Hamiltonian Dynamics

Lecture 2

David Kelliher

RAL

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Bibliography

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- Theory and Design of Charged Particle Beams - Reiser
- Accelerator Physics - Lee
- Particle Accelerator Physics II - Wiedemann
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- Beam Dynamics in High Energy Particle Accelerators - Wolski

Content

Lecture 1

- Comparison of Newtonian, Lagrangian and Hamiltonian approaches.
- Hamilton's equations, canonical transformation.
- Hamilton-Jacobi theory.

Lecture 2

- Transverse dynamics.
- Phase space, Liouville's theorem, symplectic motion.
- Perturbation theory.

Overview

- Frenet-Serret coordinates
- Change of independent variable
- Multipole magnets
- 2D linear Hamiltonian
- Action-angle transformation
- Normalised coordinates
- Sextupole resonance
- Potential in action angle coordinates

Accelerator case

Consider a circulating accelerator with particles moving around the ring at relativistic velocities.

- Start with the Hamiltonian for a relativistic particle in an electromagnetic field.
- Transform into conform into convenient coordinates (Fresnet-Serret).
- Change the independent variable from time to coordinate s.
- Convert to small dynamic variables (normalised transverse momenta and energy deviation).

Relativistic Hamiltonian

Starting with Einstein's equation in Special Relativity

$$E^2 = \mathbf{p}^2 c^2 + m^2 c^4 \tag{1}$$

we write the Hamiltonian (H = T + V, assume V=0)

$$H = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4} \tag{2}$$

where the conjugate momentum $\mathbf{p}$ equals the mechanical momentum in the absence of an electromagnetic field.

General electromagnetic fields

The Lagrangian in general EM fields $U(x, \dot{x}, t) = e(\phi - \boldsymbol{v} \cdot \boldsymbol{A})$ is given by

$$L(x, \dot{x}, t) = -mc^2\sqrt{1 - \beta^2} - e\phi + e\boldsymbol{v} \cdot \boldsymbol{A}. \tag{3}$$

the conjugate momentum is

$$P_i = \frac{\partial L}{\partial \dot{x}_i} = \frac{m\dot{x}_i}{\sqrt{1 - \beta^2}} + eA_i \tag{4}$$

i.e. the field contributes to the momentum.
The Hamiltonian

$$H(q, P, t) = \sum_i P_i \dot{q}_i - L = \frac{mc^2}{\sqrt{1 - \beta^2}} + e\phi. \tag{5}$$

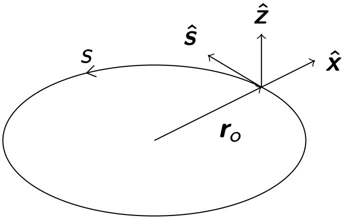
As before use

$$\frac{mc^2}{\sqrt{1-\beta^2}} = \gamma mc^2 = c\sqrt{m^2c^2 + \boldsymbol{p}^2}$$

to obtain

$$H(q, P, t) = c\sqrt{m^2c^2 + (\boldsymbol{P} - e\boldsymbol{A})^2} + e\phi. \tag{6}$$

Frenet-Serret coordinates



For the transverse plane we can specify motion with respect to a reference orbit we label $\boldsymbol{r}_0(s)$. s is the arc length along the closed orbit from some reference point.

Then the tangential unit vector

$$\hat{\boldsymbol{s}}(s) = \frac{\mathrm{d}\boldsymbol{r}_0(s)}{\mathrm{d}s}, \tag{7}$$

The principle unit normal vector, perpendicular to the tangent vector

$$\hat{\boldsymbol{x}}(s) = -\rho(s)\frac{\mathrm{d}\hat{\boldsymbol{s}}(s)}{\mathrm{d}s} \tag{8}$$

where $\rho(s)$ defines the local radius of curvature.

The unit binormal vector, orthogonal to the transverse plane

$$\hat{\boldsymbol{z}}(s) = \hat{\boldsymbol{x}}(s) \times \hat{\boldsymbol{s}}(s). \tag{9}$$

These vectors $\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}}, \hat{\boldsymbol{s}}$ form the orthonormal basis for the right handed Frenet-Serret curvilinear coordinate system. It can be shown that

$$\hat{\boldsymbol{x}}'(s) = \frac{1}{\rho(s)}\hat{\boldsymbol{s}}(s) + \tau(s)\hat{\boldsymbol{z}}(s) \tag{10}$$

$$\hat{\boldsymbol{z}}'(s) = -\tau(s)\hat{\boldsymbol{x}}(s) \tag{11}$$

where $\tau(s)$ is the torsion of the curve. Here we consider cases where $\tau = 0$, the curves are planar, so the particle orbits are

$$\boldsymbol{r}(s) = \boldsymbol{r}_0(s) + x\hat{\boldsymbol{x}}(s) + z\hat{\boldsymbol{z}}(s). \tag{12}$$

