Aspects of Diffractive Scattering at High Energy & AdS/CFT

Pulkit Agarwal[1], Richard Brower[2],
Timothy Raben[3], Chung-I Tan[4]

[1]National University of Singapore, Singapore
[2]Boston University, Boston, MA
[3]Michigan State University
[4]Brown University, Providence, RI

International Symposium on Multiparticle Dynamics 2021

Outline

Outline

Basic Approach

- Developed in the pre-QCD context, Regge theory essentially suggests that we study scattering processes in S-matrix but allow "physical" variables to be continued into a complex plane.
- The playfield is set with certain "common sense" assumptions:
    - the S-matrix is unitary (essentially a statement of conservation of probability);
    - scattering amplitudes are analytic functions (can be shown to follow from causality);
    - all discontinuities have a physical origin (true for field theories and non-relativistic scattering theory);
    - amplitudes obey crossing symmetry.

Optical Theorem

- An important implication of unitarity is the *Optical theorem*.

$$\sigma_{1\,2}^{\mathrm{Tot}} = \frac{1}{2\,|\boldsymbol{p}_1|\,\sqrt{s}}\mathrm{Im}A(s,t=0) \qquad (1)$$

- Here we have defined the scattering amplitude $A(s,t)$ as $\langle f|T|i\rangle$.
- For $f=i$, what we have essentially is scattering in the "forward" direction, meaning that the Mandelstam variable $t=0$.
- If a spin l particle is exchanged in the crossed channel process, scattering amplitudes have a leading behaviour of the form

$$A(s,t) \sim s^{l} \qquad (2)$$

Regge Trajectories

- Scattering amplitudes can be expressed in terms of partial wave expansions with some partial wave amplitude $a_l(t)$.
- Regge physics suggests that we continue $a_l(t) \rightarrow a(l,t)$ in the complex l plane
- This is then equivalent to

$$A(s,t) \sim s^{\alpha(t)} \tag{3}$$

where $\alpha(t)$ traces out the *Regge trajectory* of an exchange. For forward scattering, we can do an expansion in t to give

$$\alpha(t) = \alpha(0) + t\alpha'(0) + \ldots \tag{4}$$

Outline

Pomeronchuk Theorem

- Total cross section of a particle on a target particle becomes asymptotically equal to its total cross section on the antiparticle. For example:

$$\lim_{s \to \infty} \frac{\sigma_{pp}}{\sigma_{p\bar{p}}} \to 1 \tag{5}$$

- The pomeron is therefore defined as a resonance with vacuum quantum numbers that does not distinguish particles from antiparticles: it is CP even. For weak coupling, $\alpha(0) \sim 1$.
- It is the leading trajectory for hadronic interaction.
- There is also a subleading *Odderon* trajectory that is CP odd and therefore distinguishes matter-antimatter interactions (recently discovered by the D0+TOTEM collaboration).

Outline

Crossing Symmetry

- Crossing symmetry is the idea that the same scattering amplitude function found for, say, the s-channel can be analytically continued to give the scattering amplitude in the u-channel or the t-channel.

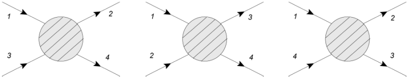


Figure A.1: (left) s-channel scattering, (middle) t-channel scattering, and (right) u-channel scattering.

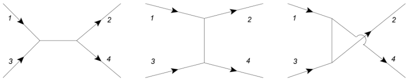


Figure A.2: s-channel scattering involving (left) an s-channel OPE, (middle) a t-channel OPE, and (right) a u-channel OPE.

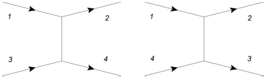
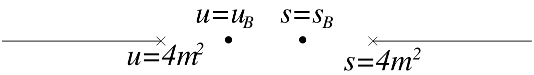


Figure A.3: The t-channel OPE for the (left) s-channel scattering process and (right) u-channel scattering process are related.

Analyticity

- The basic assumption of analyticity we make is that *all singularity structure has a dynamical origin*.
  - poles arise due to bound states
  - thresholds lead to cuts



Outline

Contour Integral

- For an analytic function $A(s,t)$, we have from Cauchy's theorem that
$$A(s,t) = \frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}s' \frac{A(s',t)}{(s'-s)} \tag{6}$$
- What we have done essentially is introduce a function that has a pole, $(s'-s)^{-1}A(s',t)$ and used the residue theorem.

