# The QCD Equation of State in Small Systems

**W. A. Horowitz** and Alexander Rothkopf
University of Cape Town, South Africa and University of Stavanger, Norway
wa.horowitz@uct.ac.za and alexander.rothkopf@uis.no

## 1. Introduction

The goal of heavy ion collision phenomenology is the quantitative understanding of the non-trivial, emergent, many-body dynamics of quantum chromodynamics. Viscous, relativistic hydrodynamics has proven to be an incredibly valuable tool in this endeavor: important statements about the bulk properties of quark-gluon plasma, such as the equation of state and shear and bulk viscosities, have been deduced by describing the measured distribution of low momentum particles ($\lesssim 2$ GeV) emerging from nuclear collisions [1].

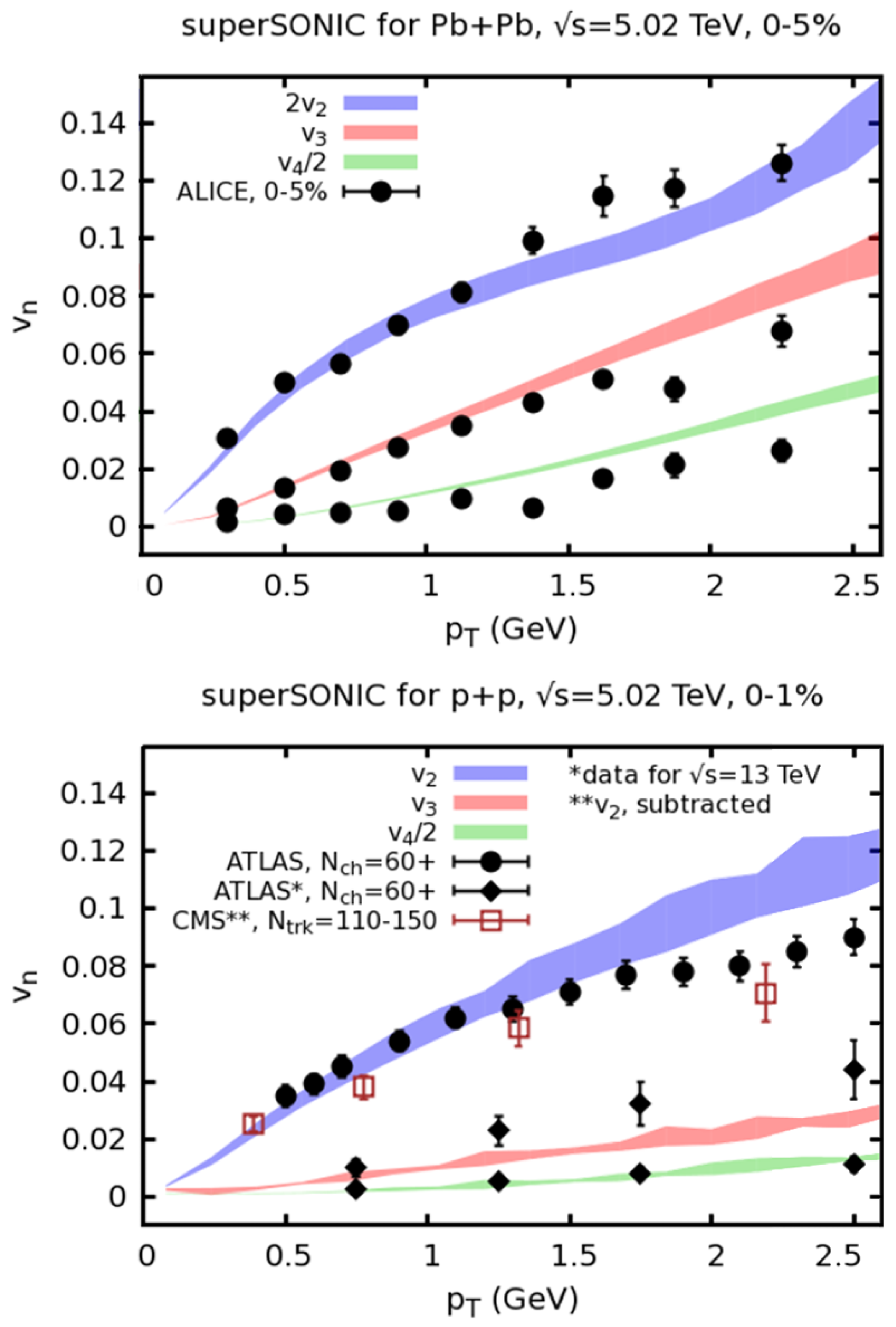


**Figure 1:** *Remarkably successful relativistic viscous hydrodynamics description of the distribution of low-$p_T$ particles in hadronic collisions from A+A (top) down to even p+p (bottom) collision systems. (Figure adapted from [1].)