Tracking Topological phase transitions

Syed Tahir Amin

Center of Physics and Engineering of Advanced Materials (CeFEMA)
Department of Physics
Instituto Superior Tecnico

tahiramin@tecnico.ulisboa.pt

Sep 16 2019

Overview

Quantities of interest:

- Fidelity
- Uhlmann factor
- Uhlmann quantity
- Fidelity susceptibility

Results and discussions

Fidelity

Fidelity in terms of different quantum states:

- Fidelity for two generic pure states $|\psi\rangle$ and $|\phi\rangle$ is

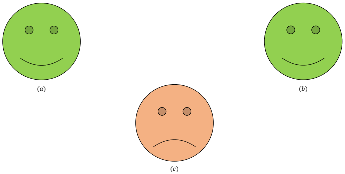
$$\mathrm{F} = |\langle\psi|\phi\rangle|$$

- In case of one pure state $|\psi\rangle$ and one mixed state ρ

$$\mathrm{F} = \sqrt{\langle\psi|\rho|\psi\rangle}$$

- While in case of two mixed states ρ' and ρ

$$\mathrm{F} = \mathrm{Tr}\sqrt{\sqrt{\rho}\rho'\sqrt{\rho}}$$



Uhlmann factor

- We take two density matrices $\rho(\tau)$ and $\rho(\tau+\delta\tau)$, by polar decomposition

$$\sqrt{\rho(\tau+\delta\tau)}\sqrt{\rho(\tau)} = |\sqrt{\rho(\tau+\delta\tau)}\sqrt{\rho(\tau)}|\mathrm{V}$$

- V is the Uhlmann factor which characterizes the Uhlmann parallel transport.
- If the density matrices $\rho(\tau)$ and $\rho(\tau+\delta\tau)$, belong to the same phase then they commute and thus $\mathrm{V} \approx \mathrm{I}$
- On the other hand if density matrices $\rho(\tau)$ and $\rho(\tau+\delta\tau)$, belong to the different phases then they are drastically different and thus $\mathrm{V} \neq \mathrm{I}$

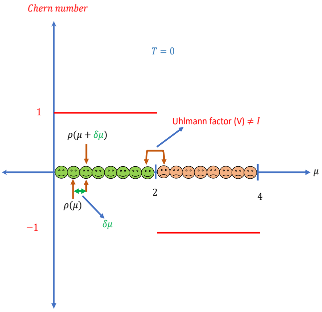
Uhlmann quantity

- To quantify the difference between a nontrivial Uhlmann factor and identity, the following quantity is introduced

$$\Delta(\rho(\tau), \rho(\tau+\delta\tau)) := \mathrm{Tr}\{|\sqrt{\rho(\tau+\delta\tau)}\sqrt{\rho(\tau)}|(\mathrm{I}-\mathrm{V})\}$$

Fidelity and Uhlmann quantity

- If the density matrices $\rho(\tau)$ and $\rho(\tau+\delta\tau)$ belong to same phase then they are almost the same and thus $\mathrm{F} \approx 1$.
- On the other hand if density matrices $\rho(\tau)$ and $\rho(\tau+\delta\tau)$, belong to different phases then the they are drastically different and thus $\mathrm{F} < 1$.
- The difference between the two states can be in their spectra or their eigen basis. In the case of the latter, we also have $\color{red}{\mathrm{V}} \neq \mathrm{I}$, i.e., $\Delta > 0$.



Fidelity and Uhlmann quantity and Fidelity susceptibility

$$F(\rho(\tau), \rho(\tau+\delta\tau)) = \mathrm{Tr}\sqrt{\sqrt{\rho(\tau)}\rho(\tau+\delta\tau)\sqrt{\rho(\tau)}}$$

$$\Delta(\rho(\tau), \rho(\tau+\delta\tau)) := \mathrm{Tr}\{|\sqrt{\rho(\tau+\delta\tau)}\sqrt{\rho(\tau)}|(I - V)\}$$

$$\frac{1}{2}\chi_{\tau\tau}\delta\tau^2 = -\frac{\ln[F(\rho(\tau), \rho(\tau+\delta\tau))]}{N^2}$$

$$\chi_{\tau\tau}^{N\to\infty, T=0} \neq \chi_{\tau\tau}^{T=0, N\to\infty}$$

$$\chi_{\tau\tau'} = \int_{\text{B.Z.}} \frac{d^D k}{(2\pi)^D}\frac{1}{4}\delta_{\mu\nu}\frac{\partial n^\mu}{\partial\tau}\frac{\partial n^\nu}{\partial\tau'},$$

