

# Light-cone PDFs & GPDs from Lattice QCD: *selected results, successes and challenges*

(Focus on proton)

Martha Constantinou



Temple University

50<sup>th</sup> International Symposium on  
Multiparticle Dynamics 2021

July 13, 2021



# x-dependent distributions

# Novel Approaches

(Besides Mellin moments)

<b>Hadronic tensor</b>	[K.F. Liu, S.J. Dong, PRL 72 (1994) 1790, K.F. Liu, PoS(LATTICE 2015) 115]
<b>Auxiliary scalar quark</b>	[U. Aglietti et al., Phys. Lett. B441, 371 (1998), arXiv:hep-ph/9806277]
<b>Fictitious heavy quark</b>	[W. Detmold, C. J. D. Lin, Phys. Rev. D73, 014501 (2006) ]
<b>Auxiliary scalar quark</b>	[V. Braun & D. Mueller, Eur. Phys. J. C55, 349 (2008), arXiv:0709.1348]
<b>Higher moments</b>	[Z. Davoudi, M. Savage, Phys. Rev. D86, 054505 (2012) ]
<b>Quasi-distributions (LaMET)</b>	[X. Ji, PRL 110 (2013) 262002, arXiv:1305.1539; Sci. China PPMA. 57, 1407 (2014)]
<b>Compton amplitude and OPE</b>	[A. Chambers et al. (QCDSF), PRL 118, 242001 (2017), arXiv:1703.01153]
<b>Pseudo-distributions</b>	[A. Radyushkin, Phys. Rev. D 96, 034025 (2017), arXiv:1705.01488]
<b>Good lattice cross sections</b>	[Y-Q Ma & J. Qiu, Phys. Rev. Lett. 120, 022003 (2018), arXiv:1709.03018 ]

# Novel Approaches (Besides Mellin moments)

<b>Hadronic tensor</b>	[K.F. Liu, S.J. Dong, PRL 72 (1994) 1790, K.F. Liu, PoS(LATTICE 2015) 115]
<b>Auxiliary scalar quark</b>	[U. Aglietti et al., Phys. Lett. B441, 371 (1998), arXiv:hep-ph/9806277]
<b>Fictitious heavy quark</b>	[W. Detmold, C. J. D. Lin, Phys. Rev. D73, 014501 (2006) ]
<b>Auxiliary scalar quark</b>	[V. Braun & D. Mueller, Eur. Phys. J. C55, 349 (2008), arXiv:0709.1348]
<b>Higher moments</b>	[Z. Davoudi, M. Savage, Phys. Rev. D86, 054505 (2012) ]
<b>Quasi-distributions (LaMET)</b>	[X. Ji, PRL 110 (2013) 262002, arXiv:1305.1539; Sci. China PPMA. 57, 1407 (2014)]
<b>Compton amplitude and OPE</b>	[A. Chambers et al. (QCDSF), PRL 118, 242001 (2017), arXiv:1703.01153]
<b>Pseudo-distributions</b>	[A. Radyushkin, Phys. Rev. D 96, 034025 (2017), arXiv:1705.01488]
<b>Good lattice cross sections</b>	[Y-Q Ma & J. Qiu, Phys. Rev. Lett. 120, 022003 (2018), arXiv:1709.03018 ]

## ★ Useful reviews of methods

- A guide to light-cone PDFs from Lattice QCD: an overview of approaches, techniques and results  
**K. Cichy & M. Constantinou (invited review)** Advances in HEP 2019, 3036904, arXiv:1811.07248
- Large Momentum Effective Theory  
**X. Ji, Y.-S. Liu, Y. Liu, J.-H. Zhang, and Y. Zhao (2020)**, 2004.03543
- The  $x$ -dependence of hadronic parton distributions: A review on the progress of lattice QCD  
**M. Constantinou (invited review)** Eur. Phys. J. A 57 (2021) 2, 77, arXiv:2010.02445

# Access of PDFs & GPDs on a Euclidean Lattice

Matrix elements of non-local operators (space-like separated fields) with boosted hadrons

$$\mathcal{M}(P_f, P_i, z) = \langle N(P_f) | \bar{\Psi}(z) \Gamma \mathcal{W}(z, 0) \Psi(0) | N(P_i) \rangle_\mu$$

# Access of PDFs & GPDs on a Euclidean Lattice

Matrix elements of non-local operators (space-like separated fields) with boosted hadrons

$$\mathcal{M}(P_f, P_i, z) = \langle N(P_f) | \bar{\Psi}(z) \Gamma \mathcal{W}(z, 0) \Psi(0) | N(P_i) \rangle_\mu$$

[X. Ji, Phys. Rev. Lett. 110 (2013) 262002] [A. Radyushkin, PRD 96, 034025 (2017)]  
[X. Ji, Sci. China Phys. M.A. 57 (2014) 1407]

$$\tilde{q}_\Gamma^{\text{GPD}}(x, t, \xi, P_3, \mu) = \int \frac{dz}{4\pi} e^{-ixP_3z} \mathcal{M}(P_f, P_i, z)$$

pseudo-ITD



$$\mathfrak{M}(\nu, z^2) = \frac{\mathcal{M}(\nu, z^2) / \mathcal{M}(\nu, 0)}{\mathcal{M}(0, z^2) / \mathcal{M}(0, 0)} \quad (\nu = z \cdot p)$$

# Access of PDFs & GPDs on a Euclidean Lattice

Matrix elements of non-local operators (space-like separated fields) with boosted hadrons

$$\mathcal{M}(P_f, P_i, z) = \langle N(P_f) | \bar{\Psi}(z) \Gamma \mathcal{W}(z, 0) \Psi(0) | N(P_i) \rangle_\mu$$

[X. Ji, Phys. Rev. Lett. 110 (2013) 262002] [A. Radyushkin, PRD 96, 034025 (2017)]  
[X. Ji, Sci. China Phys. M.A. 57 (2014) 1407]

$$\tilde{q}_\Gamma^{\text{GPD}}(x, t, \xi, P_3, \mu) = \int \frac{dz}{4\pi} e^{-ixP_3z} \mathcal{M}(P_f, P_i, z)$$

$$\mathfrak{M}(\nu, z^2) = \frac{\mathcal{M}(\nu, z^2) / \mathcal{M}(\nu, 0)}{\mathcal{M}(0, z^2) / \mathcal{M}(0, 0)} \quad (v = z \cdot p)$$

Matching in momentum space  
(Large Momentum Effective Theory )

Matching in coordinate space

Light-cone PDFs & GPDs

# Access of PDFs & GPDs on a Euclidean Lattice

Matrix elements of non-local operators (space-like separated fields) with boosted hadrons

$$\mathcal{M}(P_f, P_i, z) = \langle N(P_f) | \bar{\Psi}(z) \Gamma \mathcal{W}(z, 0) \Psi(0) | N(P_i) \rangle_\mu$$

[X. Ji, Phys. Rev. Lett. 110 (2013) 262002] [A. Radyushkin, PRD 96, 034025 (2017)]  
[X. Ji, Sci. China Phys. M.A. 57 (2014) 1407]

$$\tilde{q}_\Gamma^{\text{GPD}}(x, t, \xi, P_3, \mu) = \int \frac{dz}{4\pi} e^{-ixP_3z} \mathcal{M}(P_f, P_i, z)$$

$$\mathfrak{M}(\nu, z^2) = \frac{\mathcal{M}(\nu, z^2) / \mathcal{M}(\nu, 0)}{\mathcal{M}(0, z^2) / \mathcal{M}(0, 0)} \quad (v = z \cdot p)$$

Matching in momentum space  
(Large Momentum Effective Theory )

Matching in coordinate space

Light-cone PDFs & GPDs

Calculation very taxing!

- length of the Wilson line ( $z$ )
- nucleon momentum boost ( $P_3$ )
- momentum transfer ( $t$ )
- skewness ( $\xi$ )

# Access of PDFs & GPDs on a Euclidean Lattice

Matrix elements of non-local operators (space-like separated fields) with boosted hadrons

$$\mathcal{M}(P_f, P_i, z) = \langle N(P_f) | \bar{\Psi}(z) \Gamma \mathcal{W}(z, 0) \Psi(0) | N(P_i) \rangle_\mu$$

[X. Ji, Phys. Rev. Lett. 110 (2013) 262002]  
[X. Ji, Sci. China Phys. M.A. 57 (2014) 1407] **quasi-PDFs** [A. Radyushkin, PRD 96, 034025 (2017)]  
[X. Ji, Sci. China Phys. M.A. 57 (2014) 1407]

$$\tilde{q}_\Gamma^{\text{GPD}}(x, t, \xi, P_3, \mu) = \int \frac{dz}{4\pi} e^{-ixP_3z} \mathcal{M}(P_f, P_i, z)$$

$$\mathfrak{M}(\nu, z^2) = \frac{\mathcal{M}(\nu, z^2) / \mathcal{M}(\nu, 0)}{\mathcal{M}(0, z^2) / \mathcal{M}(0, 0)} \quad (v = z \cdot p)$$

Matching in momentum space  
(Large Momentum Effective Theory )

Matching in coordinate space

Light-cone PDFs & GPDs

Calculation very taxing!

- length of the Wilson line ( $z$ )
  - nucleon momentum boost ( $P_3$ )
  - momentum transfer ( $t$ )
  - skewness ( $\xi$ )
- } PDFs, GPDs
- } GPDs

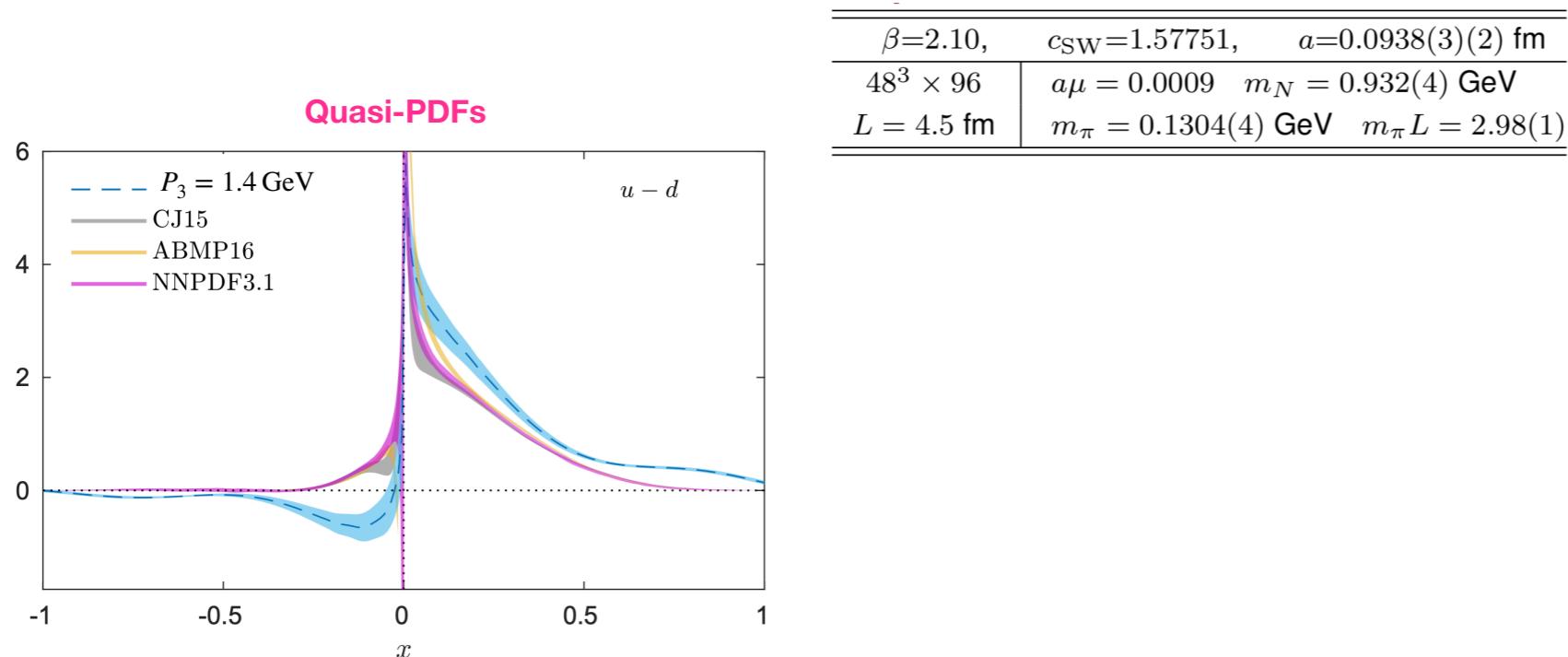
# Successes

## Selected Results

# PDFs at physical pion mass

# PDFs at physical pion mass

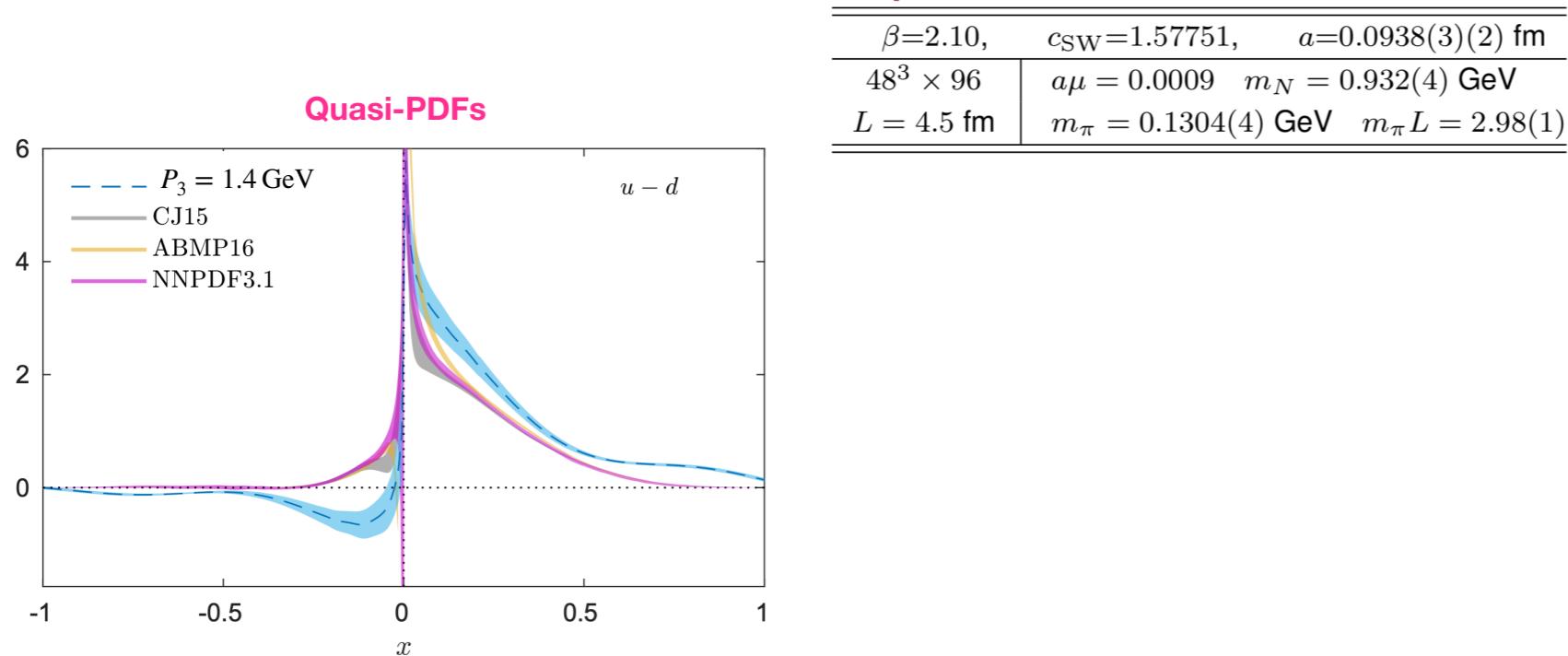
- ★ First complete study of quasi-PDFs including all steps



[C. Alexandrou et al. (ETMC), PRL 121 (2018) 112001, arXiv:1803.02685]

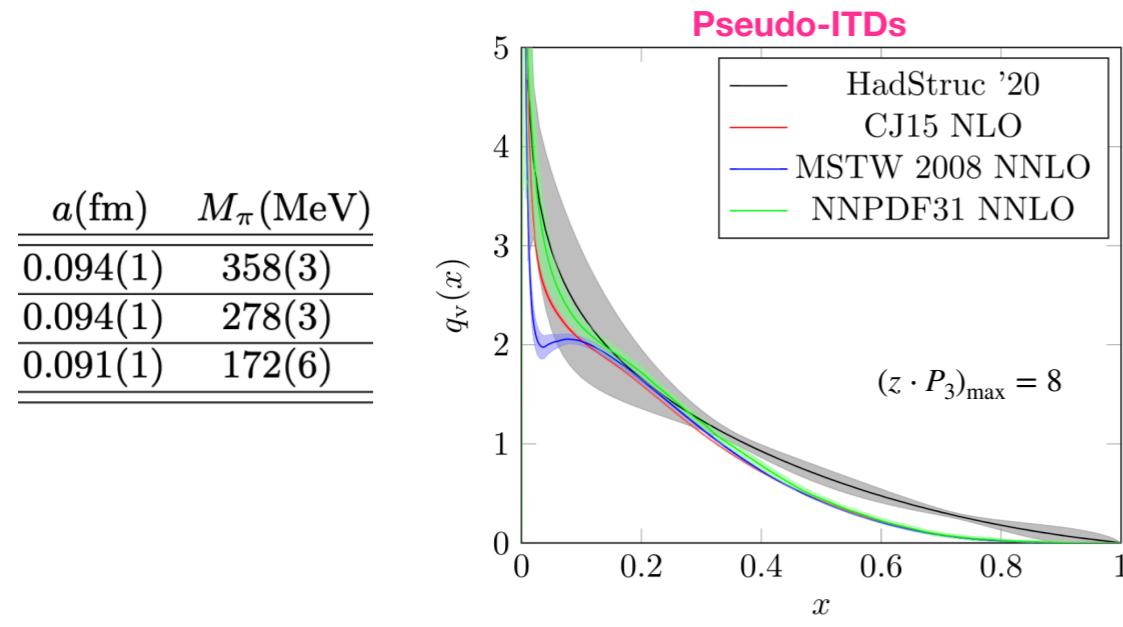
# PDFs at physical pion mass

- ★ First complete study of quasi-PDFs including all steps



[C. Alexandrou et al. (ETMC), PRL 121 (2018) 112001, arXiv:1803.02685]

- ★ First complete study of pseudo-PDFs

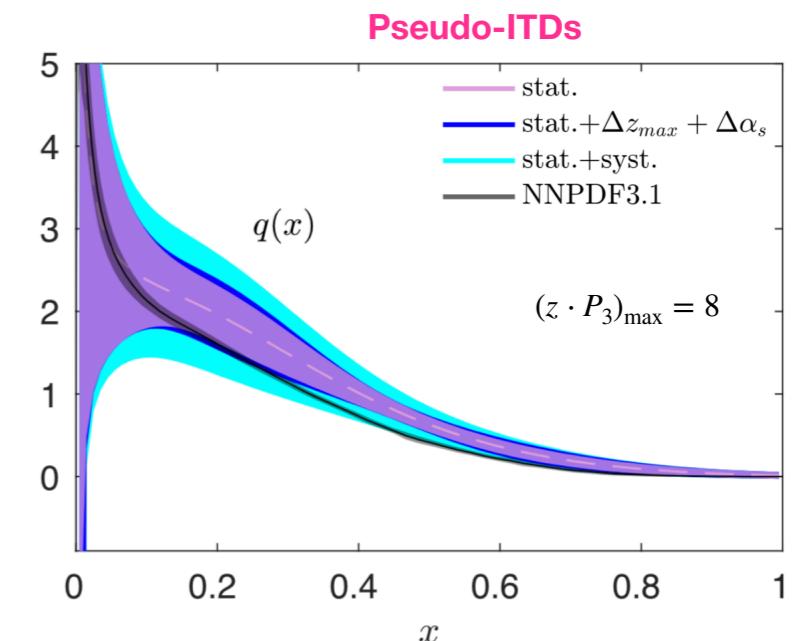
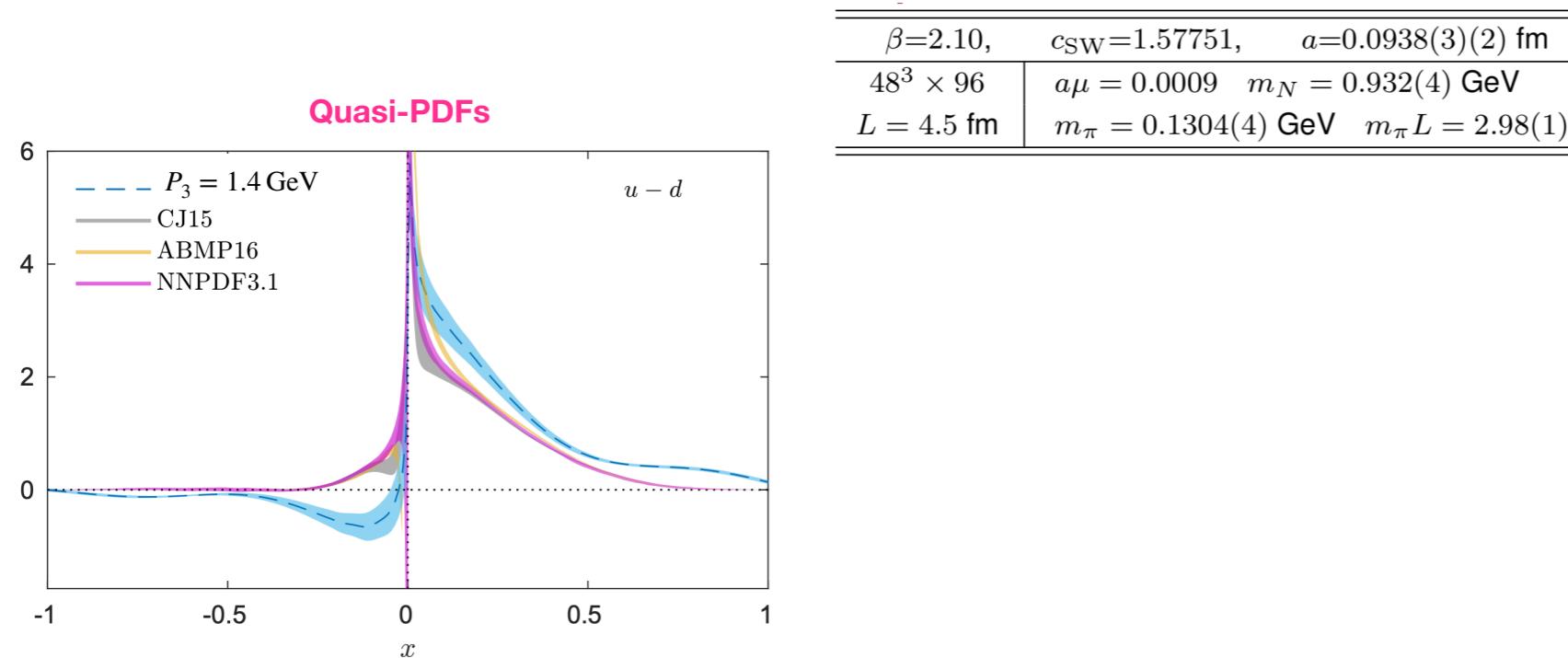


[B. Joo et al. (HasStruc), PRL 125, (2020) 23, arXiv:2004.01687]

- Approaching the physical point and increasing momentum leads to large statistical uncertainties.
- Regions with large errors: few data

