Third order correction to the muon lifetime

16th International Workshop on Tau Lepton Physics

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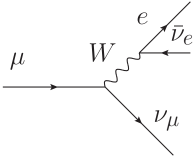
in collaboration with K. Schönwald, M. Steinhauser, Phys.Rev.D 104 (2021) 016003

Muon Decay



- Fundamental process to study weak interaction.
- Fermi Constant G_F extracted from muon lifetime

$$\frac{\sim}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3} F(\rho) \left[1 + H_1(\rho) \frac{\hat{\alpha}(m_\mu)}{\pi} + H_2(\rho) \left(\frac{\hat{\alpha}(m_\mu)}{\pi} \right)^2 + H_3(\rho) \left(\frac{\hat{\alpha}(m_\mu)}{\pi} \right)^3 \right]$$

with $\rho = m_e/m_\mu$ ' $1/210$.

- $O(\alpha)$ Behrends, Finkelstein, Sirlin Phys.Rev.101 (1956) 866.
- $O(\alpha^2)$
  van Ritbergen, Stuart Phys.Rev.Lett. 82 (1999) 488 $(m_e = 0)$
  Czarnecki, Pak, Phys.Rev.Lett. 100 (2008) 241807 $(m_e \ll m_\mu)$
- $O(\alpha^3)$ **NEW**
  MF, Schönwald, Steinhauser, Phys.Rev.D 104 (2021) 016003
  $(m_e \sim m_\mu)$
  parts of the calculation also confirmed in
  Czakon, Czarnecki, Dowling Phys.Rev.D 103 (2021) L111301.

Input Parameters of SM

- Fine structure constant CODATA

$$1/\alpha = 137.035\,999\,084\,(21)\ (0.15\ \mathrm{ppb})$$

from a_e: 0.35 ppb Aoyama, Hayakawa, Kinoshita, Nio, PRL 109 (11) 111807
from h/m in ^{87}Rb: (0.08ppb) Morel, Yao, Cladé, Guellati-Khélifa, Nature 588 61

- Fermi Constant

$$G_F = 1.166\,378\,7\,(6) \quad 10^{\ 5}\ \mathrm{GeV}^{\ 2}\ (0.5\ \mathrm{ppm})$$

Webber,et al., 2011, Phys.Rev.Lett.106, 041803; Phys.Rev.Lett.106, 079901

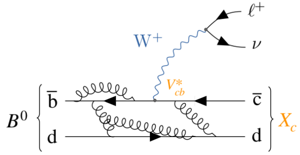
- Z-boson mass

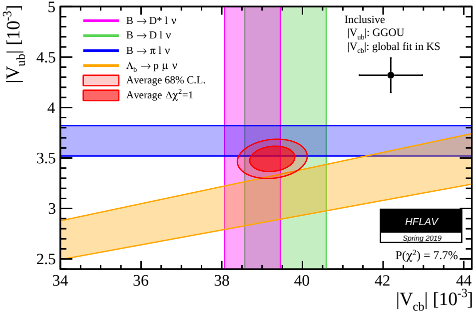
$$M_Z = 91.1876 \quad 0.0021\ \mathrm{GeV}\ (23\ \mathrm{ppm})$$

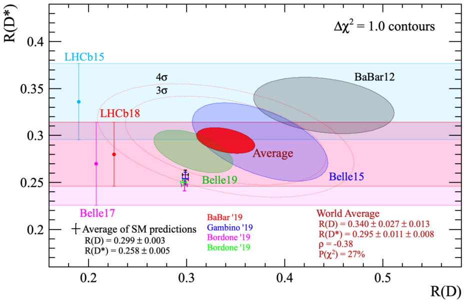
P.A. Zyla et al. (Particle Data Group), Prog. Theor. Exp. Phys. 2020, 083C01 (2020)
FCC-ee aims at improving M_Z by factor 400 (stat).