Main ingredient

- Here we treat temperature and parameters of Hamiltonian on equal footing.
- Fidelity (F) and the Uhlmann quantity (Δ) are evaluated for different Topological superconductors and Insulators.
- Analytic expression for previous quantities are calculated with respect to the many-body thermal states

$$\rho = \exp[-\beta\mathcal{H}]/\mathrm{Z}; \quad \mathrm{Z} = \mathrm{Tr}(\exp[-\beta\mathcal{H}])$$
$$\mathcal{H} = \frac{1}{2}\widetilde{\psi}^{\dagger} H_{\mathrm{BdG}}\widetilde{\psi}.$$

2D Topological Superconductor

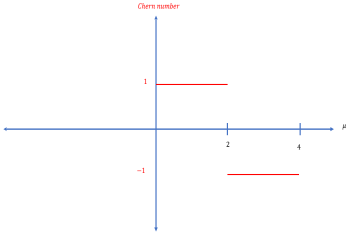
The Hamiltonian of topological superconductor

$$\mathcal{H} = \sum_{ij}[-t(c^\dagger_{i+1,j}c_{i,j} + c^\dagger_{i,j+1}c_{i,j}) - \frac{1}{2}(\mu - 4t)c^\dagger_{i,j}c_{i,j}$$
$$+S(c^\dagger_{i+1,j}c^\dagger_{i,j} + ic^\dagger_{i,j+1}c^\dagger_{i,j})] + \mathrm{H.c.}].$$

- we fix $|S| = t = \frac{1}{2}$, and translation invariance is taken in both directions

$$H(\mathbf{k}) = -\{\sin(k_y)\sigma_x + \sin(k_x)\sigma_y$$
$$+[\mu - 2 + \cos(k_x) + \cos(k_y)]\sigma_z\}$$

- So topological phase transitions should occur at $\mu = 2$



2D Topological superconductor

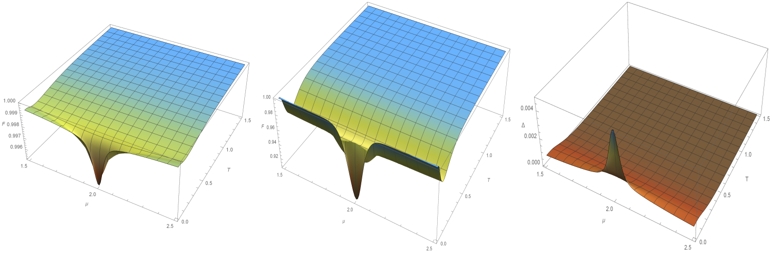


Figure: Fidelity (F) for Gibbs states ρ, (Left) probing parameter of Hamiltonian $\delta\mu = 0.01$, (Middle) when probing temperature $\delta\mathrm{T} = 0.01$, and (Right) Uhlmann quantity Δ while probing Hamiltonian parameter with $\delta\mu = 0.01$, for 2D Topological superconductor.

2D Topological insulator

- Following topological insulator Hamiltonian is considered

$$\mathcal{H} = \sum_{ij}[c^\dagger_{i+1,j}(t_1\sigma_x + it_3\sigma_z)c_{i,j} + c^\dagger_{i,j+1}(t_1\sigma_y + it_3\sigma_z)c_{i,j} \\ +c^\dagger_{i+1,j+1}(t_2\sigma_z)c_{i,j} + \mathrm{H.c.}]$$

- for $|t_1| = |t_3| = 1$, again going to momentum space we get

$$H(\mathbf{k}) = 2\cos(k_x)\sigma_x + 2\cos(k_y)\sigma_y \\ +\{2t_2\cos(k_x + k_y) + 2[\sin(k_x) + \sin(k_y)]\}\sigma_z.$$