# PDFs at physical pion mass

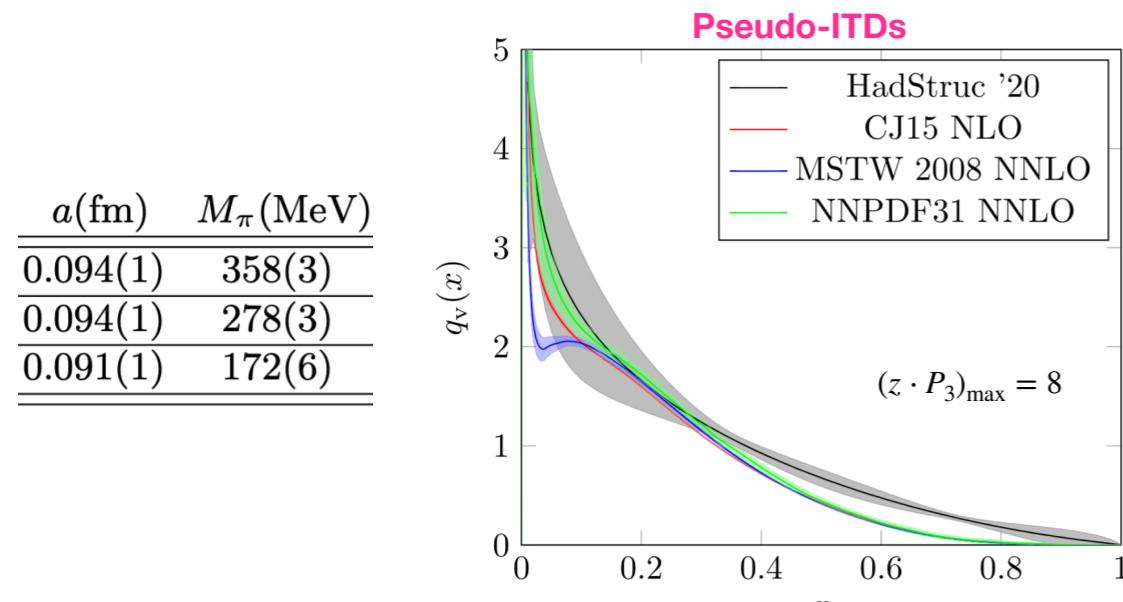
## ★ First complete study of quasi-PDFs including all steps



[C. Alexandrou et al. (ETMC), PRL 121 (2018) 112001, arXiv:1803.02685]

[M. Bhat et al. (ETMC), arXiv:2005.02102]

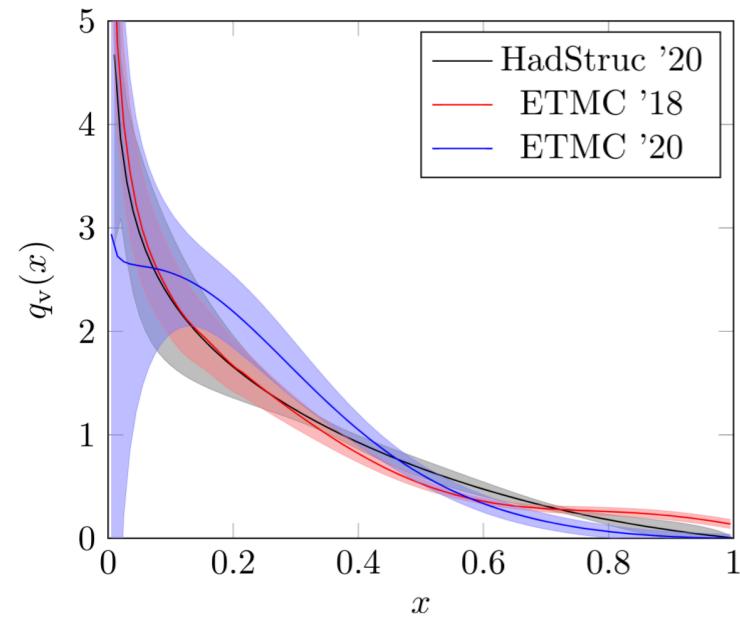
## ★ First complete study of pseudo-PDFs



[B. Joo et al. (HasStruc), PRL 125, (2020) 23, arXiv:2004.01687]

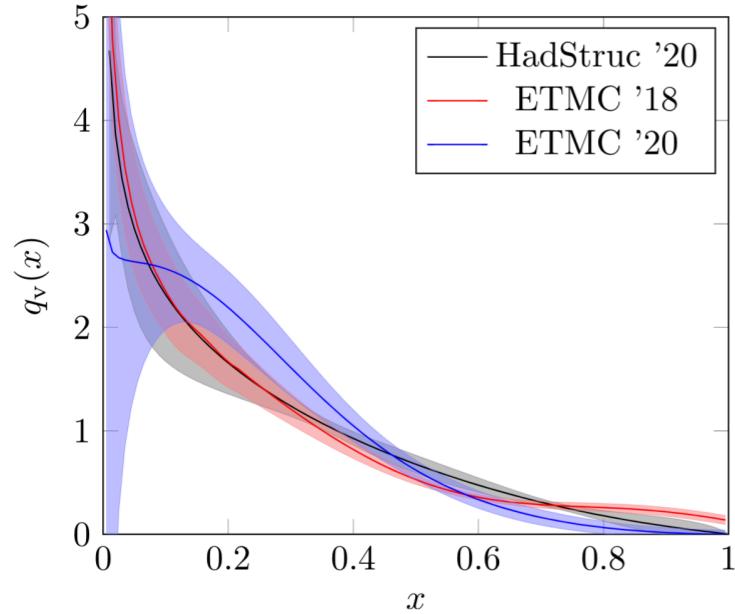
- Approaching the physical point and increasing momentum leads to large statistical uncertainties.
- Regions with large errors: few data

# Lattice data comparison



- Different formulations & same method  
**(clover, twisted mass)**
- Same formulation for different methods  
**(quasi, pseudo)**

# Lattice data comparison

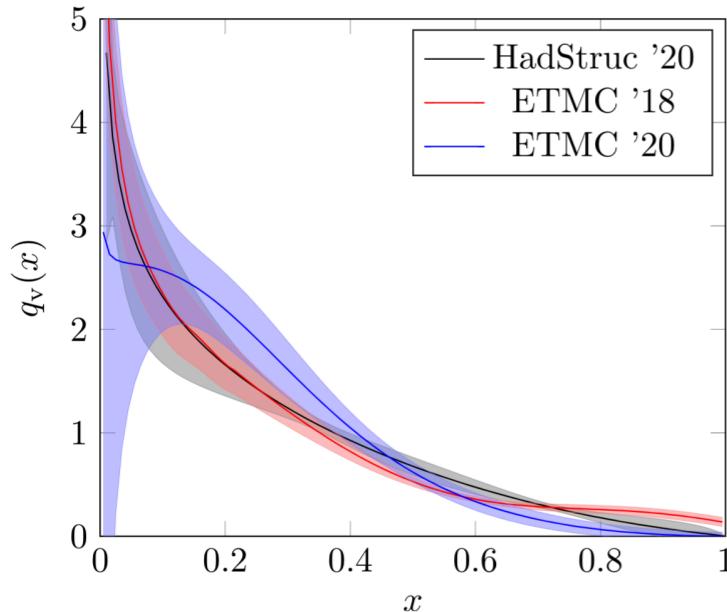


- Different formulations & same method  
(clover, **twisted mass**)
- Same formulation for different methods  
(**quasi**, **pseudo**)

## Incorporating lattice PDFs in global analyses

*Can lattice data assist in regions where experimental data are sparse, imprecise, or non-existing?*

# Lattice data comparison

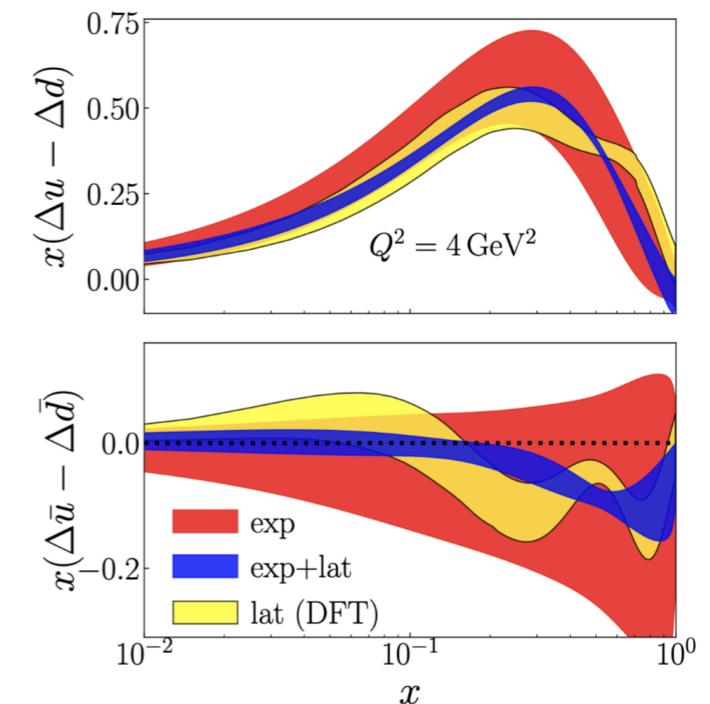


- Different formulations & same method  
**(clover, twisted mass)**
- Same formulation for different methods  
**(quasi, pseudo)**

## Incorporating lattice PDFs in global analyses

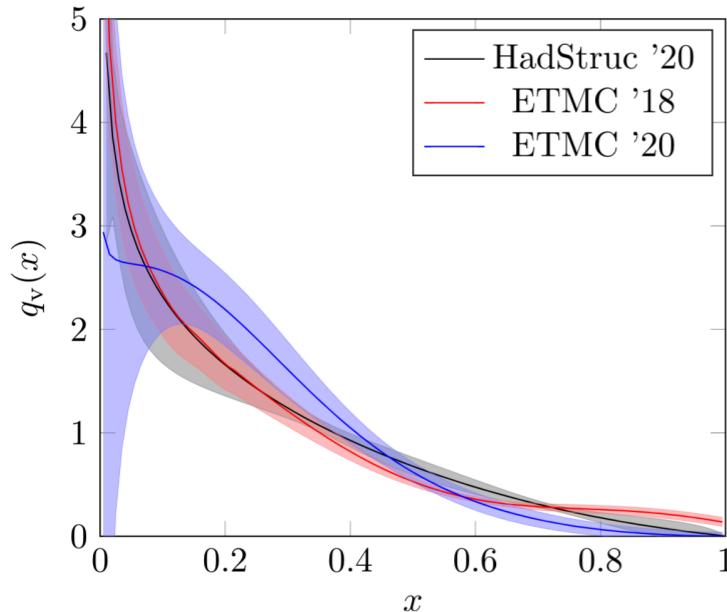
*Can lattice data assist in regions where experimental data are sparse, imprecise, or non-existing?*

- ★ Lattice and exp. data sets data within the same global analysis
- ★ Significant impact for helicity PDF
- ★ Consistent picture with JAM for unpolarized PDF



[J. Bringewatt, N. Sato, W. Melnitchouk, J. Qiu, F. Steffens, M. Constantinou, PRD 103 (2021) 016003, arXiv:2010.00548]

# Lattice data comparison



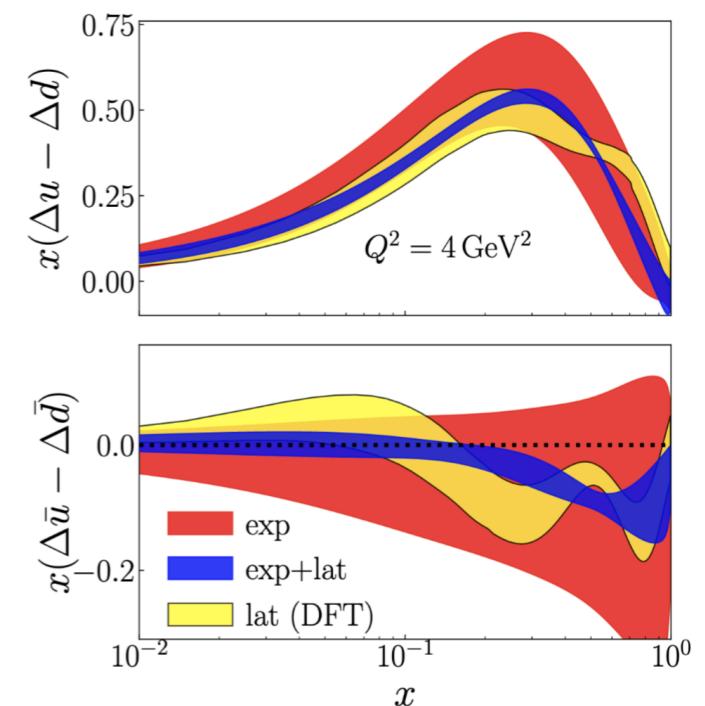
- Different formulations & same method  
(clover, twisted mass)
- Same formulation for different methods  
(quasi, pseudo)

## Incorporating lattice PDFs in global analyses

*Can lattice data assist in regions where experimental data are sparse, imprecise, or non-existing?*

- ★ Lattice and exp. data sets data within the same global analysis
- ★ Significant impact for helicity PDF
- ★ Consistent picture with JAM for unpolarized PDF

[J. Bringewatt, N. Sato, W. Melnitchouk, J. Qiu, F. Steffens, M. Constantinou, PRD 103 (2021) 016003, arXiv:2010.00548]



## ★ Other efforts from NNPDF collaboration:

[K. Cichy, L. Del Debbio, T. Giani, JHEP 10 (2019) 137, arXiv:1907.06037]

[L. Del Debbio, T. Giani, Karpie, K. Orginos, A. Radyushkin, S. Zafeiropoulos, arXiv:1907.06037]

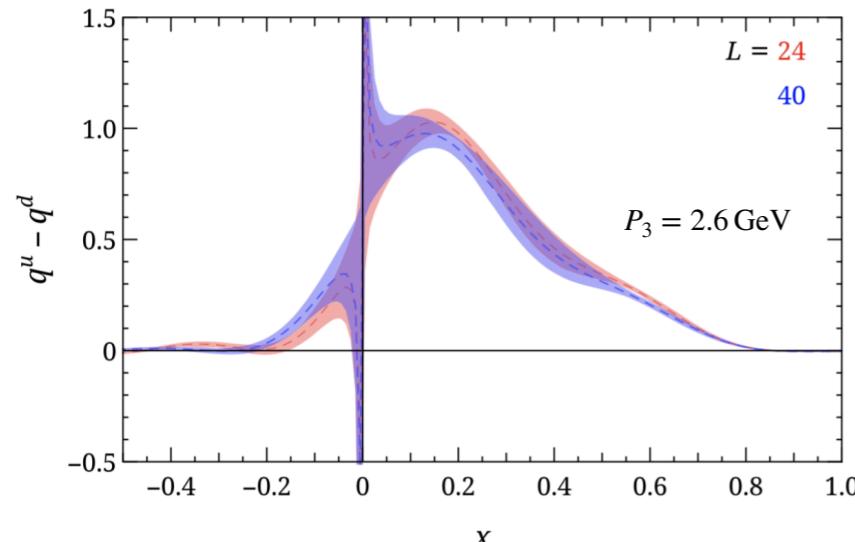
# Recent focus: Systematic Uncertainties

# Recent focus: Systematic Uncertainties

## ★ Finite-volume effects (Could be sizable)

[R. Briceno et al., PRD 98 (2018) 014511, arXiv:1805.01034]

Ensembles:  $a=0.12 \text{ fm}$ ,  $L=2.88, 3.84, 4.8 \text{ fm}$  ( $m_\pi=220 \text{ MeV}$ )



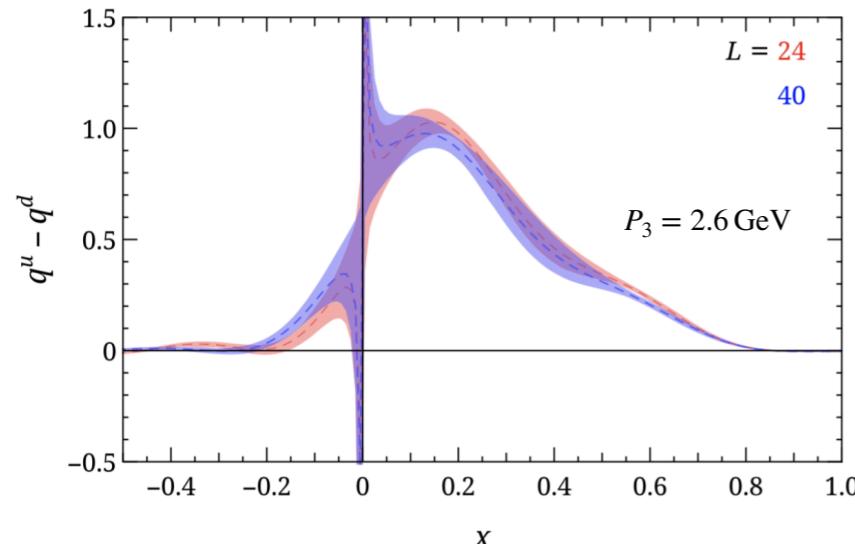
[Lin & Zhang, PRD 100 (2019) 074502]

# Recent focus: Systematic Uncertainties

## ★ Finite-volume effects (Could be sizable)

[R. Briceno et al., PRD 98 (2018) 014511, arXiv:1805.01034]

Ensembles:  $a=0.12$  fm , $L=2.88, 3.84, 4.8$  fm ( $m_\pi=220$  MeV)

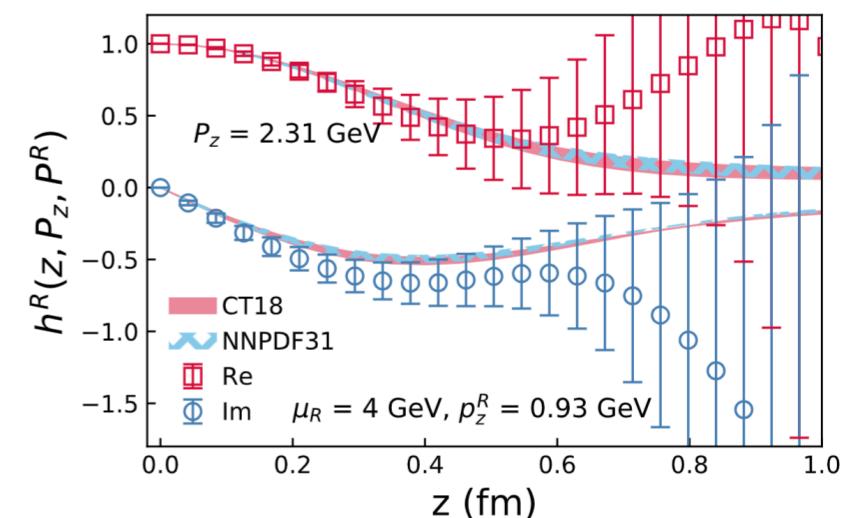


[Lin & Zhang, PRD 100 (2019) 074502]

## ★ Discretization effects

[Green et al., PRD 101 (2020) 7, 074509, arXiv:2002.09408]

Ensembles:  $a=0.042$  fm,  $L=2.7$  fm ( $m_\pi=310$  MeV)



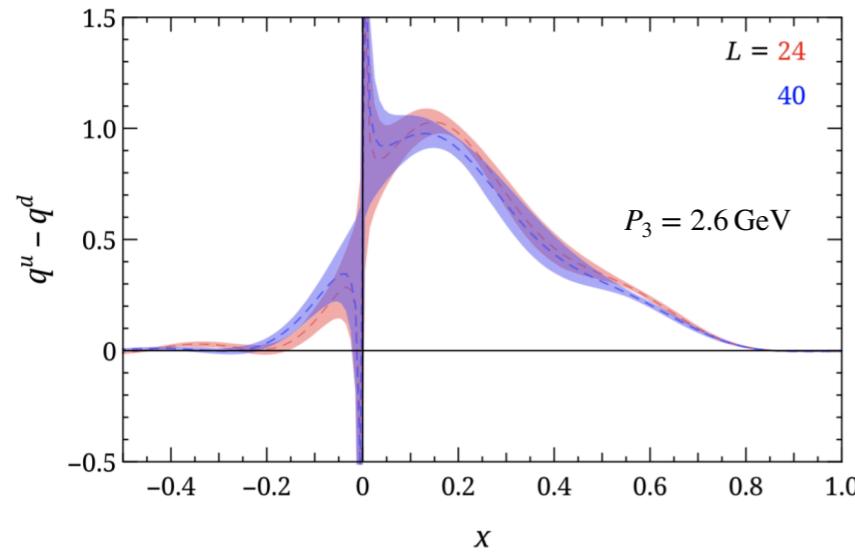
[Fan et al., PRD 102 (2020) 074504, arXiv:2005.12015]

# Recent focus: Systematic Uncertainties

## ★ Finite-volume effects (Could be sizable)

[R. Briceno et al., PRD 98 (2018) 014511, arXiv:1805.01034]

Ensembles:  $a=0.12$  fm , $L=2.88, 3.84, 4.8$  fm ( $m_\pi=220$  MeV)

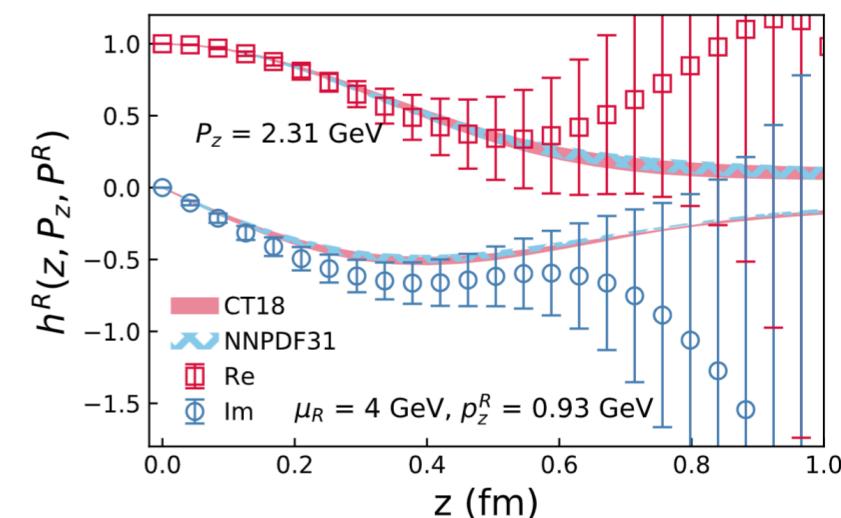


[Lin & Zhang, PRD 100 (2019) 074502]

## ★ Discretization effects

[Green et al., PRD 101 (2020) 7, 074509, arXiv:2002.09408]

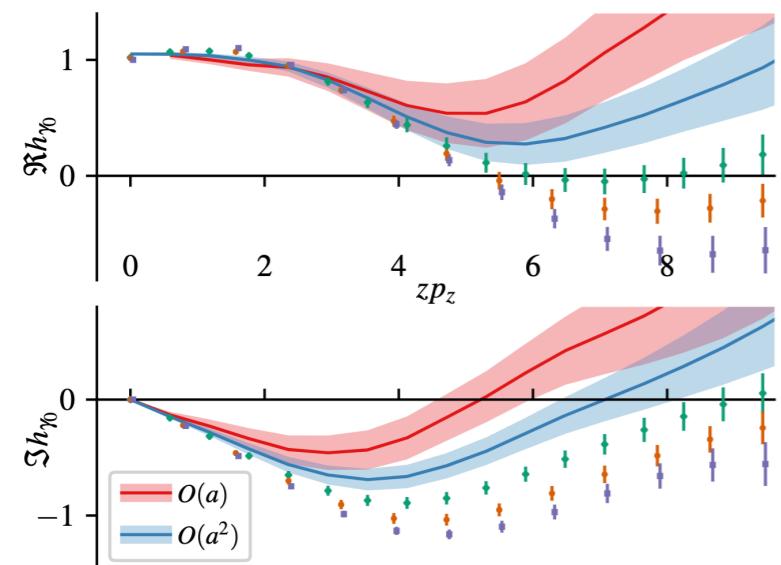
Ensembles:  $a=0.042$  fm,  $L=2.7$  fm ( $m_\pi=310$  MeV)