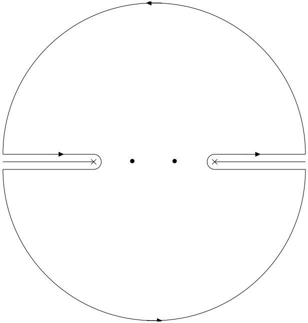


Figure: Integration Contour

Discontinuities

- Writing piece-wise,

$$A(s,t) = \frac{1}{2\pi i}\int_0^\infty ds' \frac{D_s(s',t)}{s'-s} + \frac{1}{2\pi i}\int_0^\infty du' \frac{D_u(u',t)}{u'-u} \qquad (7)$$

where we have defined the discontinuity across the cut as

$$D(s,t) = A(s+i\varepsilon,t) - A(s-i\varepsilon,t) = 2i\, \mathrm{Im}A(s+i\varepsilon,t) \qquad (8)$$

- In extending the integral over $(0,\infty)$, we have included the poles into the definition of the discontinuities. As is evident, this is a very generic statement, it is nothing more than Cauchy's theorem.
- It's power lies in the fact that we can construct the entire function with just a knowledge of it's imaginary parts!

Subtractions

- An important assumption in writing the dispersion relation is that the integral must vanish at ∞.
- If this isn't the case, and the integral is polynomial bounded at ∞, we can still salvage our discussion by making use of *subtractions*.

$$A(s,t) - A(s_1,t) = \frac{1}{2\pi \mathrm{i}}(s-s_1)\oint \mathrm{d}s' \frac{A(s',t)}{(s'-s)(s'-s_1)}$$

$$A(s,t) - A(s_1,t) - (s-s_1)\frac{\partial}{\partial s_1}A(s_1,t) = \frac{1}{2\pi \mathrm{i}}(s-s_1)^2\oint \mathrm{d}s' \frac{A(s',t)}{(s'-s)(s'-s_1)^2}$$

Outline

CFT as a model for QCD

- Soft QCD physics is hard!
- It exhibits many features of scale invariance, it is expected to have critical behaviour for high enough energy scales.

Outline

Scale Invariance to Conformal Invariance

- Consider a scale transformation $x^\mu \to \lambda x^\mu$
- Clearly not a part of the Poincaré group as $x'^\mu \eta'_{\mu\nu} x'^\nu = \lambda^2 x^\mu x_\mu \neq x^\mu x_\mu$
- For scale invariance we therefore require $\eta_{\mu\nu} \to \lambda^{-2} \eta_{\mu\nu}$
- Conformal invariance is a generalisation: $\eta_{\mu\nu} \to f(x) \eta_{\mu\nu}$
- Can be thought of as a *local* rescaling of the system

Conformal Group

- The conformal group is larger than the Poincaré group - given as $SO(d,2)$ for spacetime dimension d
- Can be organised using a $d+2$ dimensional metric $\eta_{ab} = (-1,-1,1,\dots,1)$
- In going to Euclidean signature, boosts are transformed into rotations and the group is $SO(d+1,1)$ with the metric $\eta_{ab} = (-1,1,1,\dots,1)$

Conformal Transformations

- Minkowski confromal group $SO(d,2)$ contains the following transformations:
  - a maximal compact subgroup $SO(d)\otimes SO(2)$: $\frac{d(d-1)}{2}+1$ rotations
  - a maximal abelian subgroup $SO(1,1)\otimes SO(1,1)$: 1 dilatation (scaling) and 1 boost
  - $2d-2$ nilpotent transformations that contain translations and special conformal transformations
- For a theory to be scale invariant, it needs to be independent of any intrinsic mass/length scales!

Outline

Constraints due to Conformal Invariance

- Conformal Invariance puts very strong constraints on correlation functions.
- 2-point correlators are completely determined whereas 3-point correlators are known upto a factor (CFT data).
- 4-point correlators are constrained to be a function of only 2 variables regardless of spacetime dimension d.

2-Point Function

- We define a quantity called *scaling dimension*, denoted by Δ, which determines the scaling properties of a field

$$\phi(\lambda x) = \lambda^{-\Delta}\phi(x). \tag{9}$$

- Scaling invariance therefore gives

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2}\langle \phi_1(\lambda x_1)\phi_2(\lambda x_2)\rangle \,. \tag{10}$$

- Lorentz invariance implies that

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = f(|x_1 - x_2|) \tag{11}$$

- Therefore,

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{c_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}. \tag{12}$$

2-point & 3-point Correlators

- Special conformal invariance further restricts 2-point correlator to $\Delta_1 = \Delta_2$.