- Chern number (Ch) are: $\mathrm{Ch} = \pm 2$ if $t_2 \lessgtr \mp 2$ and $\mathrm{Ch} = \pm 1$ if $\mp 2 \lessgtr t_2 \lessgtr 0$

2D Topological insulator

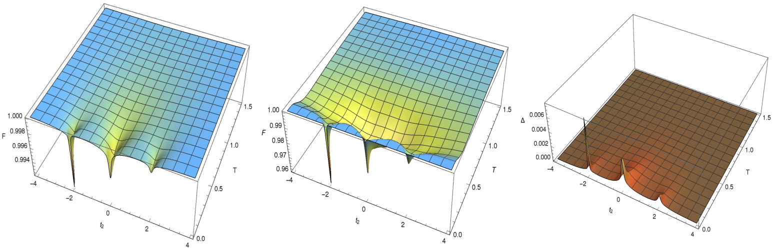


Figure: Fidelity (F) for Gibbs states ρ, (Left) parameter of Hamiltonian $\delta \mathrm{t}_2 = 0.01$, (Middle) when probing temperature $\delta \mathrm{T} = 0.01$, and (Right) Uhlmann quantity Δ while probing Hamiltonian parameter with $\delta \mathrm{t}_2 = 0.01$, for 2D Topological insulator.

2D Topological insulator

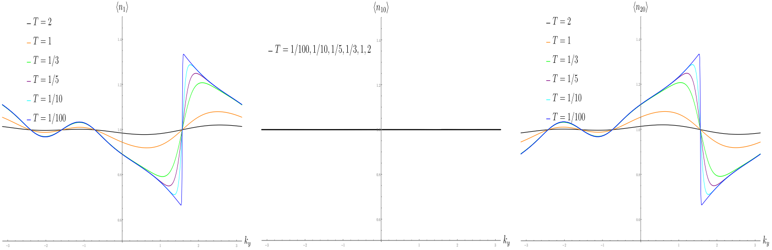
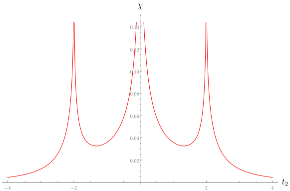


Figure: Expectation value of the occupation number at the bulk and boundary as a function of the quasi-momentum $\mathrm{k_y}$ for the 2D topological insulator.

2D Topological insulator

$$\chi_{\tau\tau}^{\mathrm{N}\to\infty,\mathrm{T}=0} \neq \chi_{\tau\tau}^{\mathrm{T}=0,\mathrm{N}\to\infty}$$

$$\chi_{\tau\tau}^{\mathrm{N}\to\infty,\mathrm{T}=0} = \int_{\mathsf{B.Z.}} \frac{\mathrm{d}^{\mathrm{D}}\mathrm{k}}{(2\pi)^{\mathrm{D}}} \frac{1}{4} \delta_{\mu\nu} \frac{\partial \mathrm{n}^{\mu}}{\partial \tau} \frac{\partial \mathrm{n}^{\nu}}{\partial \tau},$$



$$\chi_{\tau\tau} = \int_{\mathsf{B.Z.}} \frac{\mathrm{d}^{\mathrm{D}}\mathrm{k}}{(2\pi)^{\mathrm{D}}} \left\{ \frac{1}{4} \left[\frac{\cosh[\beta \mathrm{E}(\mathrm{k};\tau)] - 1}{\cosh[\beta \mathrm{E}(\mathrm{k};\tau)]} \delta_{\mu\nu} \frac{\partial \mathrm{n}^{\mu}}{\partial \tau} \frac{\partial \mathrm{n}^{\nu}}{\partial \tau} + \frac{\beta^2}{\cosh[\beta \mathrm{E}(\mathrm{k};\tau)] + 1} \left(\frac{\partial \mathrm{E}(\mathrm{k};\tau)}{\partial \tau} \right)^2 \right] \right\},$$