Semileptonic *B* decays

- $b \to c\ell\bar{\nu}_\ell$ $(\ell = e, \mu)$ sensitive to $|V_{cb}|$.
- **Inclusive decays**
  - $\overline{B}$! $X_c \ell \bar{\nu}$, with $X_c = D, D\ \ , D\pi, \ldots$

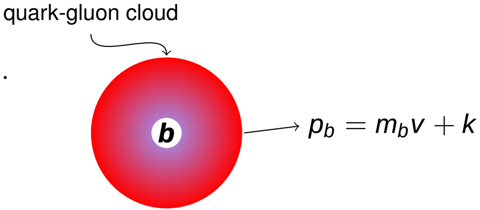






The Heavy-Quark Expansion

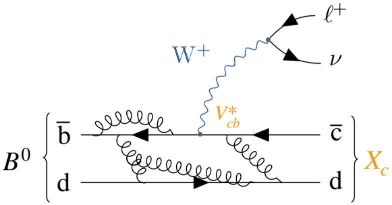
$$\Gamma_{\rm sl} = \Gamma_0 + \Gamma_{\mu_\pi} \frac{\mu_\pi^2}{m_b^2} + \Gamma_{\mu_G} \frac{\mu_G^2}{m_b^2} + \Gamma_{\rho_D} \frac{\rho_D^3}{m_b^3} + \Gamma_{\rho_{LS}} \frac{\rho_{LS}^3}{m_b^3} + \dots$$



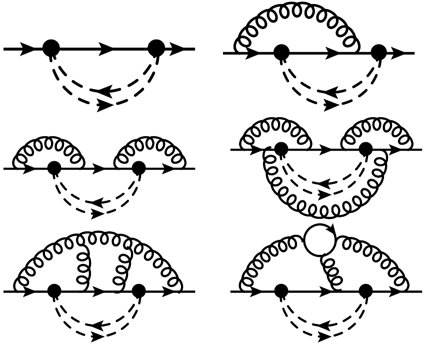
Reviews:
Benson, Bigi, Mannel, Uraltsev, Nucl.Phys. B665 (2003) 367;
Dingfelder, Mannel, Rev.Mod.Phys. 88 (2016) 035008.

- Γ_i are computed in **perturbative QCD**.
- **Γ_0: free-quark decay $b \to c\ell\nu$!**
- The HQE parameters:

  $\mu_\pi, \mu_G, \rho_D, \rho_{LS}$ $\quad \langle B| O_i^{\bar{b}b} |B\rangle$
- HQE parameters are **extracted from data or lattice**.



$$\frac{\sim}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left[X_0(\rho) + \sum_{n\geq 1}\left(\frac{\alpha}{\pi}\right)^n X_n(\rho)\right]$$



Possible strategies

- Exact result with m_μ and m_e only at $O(\alpha)$.
  Nir, Phys.Lett.B 221 (1989) 184
- Set $m_e = 0$ ($m_c = 0$).
  van Ritbergen, Stuart
- **Approximation exploiting $m_e < m_\mu$ ($m_c < m_b$).**
  Czarnecki, Pak

Computational Method

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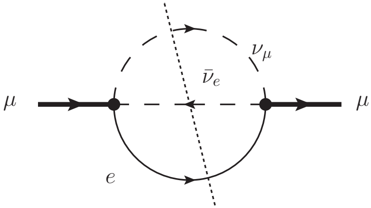
$$\frac{\sim}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left[1 \quad 8\rho^2 \quad 12\rho^4 \log(\rho^2) + 8\rho^6 \quad \rho^8\right]$$

where $\rho = m_e/m_\mu$.



- Optical theorem.
- Multi-loop diagrams with **two scales m_e and m_μ.**
- Use **method of regions:**
  Beneke, Smirnov, NPB 522 (1998) 321; Smirnov, Springer Tracts Mod. Phys. 177
  - Expansion in $\rho = m_e/m_\mu$ (or $\rho = m_c/m_b$).
    Obvious choice, too difficult to extend at $O(\alpha^3)$.
  - Expansion in $\delta = 1 - m_e/m_\mu$ (or $\delta = 1 \quad m_c/m_b$)
    Crucial decoupling and simplifications of loop integrals.