It can be shown Hamiltonian becomes

$$H(s,x,z,p_s,p_x,p_z,t) = c\sqrt{m_o^2c^2 + \frac{(p_s - eA_s)^2}{\left(1+\frac{x}{\rho}\right)^2} + (p_x - eA_x)^2 + (p_z - eA_z)^2} + e\phi, \quad (13)$$

Note: the Hamiltonian for a straight beamline is obtained in the limit $x/\rho \to 0$. The equations of motion follow

$$\dot{s} = \frac{\partial H}{\partial p_s}, \quad \dot{x} = \frac{\partial H}{\partial p_x}, \quad \dot{z} = \frac{\partial H}{\partial p_z}$$
$$\dot{p}_s = -\frac{\partial H}{\partial s}, \quad \dot{p}_x = -\frac{\partial H}{\partial x}, \quad \dot{p}_z = -\frac{\partial H}{\partial z}. \quad (14)$$

Change of independent variable

We would like to change independent variable from t to s. The action with time as the independent variable is

$$S = \int_{t_0}^{t_1} (p_x \dot{x} + p_z \dot{z} + p_s \dot{s} - H)\, dt \tag{15}$$

Changing the integration variable to path length s as the action becomes

$$S = \int_{z_0}^{z_1} \left(p_x x' + p_z z' - H t' + p_s\right) dz \tag{16}$$

Comparing the two we can see that we should take as our canonical variables

$$(x, p_x), \quad (y, p_y), \quad (-t, H) \tag{17}$$

and take as our Hamiltonian $H_1 = -p_s$.

Our new Hamiltonian is $H_1(t, x, z, -H, p_x, p_z, s) = -p_s$.
Rearrange in terms of p_s

$$\frac{(p_s - eA_s)^2}{\left(1 + \frac{x}{\rho}\right)^2} = \frac{1}{c^2}(H - e\phi)^2 - m^2c^2 - (p_x - eA_x)^2 - (p_z - eA_z)^2 \quad (18)$$

It follows

$$p_s = eA_s + \left(1 + \frac{x}{\rho}\right)\sqrt{\frac{1}{c^2}(H - e\phi)^2 - m^2c^2 - (p_x - eA_x)^2 - (p_z - eA_z)^2} \quad (19)$$

Then our new canonical equations in terms of s are

$$t' = \frac{\partial p_s}{\partial H}, \quad x' = -\frac{\partial p_s}{\partial p_x}, \quad z' = -\frac{\partial p_s}{\partial p_z}$$
$$H' = -\frac{\partial p_s}{\partial t}, \quad p_x' = \frac{\partial p_s}{\partial s}, \quad p_z' = \frac{\partial p_s}{\partial z}. \tag{20}$$

$$H_1 = -p_s =$$
$$-eA_s - \left(1 + \frac{x}{\rho}\right)\sqrt{\frac{1}{c^2}(H - e\phi)^2 - m^2c^2 - (p_x - eA_x)^2 - (p_z - eA_z)^2} \tag{21}$$

The Hamiltonian H_1 is time independent if **A** are ϕ are fixed in time.

Reference momentum

It makes sense to construct a Hamiltonian with reference to a reference momentum P_0. This allows simplification in the case of small momentum spread.
We end up with

$$\tilde{H} = -ea_s - \left(1 + \frac{x}{\rho}\right)\sqrt{\frac{(E - e\phi)^2}{P_0^2 c^2} - \frac{m^2 c^2}{P_0} - (\tilde{p}_x - ea_x)^2 - (\tilde{p}_z - ea_z)^2} \tag{22}$$

where

$$p_i \to \tilde{p}_i = \frac{p_i}{P_0}, \quad H_1 \to \tilde{H} = \frac{H_1}{P_0}, \quad A_i \to \mathbf{a} = e\frac{A_i}{P_0} \tag{23}$$

Change of longitudinal coordinates

Define new longitudinal coordinates with respect to the reference particle.

$$\delta_E = \frac{E}{P_0 c} - \frac{1}{\beta_0}, \quad S = \frac{s}{\beta_0} - ct \tag{24}$$

where δ_E is known as the energy deviation. Invoking the generating function

$$F_2(x, P_x, z, P_z, -t, \delta_E, s) = xP_x + xP_z + \left(\frac{s}{\beta_0} - ct\right)\left(\frac{1}{\beta} + \delta_E\right) \tag{25}$$

We find that the transverse variables are unchanged and the new Hamiltonian $H = \tilde{H} + \frac{\partial F_2}{\partial s}$ can be, after some manipulation, shown to be

$$\begin{aligned} H &= -(1+hx)\sqrt{(\frac{1}{\beta_0} + \delta_E - \frac{e\phi}{P_0 c})^2 - (\tilde{p}_x - ea_x)^2 - (\tilde{p}_z - ea_z)^2 - \frac{1}{\beta_0^2\gamma_0^2}} \\ &\quad -(1+hx)a_s + \frac{\delta_E}{\beta_0} \end{aligned} \tag{26}$$

where $h = \frac{1}{\rho}$ is the curvature.
The Hamiltonian for each element in an accelerator can be found by substituting the corresponding potential a_s or ϕ.