[Fan et al., PRD 102 (2020) 074504, arXiv:2005.12015]

## ★ Continuum limit

Ensembles:  $a=0.094, 0.082, 0.064$  fm , $L=2.2$  fm ( $m_\pi\sim 370$  MeV)



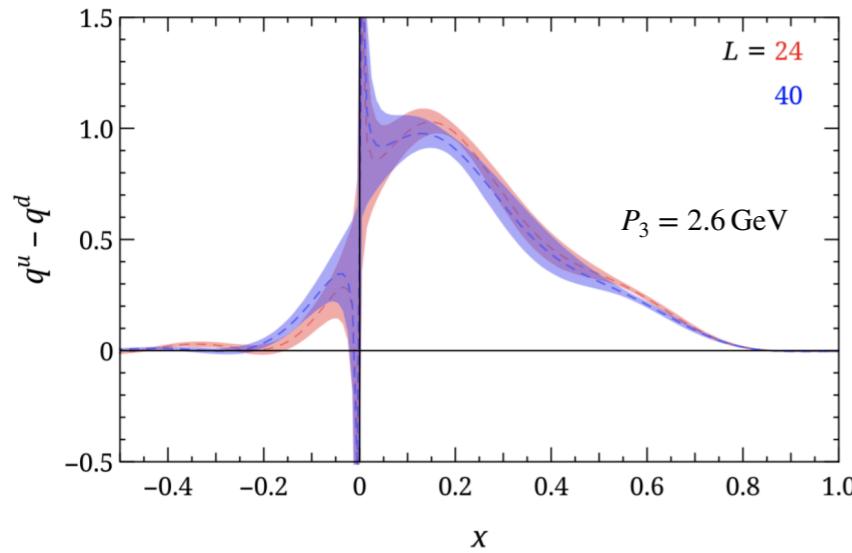
[Alexandrou et al., PRD 103 (2021) 094512, arXiv:2011..00964]

# Recent focus: Systematic Uncertainties

## ★ Finite-volume effects (Could be sizable)

[R. Briceno et al., PRD 98 (2018) 014511, arXiv:1805.01034]

Ensembles:  $a=0.12$  fm,  $L=2.88, 3.84, 4.8$  fm ( $m_\pi=220$  MeV)

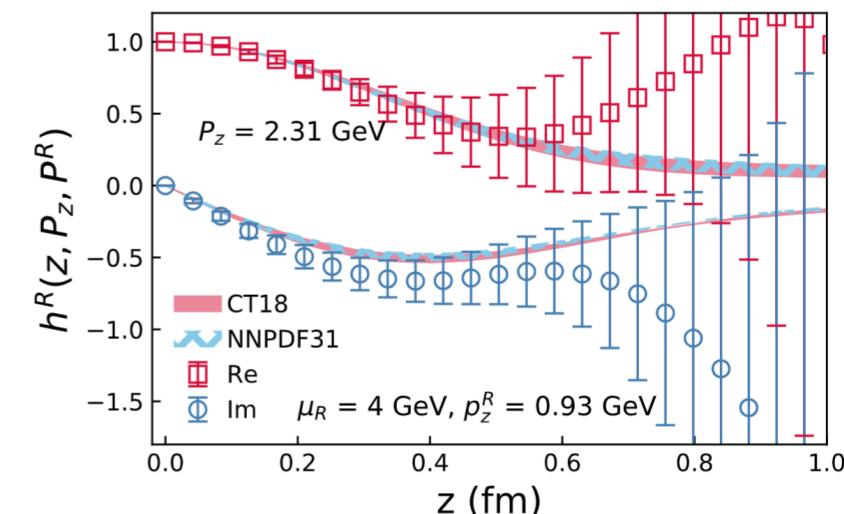


[Lin & Zhang, PRD 100 (2019) 074502]

## ★ Discretization effects

[Green et al., PRD 101 (2020) 7, 074509, arXiv:2002.09408]

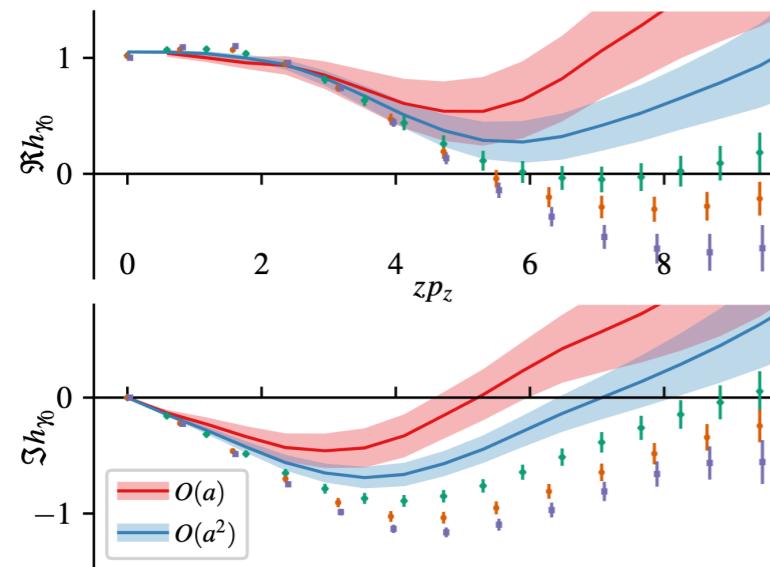
Ensembles:  $a=0.042$  fm,  $L=2.7$  fm ( $m_\pi=310$  MeV)



[Fan et al., PRD 102 (2020) 074504, arXiv:2005.12015]

## ★ Continuum limit

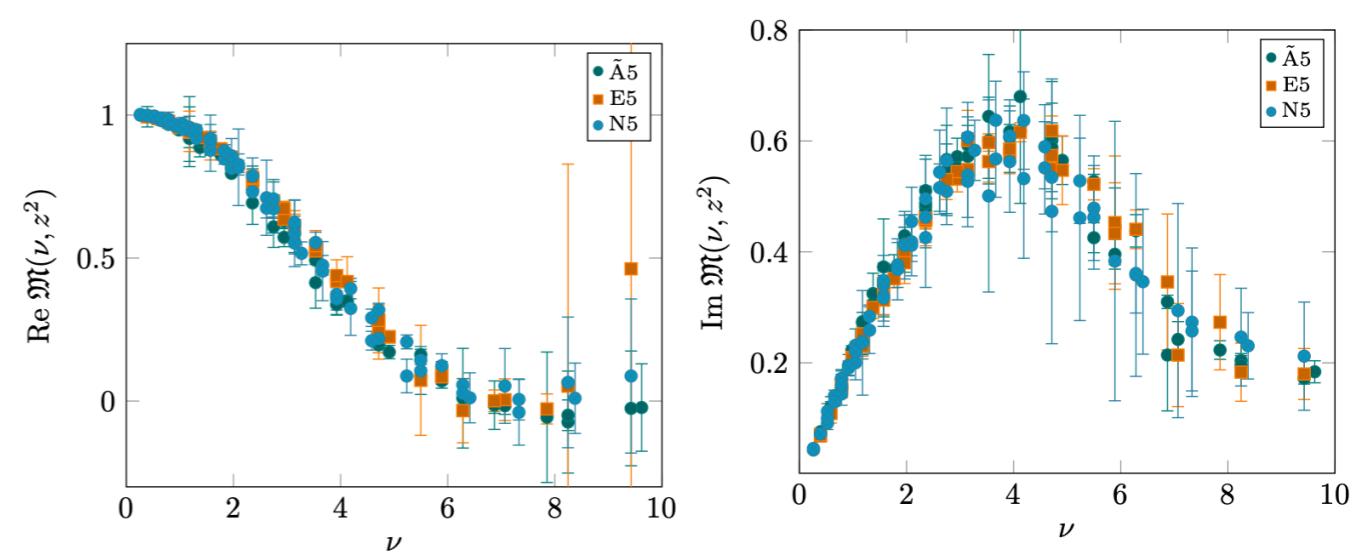
Ensembles:  $a=0.094, 0.082, 0.064$  fm,  $L=2.2$  fm ( $m_\pi \sim 370$  MeV)



[Alexandrou et al., PRD 103 (2021) 094512, arXiv:2011..00964]

## ★ Continuum limit - higher twist effects

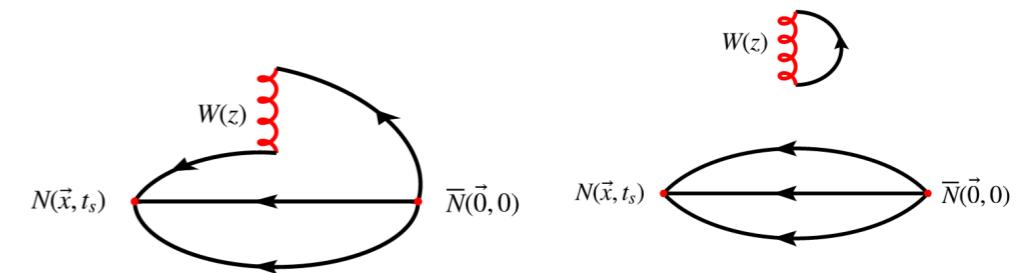
Ensembles:  $a=0.075, 0.065, 0.048$  fm,  $L=2.4$  fm ( $m_\pi \sim 440$  MeV)



[Karpie et al., arXiv:2105.13313]

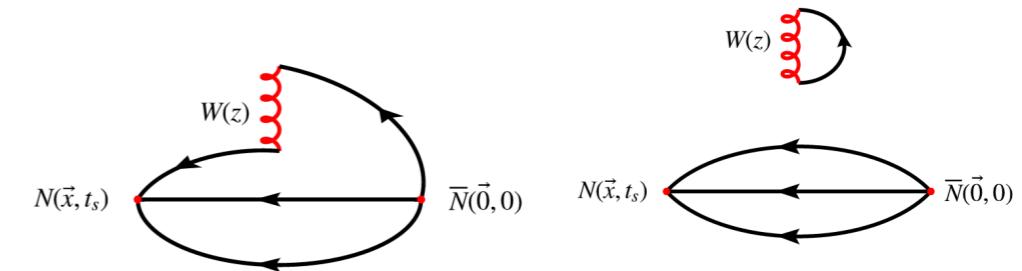
# Flavor decomposition of PDFs

- ★ Major achievement of the field  
(due to theoretical and algorithmic developments)

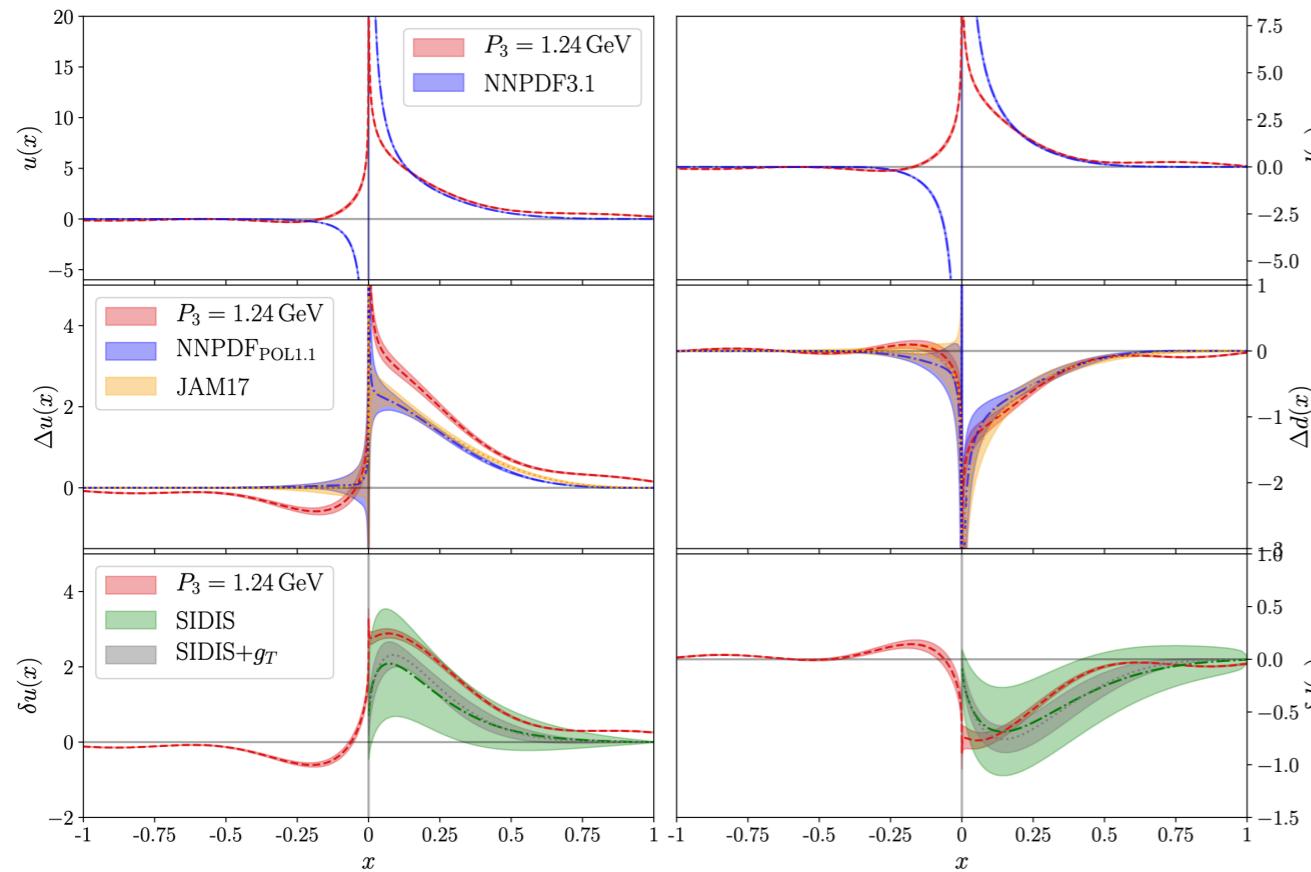


# Flavor decomposition of PDFs

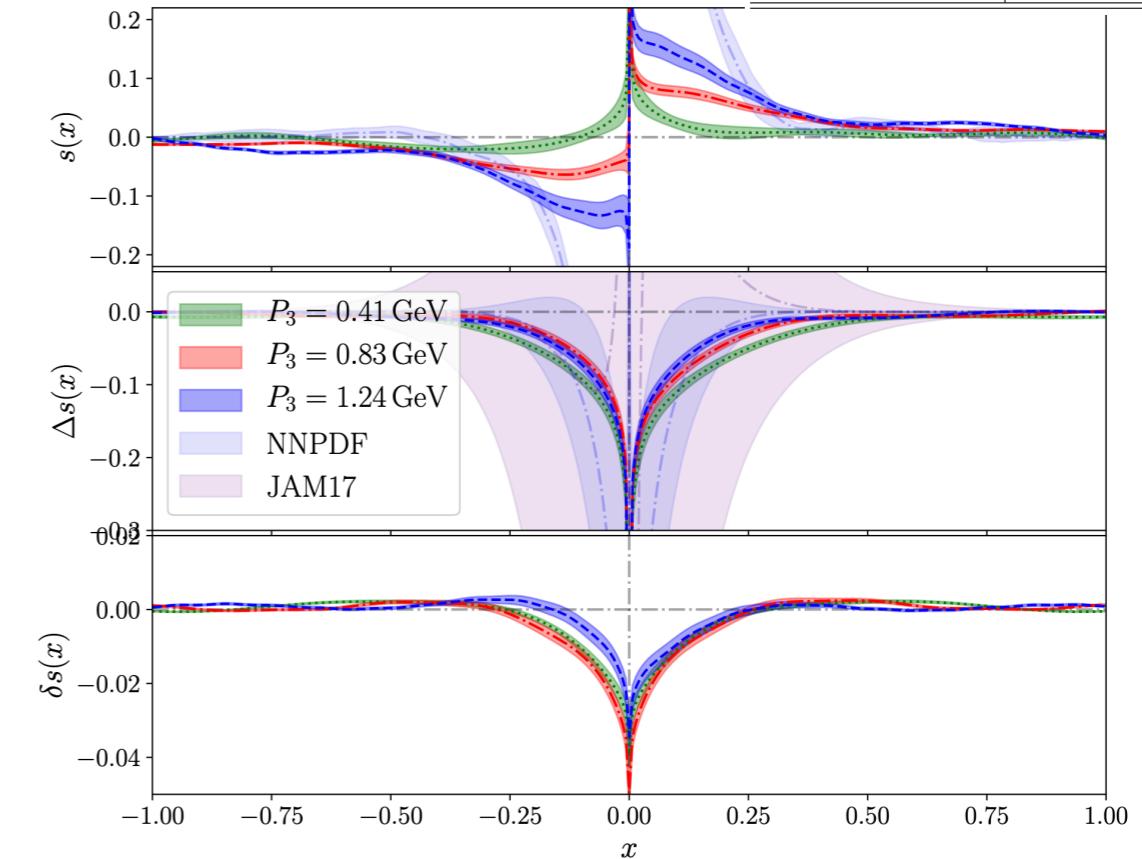
- ★ Major achievement of the field  
(due to theoretical and algorithmic developments)



- ★ Individual-quark unpolarized, helicity, transversity PDFs
- ★ At least one order higher statistics for disconnected diagrams



$\beta = 1.726, c_{\text{sw}} = 1.74, a = 0.0938(3)(2) \text{ fm}$
$a\mu_l = 0.003$
$m_\pi \approx 260 \text{ MeV}$
$m_\pi L \approx 3$
$m_N = 1.09(6) \text{ GeV}$



[C. Alexandrou et al., PRL 126 (2021) 10, arXiv:2009.13061; C. Alexandrou et al., arXiv:2106.16065]

- ★ Clear signal for strange-quark PDFs (purely disconnected)
- ★ Helicity PDF: most sizable disconnected light-quark contributions
- ★ Mixing with gluon PDFs to be addressed

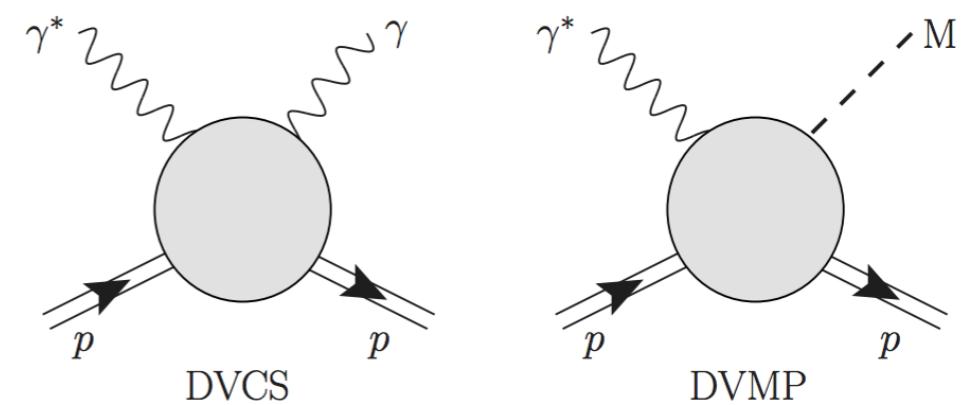
# GPDs

- ★ GPDs provide information on spatial distribution of partons inside the hadron, and its mechanical properties (OAM, pressure, etc.)  
[M. Burkardt, Phys. Rev. D62 071503 (2000), hep-ph/0005108]  
[M. V. Polyakov, Phys. Lett. B555 (2003) 57, hep-ph/0210165]

- ★ Experimentally accessed in DVCS and DVMP

[X. D. Ji, Phys. Rev. Lett. 78, 610 (1997), hep-ph/9603249]

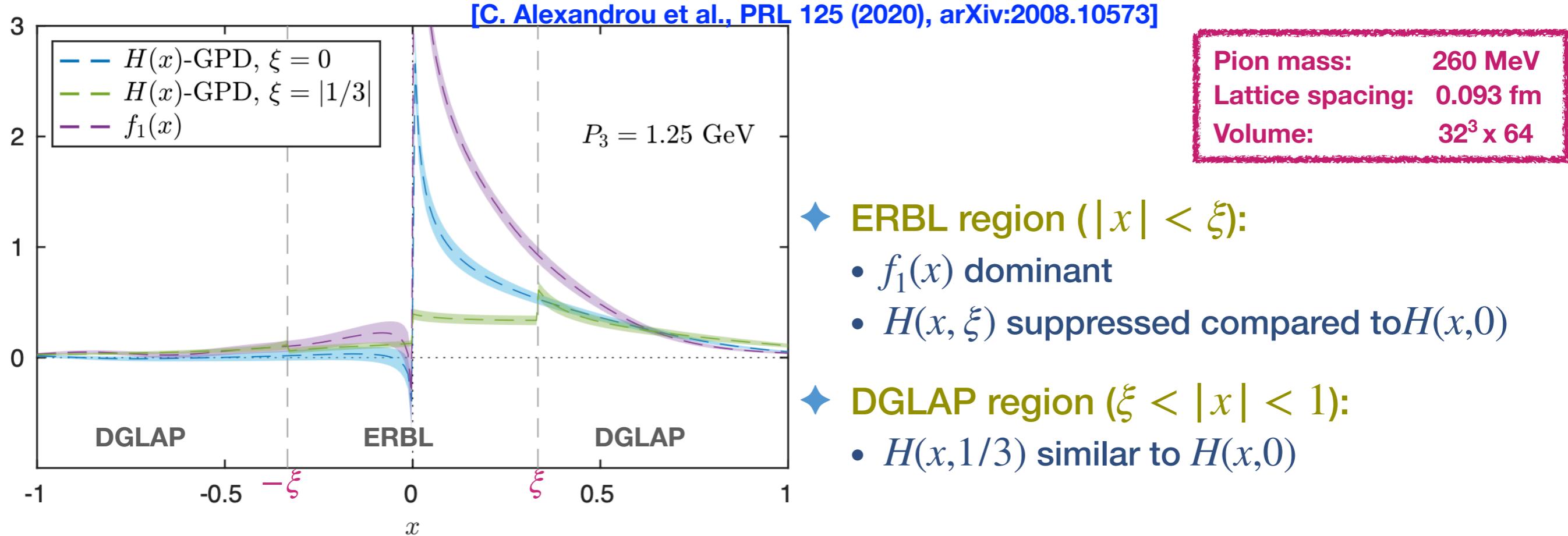
(Halls A,B,C (JLab),  
PHENIX, STAR, HERMES,  
COMPASS, GSI, BELLE, J-PARC)