A toy example

$$F(m,M,\varepsilon) = \int_0^\infty dk \frac{k^{-\varepsilon}}{(k+m)(k+M)} \overset{\varepsilon\to 0}{=} \frac{\log(M/m)}{M \quad m} = \frac{\log(M/m)}{M} \sum_{n=0}^{\infty} \left(\frac{m}{M}\right)^n$$

When k is a *hard* scale $O(M)$

$$F_{\mathrm{h}}(m,M,\varepsilon) = \int_0^\infty dk \frac{k^{-\varepsilon}}{(k+M)} \left[\frac{1}{k} \quad \frac{m}{k^2} + \dots\right] \overset{\varepsilon\to 0}{=} \left[\frac{1}{\varepsilon M} + \frac{\log M}{M}\right] \sum_{n=0}^{\infty} \left(\frac{m}{M}\right)^n$$

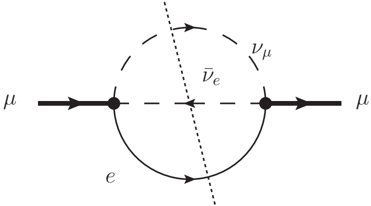
k is a *soft* scale $O(m)$

$$F_{\mathrm{s}}(m,M,\varepsilon) = \int_0^\infty dk \frac{k^{-\varepsilon}}{(k+m)} \left[\frac{1}{M} \quad \frac{k}{M^2} + \dots\right] \overset{\varepsilon\to 0}{=} \left[\frac{1}{\varepsilon M} \quad \frac{\log m}{M}\right] \sum_{n=0}^{\infty} \left(\frac{m}{M}\right)^n$$

Muon lifetime at tree level

$$\Gamma_\mu = \Gamma^{(\mathrm{hh})} + \Gamma^{(\mathrm{ss})} + \Gamma^{(\mathrm{hs})} + \Gamma^{(\mathrm{sh})}$$

$$= \Gamma_0 \left[1 \quad 8\rho^2 \quad 12\rho^4 \log(\rho^2) + 8\rho^6 \quad \rho^8 \right]$$



where $\rho = m_e/m_\mu$ and $\Gamma_0 = \dfrac{G_F^2 m_\mu^5}{192\pi^3}$

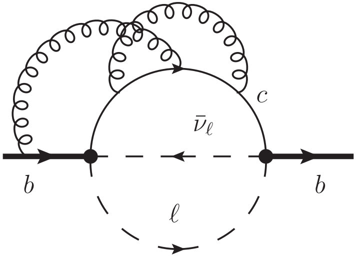
$$\Gamma^{(\mathrm{hh})} \quad \frac{\rho^4}{12\varepsilon} \quad 24\rho^4 \log\left(\frac{\mu^2}{m_\mu^2}\right) + 1 \quad 8\rho^2 \quad 24\rho^4 + 16\rho^6 \quad 2\rho^8$$

$$\Gamma^{(\mathrm{hs})} \quad + \frac{\rho^4}{12\varepsilon} + 24\rho^4 \log\left(\frac{\mu^2}{m_\mu^2}\right) \quad 12\rho^4 \log(\rho^2) + 24\rho^4 \quad 8\rho^6 + \rho^8$$