*

Shockingly, hydrodynamics provides not only an excellent description for large collision systems such as Au+Au and Pb+Pb, but also appears to provide a very good description of small collision systems such as central p+p; see Fig. (1). One may then naturally ask: do input quantities in hydrodynamics such as the equation of state depend on the size of the system? Turned the other way around, do the properties of QGP extracted from hydrodynamics' comparison to data depend on the size of the collision system?

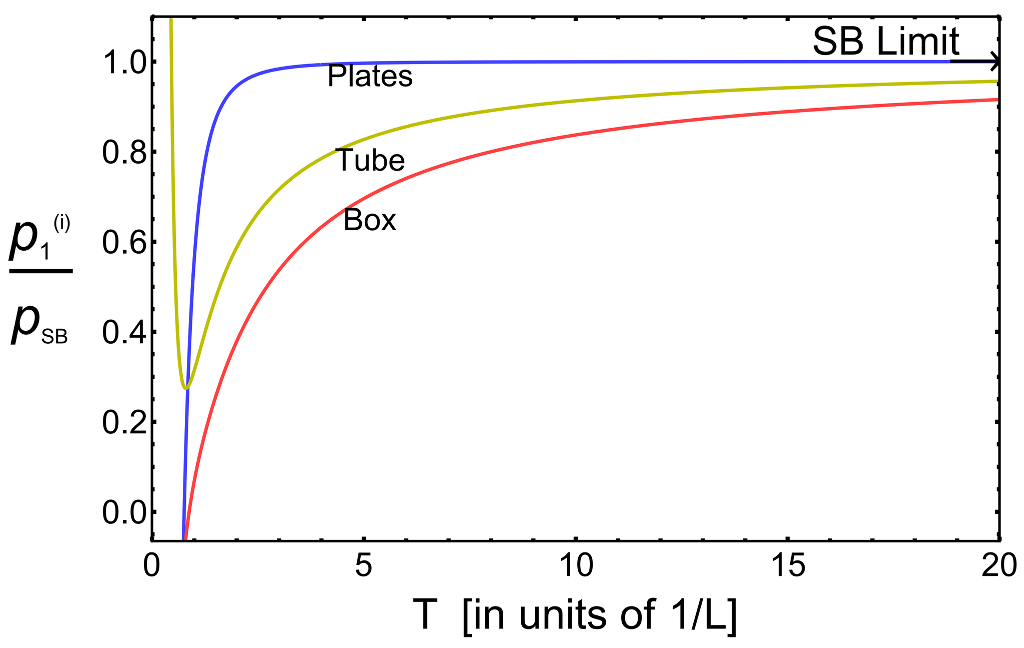


**Figure 2:** *Finite volume pressure $p$ divided by the infinite volume, Stefan-Boltzmann limit $p_{SB}$ of a free, massless scalar field with Dirichlet boundary conditions in 1, 2, and 3 dimensions as a function of temperature $T$ in units of $1/L$, where $L$ is the length of any of the fixed, compactified directions [2].*