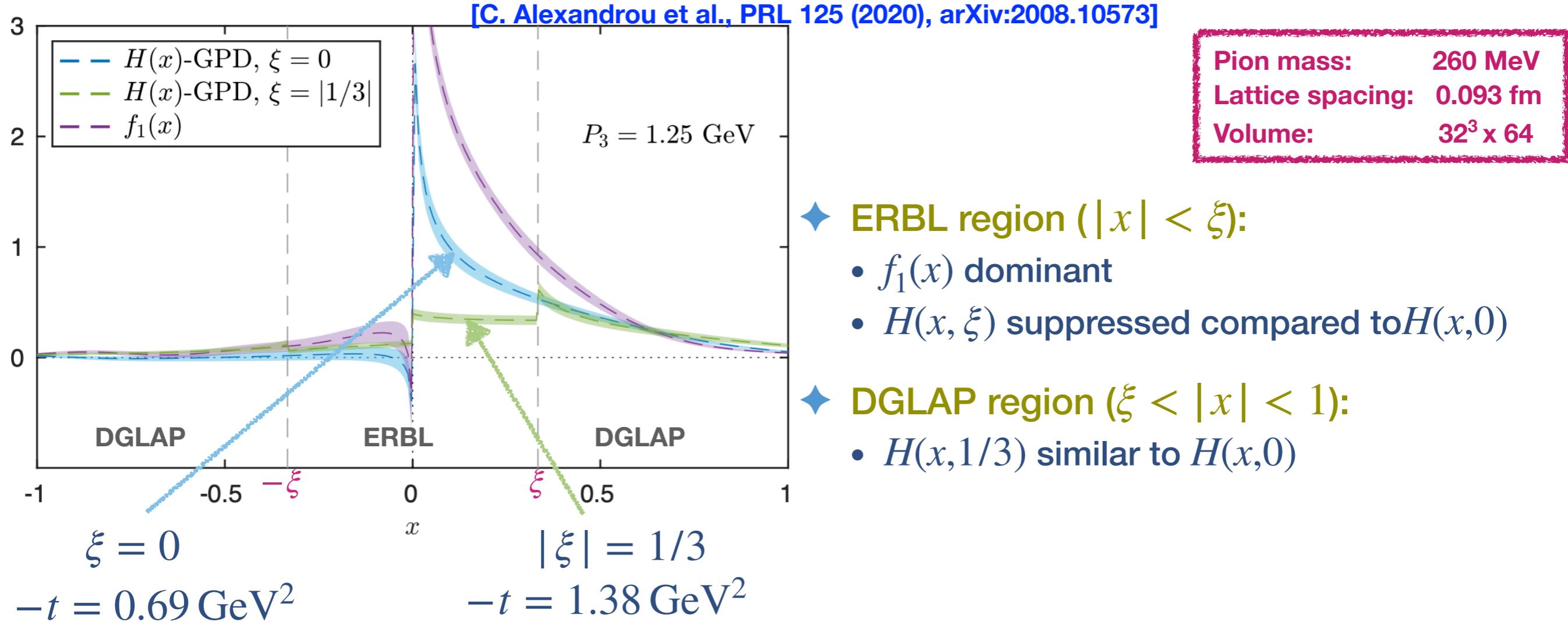
- ★ Experimentally, GPDs are not well-constrained:

- independent measurements to disentangle GPDs
- limited coverage of kinematic region
- data on certain GPDs
- indirectly related to GPDs through the Compton FFs
- GPDs phenomenology more complicated than PDFs (multi-dimensionality)

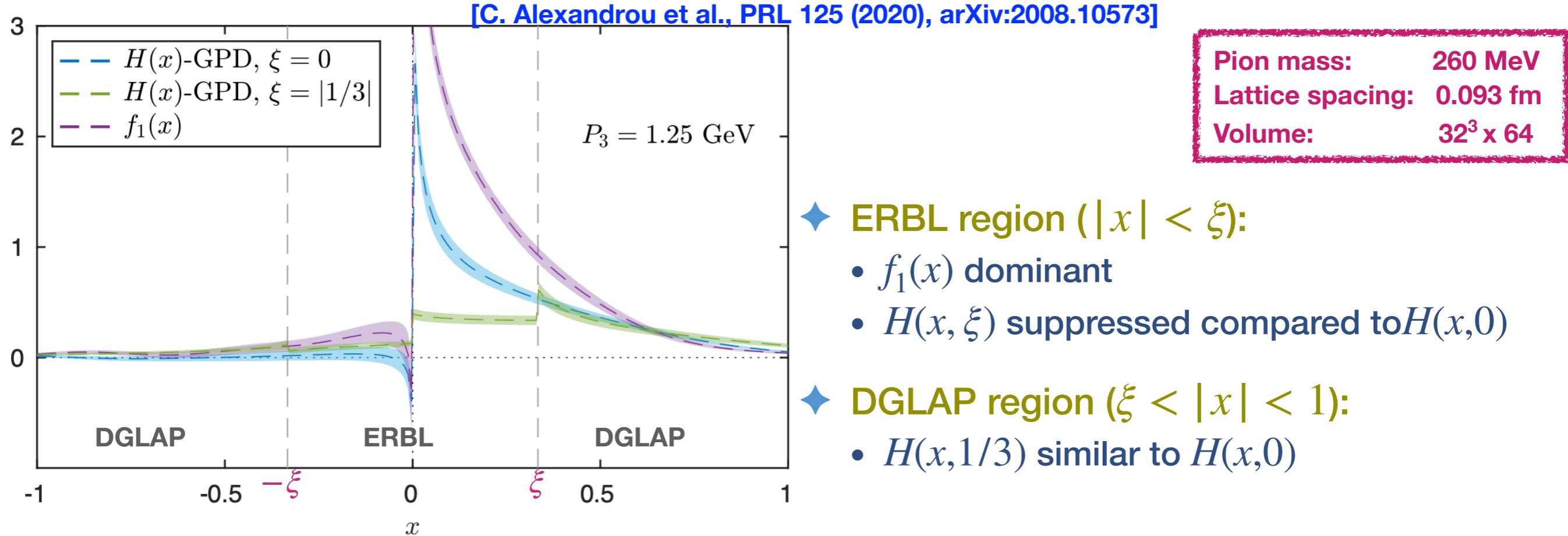
# Proton unpolarized & helicity GPDs



# Proton unpolarized & helicity GPDs

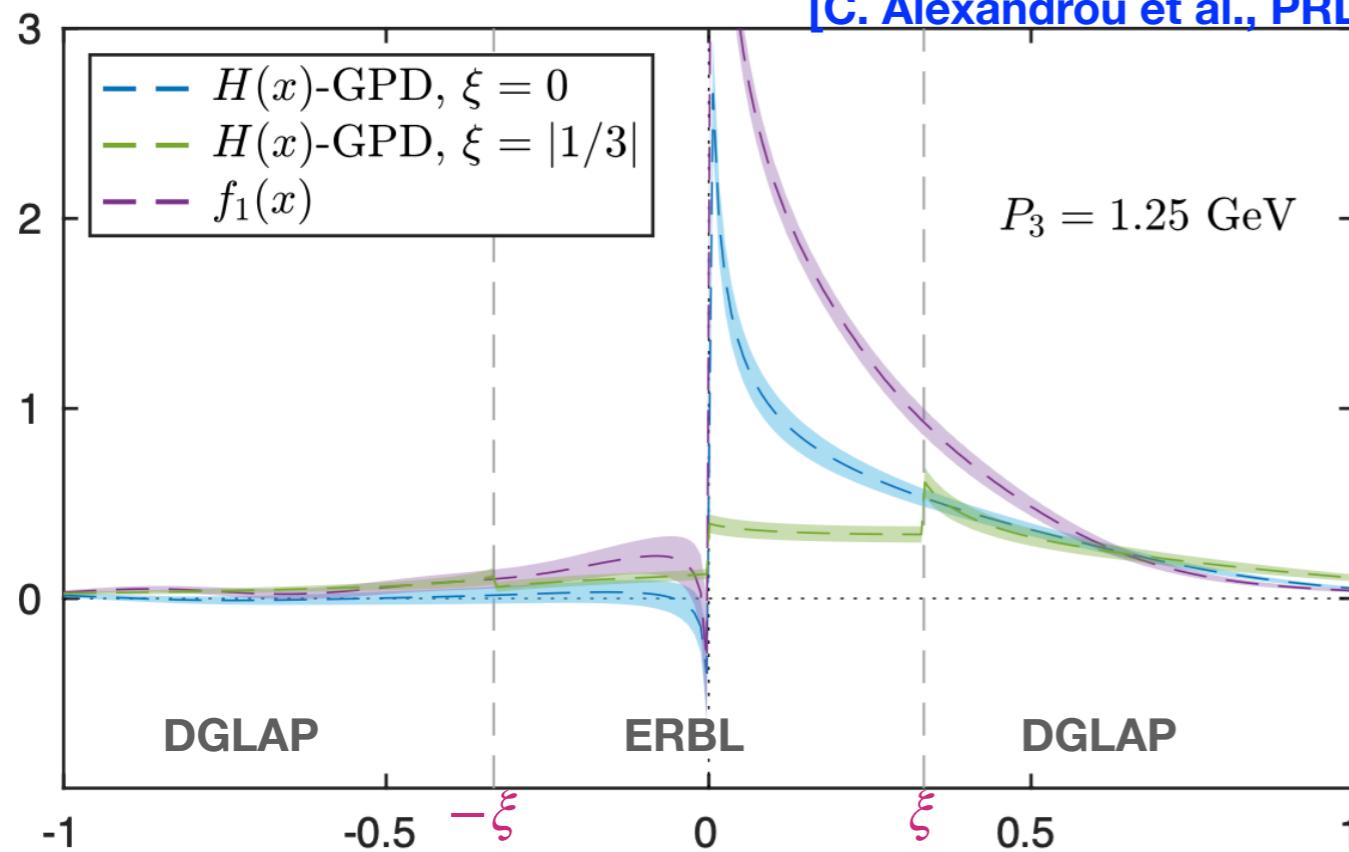


# Proton unpolarized & helicity GPDs



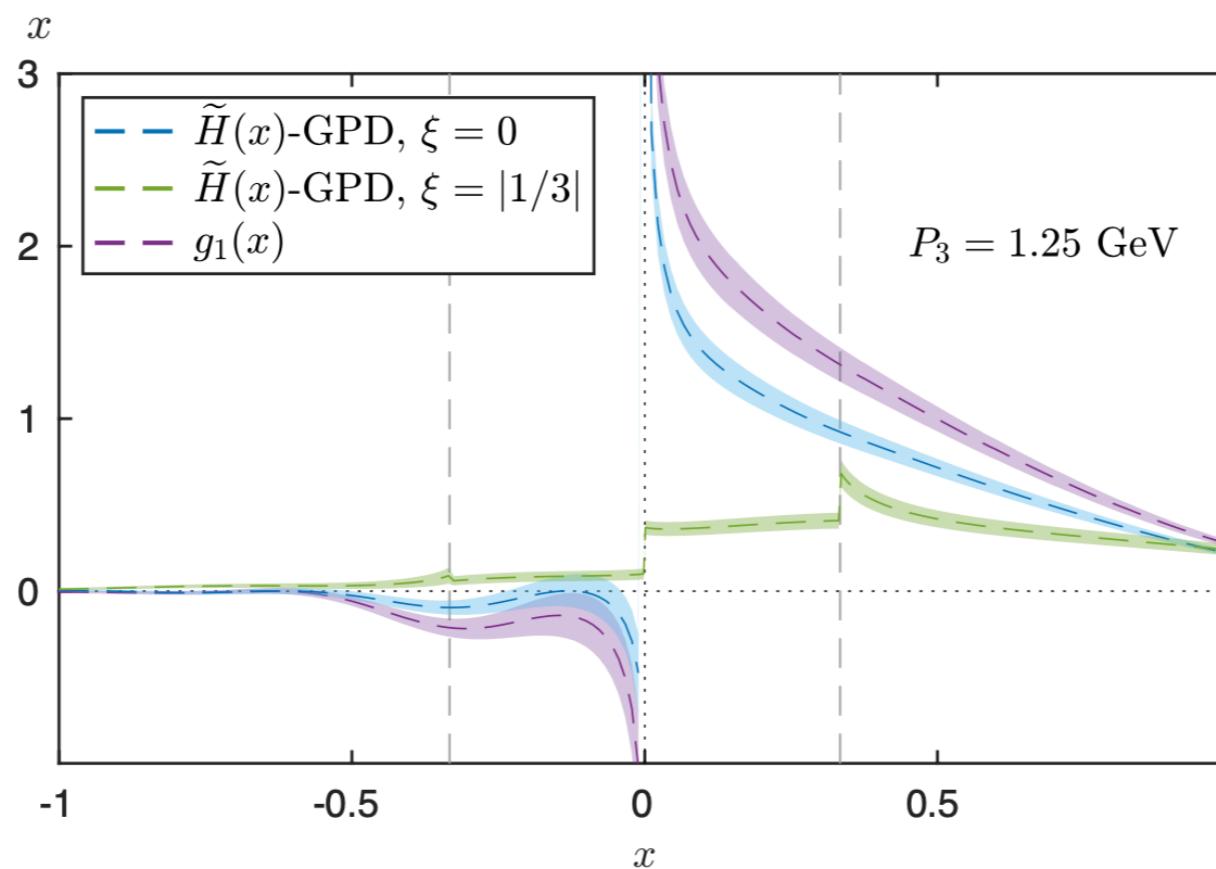
# Proton unpolarized & helicity GPDs

[C. Alexandrou et al., PRL 125 (2020), arXiv:2008.10573]



Pion mass:	260 MeV
Lattice spacing:	0.093 fm
Volume:	$32^3 \times 64$

- ◆ ERBL region ( $|x| < \xi$ ):
  - $f_1(x)$  dominant
  - $H(x, \xi)$  suppressed compared to  $H(x, 0)$
- ◆ DGLAP region ( $\xi < |x| < 1$ ):
  - $H(x, 1/3)$  similar to  $H(x, 0)$



# Twist-classification of PDFs

$$f_i = f_i^{(0)} + \frac{f_i^{(1)}}{Q} + \frac{f_i^{(2)}}{Q^2} \dots$$

## Higher-twist contributions:

- Lack density interpretation, but can be sizable
- Sensitive to soft dynamics
- challenging to probe experimentally and isolate from leading-twist  
[Defurne et al., PRL 117, 26 (2016); Defurne et al., Nature Commun. 8, 1 (2017)]

# Twist-classification of PDFs

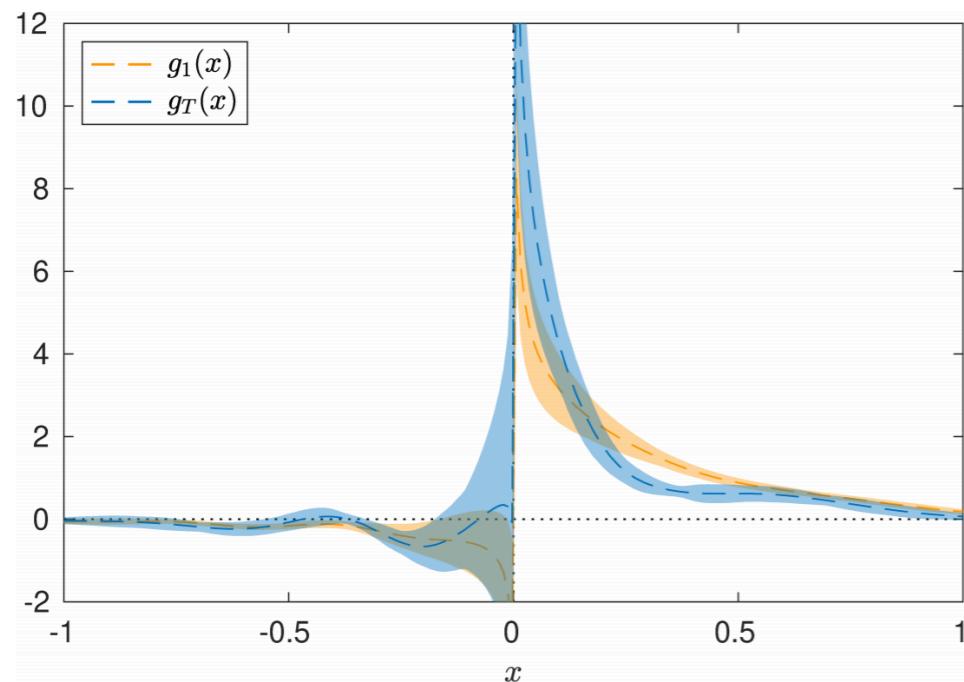
$$f_i = f_i^{(0)} + \frac{f_i^{(1)}}{Q} + \frac{f_i^{(2)}}{Q^2} \dots$$

## Higher-twist contributions:

- Lack density interpretation, but can be sizable
- Sensitive to soft dynamics
- challenging to probe experimentally and isolate from leading-twist  
[Defurne et al., PRL 117, 26 (2016); Defurne et al., Nature Commun. 8, 1 (2017)]
- $g_2(x)$  ( $g_2(x) = g_1(x) + g_T(x)$ ) related to the transverse force acting on the active quark in DIS off a transversely polarized nucleon immediately after it has absorbed the virtual photon
- $g_2(x)$  can be separated from twist-2 helicity PDF
- $h_L(x)$  accessed via di-hadron single spin asymmetries  
[Gliske et al., PRD 90 (2014) 11, 114027, arXiv:1408.5721]

# Twist-3 $g_T(x)$ & $h_L(x)$ PDF

Pion mass:	260 MeV
Lattice spacing:	0.093 fm
Volume:	$32^3 \times 64$



Twist-3 counterpart as sizable as twist-2

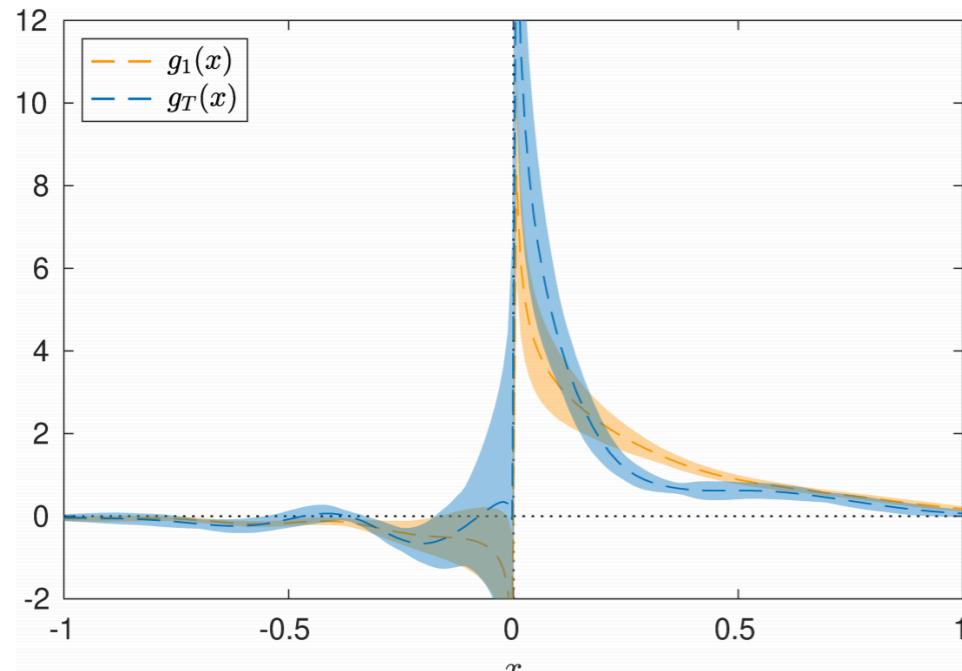
Burkhardt-Cottingham sum rule important check

$$\int_{-1}^1 dx g_1(x) - \int_{-1}^1 dx g_T(x) = 0.01(20)$$

[S. Bhattacharya et al., PRD 102 (2020) 11 (Editors Selection), arXiv:2004.04130]

# Twist-3 $g_T(x)$ & $h_L(x)$ PDF

Pion mass:	260 MeV
Lattice spacing:	0.093 fm
Volume:	$32^3 \times 64$



[S. Bhattacharya et al., PRD 102 (2020) 11 (Editors Selection), arXiv:2004.04130]

Twist-3 counterpart as sizable as twist-2

Burkhardt-Cottingham sum rule important check

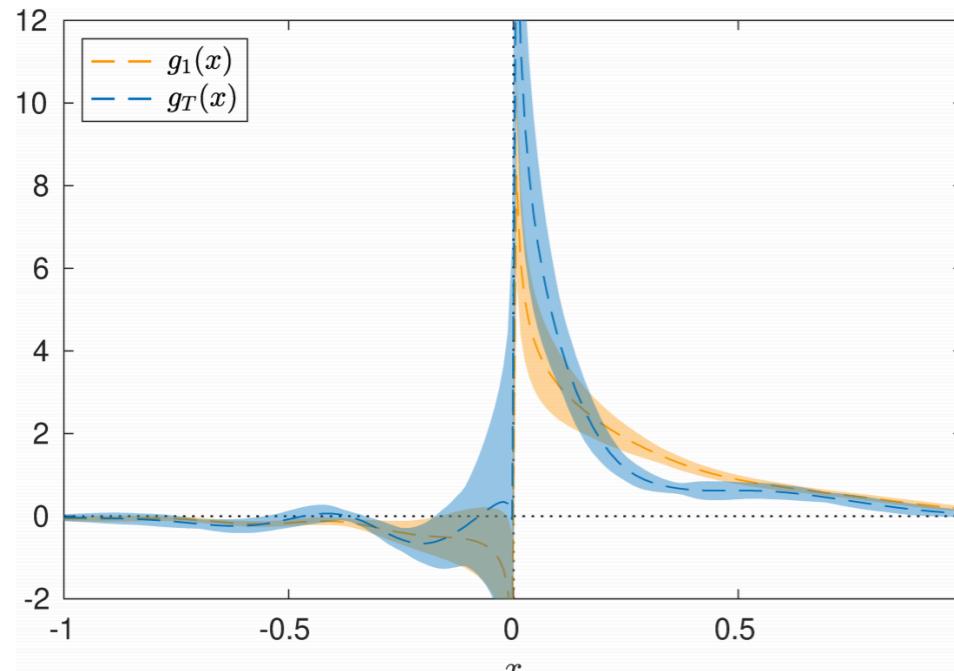
$$\int_{-1}^1 dx g_1(x) - \int_{-1}^1 dx g_T(x) = 0.01(20)$$

WW approximation:  $g_T^{\text{WW}}(x) = \int_x^1 \frac{dy}{y} g_1(y)$

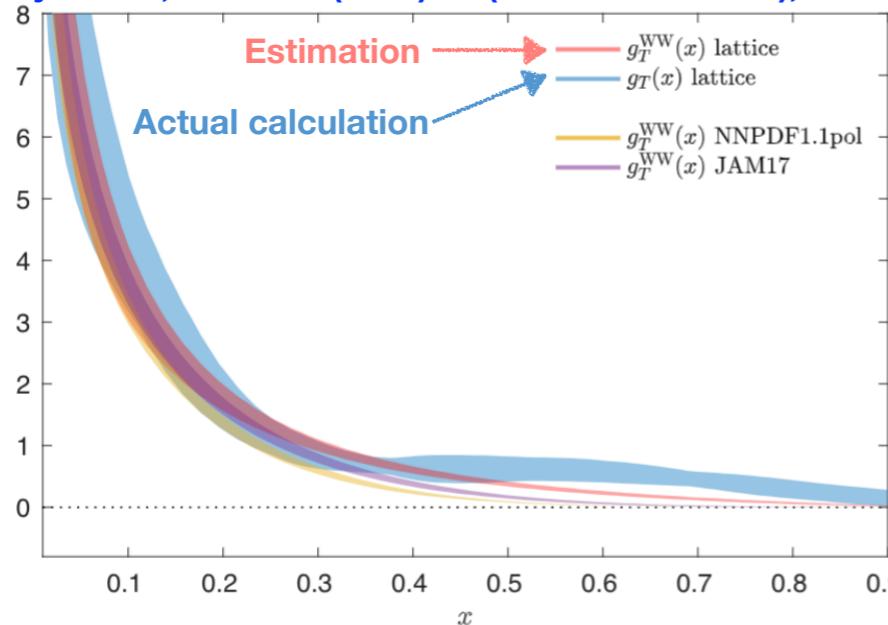
twist-3  $g_T(x)$  determined by the twist-2  $g_1(x)$