$$\Gamma^{(\mathrm{ss})} = \Gamma^{(\mathrm{sh})} = 0$$

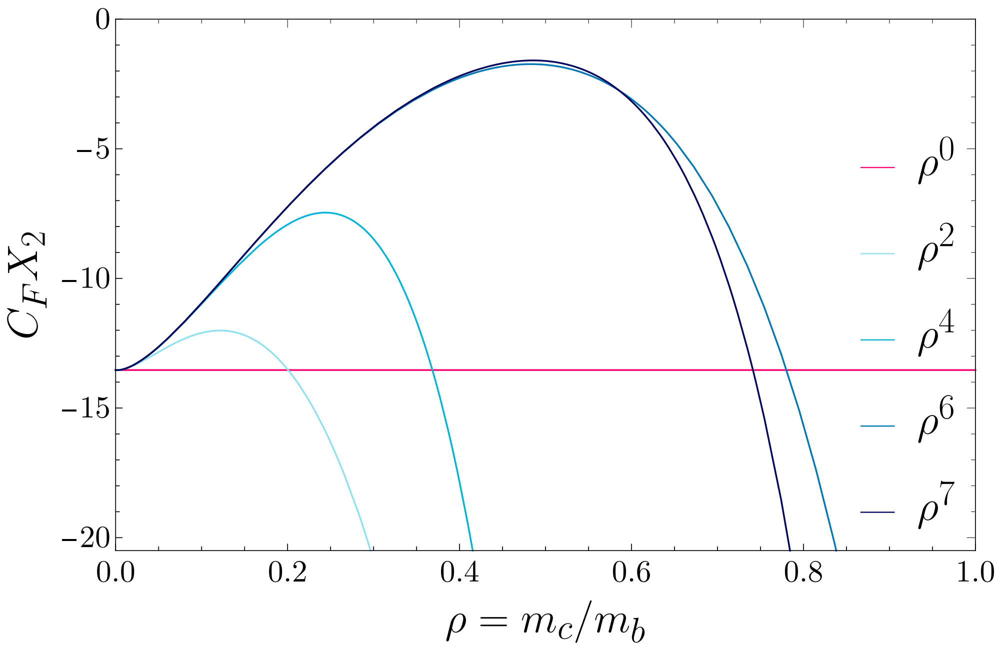
Second order corrections

$$\frac{\sim}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left[X_0 + \frac{\alpha}{\pi}X_1 + \left(\frac{\alpha}{\pi}\right)^2 X_2 + \dots\right]$$



Czarnecki, Pak, PRD 78 (2008) 114015; PRL 100 (2008) 241807.

- Four-loop diagrams, all loop momenta can scale **hard** (m_μ) or **soft** (m_e).
- Each diagram considered up to **11 different regions**.
- The *all-hard* region reduces to **33 four-loop master integrals**.
- Expansion depth: $O(\rho^7)$ ($\rho = m_e/m_\mu$).


0
−5
−10
−15
−20
0.0
0.2
0.4
0.6
0.8
1.0
$C_F X_2$
$\rho = m_c/m_b$
ρ^0
ρ^2
ρ^4
ρ^6
ρ^7

Towards the third order corrections

	α_s^2		α_s^3
n. diagrams	62	!	1450
n. loops	4	!	5
regions	11	!	O(20)
expansion depth	7	!	?
master integrals	33	!	?


RUSSIA
North Pole
Svalbard
(Norway)
NORWAY
Bering Sea
ARCTIC OCEAN
Ellesmere
Island
ICELAND
Greenland
(Denmark)
Alaska
(U.S.)
Banks
Island
ATLANTIC
OCEAN
Baffin
Bay
Victoria
Island
PACIFIC
OCEAN
Baffin
Island
CANADA
W
N
S
E
Hudson
Bay
Labrador
Northwest Passage
0
500
1,000 miles
0
500
1,000 kilometers

The heavy daughter limit

Dowling, Piclum, Czarnecki, PRD 78 (2008) 074024

- Is the most natural expansion parameter sometimes also the best one?

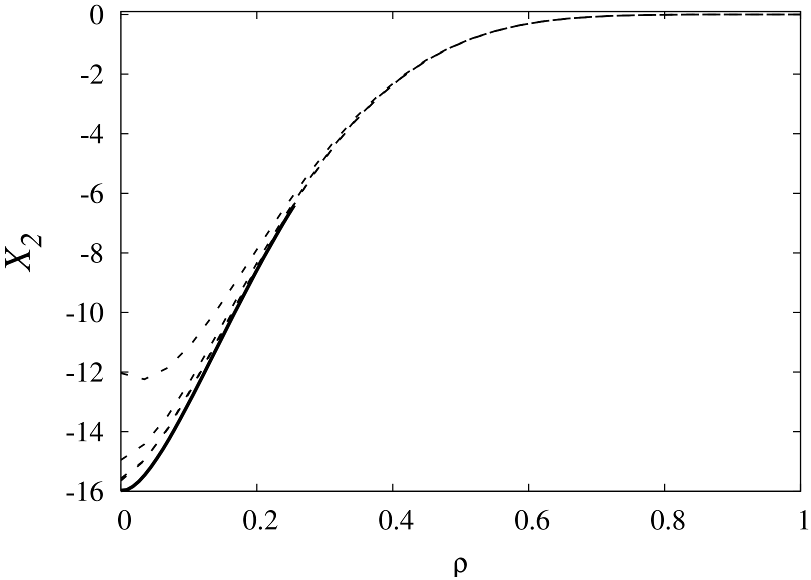
$$\frac{m_e}{m_\mu} \qquad \frac{1}{210} \qquad \frac{m_c}{m_b} \qquad 0.3$$