## 2. Analytic Results

A natural first analytic model to test the relevance of system size on thermodynamic quantities is the massless, free scalar field put into a box, i.e. $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2$ with Dirichlet boundary conditions (DBCs) in 1, 2, or 3 directions of lengths $L_1$, $L_2$, and $L_3$, respectively. One may straightforwardly compute the partition function [2], then from the free energy one may compute the pressure, energy density, etc. We show in Fig. (2) the finite system size effect on the pressure as a function of temperature: the presence of the boundary limits the modes of the field, reducing the pressure from the infinite volume, Stefan-Boltzmann (SB) limit. As the dimensionless parameter $T \times L$ increases, the finite size effects decrease, as they must. As $T \times L \to 0$, the pressure converges to the $T = 0$ Casimir pressure. One can see that the finite size effects are $\sim 10\%$ for even $T \times L \sim 20$ and grow as $T \times L$ decreases. It's worth noting that for a p+p collision, $T \times L \sim 20$ implies a phenomenologically relevant $T \sim 400$ MeV. One may—provocatively—go further and note how the general shape of the pressure as a function of temperature in the finite system is reminiscent of the shape of $p(T)$ in the infinite volume limit from lattice QCD simulations [3].

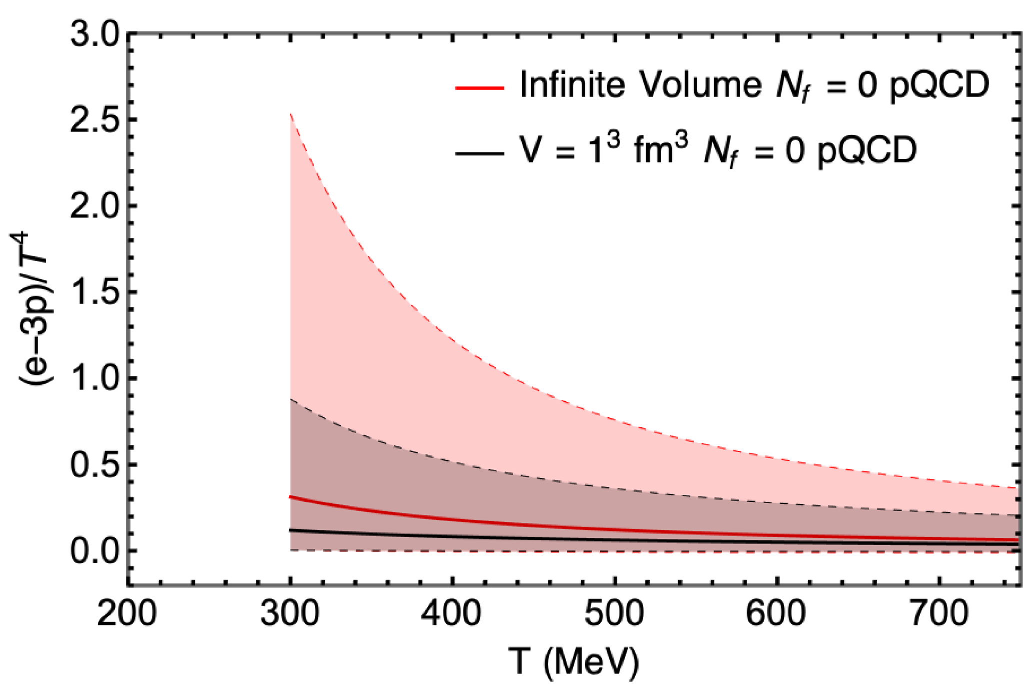


**Figure 3:** *The scaled trace anomaly $\Delta/T^4 = (\epsilon - 3p)/T^4$ as a function of temperature $T$ from $N_f = 0$ pQCD to order $g_s^5$ with scale set by $\pi T$, $2\pi T$, and $4\pi T$ in infinite volume as given in [4] and then with finite size running coupling Eq. (1).*

One may then examine the trace anomaly $\Delta \equiv \epsilon - 3p$ in this compactified massless free scalar field. One may show that the trace anomaly in this case is identically 0, independent of any choice of conformality-breaking non-trivial finite geometry. One may further see that for a coupled $\lambda\phi^4$ theory, the trace anomaly in infinite volume is identically 0 up to order fixed order $\lambda^2$; note that the dynamically generated Debye mass $m_D \sim \lambda T$ impacts the pressure and energy density at order $\lambda^{3/2}$ [5].

