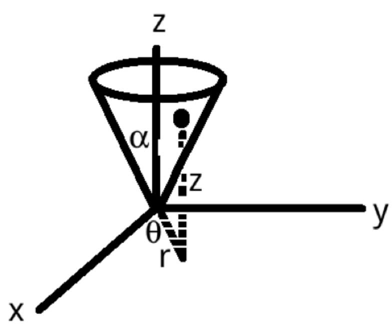
Hamiltonian Dynamics - Problem Set

David Kelliher

December 5th, 2019

Problem 1 Particle on a cone



Consider a particle rolling due to gravity in a frictionless cone. The cone's opening angle α places a constraint on the coordinates $\tan\alpha = \sqrt{x^2+y^2}/z$. It is convenient to write the kinetic and potential energies in cylindrical coordinates

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right), \quad V = mgz \tag{1}$$

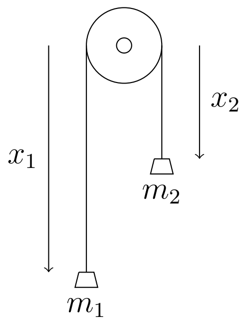
Use the constraint to reduce the number of generalised coordinates. Write down the Lagrangian and find the equation of motion.

Problem 2 Kepler problem

Write down Hamilton's equation for the following Hamiltonian

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) - \frac{\mu m}{r} \tag{2}$$

Problem 3 Atwood's machine



Two masses are hanging via a massless string from a frictionless pulley, The kinetic energy of the masses is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2, \tag{3}$$

while the potential energy is

$$V = -m_1 g x_1 - m_2 g x_2. \tag{4}$$

We selected $V = 0$ at the centre of the pulley. The system is subjected to the constraint $x_1 + x_2 = l =$ constant. Write down the Lagrangian, convert to the Hamiltonian and write down Hamilton's equations.

Problem 4 Canonical transformation

By evaluating the Poisson bracket show that, for a system of one degree of freedom, the transformation

$$Q = \arctan\frac{\alpha q}{p}, \qquad P = \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha^2 q^2}\right)$$

is canonical (recall for a canonical transformation the Poisson bracket must satisfy $[Q, P]_{q,p} = 1$).

Problem 5 Transfer matrix of a drift

The Hamiltonian of a drift (field free region) of length L is given by

$$H = \frac{\delta_E}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta_E\right) - p_x^2 - p_z^2 - \frac{1}{\beta_0^2\gamma_0^2}} \tag{5}$$

where δ_E is the energy deviation, p_x, p_z are the normalised transverse momenta and β_0, γ_0 are the relativistic parameters at the reference momentum. Write down Hamilton's equations and show that in the limit $p_x \ll 1, p_z \ll$

$1, \delta_E \ll 1$ and $\gamma_0 \gg 1$, the transfer matrix is given by

$$M = \begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6}$$

Problem 6 Amplitude dependent tune shift

The Hamiltonian for a normal octupole magnet can be written

$$H = \frac{q}{p}\frac{b_4}{4}\left(x^4 - 6x^2z^2 + z^4\right) \tag{7}$$

where q is the particle charge, p is the momentum, (x, z) are the horizontal and vertical coordinates and the octupole strength b_4 is constant along the magnet length. The coordinates can be written in terms of action-angle coordinates (J_x, ϕ_x), (J_y, ϕ_y) via

$$x(s) = \sqrt{2J_x\beta_x(s)}\cos\phi_x \tag{8}$$

$$y(s) = \sqrt{2J_y\beta_y(s)}\cos\phi_y \tag{9}$$

where $\beta_{x,y}$ are betatron functions. Find the averaged Hamiltonian $\langle H\rangle$ by evaluating

$$\langle H\rangle = \frac{1}{2\pi}\oint H(J,\phi)d\phi \tag{10}$$

Note, integrals of the cosine function such as $\oint \cos^4\phi \, d\phi = \frac{3\pi}{4}$ will be useful. Then work out the tune shift in each transverse plane $\Delta Q_{x,y}$ using the relation

$$\Delta Q_{x,y} = \frac{1}{2\pi}\oint \frac{\partial}{\partial J_{x,y}}\langle H\rangle \, ds \tag{11}$$