- Perform the expansion in the limit $m_e \quad m_\mu$ ($m_c \quad m_b$):

$$\delta = 1 \qquad \rho = 1 \qquad \frac{m_e}{m_\mu} \qquad 1$$

- The width must behave in the the $m_e \;!\; m_\mu$ limit as:

$$\Gamma_\mu^{\;m_e \,!,\, m_\mu} \; \frac{G_F^2}{192\pi^3}(m_\mu \quad m_e)^5 = \frac{G_F^2 m_\mu^5}{192\pi^3}\delta^5$$

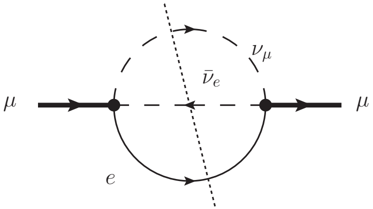


Dowling, Piclum, Czarnecki, PRD 78 (2008) 074024

Muon Lifetime Reloaded

$$\Gamma_\mu = \Gamma^{(\mathrm{hh})} + \Gamma^{(\mathrm{ss})} + \Gamma^{(\mathrm{sh})} + \Gamma^{(\mathrm{hs})}$$

$$= \Gamma_0 \left[\frac{64}{5}\delta^5 \quad \frac{96}{5}\delta^6 + \frac{288}{35}\delta^7 + \dots \right]$$



where $\rho = m_e/m_\mu$ and $\Gamma_0 = \frac{G_F^2 m_\mu^5}{192\pi^3}$

- **Crucial simplifications** in the heavy daughter limit!
- At least one electron's propagator must scale *soft* to generate an imaginary part $\log(\quad \delta)$.
- **Much smaller number of regions**, e.g. $\Gamma^{(\mathrm{hh})} = \Gamma^{(\mathrm{sh})} = \Gamma^{(\mathrm{hs})} = 0$

Divide et Impera

- The heavy daughter limit converts **5 loops** ! **3 loops**!
- Loop integrals associated to ν and $\bar{\nu}$ momenta decouple and are integrated analytically at once.
- Explanation is rather technical, connected to the appearance of linear propagators:

$$\frac{1}{(p+k)^2 \quad m_c^2} \; ! \; \frac{1}{2p \quad k \quad \delta}$$

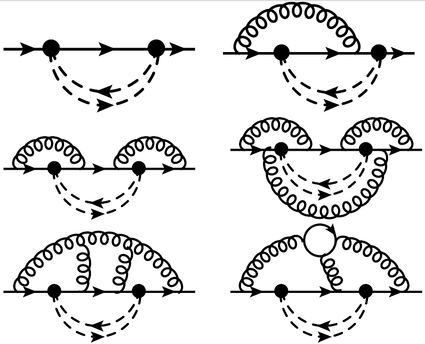
	scaling	n. regions
$O(\alpha)$	h, u	2
$O(\alpha^2)$	**hh**, hu, **uu**	4
$O(\alpha^3)$	**hhh**, **uuu**, h**uu**, **hh**u	8

Computational Challenges

- 1450 five-loop diagrams.
- Several subtleties with `FORM`
  - Propagators expanded up to **10th - 12th order.**
  - Major obstacle is to keep as small as possible **the size of intermediate expressions**.
  - Efficient expansion of propagators and memory management.
- Intermediate `FORM` expressions up to $O(100)$ GB.
- Master integrals:
  - $O(\alpha^2)$: 3 (ss) and 3 (hh).
  - $O(\alpha^3)$: 20 (sss) and 19 (hhh).
    Melnikov, van Ritbergen, Nucl.Phys.B 591 (2000) 515;
    MF, Schönwald, Steinhauser, Phys.Rev.Lett. 125 (2020) 5.
- Renormalization constants at 3 loops with two massive fermions.