1D Topological superconductor with long-range amplitudes

- Following topological superconductor Hamiltonian is considered

$$\mathcal{H} = \sum_{i=1}^{N} \left(\frac{1}{2} \sum_{l=1}^{N-1} (-\frac{1}{r_{l,\beta}} c_{i+1}^{\dagger} c_i + \frac{1}{R_{l,\alpha}} c_{i+1}^{\dagger} c_i^{\dagger} + h.c) - \mu (c_i^{\dagger} c_i - \frac{1}{2}) \right),$$

- with $\frac{1}{R_{l,\alpha}} = \delta_{1,l}$ and $r_{l,\beta} = \exp[(l-1)/\beta]$.
- Winding number (Ω) are: $\Omega = 1$ if $-1 < \mu < 1$ and $\Omega = 0$ for all other values of μ at $\beta = 0$.

1D Topological superconductor with long-range amplitudes

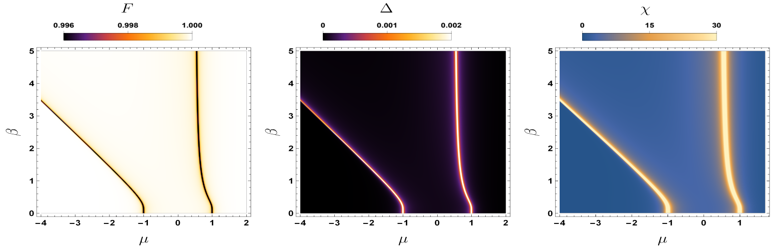


Figure: (Left) Fidelity (F), (Middle) Uhlmann quantity Δ while probing Hamiltonian parameter with $\delta\mu = 0.01$, and (Right) Fidelity susceptibility ($\chi_{\mu\mu}$) for 1D topological superconductor with long-range hopping amplitude

2D Topological superconductor with long-range amplitudes

- Following topological superconductor Hamiltonian is considered

$$\mathcal{H} = \sum_{\mathrm{l,m}}\{{\sum}'_{\mathrm{r,s}}(-\frac{\mathrm{t}}{\mathrm{d}_{\mathrm{r,s}}^{\beta}}\mathrm{c}^{\dagger}_{\mathrm{l+r,m+s}}\mathrm{c}_{\mathrm{l,m}} + \frac{\Delta}{\mathrm{d}_{\mathrm{r,s}}^{\alpha+1}}(\mathrm{r}+\mathrm{i}\ \mathrm{s})\mathrm{c}^{\dagger}_{\mathrm{l+r,m+s}}\mathrm{c}^{\dagger}_{\mathrm{l,m}}) \\ -(\mu - 4\mathrm{t})\mathrm{c}^{\dagger}_{\mathrm{l,m}}\mathrm{c}_{\mathrm{l,m}}) + \mathrm{H.c.}\}$$

- with $\mathrm{d}_{\mathrm{r,s}} = \sqrt{\mathrm{r}^2+\mathrm{s}^2}$ and $\Delta = \mathrm{t} = \frac{1}{2}$.

2D Topological superconductor with long-range amplitudes

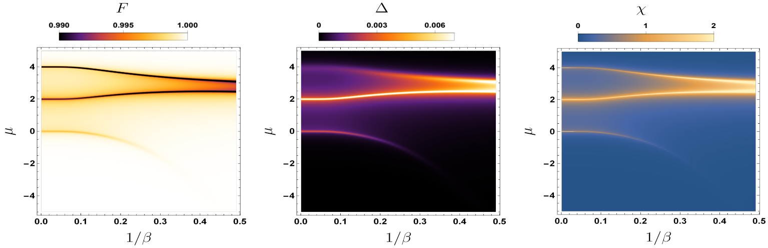


Figure: (Left) Fidelity (F), (Middle) Uhlmann quantity Δ while probing Hamiltonian parameter with $\delta\mu = 0.01$, and (Right) Fidelity susceptibility ($\chi_{\mu\mu}$) for 2D topological superconductor with long-range amplitudes.

Summary

- The fidelity and Uhlmann connection analysis of phase transitions are applied to different free-fermion topological insulators and superconductors
- Both quantities are detecting zero-temperature topological quantum phase transitions.
- Moreover, both quantities behave differently with respect to changes in eigenvalues and eigenbasis
- we also observe that with increasing temperature the non-trivial behavior of both quantities around the gap-closing point at zero-temperature is smeared out.
- Detailed study of the edge states of the systems at finite temperature was carried out.