# Twist-3 $g_T(x)$ & $h_L(x)$ PDF

Pion mass:	260 MeV
Lattice spacing:	0.093 fm
Volume:	$32^3 \times 64$



[S. Bhattacharya et al., PRD 102 (2020) 11 (Editors Selection), arXiv:2004.04130]



- $g_T(x)$  agrees with  $g_T^{WW}(x)$  for  $x < 0.5$   
(violations up to 30-40% possible)
- Violations of 15-40% expected  
from experimental data

[A. Accardi et al., JHEP 11 (2009) 093]

Twist-3 counterpart as sizable as twist-2

Burkhardt-Cottingham sum rule important check

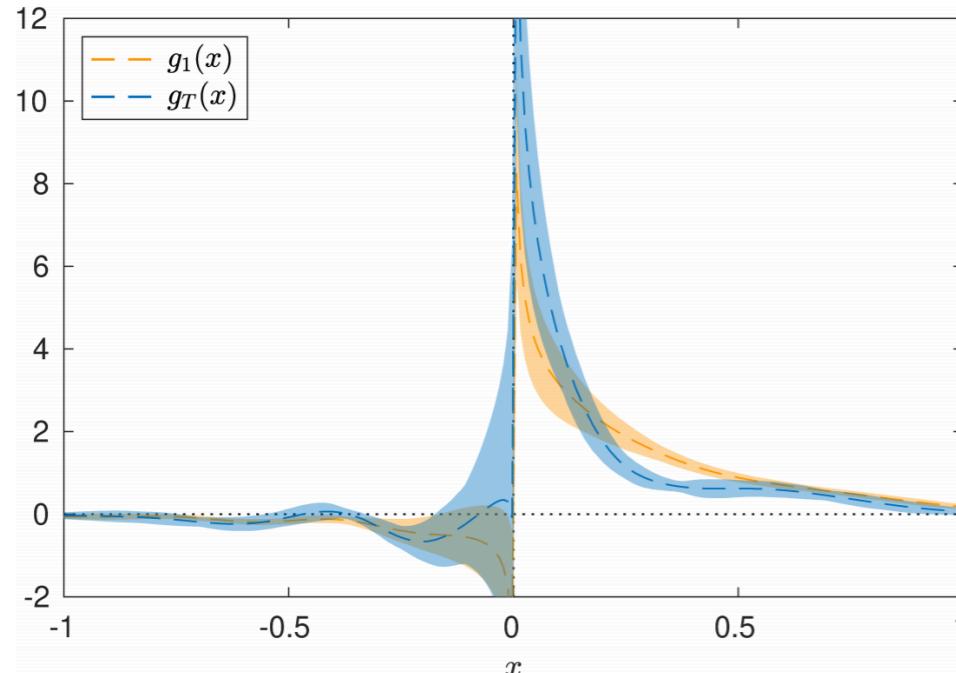
$$\int_{-1}^1 dx g_1(x) - \int_{-1}^1 dx g_T(x) = 0.01(20)$$

**WW approximation:**  $g_T^{WW}(x) = \int_x^1 \frac{dy}{y} g_1(y)$

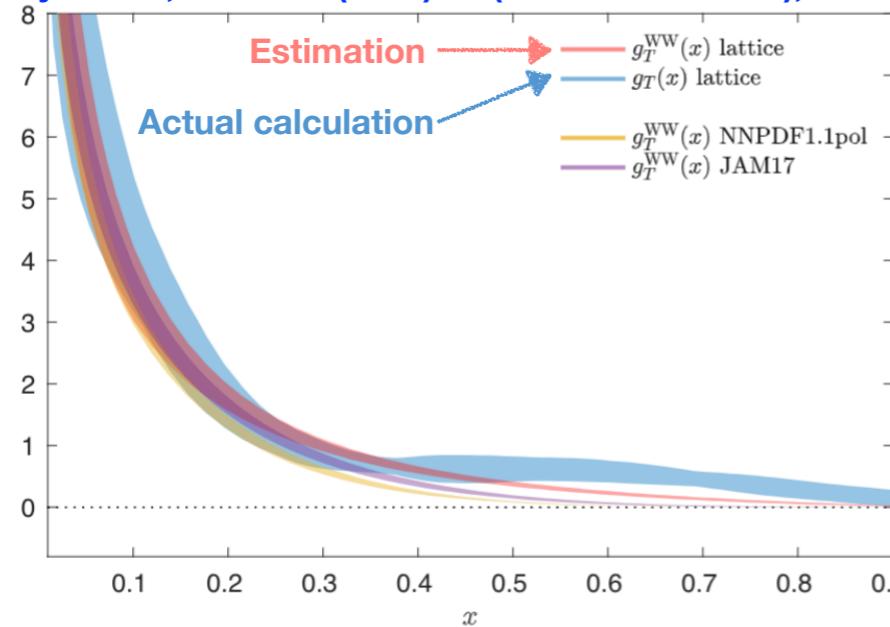
twist-3  $g_T(x)$  determined by the twist-2  $g_1(x)$

# Twist-3 $g_T(x)$ & $h_L(x)$ PDF

Pion mass: 260 MeV  
 Lattice spacing: 0.093 fm  
 Volume:  $32^3 \times 64$



[S. Bhattacharya et al., PRD 102 (2020) 11 (Editors Selection), arXiv:2004.04130]



- $g_T(x)$  agrees with  $g_T^{WW}(x)$  for  $x < 0.5$   
 (violations up to 30-40% possible)

[A. Accardi et al., JHEP 11 (2009) 093]

Twist-3 counterpart as sizable as twist-2

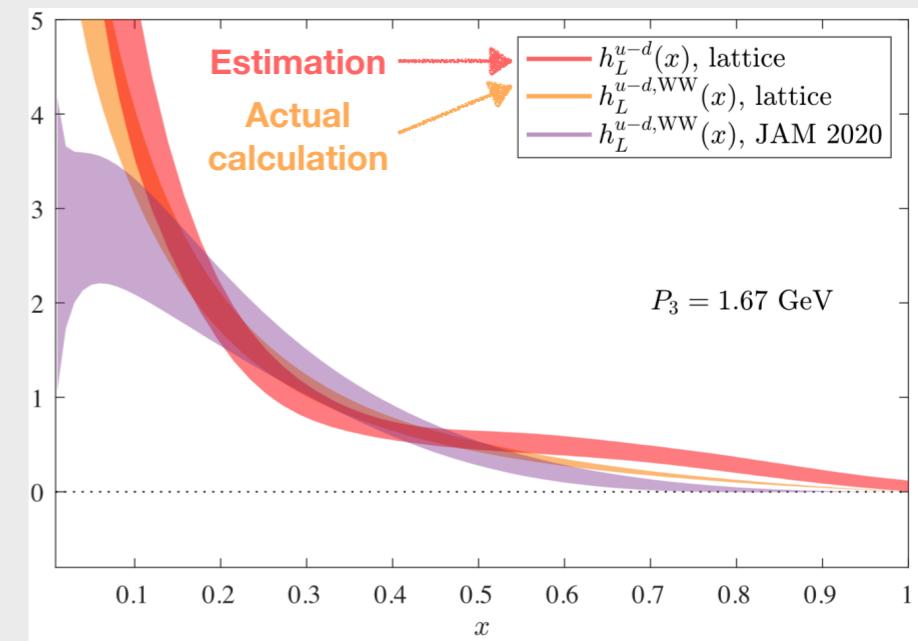
Burkhardt-Cottingham sum rule important check

$$\int_{-1}^1 dx g_1(x) - \int_{-1}^1 dx g_T(x) = 0.01(20)$$

WW approximation:  $g_T^{WW}(x) = \int_x^1 \frac{dy}{y} g_1(y)$

twist-3  $g_T(x)$  determined by the twist-2  $g_1(x)$

$$h_L^{WW}(x) = 2x \int_x^1 dy \frac{h_1(y)}{y^2}$$



[S. Bhattacharya et al., arXiv:2107.02574]

- Lattice data suggest that twist-3  $h_L(x)$  determined from twist-2 counterpart within uncertainties

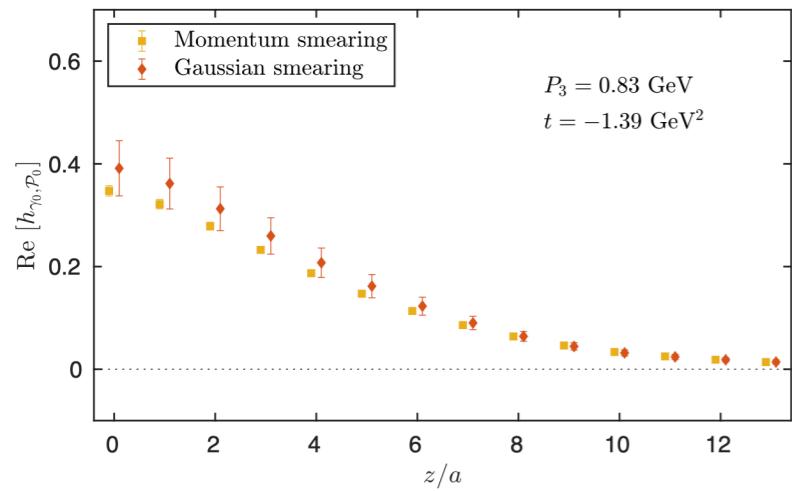
# Challenges

# Challenges of calculations

- ★ Statistical noise increases with  $P_3, t$   
use of momentum smearing method

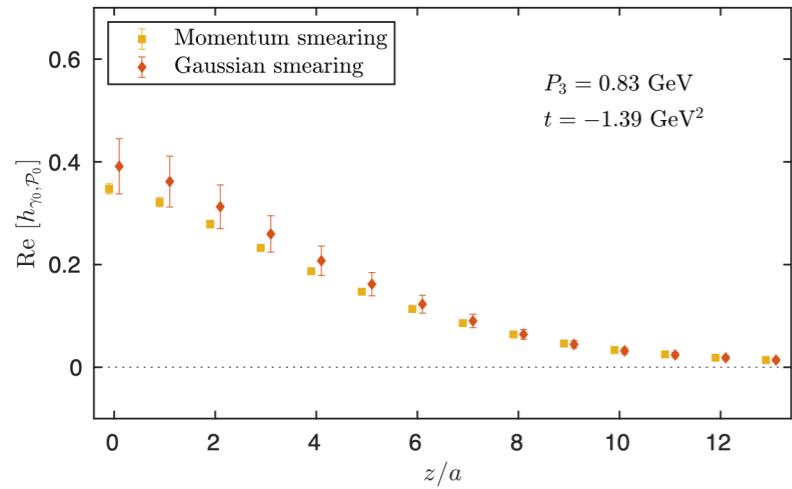
# Challenges of calculations

- ★ Statistical noise increases with  $P_3, t$   
use of momentum smearing method



# Challenges of calculations

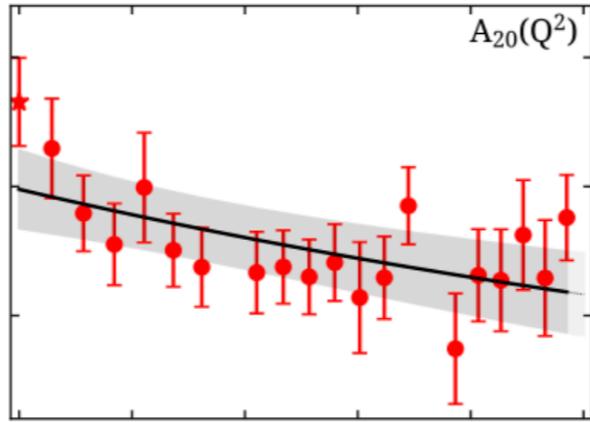
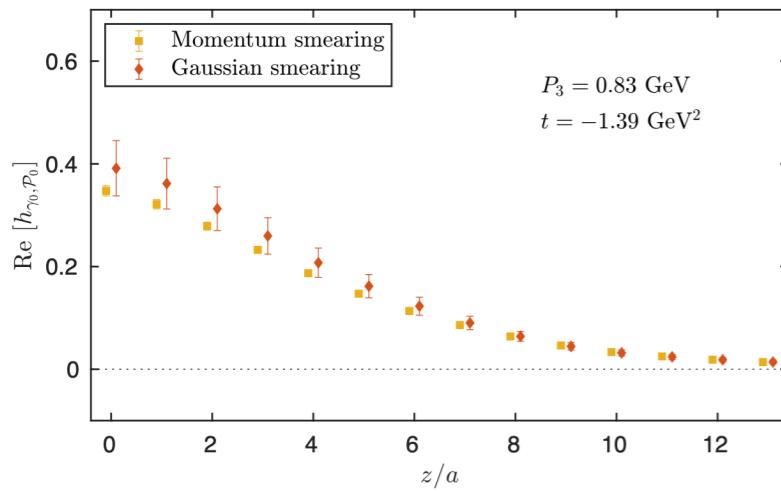
- ★ Statistical noise increases with  $P_3, t$   
use of momentum smearing method



- ◆ Implementation in GPDs nontrivial due to momentum transfer
- ◆ Standard definition of GPDs in Breit (symmetric) frame  
separate calculations at each  $t$
- ◆ Matrix elements decompose into more than one GPDs  
at least 2 parity projectors are needed to disentangle GPDs
- ◆ Nonzero skewness  
nontrivial matching
- ◆  $P_3$  must be chosen carefully due to UV cutoff ( $a^{-1} \sim 2 \text{ GeV}$ )

# Challenges of calculations

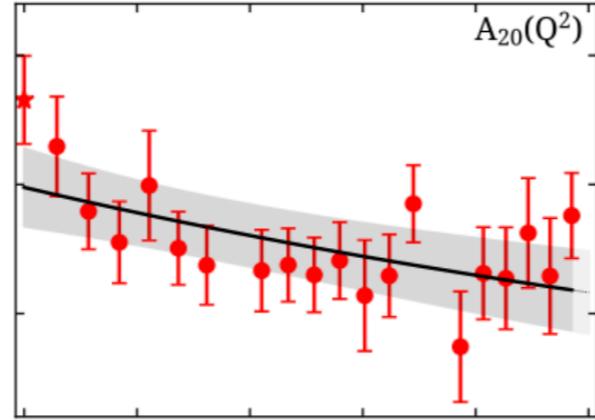
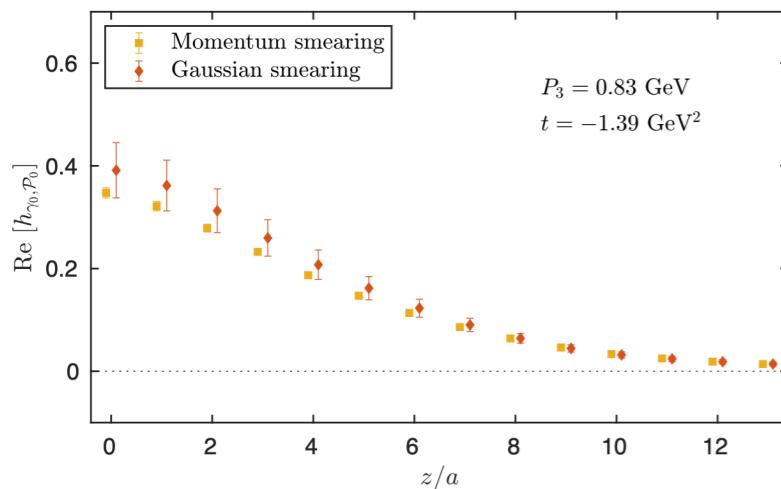
- ★ Statistical noise increases with  $P_3, t$   
use of momentum smearing method



- ◆ Implementation in GPDs nontrivial due to momentum transfer
- ◆ Standard definition of GPDs in Breit (symmetric) frame  
separate calculations at each  $t$
- ◆ Matrix elements decompose into more than one GPDs  
at least 2 parity projectors are needed to disentangle GPDs
- ◆ Nonzero skewness  
nontrivial matching
- ◆  $P_3$  must be chosen carefully due to UV cutoff ( $a^{-1} \sim 2 \text{ GeV}$ )

# Challenges of calculations

- ★ Statistical noise increases with  $P_3, t$   
use of momentum smearing method



Ref.	$m_\pi$ (MeV)	$P_3$ (GeV)	$\frac{n}{s} _{z=0}$
quasi/pseudo [59, 95]	130	1.38	6%
pseudo [92]	172	2.10	8%
current-current [98]	278	1.65	19% *
quasi [72]	300	1.72	6% †
quasi/pseudo [77]	300	2.45	8% †
quasi/pseudo [70]	310	1.84	3% †
—	260	1.67	15%
—	260	1.24	31%
s-quark quasi [112]	310	1.30	43% **
gluon pseudo [134]	310	1.73	39%
—	260	1.67	23%
quasi-GPDs [169] $-t=0.92 \text{ GeV}^2$	310	1.74	59%

† At  $T_{\text{sink}} < 1 \text{ fm}$ .

\* At smallest  $z$  value used,  $z = 2$ .

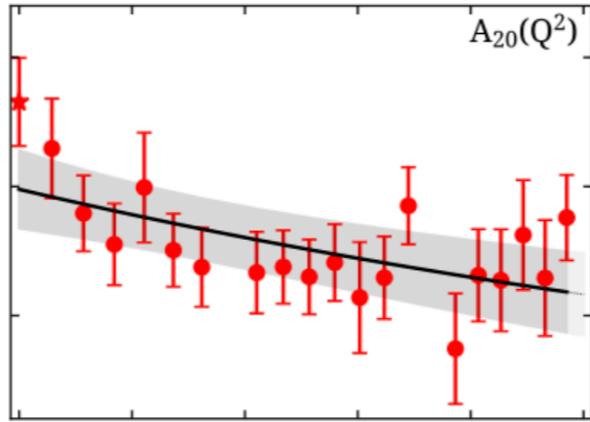
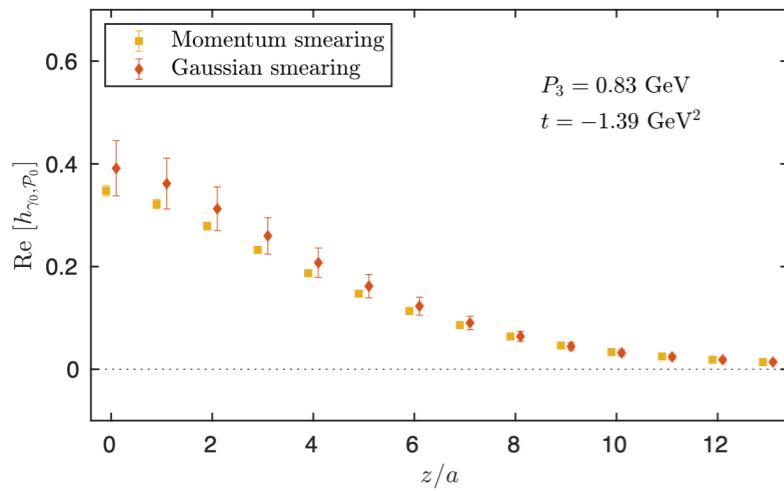
\*\* At maximum value of imaginary part,  $z = 4$ .

[M. Constantinou, (Invited review) EPJA 57 (2021) 77]

- ◆ Implementation in GPDs nontrivial due to momentum transfer
- ◆ Standard definition of GPDs in Breit (symmetric) frame  
separate calculations at each  $t$
- ◆ Matrix elements decompose into more than one GPDs  
at least 2 parity projectors are needed to disentangle GPDs
- ◆ Nonzero skewness  
nontrivial matching
- ◆  $P_3$  must be chosen carefully due to UV cutoff ( $a^{-1} \sim 2 \text{ GeV}$ )

# Challenges of calculations

- ★ Statistical noise increases with  $P_3, t$   
use of momentum smearing method



Ref.	$m_\pi$ (MeV)	$P_3$ (GeV)	$\frac{n}{s} _{z=0}$
quasi/pseudo [59, 95]	130	1.38	6%
pseudo [92]	172	2.10	8%
current-current [98]	278	1.65	19% *
quasi [72]	300	1.72	6% †
quasi/pseudo [77]	300	2.45	8% †
quasi/pseudo [70]	310	1.84	3% †
—	260	1.67	15%
—	260	1.24	31%
s-quark quasi [112]	310	1.30	43% **
gluon pseudo [134]	310	1.73	39%
—	260	1.67	23%
quasi-GPDs [169] $-t=0.92 \text{ GeV}^2$	310	1.74	59%

\* At  $T_{\text{sink}} < 1 \text{ fm}$ .

† At smallest  $z$  value used,  $z = 2$ .

\*\* At maximum value of imaginary part,  $z = 4$ .