Multipole magnets

The vector potential for a straight multipole magnet with axial symmetry is

$$A_x = 0, \quad A_z = 0, \quad A_l = -\mathcal{R}\sum_{n=1}^{\infty}(b_n + ia_n)\frac{(x+iz)^n}{nr_0^{n-1}} \tag{27}$$

giving magnetic field ($\mathbf{B} = \nabla \times \mathbf{A}$)

$$B_z + iB_x = -\frac{\partial A_l}{\partial x} + i\frac{\partial A_l}{\partial y} = \mathcal{R}\sum_{n=1}^{\infty}(b_n + ia_n)\frac{(x+iz)^{n-1}}{r_0} \tag{28}$$

Curl in curvilinear coordinates

The curl in curvilinear coordinates is

$$B_x = [\nabla \times A]_x = \frac{\partial A_s}{\partial z} - \frac{1}{1+hx}\frac{\partial A_z}{\partial s} \tag{29}$$

$$B_z = [\nabla \times A]_z = \frac{1}{1+hx}\frac{\partial A_x}{\partial s} - \frac{h}{1+hx}A_s - \frac{\partial A_s}{\partial x} \tag{30}$$

$$B_s = [\nabla \times A]_s = \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \tag{31}$$

Dipole magnet (n=1)

Starting with the following vector potential components

$$A_x = 0, \quad A_z = 0, \quad A_s = -B_0 \left(x - \frac{hx^2}{2(1+hx)} \right) \tag{32}$$

using the curl equations one finds the field components

$$B_x = 0, \quad B_z = B_0, \quad B_s = 0 \tag{33}$$

Dipole magnet: Hamiltonian

Using the vector potential for a dipole, the following Hamiltonian results

$$
\begin{aligned}
H &= -(1+hx)\sqrt{\left(\frac{1}{\beta_0}+\delta_E\right)^2 - \tilde{p}_x^2 - \tilde{p}_z^2 - \frac{1}{\beta_0^2\gamma_0^2}} \\
&\quad + (1+hx)\,k_0\left(x - \frac{hx^2}{2(1+hx)}\right) + \frac{\delta_E}{\beta_0} \qquad (34)
\end{aligned}
$$

where the normalised dipole field strength is $k_0 = \frac{e}{P_0} B_0$.

As long as the dynamical variables are small the Hamiltonian can be expanded to second order as

$$H_2 = \frac{p_x^2}{2} + \frac{p_z^2}{2} + (k_0 - h)x + \frac{hk_0x^2}{2} - \frac{hx\delta_E}{\beta_0} + \frac{\delta_E^2}{2\beta_0^2\gamma_0^2} \qquad (35)$$

The following observations can be made:

- The $(k_0 - h)x$ term results in a change in p_x. It is zero if $k_0 = h$, i.e. when the dipole field bends with the design curvature.
- The $\frac{1}{2}hk_0x^2$ term is the weak focusing term.
- The $\frac{hx\delta_E}{\beta_0}$ term represents first order dispersion.

From Hamilton equations, and setting $k_0 = h$, we find for the transverse coordinates

$$
\begin{aligned}
x(s) &= x(0)\cos\omega s + p_x(0)\frac{sin\omega s}{\omega} + \delta_E(0)\frac{h}{\beta_0}\left(\frac{1-\cos\omega s}{\omega^2}\right) && (36)\\
p_x(s) &= -x(0)\omega\sin\omega s + p_x(0)cos\omega s + \delta_E(0)\frac{h}{\beta_0}\frac{\sin\omega s}{\omega} && (37)\\
z(s) &= z(0) + p_z(0)s && (38)\\
p_z(s) &= p_z(0) && (39)
\end{aligned}
$$

where $\omega = \sqrt{hk}$ and $(x(0), p_x(0), z(0), p_z(0))$ are the initial transverse coordinates. Note the oscillatory terms in the horizontal plane - the effect of weak focusing.

Dipole magnet: Transfer Matrix

It is convenient to express the map of a dipole magnet in the form of a transfer matrix

$$M = \begin{pmatrix} \cos\omega L & \frac{\sin\omega L}{\omega} & 0 & 0 & 0 & \frac{1-\cos\omega L}{\omega\beta_0} \\ -\omega\sin\omega L & \cos\omega L & 0 & 0 & 0 & \frac{\sin\omega L}{\beta_0} \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{\sin\omega L}{\beta_0} & -\frac{1-\cos\omega L}{\omega\beta_0} & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} - \frac{\omega L-\sin\omega L}{\omega\beta_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{40}$$

where L is the dipole length.