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \begin{cases} \frac{c_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} & \text{if } \Delta_1 = \Delta_2 \\ 0 & \text{if } \Delta_1 \neq \Delta_2 \end{cases}$$

- The constant c_{12} can be set by normalisation.
- Similar analysis reveals the 3-point function as

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{e_{123}}{x_{12}^{\Delta-2\Delta_3} x_{23}^{\Delta-2\Delta_1} x_{13}^{\Delta-2\Delta_2}} \tag{13}$$

where $\Delta = \Sigma_i \Delta_i$ and e_{123} are the CFT data mentioned before.

4-Point Functions

- 4-point functions are constrained by CI as

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \frac{1}{(x_{12}^2)^{\Delta_1}(x_{34}^2)^{\Delta_3}} F^{(M)}(u,v) \quad (14)$$

where we assume $\Delta_1 = \Delta_2$ and $\Delta_3 = \Delta_4$ and

$$u \equiv \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}; \quad v \equiv \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2} \quad (15)$$

are called *cross ratios*. Note that these are conformally invariant.

Outline

Definition

- The 4-point function is known to have what is called a *Conformal Block* expansion

$$F^{(M)}(u,v) = \Sigma_\alpha a_\alpha^{(12;34)} G_\alpha^{(M)}(u,v) \qquad (16)$$

where $G_\alpha^{(M)}(u,v)$ are called conformal blocks and a_α are some constants.

- An important property of these functions is that they are eigenfunctions of the conformal Casimir operator:

$$\begin{aligned}\mathscr{D} =& (1-u-v)\partial_v(v\partial_v) + u\partial_u(u\partial_u - d)\\ &-(1+u-v)(u\partial_u + v\partial_v)(u\partial_u + v\partial_v).\end{aligned}$$

Partial Wave Analysis

- This can be interpreted in terms of the usual partial wave analysis: recall in quantum mechanics, if we assume the scattering potential to be spherically symmetric, we can expand the scattering amplitude in terms of spherical harmonics $Y_l^0(\theta,\phi) \sim P_l(\cos\theta)$:

$$A(\theta,\phi) = \Sigma_l (2l+1) a_l P_l(\cos\theta) \tag{17}$$

- We have the same situation here, but our symmetry group is now the larger $SO(d,2)$ instead of the rotation group $SO(d-1)$.
- For $SO(d-1)$, the quadratic Casimir is the angular momentum squared L^2 and the partial waves are eigenstates of this operator.

Previous Work

- Historically, people like working in the Euclidean signature. Our collaborators realised that a direct treatment of the Minkowski signature is important.

PHYSICAL REVIEW D **98,** 086009 (2018)

Minkowski conformal blocks and the Regge limit for Sachdev-Ye-Kitaev-like models

Timothy G. Raben*

University of Kansas, Department of Physics & Astronomy 1082 Malott, 1251 Wescoe Hall Dr. Lawrence, Kansas 66045, USA

Chung-I Tan†

Department of Physics, Brown University Box 1843 182 Hope Street Providence, Rhode Island 02912, USA

(Received 13 July 2018; revised manuscript received 18 September 2018; published 8 October 2018)

- In this paper, one of their many results is that they solve the conformal Casimir in the Minkowski signature
  - exactly for all even d
  - for leading order in general d

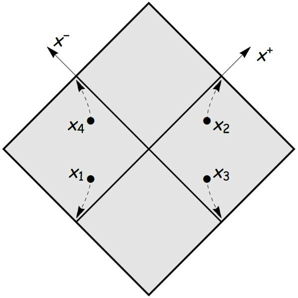
Current Work

- Constructed a unitary irreducible representation of the conformal group.
- Studied the singularity structure of the amplitude.

Outline

The DLC Limit

- To make use of the Optical Theorem (as well as study Regge behaviour), we wish to study the forward scattering limit of $1+3 \to 2+4$.
- This is also called the *Double Light-Cone* limit as we wish for $(1,2)$ to remain close to the forward lightcone and $(3,4)$ close to the backward lightcone. In terms of (u,v), this translates to the limit $(0,1)$.