[M. Constantinou, (Invited review) EPJA 57 (2021) 77]

- ◆ Implementation in GPDs nontrivial due to momentum transfer
- ◆ Standard definition of GPDs in Breit (symmetric) frame  
separate calculations at each  $t$
- ◆ Matrix elements decompose into more than one GPDs  
at least 2 parity projectors are needed to disentangle GPDs
- ◆ Nonzero skewness  
nontrivial matching
- ◆  $P_3$  must be chosen carefully due to UV cutoff ( $a^{-1} \sim 2 \text{ GeV}$ )

Further increase of momentum  
at the cost of credibility

# Challenges of calculations

## ★ x-dependence reconstruction: Inverse problem

[J. Karpie et al., JHEP 11 (2018) 178, arXiv:1807.10933]

# Challenges of calculations

## ★ x-dependence reconstruction: Inverse problem

[J. Karpie et al., JHEP 11 (2018) 178, arXiv:1807.10933]

### ◆ Standard Fourier transform ill-defined

$$\tilde{q}(x, P_3) = \frac{2P_3}{4\pi} \sum_{z=-z_{\max}}^{z_{\max}} e^{-ixP_3 z} h_\Gamma(P_3, z)$$

### ◆ Derivative method problematic

$$\tilde{q}(x) = h(z) \frac{e^{ixzP_3}}{2\pi ix} \Big|_{-z_{\max}}^{z_{\max}} - \int_{-z_{\max}}^{z_{\max}} \frac{dz}{2\pi} \frac{e^{ixzP_3}}{ix} h'(z)$$

# Challenges of calculations

## ★ x-dependence reconstruction: Inverse problem

[J. Karpie et al., JHEP 11 (2018) 178, arXiv:1807.10933]

### ◆ Standard Fourier transform ill-defined

$$\tilde{q}(x, P_3) = \frac{2P_3}{4\pi} \sum_{z=-z_{\max}}^{z_{\max}} e^{-ixP_3 z} h_\Gamma(P_3, z)$$

### ◆ Derivative method problematic

$$\tilde{q}(x) = h(z) \frac{e^{ixzP_3}}{2\pi ix} \Big|_{-z_{\max}}^{z_{\max}} - \int_{-z_{\max}}^{z_{\max}} \frac{dz}{2\pi} \frac{e^{ixzP_3}}{ix} h'(z)$$

## Advanced PDF reconstructions

- Backus-Gilbert Method
- Neural Network Reconstruction
- Bayesian PDF reconstruction
- Bayes-Gauss-Fourier transform

# Challenges of calculations

## ★ **x-dependence reconstruction: Inverse problem**

[J. Karpie et al., JHEP 11 (2018) 178, arXiv:1807.10933]

- ◆ Standard Fourier transform ill-defined

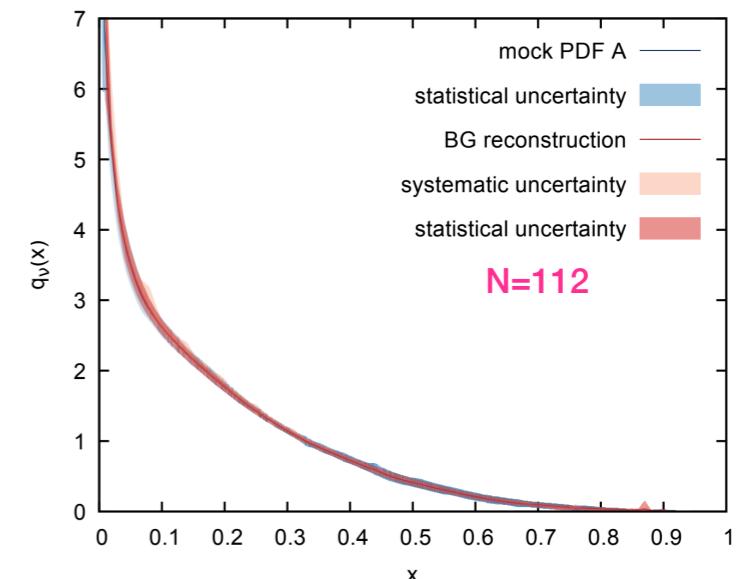
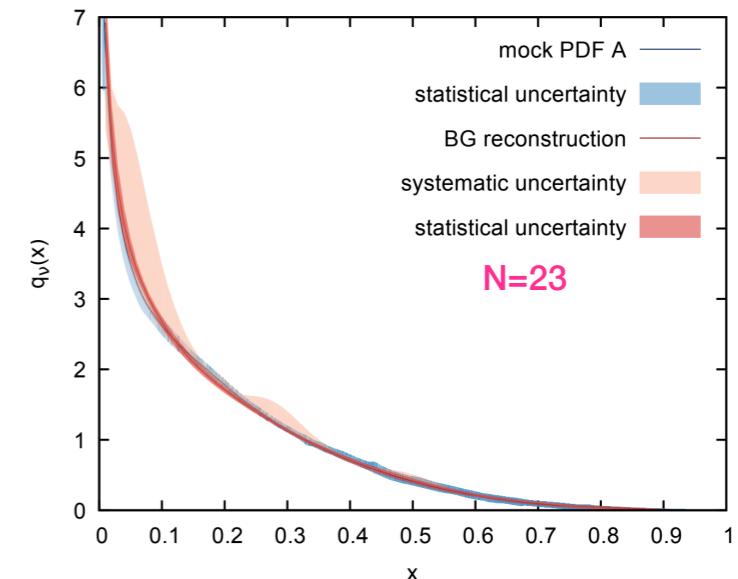
$$\tilde{q}(x, P_3) = \frac{2P_3}{4\pi} \sum_{z=-z_{\max}}^{z_{\max}} e^{-ixP_3 z} h_{\Gamma}(P_3, z)$$

- ◆ Derivative method problematic

$$\tilde{q}(x) = h(z) \frac{e^{ixzP_3}}{2\pi ix} \Big|_{-z_{\max}}^{z_{\max}} - \int_{-z_{\max}}^{z_{\max}} \frac{dz}{2\pi} \frac{e^{ixzP_3}}{ix} h'(z)$$

## Advanced PDF reconstructions

- Backus-Gilbert Method
- Neural Network Reconstruction
- Bayesian PDF reconstruction
- Bayes-Gauss-Fourier transform



# Challenges of calculations

## ★ **x-dependence reconstruction: Inverse problem**

[J. Karpie et al., JHEP 11 (2018) 178, arXiv:1807.10933]

### ◆ Standard Fourier transform ill-defined

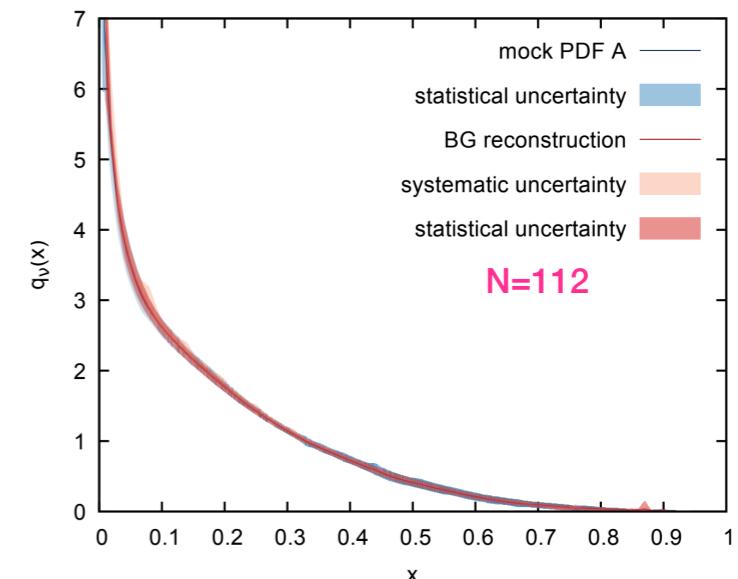
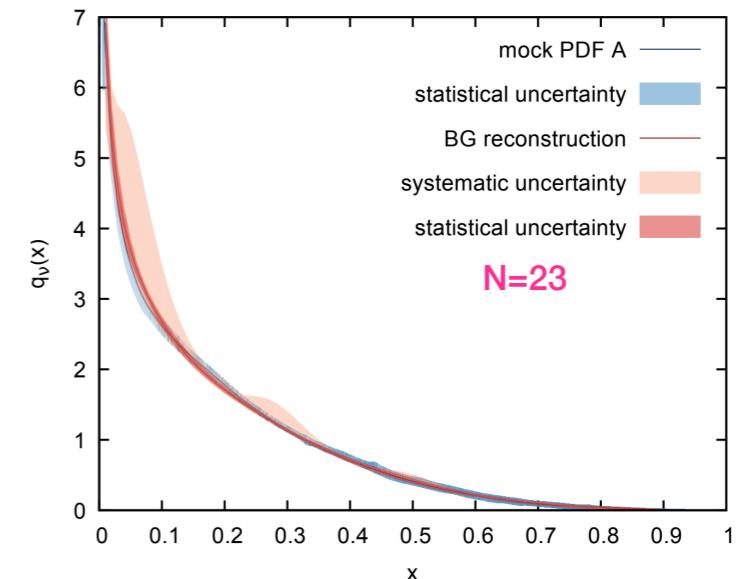
$$\tilde{q}(x, P_3) = \frac{2P_3}{4\pi} \sum_{z=-z_{\max}}^{z_{\max}} e^{-ixP_3 z} h_\Gamma(P_3, z)$$

### ◆ Derivative method problematic

$$\tilde{q}(x) = h(z) \frac{e^{ixzP_3}}{2\pi ix} \Big|_{-z_{\max}}^{z_{\max}} - \int_{-z_{\max}}^{z_{\max}} \frac{dz}{2\pi} \frac{e^{ixzP_3}}{ix} h'(z)$$

### Advanced PDF reconstructions

- Backus-Gilbert Method
- Neural Network Reconstruction
- Bayesian PDF reconstruction
- Bayes-Gauss-Fourier transform



## ★ Negative-x region: anti-quark contribution currently suffers from enhanced uncertainties

# Concluding Remarks

# Concluding Remarks

- ★ The x-dependence of distribution functions is no longer an unrealistic goal
- ★ Flavor decomposition becoming real possibility
- ★ Extension to GPDs
- ★ Extension to twist-3 PDFs

*Thank you*



U.S. DEPARTMENT OF  
**ENERGY**

Office of  
Science



**TMD** Topical Collab.



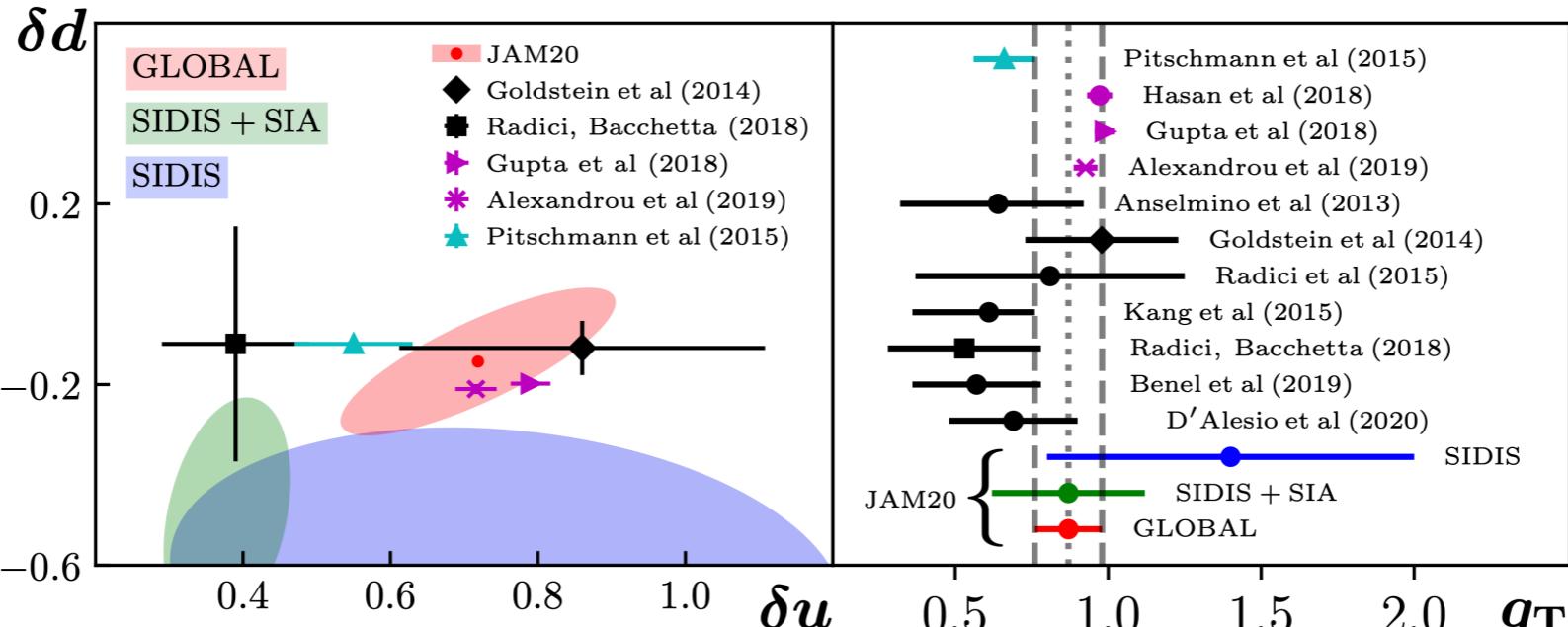
U.S. DEPARTMENT OF  
**ENERGY**

Office of  
Science

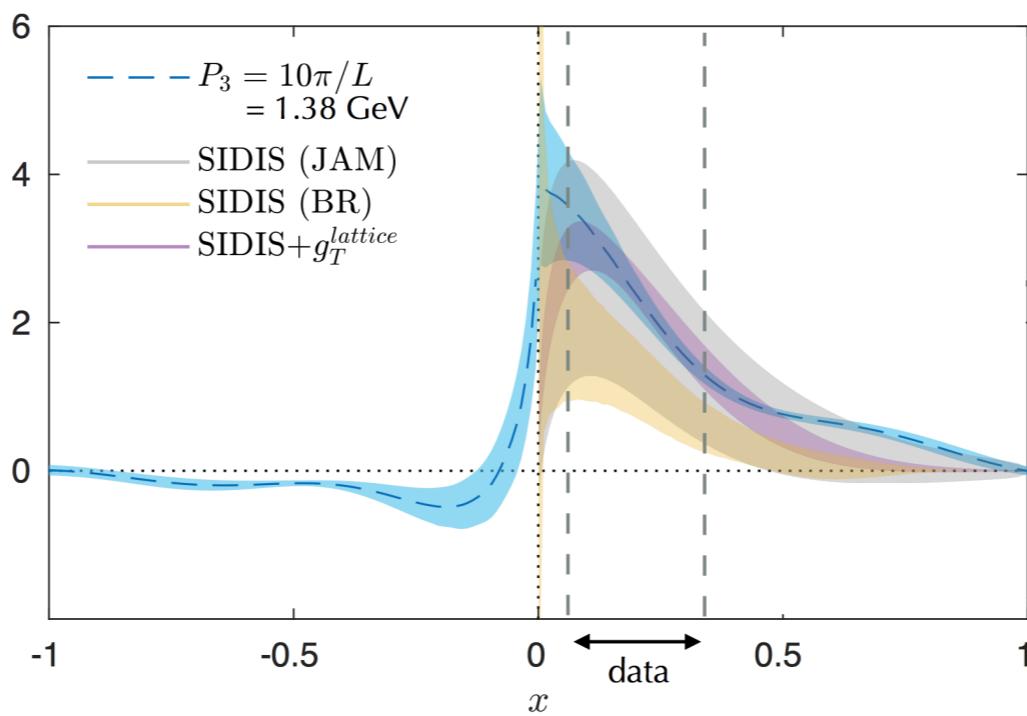
DOE Early Career Award (NP)  
Grant No. DE-SC0020405

# BACKUP SLIDES

# Transversity PDF



[J. Cammarota et al. (JAM 2020), PRD 102 (2020) 5, 054002, arXiv:2002.08384]



[C. Alexandrou et al. (ETMC), PRD 98 (2018) 091503 (R), arXiv:1807.00232]

★ Transversity: an example of the predictive power of lattice QCD

# Pseudo-PDFs

- ★ Position-space formulation, same raw matrix elements as quasi-PDFs, expressed in terms of Ioffe time  $\nu = z \cdot p$  and  $z^2$

$$\mathfrak{M}(\nu, z^2) = \frac{\mathcal{M}(\nu, z^2) / \mathcal{M}(\nu, 0)}{\mathcal{M}(0, z^2) / \mathcal{M}(0, 0)}$$

(No requirement for large hadron momentum)

- ★ Ioffe time pseudo-distribution function (pseudo-ITD).  
pseudo-PDFs: Fourier transform of pseudo-ITD (canonical support)

$$Q(\nu, \mu^2) = \int_{-1}^1 dx e^{i\nu x} q(x, \mu^2)$$

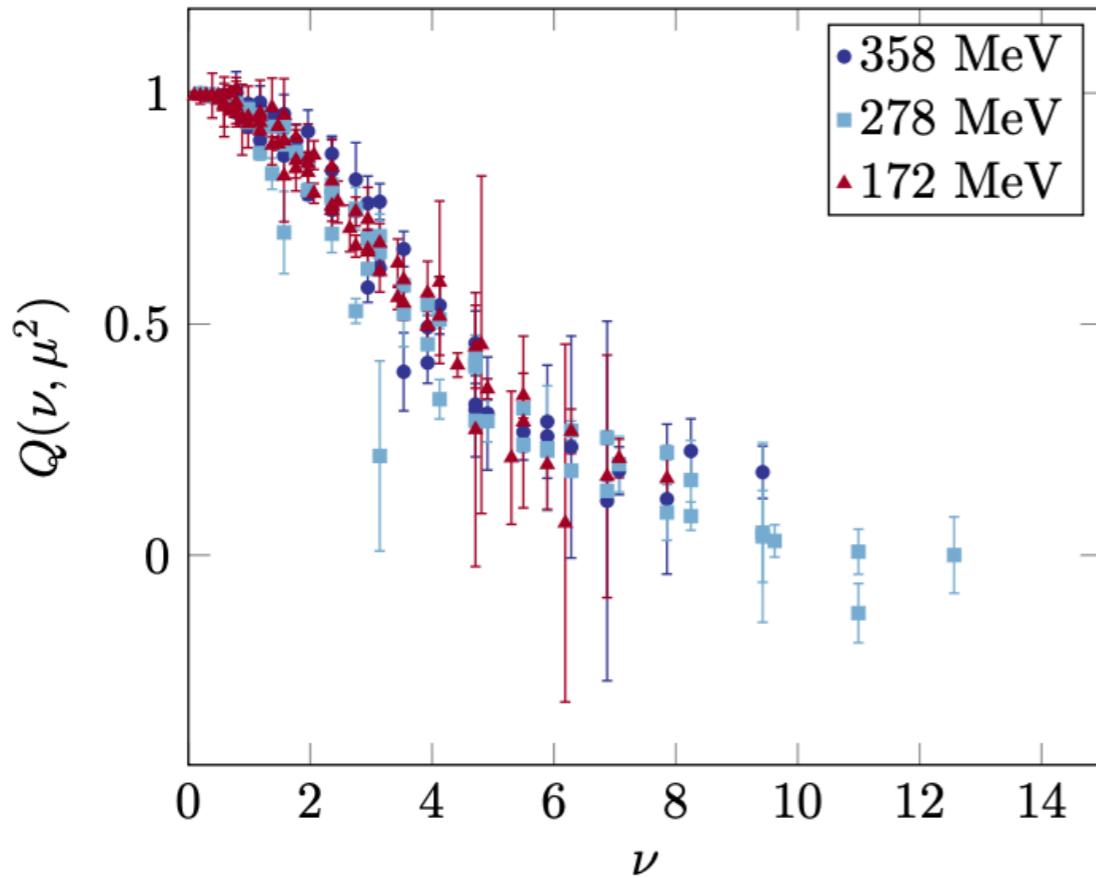
- ★ Valence  $q_v = q - \bar{q}$  and nonsinglet  $q_{v2s} \equiv q_v + 2\bar{q} = q + \bar{q}$

$$\text{Re}[Q(\nu, \mu^2)] = \int_0^1 dx \cos(\nu x) q_v(x, \mu^2), \quad \text{Im}[Q(\nu, \mu^2)] = \int_0^1 dx \sin(\nu x) q_{v2s}(x, \mu^2)$$

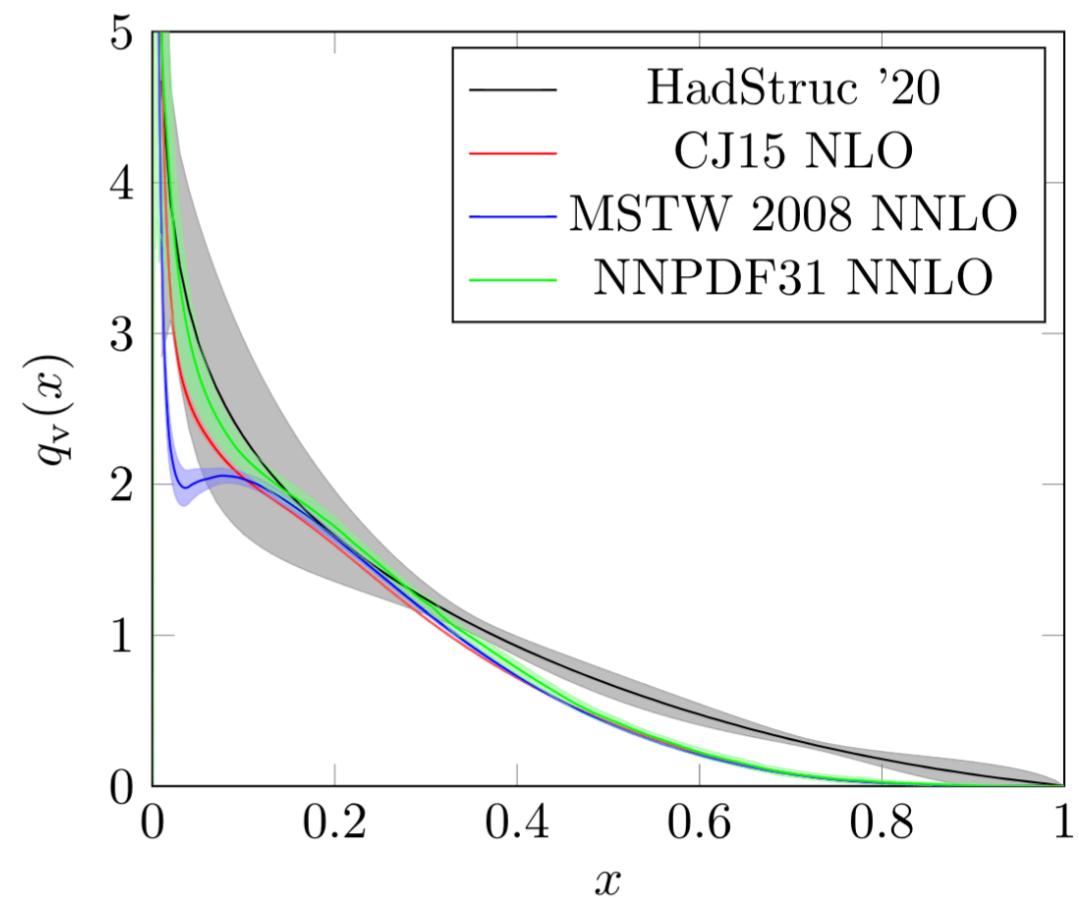
# Nucleon pseudo-PDFs

- ★  **$N_f=2+1$  clover fermions (3 ensembles):**

[B. Joo et al. (JLab-W&M), PRL 125, (2020) 23, arXiv:2004.01687]



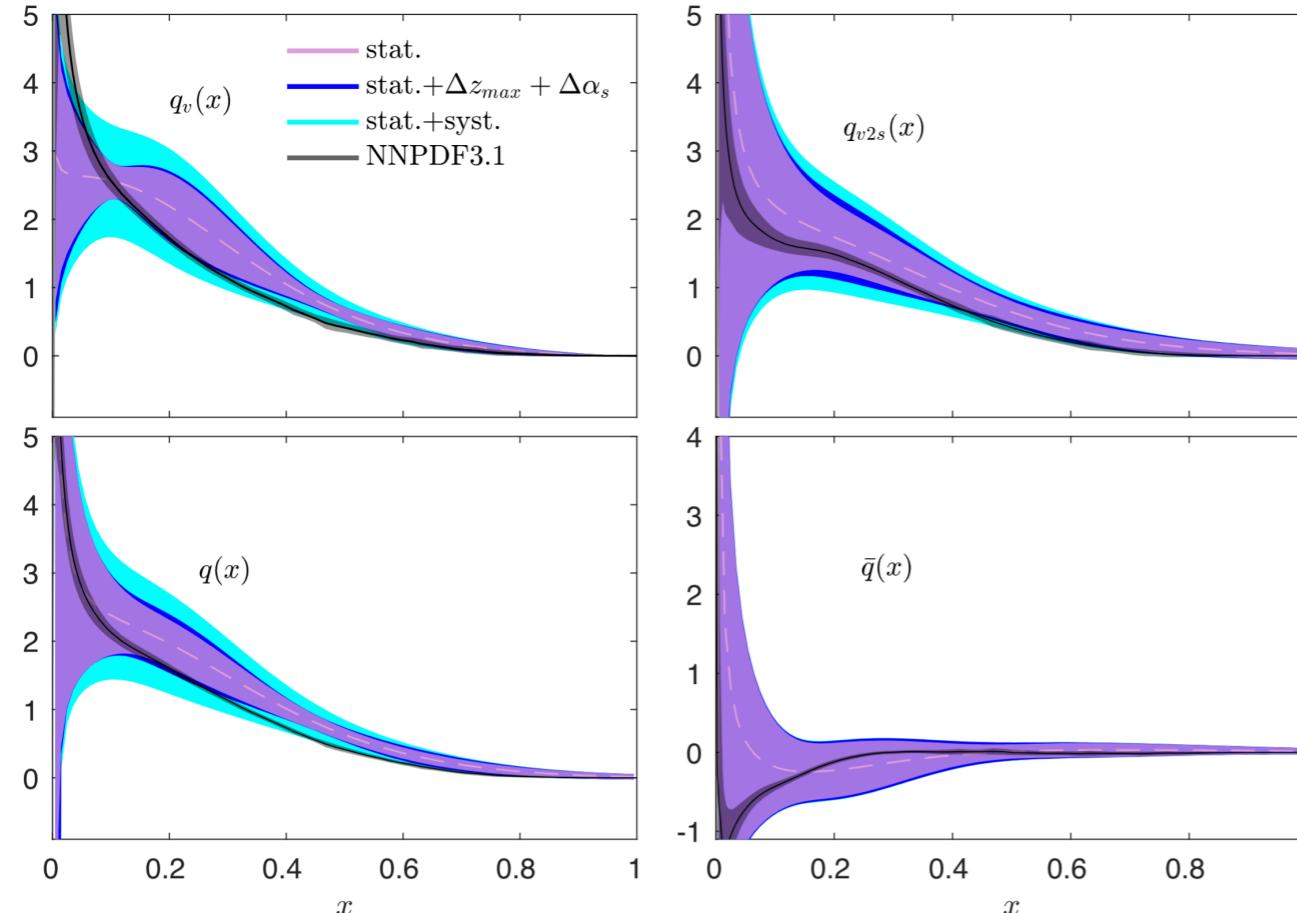
ID	$a(\text{fm})$	$M_\pi(\text{MeV})$
$a094m360$	0.094(1)	358(3)
$a094m280$	0.094(1)	278(3)
$a091m170$	0.091(1)	172(6)