Quadrupole magnet (n=2)

Starting with the following vector potential components

$$A_x = 0, \quad A_z = 0, \quad A_s = -\frac{b_2}{2r_0}\left(x^2 - z^2\right) \tag{41}$$

using the curl equations one finds the field components

$$B_x = \frac{b_2}{r_0}z, \quad B_z = \frac{b_2}{r_0}x, \quad B_s = 0 \tag{42}$$

leading to Hamiltonian

$$H = \frac{\delta_E}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta_E\right)^2 - p_x^2 - p_z^2 - \frac{1}{\beta_0^2\gamma_0^2}} + \frac{1}{2}k_1\left(x^2 - z^2\right) \tag{43}$$

where the normalised quadrupole gradient $k_1 = \frac{qb_2}{P_0 r_0}$.

To second order the Hamiltonian becomes

$$H_2 = \frac{p_x^2}{2} + \frac{p_z^2}{2} + \frac{k_1 x^2}{2} - \frac{k_1 z^2}{2} + \frac{1}{2\beta_0^2\gamma_0^2}\delta_E^2 \tag{44}$$

If $k_1 > 0$ this leads to focusing in x and defocusing in z. The transfer matrix for a "focusing" quadrupole follows

$$M = \begin{pmatrix} \cos\omega L & \frac{\sin\omega L}{\omega} & 0 & 0 & 0 & 0 \\ -\omega\sin\omega L & \cos\omega L & 0 & 0 & 0 & \frac{\sin\omega L}{\beta_0} \\ 0 & 0 & \cosh\omega L & \frac{\sinh\omega L}{\omega} & 0 & 0 \\ 0 & 0 & \omega\sinh\omega L & \cosh\omega L & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{45}$$

where $\omega = \sqrt{k_1}$.

Points of interest

If you assume that ϕ and $\boldsymbol{A}$ are independent of time then H is conserved.
If H is an invariant of the motion, then it is cyclic and t is an ignorable coordinate. The number of degrees of freedom reduces from 3 to 2.
The 6D phase space is conserved by Liouville's theorem. If the coordinates can be separated then the phase spaces of the canonical pairs (x, p_x) and (z, p_z) are mutually conserved.
We can approximate magnets by piecewise constant elements with $A_x = A_z = 0$.
We can consider only multipole perturbations.

Symplectic mapping

We can define a map that updates the system over some increment

$$(q_{i+1}, p_{i+1}) = M(q_i, p_i) \tag{46}$$

The map is symplectic if

$$M^T \Omega M = J \tag{47}$$

where

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{48}$$

Lie transformations

Symplectic maps can be created using Lie transformations.

$$\mathbf{z}(t) = e^{t:H:}\mathbf{z}_0 \tag{49}$$

with map $M = e^{:H:}$. A one turn map can be obtained from the composition of the maps of each element.

$$M = e^{:f2:}e^{:f3:}e^{:f4:}\dots \tag{50}$$

where the generator f_k is a power series of k-th order. Note: since all exponential maps are symplectic, we can truncate the factorised map at any order k and it remains symplectic (Dragt-Finn factorisation theorem).

1D Example

Starting with the generator

$$f = -\frac{L}{2}(kx^2 + p^2) \tag{51}$$

the transformation is

$$\begin{aligned} e^{:f:}x &= e^{:-\frac{L}{2}(kx^2+p^2):}x \\ e^{:f:}p_x &= e^{:-\frac{L}{2}(kx^2+p^2):}p_x \end{aligned}$$

Recall

$$e^{:f:}g = \sum_{n=0}^{n=\infty} \frac{:f:^n}{n!} g \tag{52}$$

We obtain

$$
\begin{aligned}
e^{:-\frac{L}{2}(kx^2+p^2):}x &= \sum_{n=0}^{\infty}\left(\frac{(-kL^2)^{2n}}{(2n)!}\cdot x + L\frac{(-kL^2)^{2n+1}}{(2n+1)!}\cdot p\right) \\
e^{:-\frac{L}{2}(kx^2+p^2):}p &= \sum_{n=0}^{\infty}\left(\frac{(-kL^2)^{2n}}{(2n)!}\cdot p - k\frac{(-kL^2)^{2n+1}}{(2n+1)!}\cdot x\right)
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
e^{:f:}x &= x\cos\sqrt{k}L + \frac{p_x}{\sqrt{k}}\sin\sqrt{k}L \\
e^{:f:}p_x &= -\sqrt{k}x\sin\sqrt{k}L + p\cos\sqrt{k}L
\end{aligned}
$$

It is clear that $e^{:f:}$ is the transfer matrix of a quadrupole.