Summary

- Our results clearly show that, as the temperature is increased, the edge states at $T = 0$ start mixing with the bulk states. Thus confirming the conclusion of fidelity and Δ analysis.

References

- Syed Tahir Amin, Bruno Mera, Chrysoula Vlachou, Nikola Paunković, and Vítor R. Vieira (Phys. Rev. B 98, 245141)
- Syed Tahir Amin, Bruno Mera, Nikola Paunković, Vítor R. Vieira J. Phys.: Condens. Matter 31 485402.

Thank you all :-)

I acknowledge the support from CeFEMA, DP-PMI and FCT(Portugal) through scholarship PD/ BD/113651/2015.

Uhlmann parallel Transport

- Geometrical phases were generalized by Uhlmann from pure states to mixed states.
- A Hilbert space consisting of a set of density matrices ($\rho = \mathrm{w}\mathrm{w}^\dagger$) and corresponding amplitudes w.
- Having gauge freedom, i.e., w and $\mathrm{w}' = \mathrm{w}\mathrm{U}$ belongs to same density matrix ρ: such that $\rho = \mathrm{w}\mathrm{w}^\dagger = \mathrm{w}\mathrm{U}\mathrm{U}^\dagger\mathrm{w}^\dagger = \mathrm{w}'(\mathrm{w}')^\dagger$
- Two amplitudes w_1 and w_2 are parallel in Uhlmann sense iff they minimize the Hilbert-Schmidt distance $||\mathrm{w}_2 - \mathrm{w}_1||^2$.
- After a few steps of simple calculation

$$||\mathrm{w}_2 - \mathrm{w}_1||^2 = 2(1 - \mathrm{Re}\langle \mathrm{w}_2, \mathrm{w}_1 \rangle).$$

Uhlmann parallel Transport

- It turns out that

$$\mathrm{Re}\langle \mathrm{w}_2, \mathrm{w}_1\rangle = \mathrm{F}(\rho_2, \rho_1).$$

- By writing $\mathrm{w_j} = \sqrt{\rho_\mathrm{j}}\mathrm{U_j}$, with j ϵ $\{1, 2\}$

$$\begin{aligned}
\mathrm{Re}\langle \mathrm{w}_2, \mathrm{w}_1\rangle \quad &\leq |\langle \mathrm{w}_2, \mathrm{w}_1\rangle| = |\mathrm{Tr}(\mathrm{w}_2^\dagger, \mathrm{w}_1)| \\
&= |\mathrm{Tr}(\mathrm{U}_2^\dagger \sqrt{\rho_2}\sqrt{\rho_1}\mathrm{U}_1)| \\
&= |\mathrm{Tr}(|\sqrt{\rho_2}\sqrt{\rho_1}|\mathrm{V}\mathrm{U}_1\mathrm{U}_2^\dagger)| \\
&\leq \mathrm{Tr}|\sqrt{\rho_2}\sqrt{\rho_1}| \\
&= \mathrm{Tr}(\sqrt{\rho_2}\rho_1\sqrt{\rho_2})^{\frac{1}{2}} \\
&= \mathrm{F}(\rho_2, \rho_1)
\end{aligned}$$

Uhlmann factor

- We take two density matrices $\rho(\tau)$ and $\rho(\tau+\delta\tau)$, by polar decomposition

$$\sqrt{\rho(\tau+\delta\tau)}\sqrt{\rho(\tau)} = |\sqrt{\rho(\tau+\delta\tau)}\sqrt{\rho(\tau)}|\mathrm{V}$$

- V is the Uhlmann factor which characterizes the Uhlmann parallel transport.
- For two close points τ and $\tau+\delta\tau$ on a curve of density matrices given by $\rho(\tau)$, $\tau \epsilon [0,1]$.
- The Uhlmann parallel transport implies that two amplitudes $\mathrm{w}(\tau)$ and $\mathrm{w}(\tau+\delta\tau)$ are parallel for infinitesimal $\delta\tau$.