Kinematics

- In terms of rapidity y, we have

$$t' = t\cosh y - x\sinh y$$
$$x' = x\cosh y - t\sinh y$$

- We can define *lightcone coordinates* $x_i^{\pm} = t \pm x$ so that we have

$$x_i^{\pm} = \pm\varepsilon_i r_i e^{\pm y_i} \tag{18}$$

where $r_i = \sqrt{-x_i^+ x_i^-}$ which we can assume to be > 0 without loss of generality. Clearly, $r^2 = x^2 = -t^2 + x^2$.

- ε_i is some sign factor that depends on the particle we are talking about. Explicitly,

$$x_1^{\pm} = \mp r_1 e^{\pm y_1};\; x_2^{\pm} = \pm r_2 e^{\pm y_2}\; x_3^{\pm} = \pm r_3 e^{\mp y_3}\; x_4^{\pm} = \mp r_4 e^{\mp y_4} \tag{19}$$

Kinematics

- One simplifying assumption we can make is $y = y_i$ and $r_1 = r_2$ and $r_3 = r_4$. This gives

$$u = \frac{16}{(e^{2y} + 2R(1,3) + e^{-2y})^2}; \quad v = \frac{(e^{2y} - 2R(1,3) + e^{-2y})^2}{(e^{2y} + 2R(1,3) + e^{-2y})^2} \tag{20}$$

where

$$R(i,j) = \frac{r_i^2 + r_j^2 + b_\perp^2}{2r_i r_j} \tag{21}$$

Kinematics

$$u = \frac{16}{(e^{2y} + 2R(1,3) + e^{-2y})^2}; \quad v = \frac{(e^{2y} - 2R(1,3) + e^{-2y})^2}{(e^{2y} + 2R(1,3) + e^{-2y})^2} \quad (22)$$

- The limit $(u, v) \to (0, 1)$ can be taken in two ways:
  - Global Lorentz boost - $y \to \infty$
  - Dilatation - $b_\perp \to \infty$
- In the Euclidean region, $y \to \pm iy$. This means *Boosts become Rotations!*
- **Important Conclusion!** Euclidean signature has just one scaling limit whereas Minkowski has two.

[This relates to the fact that the maximum abelian subgroup of $SO(d,2)$ is $SO(1,1) \times SO(1,1)$ whereas that for $SO(d+1,1)$ is $SO(1,1)$.]

Crossing Symmetric Variables

$$u \equiv \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}; \quad v \equiv \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2} \tag{23}$$

- These definitions are for the physical s-channel scattering region where $0 < u < 1$ and $0 < v < 1$.
- Under crossing, we can go to the u-channel by either $1 \leftrightarrow 2$ or $4 \leftrightarrow 3$ which gives

$$u' = \frac{x_{12}^2 x_{34}^2}{x_{23}^2 x_{14}^2} = \frac{u}{v}; \quad v' = \frac{x_{13}^2 x_{24}^2}{x_{23}^2 x_{14}^2} = \frac{1}{v}. \tag{24}$$

- In the u-channel region, we have $0 < u' < 1$ and $0 < v' < 1$ which corresponds to $0 < u < \infty$ and $1 < v < \infty$.
- (u, v) are therefore **not** crossing symmetric.

Crossing Symmetric Variables

- Define a new set of variables (w, σ) such that

$$w\sigma = \frac{1 - v}{u}. \tag{25}$$

- There are 2 important issues we are trying to resolve with this new set:
  - the variables be symmetric under s-u crossing
  - the variables have a clear distinction in what is meant by Euclidean region and Minkowski region.

Crossing Symmetric Variables

- Choose:

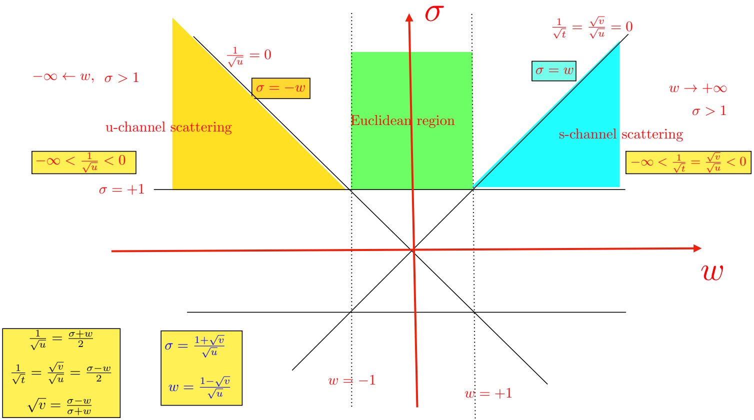
$$w = \cosh Y = \frac{1-\sqrt{v}}{\sqrt{u}}; \quad \sigma = \cosh \eta = \frac{1+\sqrt{v}}{\sqrt{u}},$$

where we have defined $2y = Y$ and $R(1,3) = \cosh \eta$.