Symplectic integration of a Harmonic oscillator

The Hamiltonian for a harmonic oscillator in one dimension is

$$H(p, q) = \frac{1}{2}\left(p^2 + q^2\right) \tag{53}$$

where the potential energy is $U(q) = \frac{q^2}{2}$. The equations of motion are

$$\begin{aligned}\dot{q} &= p \\ \dot{p} &= q\end{aligned}$$

The exact evolution is given by

$$\begin{pmatrix} q(\tau) \\ p(\tau) \end{pmatrix} = \begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix} \begin{pmatrix} q(0) \\ p(0) \end{pmatrix} \tag{54}$$

Note the symplectic condition is met

$$\begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (55)$$

This condition must be satisfied to preserve the phase space volume under evolution (Liouville). Next, expand the cosine and sine to first order

$$\begin{pmatrix} q(\tau) \\ p(\tau) \end{pmatrix} = \begin{pmatrix} 1 & \tau \\ -\tau & 1 \end{pmatrix} \begin{pmatrix} q(0) \\ p(0) \end{pmatrix} \quad (56)$$

The symplectic condition is not satisfied in this case and furthermore

$$\left| det \begin{pmatrix} 1 & \tau \\ -\tau & 1 \end{pmatrix} \right| = 1 + \tau^2 \quad (57)$$

The energy after one timestep

$$H_{integrated} = \frac{1}{2}\left(p(\tau)^2 + q(\tau)^2\right) = \frac{1}{2}(1+\tau^2)\left(p^2+q^2\right) \tag{58}$$

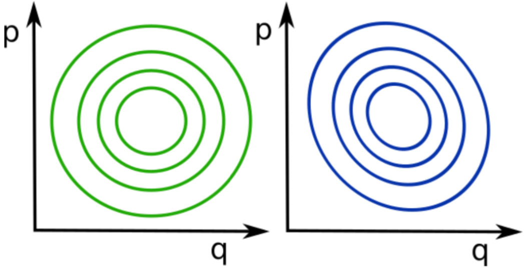
The increase in energy will cause the trajectory to spiral outwards. A *symplectic* integration scheme (one the preserves phase space volume) can be created as follows

$$\begin{pmatrix} q(\tau) \\ p(\tau) \end{pmatrix} = \begin{pmatrix} 1 & \tau \\ -\tau & 1-\tau^2 \end{pmatrix} \begin{pmatrix} q(0) \\ p(0) \end{pmatrix} \tag{59}$$

Although the symplectic condition is met we find after one time step

$$H_{integrated} = \frac{1}{2}\left(p^2+q^2\right) + \frac{\tau}{2}pq \tag{60}$$

The integrated Hamiltonian differs from the true one.



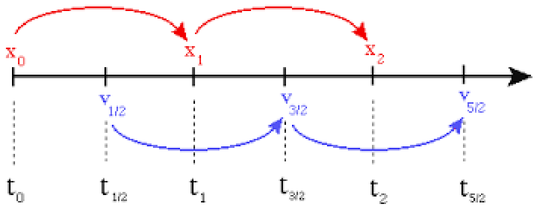
Since $H_{integrated}$ is conserved, the difference between it and the true Hamiltonian H_{true} is constant and the trajectory is *bounded*. The figure on the left shows *level curves* for H_{true} and on the right for $H_{integrated}$.

Leapfrog integration

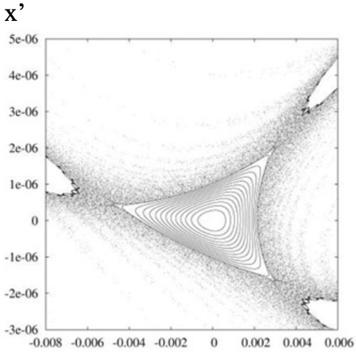
The leapfrog scheme is a second order symplectic integrator. In simplified terms

$$x_{n+1} = x_n + \tau v_{n+\frac{1}{2}} \tag{61}$$

$$v_{n+\frac{3}{2}} = v_{n+\frac{1}{2}} + \tau F(x_{n+\frac{1}{2}}) \tag{62}$$



Dynamic Aperture



- The dynamic aperture is largest amplitude in phase space inside of which the motion is regular and bounded in the time range of interest.
- Outside the dynamic aperture there is chaotic motion (but there may also be regular motion - islands of stability).