The trace anomaly becomes non-zero when the coupling is allowed to run and when the order of the expansion of the partition function yields logs of the temperature. Inspired by the finite size correction to the running coupling in $\lambda\phi^4$ theory [6], we took as an ansatz

$$g_s(\mu, L) = g_s(\mu) - \frac{3}{2}g_s^2(\mu)\frac{3}{2}\big[(2\pi)^3 m_D(\mu)L\big]^{-1/2} \qquad (1)$$

in the $g_s^5$ QCD partition function as given in [4]. We plotted the results for the scaled trace anomaly $\Delta/T^4 \equiv (\epsilon - 3p)/T^4$ in infinite volume and in a box of sides $L = 1$ fm in Fig. (3). We conservatively estimated the uncertainty in the result by varying the scale $\mu$ from $\pi T$ to $4\pi T$. One can see that the reduction in the coupling from the infinite volume limit due to the finite system size significantly reduces the trace anomaly, which will lead to a faster speed of sound.

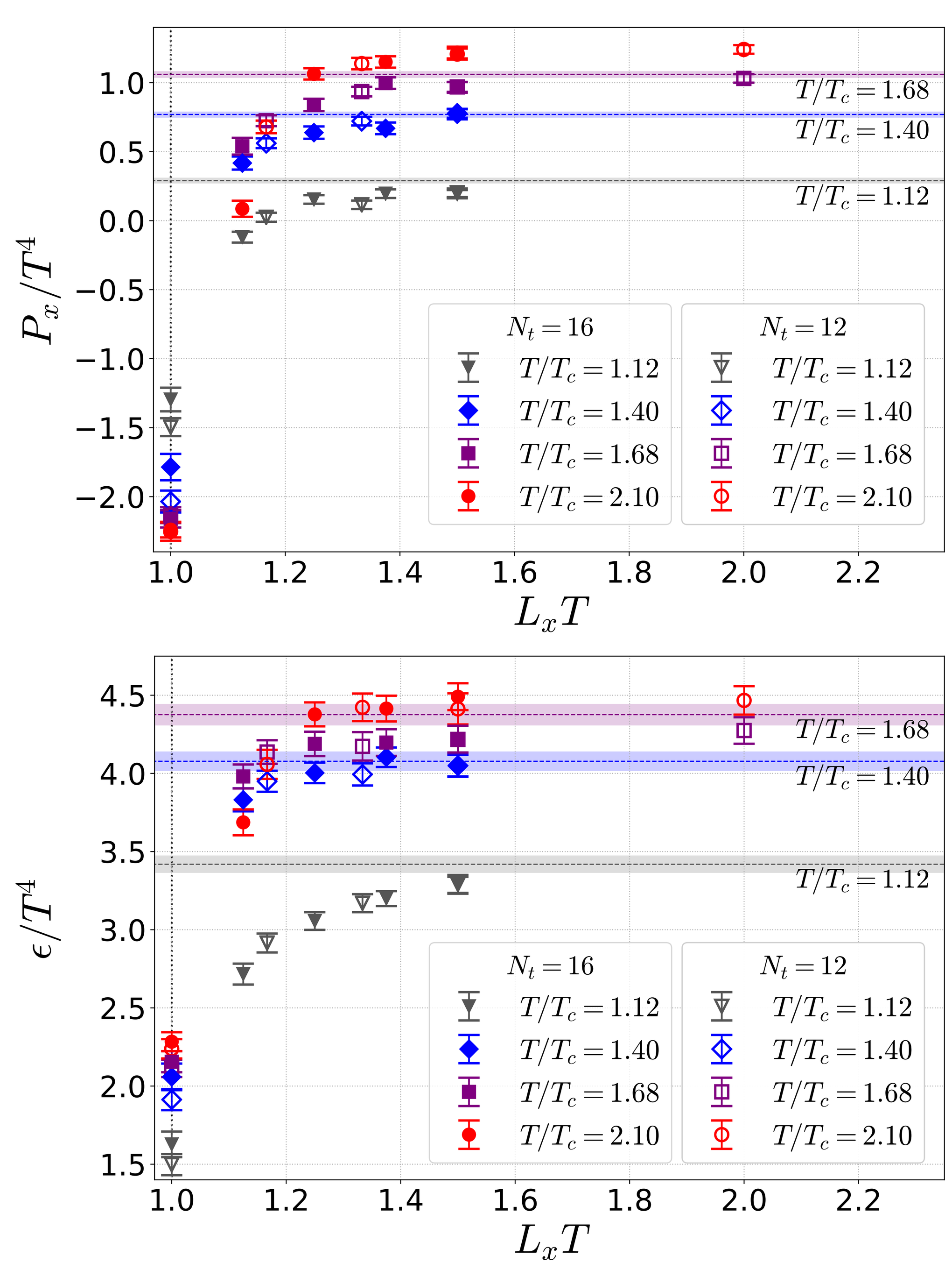


**Figure 4:** (Top) *Scaled pressure $p/T^4$ and* (bottom) *energy density $\epsilon/T^4$ of quenched QCD with periodic boundary conditions with one side of fixed (small) length L [7].*

## 3. Lattice QCD Results

With the intuition from our analytic calculations in hand, we may now quantify the finite system size effects on thermodynamic quantities using the lattice. Quenched QCD pure SU(3) gauge lattice simulations were performed on lattices with anisotropic spatial volumes with periodic boundary conditions (PBCs) [7]. The energy-momentum tensor defined through the gradient flow was used for the analysis of the stress tensor on the lattice. We show in Fig. (4) clear finite-size effects in the pressure and the energy density. Note that in quenched QCD $T_c \approx 270$ MeV. As anticipated from the analytic results, the pressure and energy density are reduced compared to their infinite volume limit as the dimensionless parameter $T \times L_x$ decreases.

We performed additional lattice calculations in quenched QCD with PBCs to examine the trace anomaly as a function of the temperature alone [8]. One can see in Fig. (5) that as the system size decreases, the phase transition broadens and the trace anomaly is reduced, as expected from our analytic results.