Here, s-channel physical region $1 < w < \infty$ and $1 < \sigma < \infty$ with $(w, \sigma) \to (-w, \sigma)$ under s-u crossing!

- Clearly, $w \to$ boosts, $\sigma \to$ dilatations.
- In the Euclidean region, $y \to \pm iy \Rightarrow$ boosts $\to$ rotations.

(w, σ) Plane



Conformal Casimir

- In our variables, conformal Casimir operator is

$$\tilde{\mathscr{C}} = -\partial_y^2 - \partial_\eta^2 + V(y,\eta) \tag{26}$$

where

$$\begin{aligned} V(y,\eta) = & -\frac{2w^2\sigma^2 - w^2 - \sigma^2}{(w^2-\sigma^2)^2} + V_0 \\ & + \frac{(d-4)(d-2)}{2}\left(\frac{1}{w^2-1} + \frac{1}{\sigma^2-1}\right) \end{aligned}$$

[A detailed analysis is to appear:
Pulkit Agarwal, Richard Brower, Timothy Raben and Chung-I Tan,
"CFT in Lorentzian Limit and Principal Series Representation"]

Conformal Casimir

- This is a *pseudo* potential scattering problem in classical mechanics. The solution can be expanded in terms of plane waves with the eigenvalue

$$\lambda = -\tilde{\ell}^2 - \tilde{\Delta}^2 \tag{27}$$

where we have subtracted the "ground state energy" $V_0 = (d/2)^2 + (d/2 - 1)^2$.

- Solution to this equation gives us our representation for $SO(d,2)$ where $(\tilde{\ell}, \tilde{\Delta})$ serve as our representation labels.

Outline

TMDs

- One of the most interesting and direct phenomenological applications of our results is to studying TMDs.
- Most notable is the work of Xiangdong Ji (PRL 110, 262002 (2013)) who realised that equal time correlators in large momentum limit can be used to study parton physics!
- This is identical to taking 4-point functions in our setup to study parton distribution functions.
- Most studies in this domain are currently done using lattice gauge theory methods, which requires an additional step of going to the Euclidean region of the phase space and relating back to Minkowski. Our direct treatment of Minkowski scattering helps avoid this.
- It can also be further generalised in our CFT context for broader applications!

CFT Applications to Phenomenology

- Of course, applications of a scale invariant theory to real world physics requires the introduction of a scale parameter, the most obvious being a confinement scale.
- Many such models have been introduced, that impose a cut off in the dual AdS geometry and study the implications on a boundary CFT.

Regge Physics & Pomeron

1. S. Donnachie, G. Dosch, P. Landshoff, O. Nashtmann, *Pomeron Physics and QCD*, Cambridge University Press
2. P. D. B. Collins, *An Introduction to Regge Theory and High Energy Physics*, Cambridge Monographs on Mathematical Physics 4
3. ...

Conformal Field Theory

1. P. D. Francesco, P. Mathieu, D. Senechal, *Conformal Field Theory*, Springer
2. Joshua D. Qualls, *Lectures on Conformal Field Theory*, arXiv:1511.04074v2
3. T. Raben, C-I Tan, *Minkowski Conformal Blocks and the Regge Limit for SYK-like Models*, arXiv:1801.04208v2
4. ...

Applications to High Energy Scattering

1. Xiangdong Ji, *Parton Physics on a Euclidean Lattice*, PRL 110, 262002 (2013)
2. A. V. Radyushkin, *Quasi-PDFs, momentum distributions and pseudo-PDFs*, arXiv:1705.01488
3. Miguel S. Costa, Marko Djurić, *Deeply virtual Compton scattering from gauge/gravity duality*, arXiv:1201.1307
4. Richard C. Brower, Marko Djurić, Ina Sarčević, Chung-I Tan, *String-Gauge Dual Description of Deep Inelastic Scattering at Small-x*, arXiv:1007.2259
5. ...