Chaotic motion

One can test whether the motion is chaotic by calculating the rate of divergence between two initially close points in phase space. For regular motion the distance d between the two tracks grows linearly with the number of turns N

$$d(N) \propto N \tag{63}$$

while for chaotic motion the separation increases exponentially

$$d(N) \propto e^{\lambda N} \tag{64}$$

where λ is the Lyapunov exponent formally defined as

$$\lambda = \lim_{N \to \infty} \lim_{d(0) \to 0} \frac{1}{N} d(N) d(0) \tag{65}$$

Action-angle transformation

Recall

$$y = a\sqrt{\beta(s)}\cos(\phi + \zeta) \tag{66}$$

$$y' = -\frac{a}{\sqrt{\beta(s)}}\left(\sin(\phi + \zeta) + \alpha(s)\cos(\phi + \zeta)\right) \tag{67}$$

where $\alpha(s) = -\frac{\beta'(s)}{2}$ and $\phi = \int_0^s \frac{\mathrm{d}s}{\mathrm{d}\beta(s)}$.

$$\frac{y'}{y} = -\frac{1}{\beta(s)}(\tan\psi + \alpha(s)) \tag{68}$$

This suggests a generating function of the type

$$F_1(y, \psi) = \int_0^y y'\,\mathrm{d}y = -\frac{y^2}{2\beta(s)}(\tan\psi + \alpha(s)). \tag{69}$$

$$y' = \frac{\partial F_1}{\partial y} \quad \text{and} \quad J = \frac{\partial F_1}{\partial \psi} = \frac{y^2}{2\beta(s)} \sec^2 \psi = \frac{y^2}{2\beta(s)}(1 + \tan^2 \psi) \tag{70}$$

$$J = \frac{1}{2\beta(s)}(y^2 + (\beta(s)y' + \alpha(s)y)^2) \tag{71}$$

The action J is therefore equal to half the Courant-Snyder invariant. So

$$y = \sqrt{2\beta J} \cos \psi \quad \text{and} \quad y' = -\sqrt{\frac{2J}{\beta}}(\sin \psi + \alpha \cos \psi). \tag{72}$$

The canonical transformation is

$$\bar{H} = H + \left| \frac{\partial F_1}{\partial s} \right|_{y,\psi} \tag{73}$$

where

$$H = \frac{y'^2}{2} + \frac{1}{2}K(s)y^2 = \frac{J}{\beta(s)}(\sin\psi + \alpha\cos\psi)^2 + K(s)\beta(s)J\cos^2\psi \tag{74}$$

It can be shown that this reduces to

$$\bar{H} = \frac{J}{\beta} \tag{75}$$

Hamilton's equations are therefore

$$\frac{d\psi}{ds} = \frac{\partial\bar{H}}{\partial J} = \frac{1}{\beta}, \quad \frac{dJ}{ds} = -\frac{\partial\bar{H}}{\partial\psi} = 0. \tag{76}$$

i.e. the action J is invariant.

Just one more transformation...

A further simplification is obtained by canonical transformation of the independent coordinate from s to θ.

$$F_2(\psi, \bar{J}) = \left(\psi - \int_0^s \frac{\mathrm{d}s}{\beta} + Q\theta\right)\bar{J} \tag{77}$$

where $\theta = \frac{s}{R}$. We have a new conjugate pair $(\bar{\psi}, \bar{J})$.

$$\bar{\psi} = \frac{\partial F_2}{\partial \bar{J}} = \psi - \int_0^s \frac{\mathrm{d}s}{\beta} + Q\theta \tag{78}$$

and

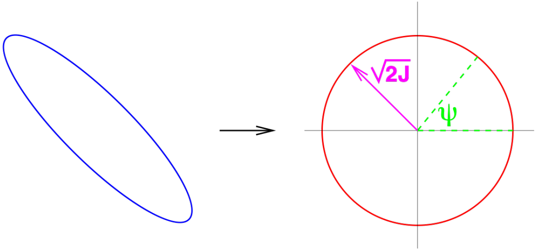
$$J = \frac{\partial F_2}{\partial \psi} = \bar{J}. \tag{79}$$

Thus

$$\widehat{H} = \bar{H} + \left|\frac{\partial F_2}{\partial s}\right|_{\psi,\bar{J}} = \frac{\bar{J}}{\beta} - \frac{\bar{J}}{\beta} + \frac{Q}{R}\bar{J} = \frac{Q}{R}\bar{J}. \tag{80}$$

Scale transformation, multiply by R, drop bars

$$H = R\widehat{H} = Q\bar{J}. \tag{81}$$



This transformation from betatron phase coordinates to action angle variables is given by

$$Y = \sqrt{2\beta J}\cos\left(\bar{\psi} + \chi(s) - Q\theta\right) \tag{82}$$

and

$$Y' = \alpha y + \beta y' = -\sqrt{2\beta J}\sin\left(\bar{\psi} + \chi(s) - Q\theta\right) \tag{83}$$

where the phase advance

$$\chi(s) = \int_0^s \frac{\mathrm{d}s}{\beta}. \tag{84}$$

Normalised coordinates

Action angle variables are closely related to normalised coordinates used in accelerator physics (Y, Y')

$$Y = y = \sqrt{2\beta J}\cos\psi \tag{85}$$

$$Y' = \alpha y + \beta y' = -\sqrt{2\beta J}\sin\psi. \tag{86}$$

Both turn phase space ellipses into circles of radius $\sqrt{2\beta J}$ and the normalised phase space is independent of s.

