$$

FODO lattice

Take-home exercise:

Prove that the stability condition for a FODO lattice is given by:

$$f > \frac{L}{4} \tag{30}$$

What if...

We take the FODO lattice and replace drifts by bending magnets?

You will see this in next lectures...

The end of the ideal world...

So far, we have considered ideal linear systems.

- Dispersion.
- Chromaticity.
- Misalignments.
- Magnetic errors.

Some of these topics will be covered in next lectures.

Dispersion

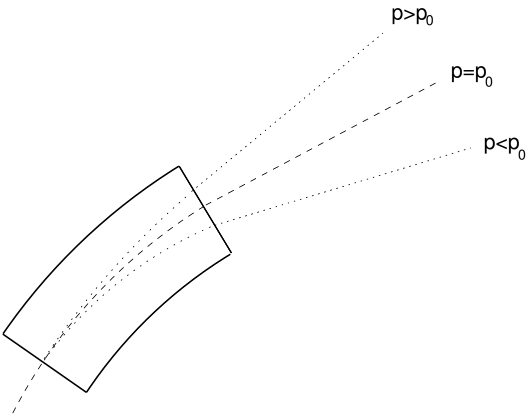
What if particles within a bunch have different momenta?
Remember beam rigidity:

$$B\rho = \frac{P}{q} \tag{31}$$

Orbit:

$$x(s) = D(s)\frac{\Delta P}{P_0} \tag{32}$$

where $D(s)$ is the dispersion function, an intrinsic propertie of the dipole magnets.



Dispersion

Inhomogeneus Hill's equation:

$$u'' + \left(\frac{1}{\rho^2} + k\right) u = \frac{1}{\rho}\frac{\Delta P}{P_0} \tag{33}$$

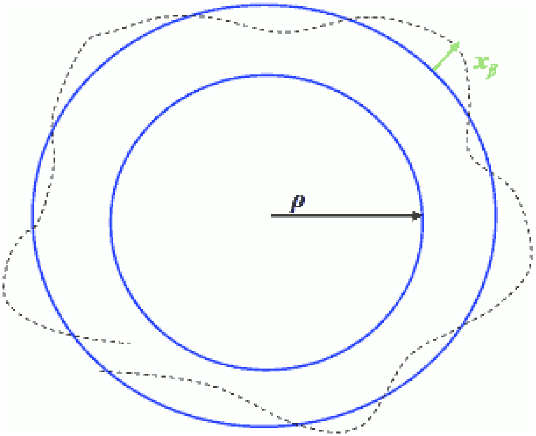
Particle trajectory:

$$u(s) = u_\beta(s) + u_D(s) = \tag{34}$$

$$= u_\beta(s) + D(s)\frac{\Delta P}{P} \tag{35}$$

where $D(s)$ is the solution of:

$$D''(s) + K(s)D(s) = \frac{1}{\rho} \tag{36}$$



Dispersion

Solution:

$$u(s) = C(s)u_0 + S(s)u_0' + D(s)\frac{\Delta P}{P_0} \quad (37)$$

so this can be added to the transfer matrix representation:

$$M = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \quad (38)$$

Dipole transfer matrix:

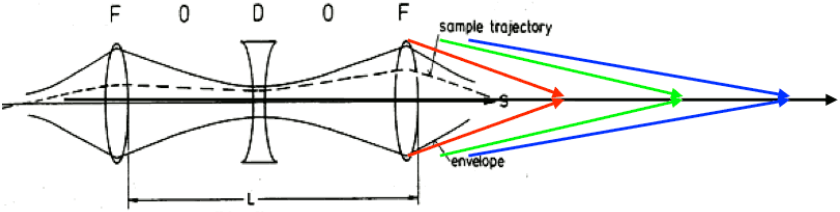
$$\begin{pmatrix} \cos(\frac{L}{\rho}) & \rho\sin(\frac{L}{\rho}) & \rho\left(1-\cos(\frac{L}{\rho})\right) \\ -\frac{1}{\rho}\sin(\frac{L}{\rho}) & \cos(\frac{L}{\rho}) & \sin(\frac{L}{\rho}) \\ 0 & 0 & 1 \end{pmatrix} \quad (39)$$

Quadrupole;

$$\begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) & 0 \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (40)$$

Chromaticity

All particles do not have exactly the same energy. Therefore, according to Eq. (??) they focalize at different points.

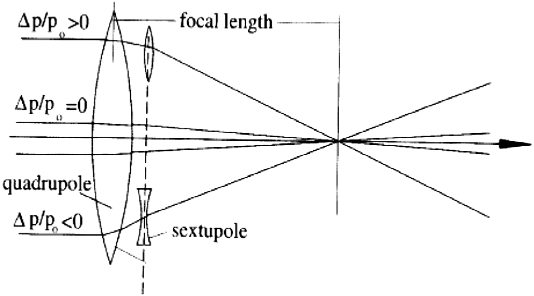


This defines chromaticity:

$$\xi = -\frac{1}{4\pi} \oint \beta(s) k(s) ds \qquad (41)$$

How to correct chromaticity

Sextuples, through a non-linear magnetic field, corrects the effect of energy spread and focuses particles at a single location.



- Located in dispersive regions
- Usually in arcs.
- Sextupole families.

Now is when the party starts...

- Sexupoles introduce non-linear fields...
- ... i.e. they induce non-linear motion.
- resonances, tune shifts, chaotic motion...

Chromaticity correction

- Chromatic aberrations must be compensated in both planes.

$$\xi_x = -\frac{1}{4\pi}\oint \beta_x(s)[k(s) - S_F D_x(s) + S_D D_x(s)]ds \tag{42}$$

$$\xi_y = -\frac{1}{4\pi}\oint \beta_y(s)[-k(s) + S_F D_x(s) - S_D D_x(s)]ds \tag{43}$$

- To minimise sextupole strength they must be located near quadrupoles where βD are large.
- For optimal independent correction S_F should be located where β_x/β_y is large and S_D where β_y/β_x is large.

Recap.

- Optics functions and Twiss parameters.
- Phase space and emittance.
- Example: FODO lattice.
- Dispersion and chromaticity.

What do we do with all this?

- We have covered the basic aspects of linear transverse dynamics.
- I skipped almost all derivations. You can follow references.
- Are you familiar with Jupyter environtment?
- Lattice design and tutorials in a couple of weeks will give a more complete picture.
- Now you are ready to take next lectures to become an accelerator phyisics experts.

Thank you very much!