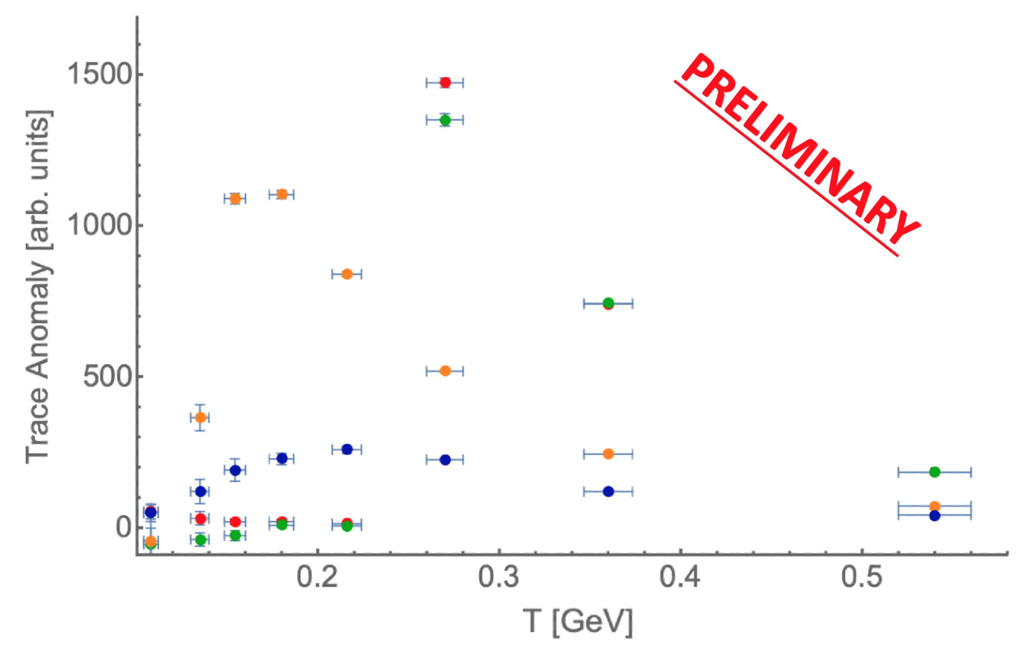


**Figure 5:** *The scaled trace anomaly $\Delta/T^4 \equiv (\epsilon - 3p)/T^4$ as a function of temperature $T$ in quenched QCD with periodic boundary conditions with sides $N_x$ = 24 (red),16 (green), 8 (orange) and 4 (blue) at fixed coupling. The temperature was varied by changing $N_t$. [8].*

## 4. Discussion and Conclusions

We showed that for a free, massless scalar field, placing the system in a finite-sized box reduces the pressure and energy density as compared to the infinite volume limit. These finite size effects persist out to surprisingly large $T \times L \sim 20$. In p+p collisions, $T \times L \sim 20$ corresponds to a QGP temperature of $\sim 400$ MeV, which suggests that finite size effects on the thermodynamics may have non-trivial implications for quantities in these small systems extracted using hydrodynamics models. We next analytically examined the finite size effects on the $N_f = 0$ QCD trace anomaly, finding that the finite system size reduced the anomaly compared to the infinite volume limit, suggesting a faster-than-expected speed of sound. Detailed quenched QCD calculations with periodic boundary conditions agreed qualitatively with the analytic results: pressure, energy density, and the trace anomaly were all reduced compared to their infinite volume limits.

Future work includes rigorously determining the finite size effects on the running coupling in QCD, implementing Dirichlet boundary conditions (DBCs) in the lattice simulations [9], and extending the lattice simulations to full QCD. We anticipate that both DBCs and the presence of fermions will increase the magnitude of the finite size corrections seen in the lattice results presented here.

## 5. Acknowledgments

The authors gratefully acknowledge financial support from the South African National Research Foundation, the SA-CERN Collaboration, the Research Council of Norway under the FRIPRO Young Research Talent grant 286883, and UNINETT Sigma2 – the National Infrastructure for High Performance Computing and Data Storage in Norway.

## References

[1] Weller R D and Romatschke P 2017 *Phys. Lett. B* **774** 351–356 (*Preprint* `1701.07145`)

[2] Mogliacci S, Kolbé I and Horowitz W A 2020 *Phys. Rev. D* **102** 116017 (*Preprint* `1807.07871`)

[3] Borsanyi S, Fodor Z, Hoelbling C, Katz S D, Krieg S and Szabo K K 2014 *Phys. Lett. B* **730** 99–104 (*Preprint* `1309.5258`)

[4] Andersen J O and Strickland M 2005 *Annals Phys.* **317** 281–353 (*Preprint* `hep-ph/0404164`)

[5] Kapusta J I and Gale C 2011 *Finite-temperature field theory: Principles and applications* Cambridge Monographs on Mathematical Physics (Cambridge University Press)

[6] Montvay I and Munster G 1997 *Quantum fields on a lattice* Cambridge Monographs on Mathematical Physics (Cambridge University Press)

[7] Kitazawa M, Mogliacci S, Kolbé I and Horowitz W A 2019 *Phys. Rev. D* **99** 094507 (*Preprint* `1904.00241`)

[8] WAH and Rothkopf A in progress

[9] Rothkopf A 2021 (*Preprint* `2102.08616`)