Sextupole-driven resonance

We will transform a perturbed Hamiltonian to action angle coordinates, leaving us with a linear Hamiltonian plus non-linear perturbation terms. These are normally the important effects in a ring, but we will neglect other terms which might have effects so it is important to check that assumptions are valid.

Our new Hamiltonian takes the form

$$H = \frac{p_x^2}{2} + \frac{p_z^2}{2} + \frac{K_x x^2}{2} + \frac{K_z z^2}{2} + f(x, z, s) \tag{87}$$

where

$$f(x, z, s) = \left(1 + \frac{x}{\rho}\right) \mathrm{Re}\left(\sum_{n>2} \frac{B^{(n)}}{n!}(x + iz)^n\right). \tag{88}$$

We will consider a single resonant term.
The vector potential of a sextupole magnet is

$$A_x = A_z = 0, \quad A_s = \left|\frac{\partial^2 B_z}{\partial x^2}\right|_{x=z=0} \frac{1}{6}(x^3 - 3xz^2). \tag{89}$$

The Hamiltonian becomes

$$H = \frac{p_x^2}{2} + \frac{p_z^2}{2} + \frac{K_x x^2}{2} + \frac{K_z z^2}{2} + \frac{1}{6}S(s)(x^3 - 3xz^2) \tag{90}$$

where the sextupole strength is defined as

$$S(s) = -\frac{1}{B\rho}\left|\frac{\partial^2 B}{\partial x^2}\right|_{x=z=0}. \tag{91}$$

Potential in action angle coordinates

Transform to action angle coordinates

$$\bar{H} = Q_x J_x + Q_z J_z + T(Q_x, J_x, Q_z, J_z) \tag{92}$$

where T is the polynomial transformed to action angle coordinates.
We can use this to study perturbations to linear motion. We could look at many effects as the complete picture is complicated - the 2D Hamiltonian includes coupling fields. To exploit action angle variables we can express the driving terms as azimuthal harmonics.
For instance for a sextupole with

$$V_3 = \frac{1}{6} S(s)(x^3 - 3xz^2), \tag{93}$$

replace x and z via

$$x = \sqrt{2\beta_x J_x} \cos(\psi_x + \chi_x(s) - Q_x\theta), \quad z = \sqrt{2\beta_z J_z} \cos(\psi_z + \chi_z(s) - Q_z\theta).$$

Resonance occurs as the variation of the cos term approaches zero. ψ_x varies by $2\pi Q_x$ and θ by 2π so the term is constant if $3Q_x = l$. If Q_x satisfies this condition and the corresponding driving strength of harmonic l dominates then the perturbed Hamiltonian

$$H = Q_x J_x + J_x^{\frac{3}{2}} \sum_l G_{3,0,l} \cos(3\psi_x - l\theta + \zeta_{3,0,l}). \tag{94}$$

We can remove the s dependence with a final canonical transformation using the generating function

$$F_2 = \left(\psi_x - \frac{l\theta}{3} + \frac{\zeta}{3}\right) \bar{J}. \tag{95}$$

The new angle ϕ is

$$\phi = \frac{\partial F_2}{\partial \bar{J}} = \psi_x - \frac{l\theta}{3} + \frac{\zeta}{3}, \quad \bar{J} = J. \tag{96}$$

Giving for the last Hamiltonian

$$\bar{H}(\phi, J) = H + \frac{\partial F_2}{\partial \theta} = Q_x J_x + J_x^{\frac{3}{2}} \sum_l G_{3,0,l} \cos(3\phi) - \frac{l}{3} J_x. \tag{97}$$

We now assume a single strong resonance in the final term, so the others average to zero

$$\bar{H} = \delta J + J^{\frac{3}{2}} G_{3,0,l} \cos(3\phi), \quad \delta = Q_x - \frac{l}{3}. \tag{98}$$

The equations of motion are

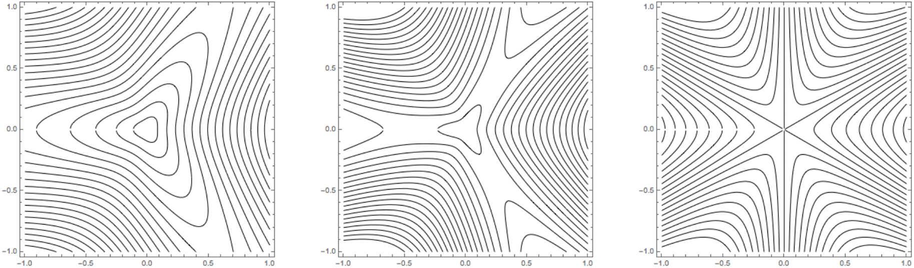
$$\dot{\phi} = \frac{\partial \bar{H}}{\partial J} = \delta + \frac{3}{2} J^{\frac{1}{2}} G_{3,0,l} \cos(3\phi) \tag{99}$$

$$\dot{J} = -\frac{\partial \bar{H}}{\partial \phi} = 3 J^{\frac{3}{2}} G_{3,0,l} \sin(3\phi). \tag{100}$$

What does this look like?