- Approaching the physical point and increasing momentum leads to large statistical uncertainties.
- Lattice data fitted similar to CJ and MSTW
- Regions with large errors: few data

# Nucleon pseudo-PDFs

## ★ $N_f=2$ twisted-mass fermions at the physical point:

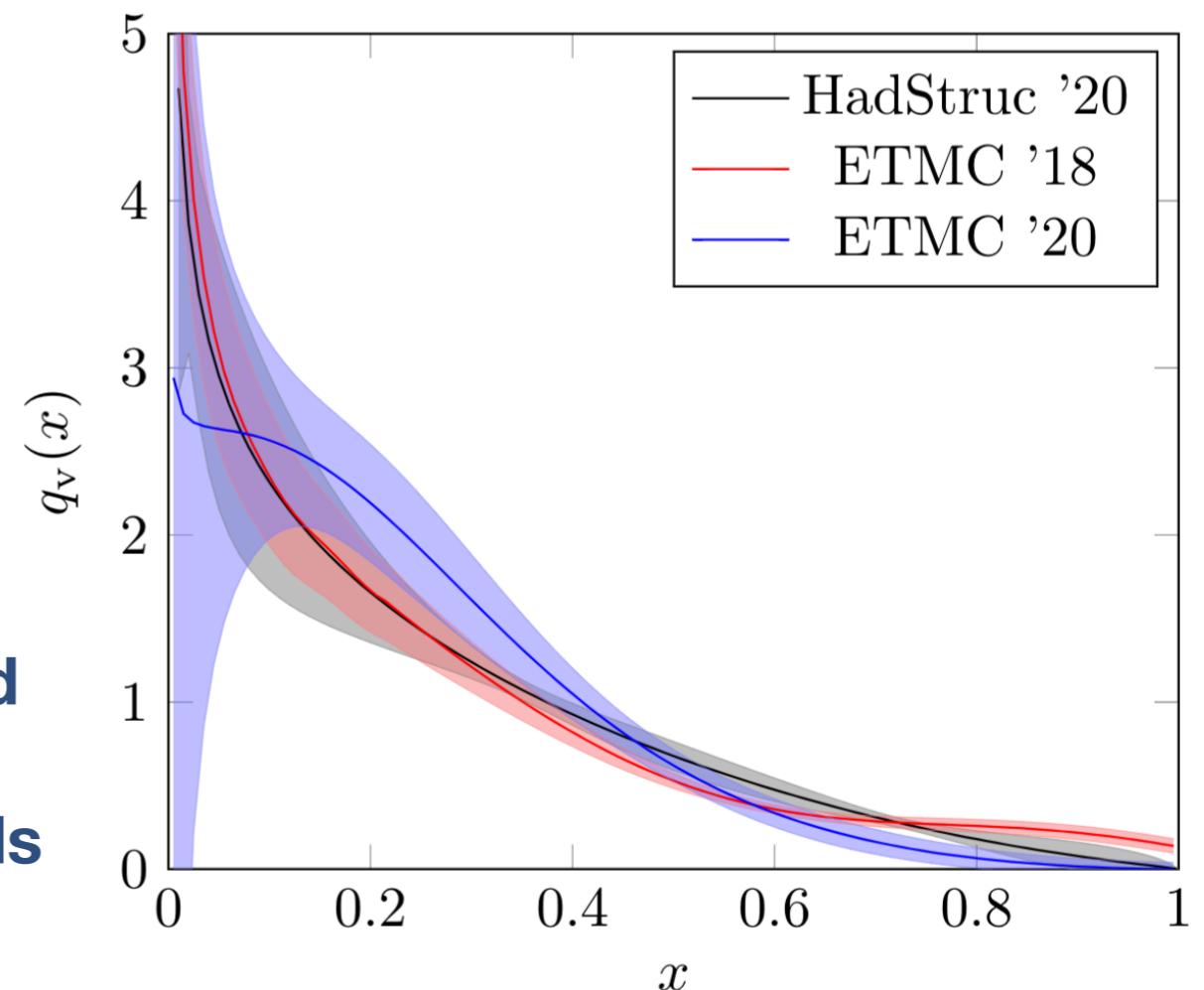


[M. Bhat et al. (ETMC), arXiv:2005.02102]

$P_3$	$P_3$ [GeV]	$N_{\text{confs}}$	$N_{\text{meas}}$
0	0	20	320
$2\pi/L$	0.28	19	1824
$4\pi/L$	0.55	18	1728
$6\pi/L$	0.83	50	4800
$8\pi/L$	1.11	425	38250
$10\pi/L$	1.38	811	72990

- Statistical accuracy sufficient and gives clear signal for both valence and sea quark contributions

- Different formulations & same method  
(clover, twisted mass)
- Same formulation for different methods  
(quasi, pseudo)

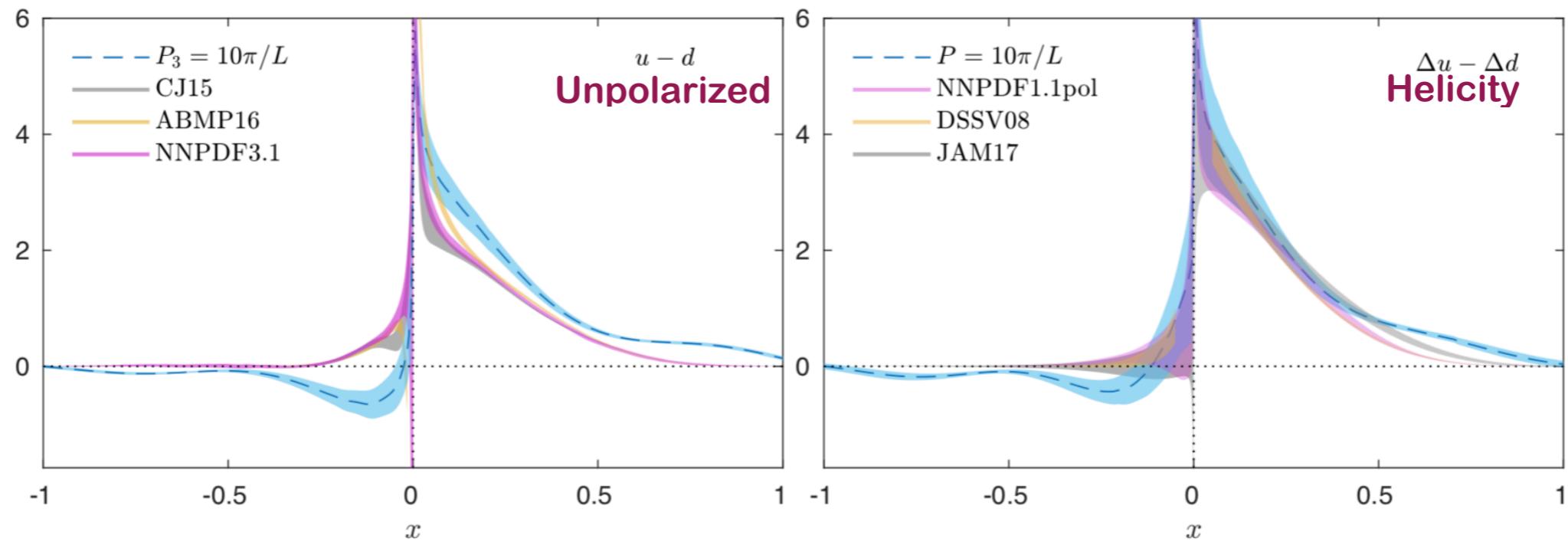


# PDFs at physical pion mass

- ★ First complete study including all steps

[C. Alexandrou et al. (ETMC), PRL 121 (2018) 112001, arXiv:1803.02685; PRD 99 (2019) 114504, arXiv:1902.00587]

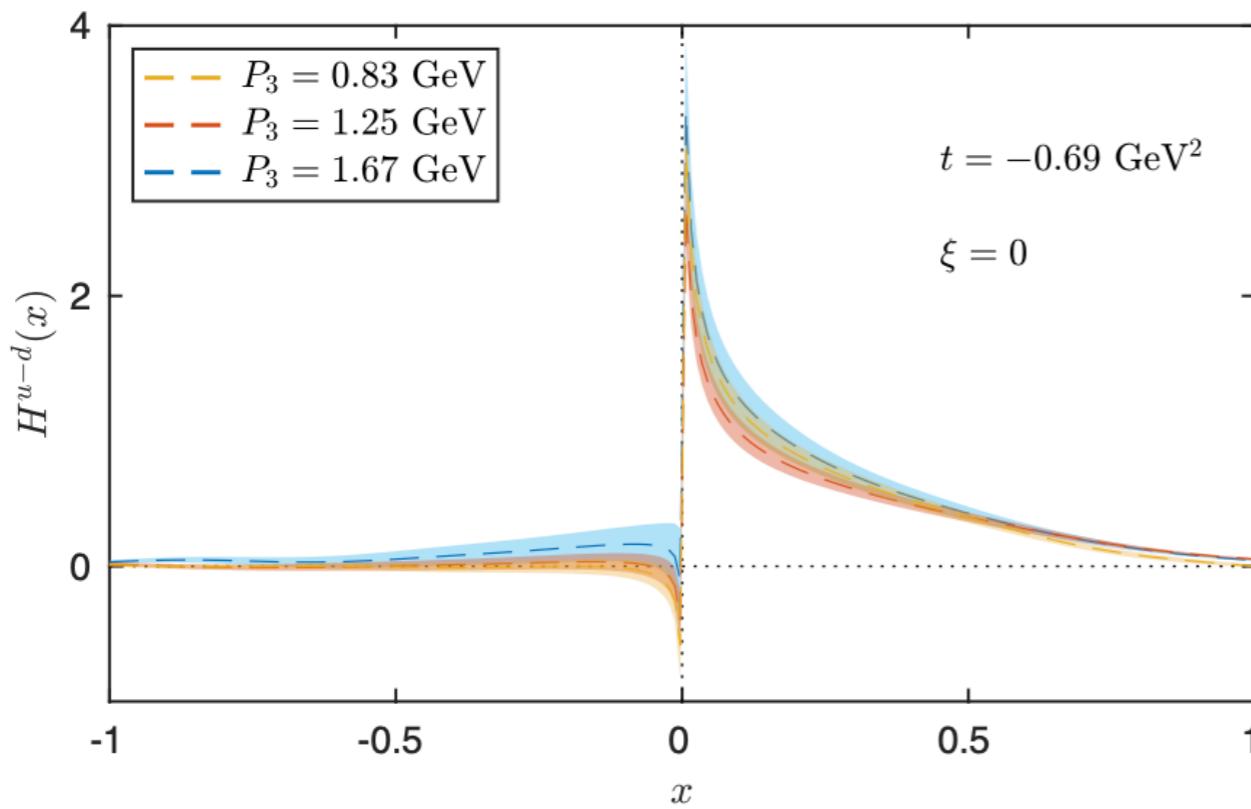
$P = \frac{6\pi}{L}$ (0.83 GeV)			$P = \frac{8\pi}{L}$ (1.11 GeV)			$P = \frac{10\pi}{L}$ (1.38 GeV)		
Ins.	$N_{\text{conf}}$	$N_{\text{meas}}$	Ins.	$N_{\text{conf}}$	$N_{\text{meas}}$	Ins.	$N_{\text{conf}}$	$N_{\text{meas}}$
$\gamma_3$	100	9600	$\gamma_3$	425	38250	$\gamma_3$	811	72990
$\gamma_0$	50	4800	$\gamma_0$	425	38250	$\gamma_0$	811	72990
$\gamma_5 \gamma_3$	65	6240	$\gamma_5 \gamma_3$	425	38250	$\gamma_5 \gamma_3$	811	72990



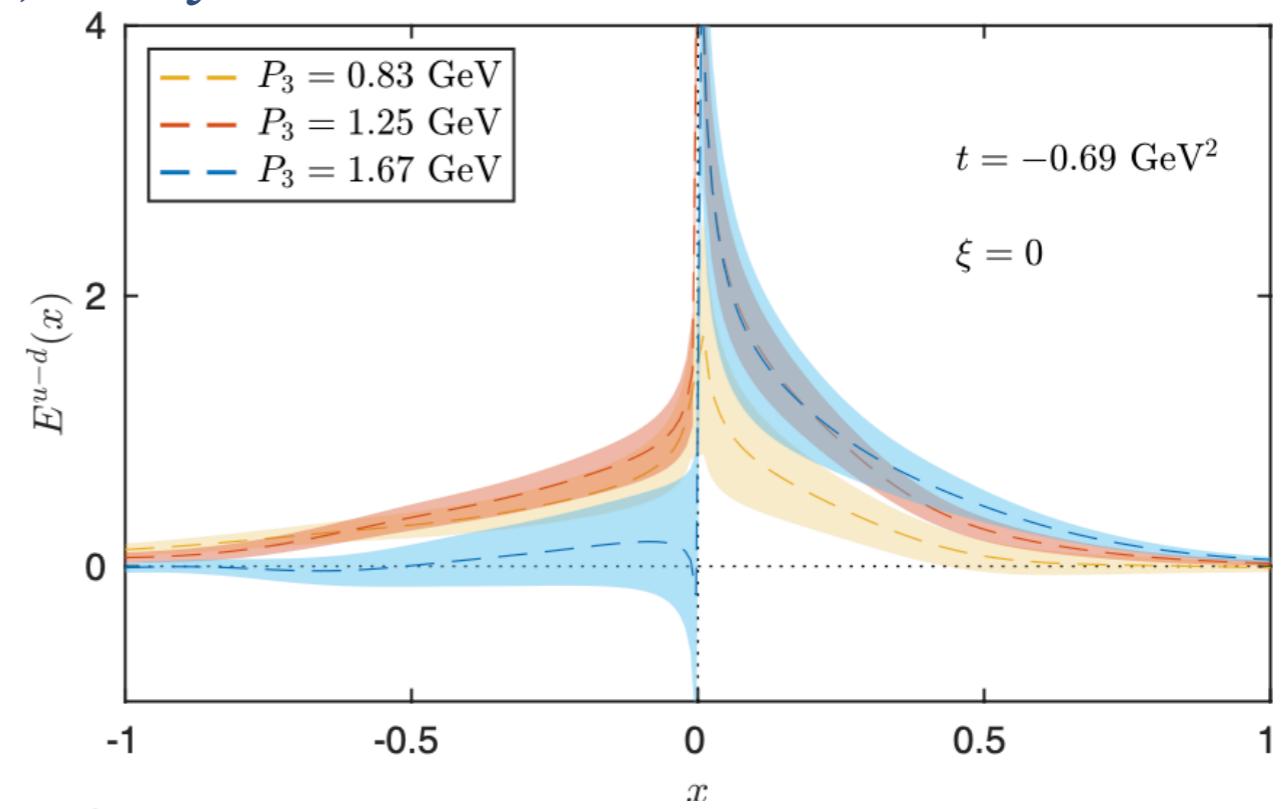
- ★ Lattice data at  $P_3=1.4$  GeV approach global fits
- ★  $P_3$  must be chosen carefully due to UV cutoff ( $a^{-1} \sim 2$  GeV)
- ★ Negative- $x$  region: anti-quark contribution currently suffers from enhanced uncertainties

# Momentum Dependence of GPDs

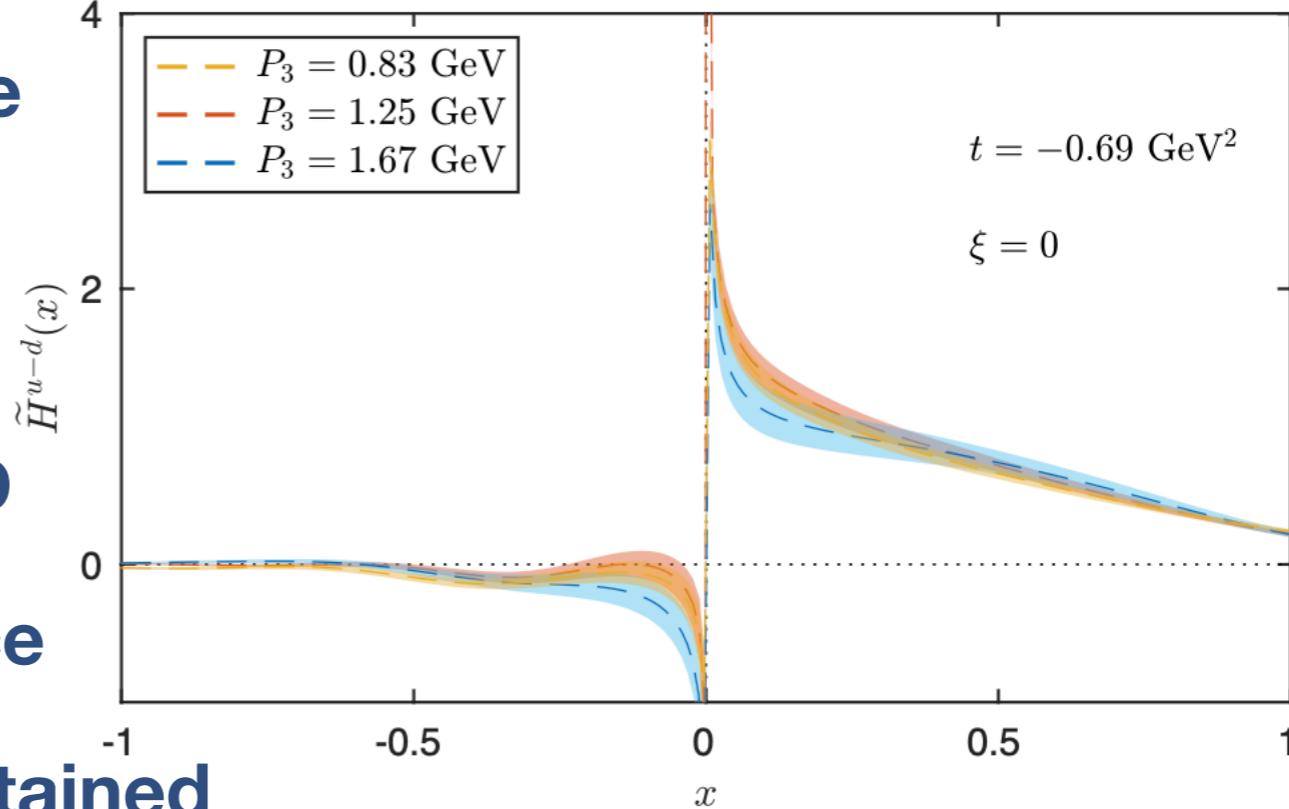
$-t = 0.69 \text{ GeV}^2$ ,



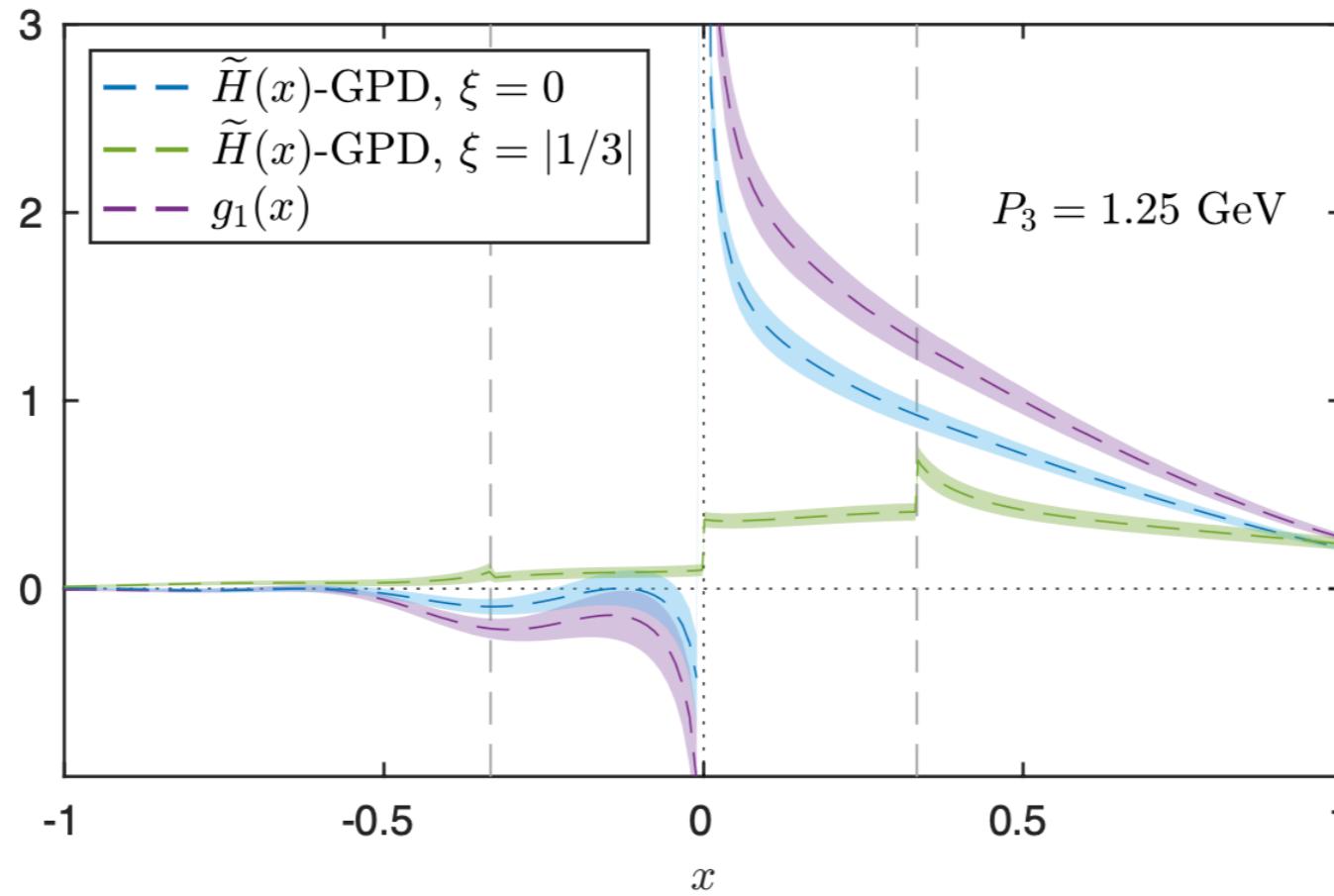
$\xi = 0$



- ◆  **$H$ -GPD: negligible  $P_3$ -dependence**
- ◆  **$E$ -GPD: convergence between  $P_3 = 1.25 \text{ GeV}$  and  $P_3 = 1.78 \text{ GeV}$**
- ◆  **$E$ -GPD less accurate than  $H$ -GPD**
- ◆  **$\tilde{H}$ -GPD: negligible  $P_3$ -dependence**
- ◆ **For  $\xi = 0$  only  $\tilde{H}$ -GPD can be obtained**