MF, Schönwald, Steinhauser, JHEP 10 (2020) 087.

$$\frac{\sim}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left[X_0 + \sum_{n\geq1}\left(\frac{\hat{\alpha}(m_\mu)}{\pi}\right)^n X_n\right]$$

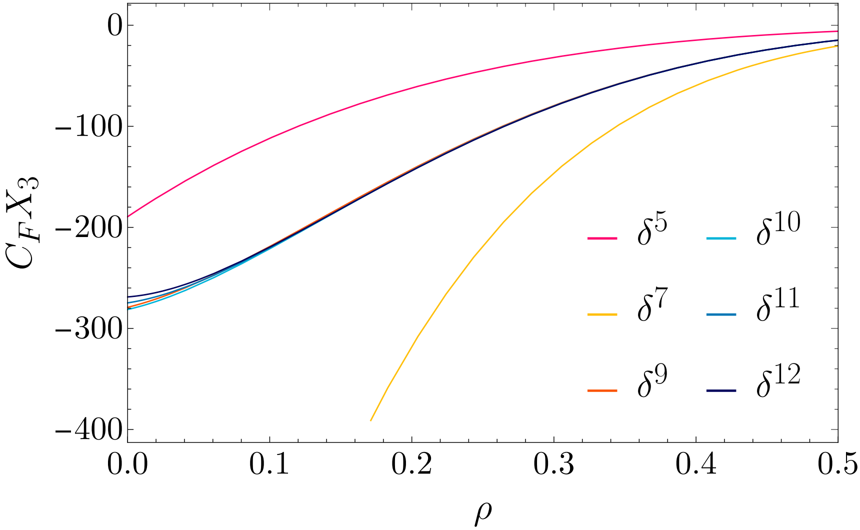


- **NEW**: X_3 up to δ^{12} (first 8 terms).

$$X_3 = \sum_{m \quad 5} x_{3,m}\delta^m = \delta^5\left[\frac{256}{3}\mathrm{a4} \quad 128\zeta(5) + \frac{56}{5}\pi^2\zeta_3 + \frac{1984}{45}\zeta_3 + \frac{452}{675}\pi^4 \quad \frac{3652}{405}\pi^2 \right.$$
$$\left. \frac{18451}{270} + \frac{32}{9}\log^4(2) \quad \frac{352}{45}\pi^2\log^2(2) \quad \frac{32}{45}\pi^2\log(2)\right] + O(\delta^6)$$

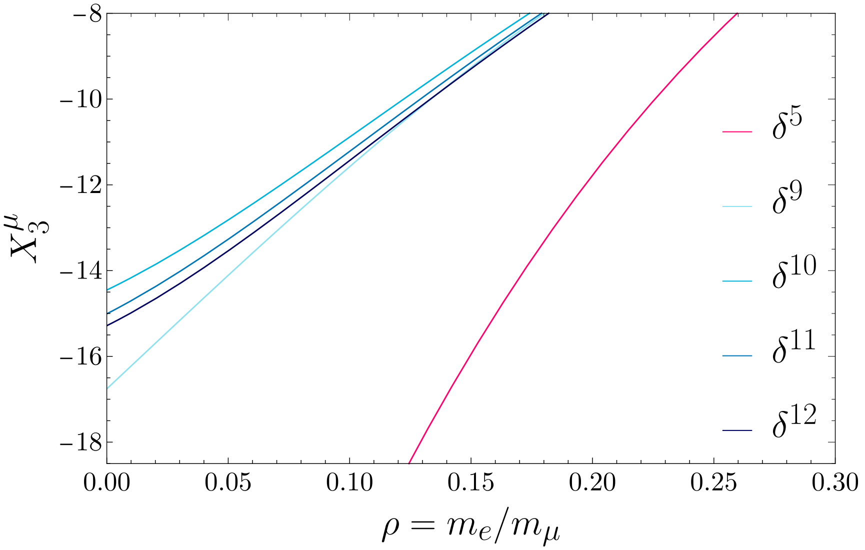
MF, Schönwald, Steinhauser, Phys.Rev.D 104 (2021) 016003

confirmed C_F^3, $C_F^2 N_H$ and $C_F N_H^2$ color factors up to δ^9 Czakon, Czarnecki, Dowling, Phys.Rev.D 103 (2021) L111301



$C_F X_3(\rho = 0.28) =$ 91.2 0.4 (0.4%)

MF, Schönwald, Steinhauser, Phys.Rev.D 104 (2021) 1, 016003



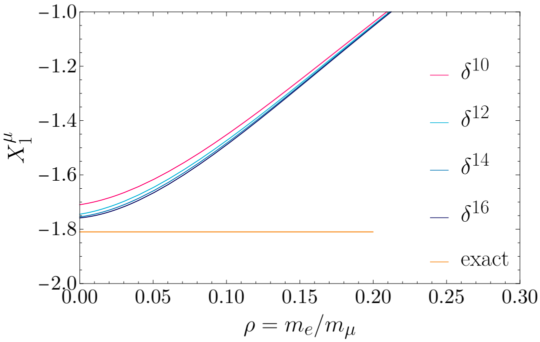
$X_3^\mu =$ 15.3 2.3 (15%)