Moving towards resonance

→



Linear Integrable systems

- The ideal linear Hamiltonian

$$H = Q_x J_x + Q_y J_y \tag{101}$$

has two invariants of motion, the transverse actions J_x, J_y. This ensures the system is integrable.

- However, the addition of nonlinearities may compromise this integrability and lead to a reduction in the dynamic aperture.
- Nonlinear magnets may be added intentionally, for example sextupole magnets to correct chromaticity, or arise from magnet imperfections or other sources.

Nonlinear Integrable systems

- It has been proposed to build an accelerator based on a nonlinear integrable Hamiltonian.
- As well as reducing chaos in single particle motion, the strong tune spread in such a machine may help stem collective instabilities via *Landau damping*.
- As before, the Hamiltonian needs to possess two integrals of motion. A solution was found by Danilov and Nagaitsev (2010).

Start with the Hamiltonian

$$H = \frac{p_x^2}{2} + \frac{p_z^2}{2} + k(s)\left(\frac{x^2}{2} + \frac{z^2}{2}\right) + V(x, z, s) \tag{102}$$

Choose s-dependence of nonlinear potential V so that the Hamiltonian is time-independent in normlised variables $(x_N, p_{xN}, z_N, p_{zN})$.

$$\begin{aligned} H_N &= \frac{p_{xN}^2 + p_{zN}^2}{2} + \frac{x_N^2 + z_N^2}{2} + \beta(\psi)V(x_N\sqrt{\beta(\psi)}, z_N\sqrt{\beta(\psi)}), s(\psi)) \\ &= \frac{p_{xN}^2 + p_{zN}^2}{2} + \frac{x_N^2 + z_N^2}{2} + U(x_N, z_N, \psi) \end{aligned}$$

H_N is an integral of motion for any choice of V(x,z,s) so long as it scales with β appropriately.

Octupole case

If we use an octupole for the nonlinear element then the potential should be scaled by $1/\beta^3$.

$$V(x, z, s) = \frac{\alpha}{\beta(s)^3}\left(\frac{x^4}{4} + \frac{z^4}{4} - \frac{3x^3y^3}{2}\right) \tag{103}$$

where α sets the octupole strength. Then the normalised Hamiltonian becomes

$$H_N = \frac{p_{xN}^2 + p_{zN}^2}{2} + \frac{x_N^2 + z_N^2}{2} + \alpha\left(\frac{x_N^4}{4} + \frac{z_N^4}{4} - \frac{3x_N^3y_N^3}{2}\right) \tag{104}$$

In this case H_N is the only integral of motion. This solution is known as quasi-integrable.

Special potential

A nonlinear potential that results in a second integral of motion arises from the Bertrand-Darboux partial differential equation[1].

$$xz(U_{xx} - U_{zz}) + (z^2 - x^2 + c^2)U_{xz} + 3zU_x - 3xU_z = 0 \qquad (105)$$

The equation has general solution

$$U(x, z) = \frac{f(\xi) + g(\eta)}{\xi^2 - \eta^2} \qquad (106)$$

where f and g are arbitrary functions of the elliptic coordinates

$$\xi = \frac{\sqrt{(x+c)^2 + z^2} + \sqrt{(x-c)^2 + z^2}}{2c}$$
$$\eta = \frac{\sqrt{(x+c)^2 + z^2} - \sqrt{(x-c)^2 + z^2}}{2c}$$

[1]The coordinates are normalised but the N is omitted

As before, the normalised Hamiltonian is one invariant

$$H = \frac{p_x^2 + p_z^2}{2} + \frac{x^2 + z^2}{2} + \frac{f(\xi) + g(\eta)}{\xi^2 - \eta^2} \tag{107}$$

but there is now a second invariant

$$I(x, z, p_x, p_z) = (xp_z - zp_x)^2 + c^2 p_x^2 + 2c^2 \frac{f(\xi)\eta^2 + g(\eta)\xi^2}{\xi^2 - \eta^2} \tag{108}$$

See *V. Danilov and S. Nagaitsev, PRST-AB 13 084002 (2010)* for details.

IOTA

The concept is currently being investigated at the Integrable Optics Test Accelerator (IOTA), Fermilab.