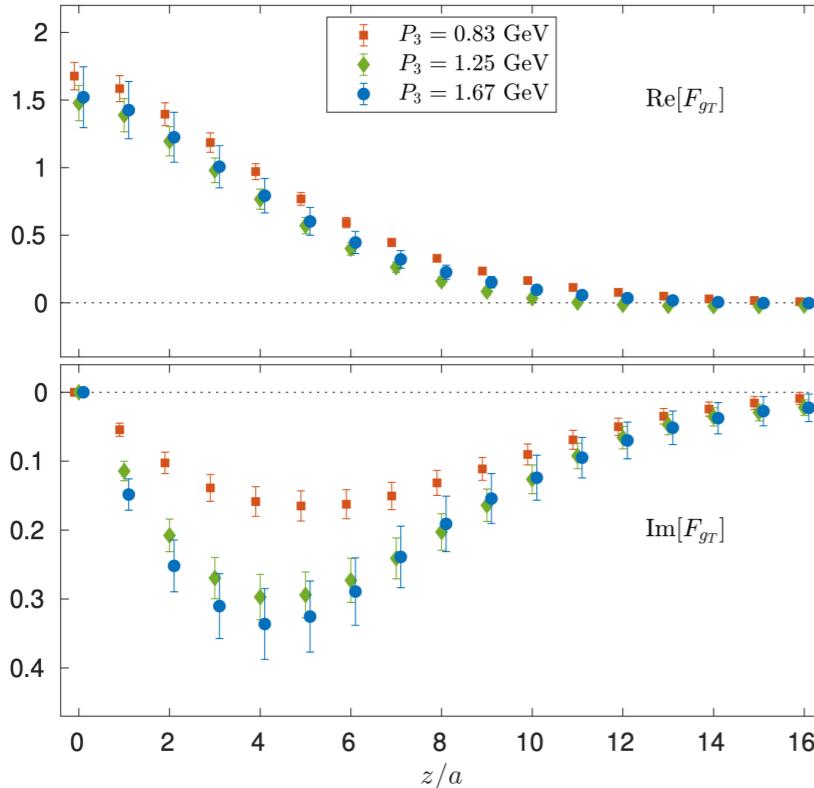
# Comparison of zero and nonzero skewness



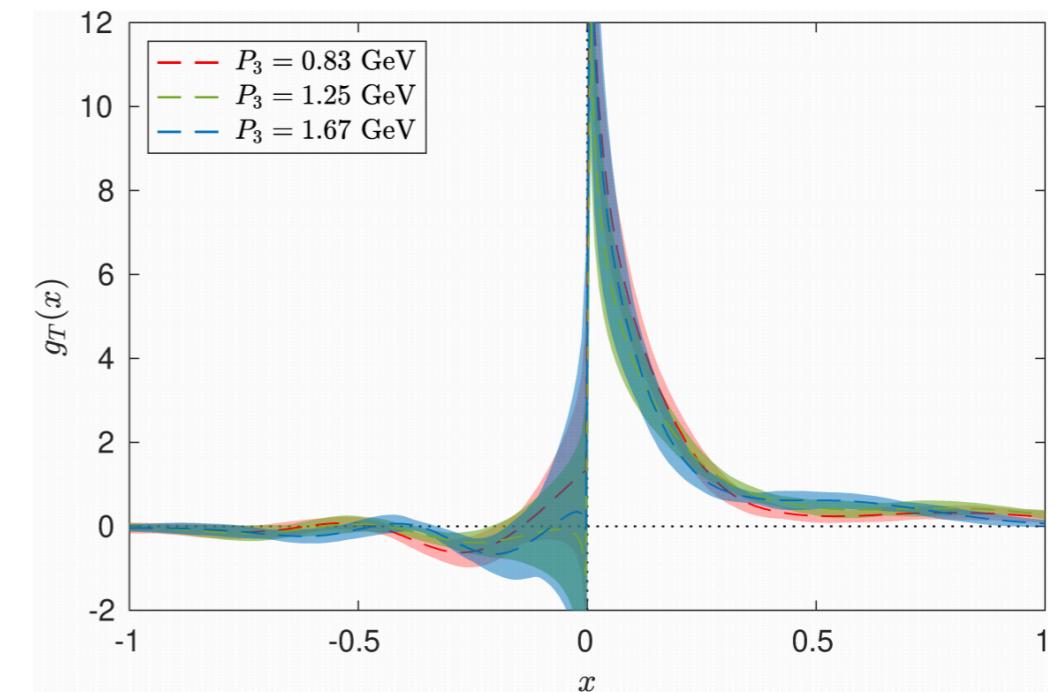
- ◆ ERBL-region behavior similar in the unpolarized and polarized case
- ◆ DGLAP-region exhibits hierarchy between PDF and GPD
- ◆ Power-counting analysis less trivial for u-d, but could reveal interesting conclusions for the d-quark distribution

[H. Avakian et al. PRL 99 (2007) 082001, arXiv:0705.1553]

# Twist-3 $g_T(x)$ PDF



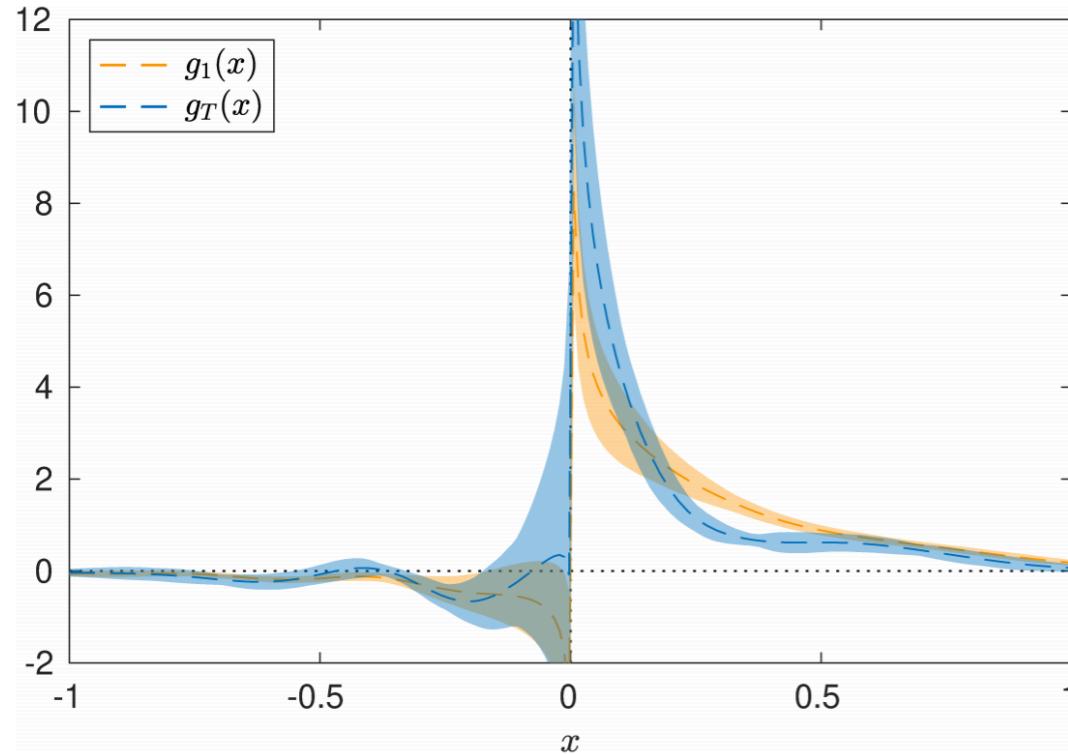
F.T. using BG  
&  
Matching



Clear momentum dependence in matrix elements



Convergence for all momenta



Twist-3 counterpart as sizable as twist-2

$$f_i = f_i^{(0)} + \frac{f_i^{(1)}}{Q} + \frac{f_i^{(2)}}{Q^2} \dots$$

Burkhardt-Cottingham sum rule important check

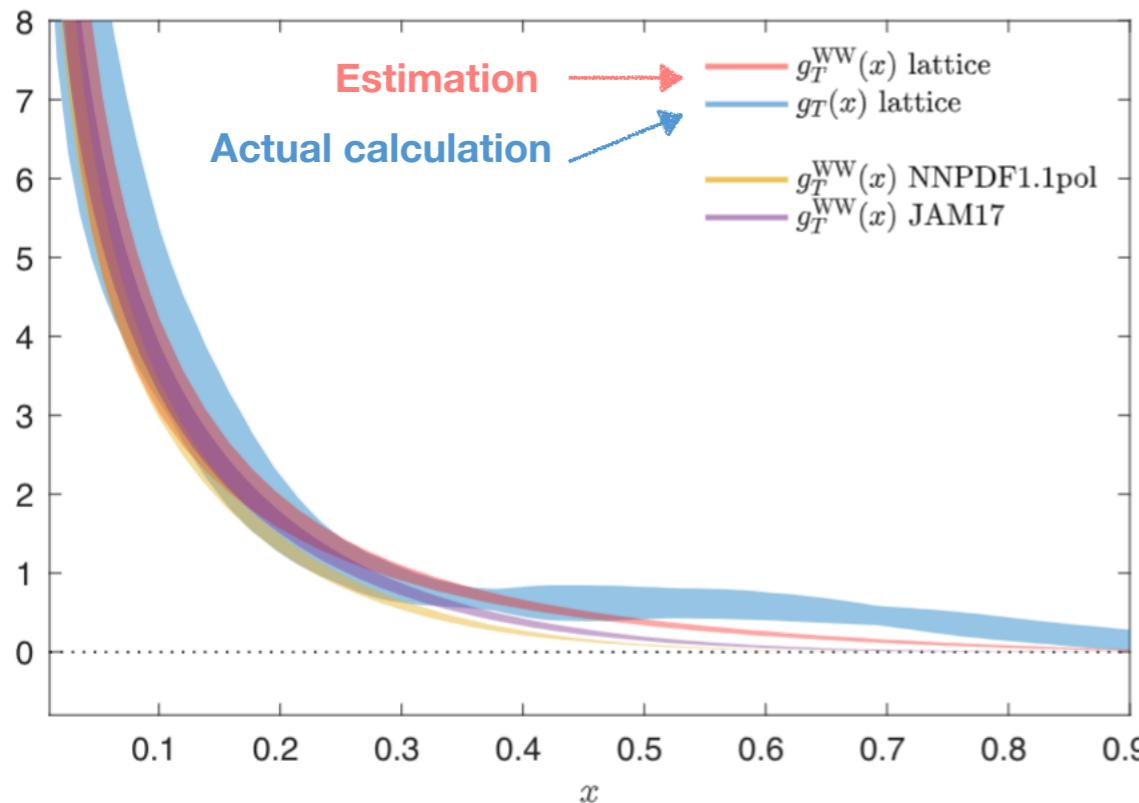
$$\int_{-1}^1 dx g_1(x) - \int_{-1}^1 dx g_T(x) = 0.01(20)$$

# Twist-3 $g_T(x)$ PDF

## WW approximation:

- twist-3  $g_T(x)$  determined by the twist-2  $g_1(x)$ :

$$g_T^{\text{WW}}(x) = \int_x^1 \frac{dy}{y} g_1(y)$$



- $g_T(x)$  agrees with  $g_T^{\text{WW}}(x)$  for  $x < 0.5$  (violations up to 30-40% possible)
- Violations of 15-40% expected from experimental data

[A. Accardi et al., JHEP 11 (2009) 093]

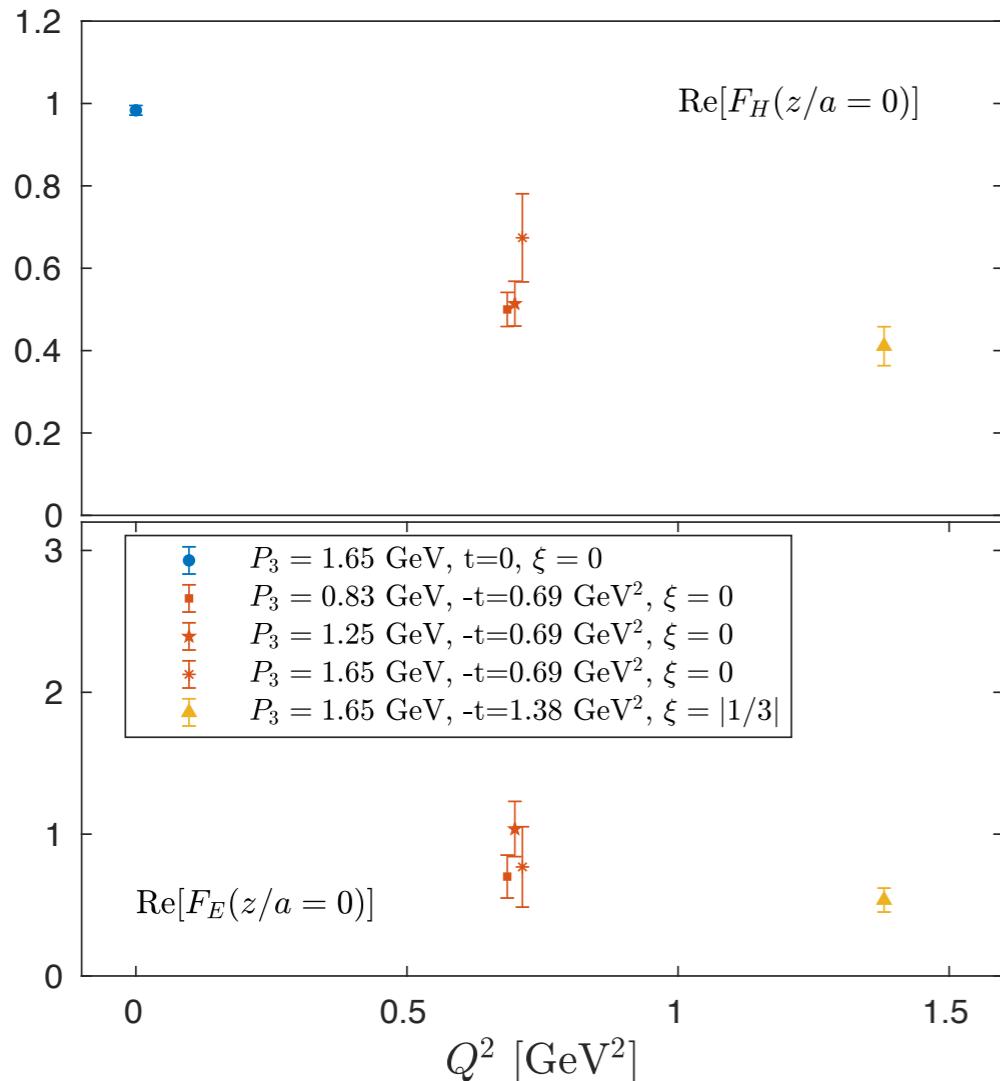
## $d_2$ -moment:

$$d_2 = \int dx 3x^2 [g_T(x) - g_T^{\text{WW}}(x)]$$

- Investigated experimentally (JLab Hall A) and found  $\mathcal{O}(10^{-3})$   
[D. Flay et al., PRD 94, 5 (2016) 052003, arXiv:1603.03612]
- Similar order of magnitude indicated in our results

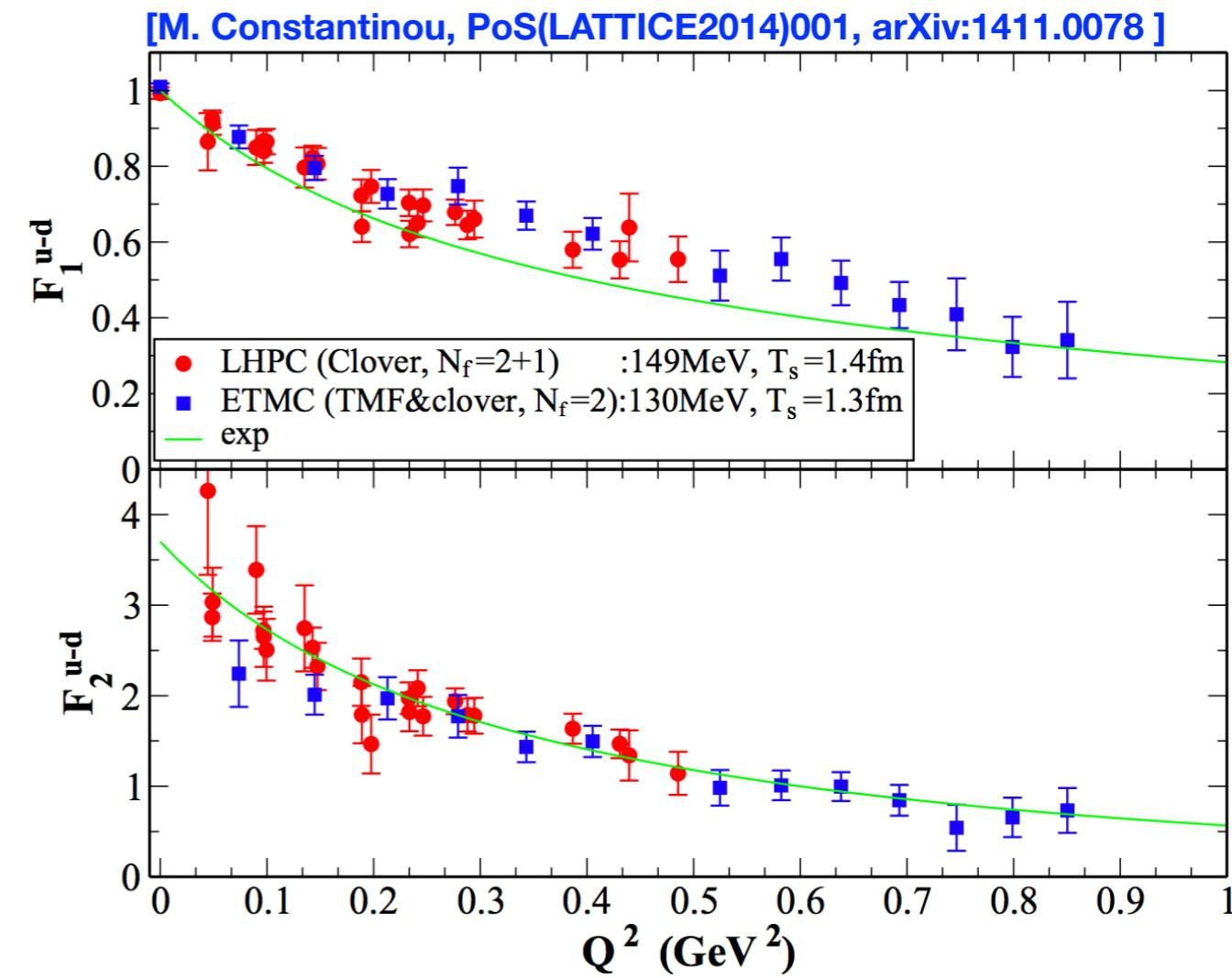
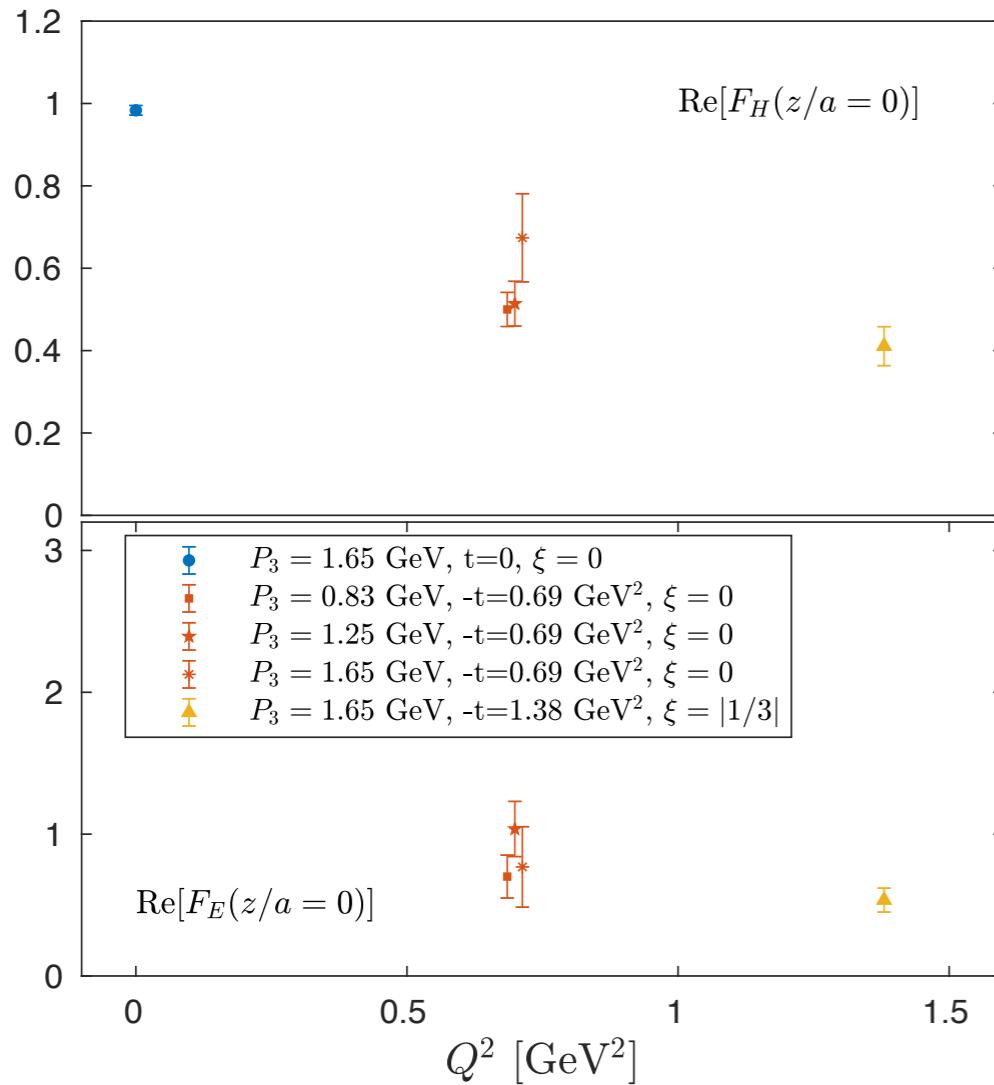
# Comparison with Dirac & Pauli Form Factors

## ♦ Important check: z=0 behavior



# Comparison with Dirac & Pauli Form Factors

- Important check:  $z=0$  behavior



- No quantitative comparison
- small pion mass dependence

- $Q^2$ -behavior of  $F_H(z=0)$ ,  $F_E(z=0)$  qualitatively similar to  $F_1, F_2$
- Rapid increase of statistical errors with increase of  $P_3$  for same  $t$