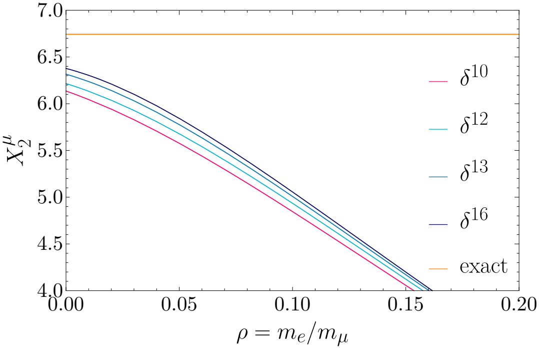
MF, Schönwald, Steinhauser, Phys.Rev.D 104 (2021) 1, 016003

Previous estimate: X_3^μ ′ 20

Ferroglia, Ossola, Sirlin, Nucl.Phys.B 560 (1999) 23

Theoretical uncertainties





order	X_n/X_{exact}	$x_{n,12}/X_n$
α	0.96	0.6 %
α^2	0.92	1.0%
α^3	0.85	1.8 %

Final Error Budget

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$$\frac{\sim}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^2} F(\rho)\,(1 + \Delta q)$$

$$\Delta q = (4\,234\,530 j_{\hat\alpha} + 36\,332 j_{\hat\alpha^2} + 200 j_{\hat\alpha^3} \quad 29 j_{\delta\hat\alpha^3} \quad 11 j_{\delta\mathrm{had}}) \quad 10^{-9}$$

- $\delta(1 + \Delta q) = 0.031$ ppm
- $\frac{1}{2}\frac{\delta\tau_\mu}{\tau_\mu} = 0.5$ ppm
- $\frac{5}{2}\frac{\delta m_\mu}{m_\mu} = 0.05$ ppm

- at $O(\alpha^2)$: $\delta(1 + \Delta q) = 0.17$ ppm
  van Ritbergen, Stuart, Nucl.Phys.B 564 (2000) 343
  Sirlin, Ferroglia, Rev.Mod.Phys. 85 (2013) 1
- Precise $\overline{\mathrm{MS}}$-on shell conversion of α up to 4 loops
  Baikov, Chetyrkin, Kuhn, Sturm, Nucl.Phys.B 867 (2013) 182

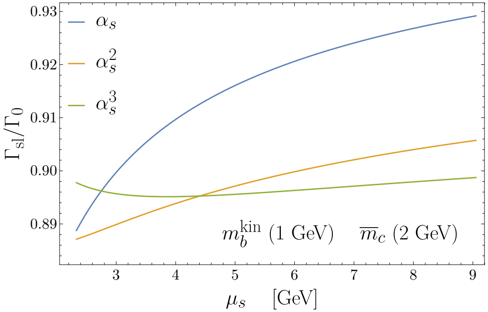
Lessons for $B \to X_c \ell \nu$

$$\Gamma_{\rm sl} = \frac{G_F^2 m_b^5 |V_{cb}|^2}{192\pi^3} F(\rho) \left[1 + \sum_n Y_n \left(\frac{\alpha_s(m_b)}{\pi} \right)^n \right]$$

n=1 Jezabek, Kühn, Jezabek, Kuhn, NPB 314 (1989) 1

n=2 Melnikov, PLB 666 (2008) 336; Pak, Czarnecki, PRD 78 (2008) 114015.

n=3 Fael, Schönwald, Steinhauser, hep-ph/2011.13654



$$m_b^{\rm OS} : m_c^{\rm OS} \quad 1 \quad 1.78 \left(\frac{\alpha_s}{\pi}\right) \quad 13.1 \left(\frac{\alpha_s}{\pi}\right)^2 \quad 163.3 \left(\frac{\alpha_s}{\pi}\right)^3$$

$$\overline{m}_b(\overline{m}_b) : \overline{m}_c(3\text{ GeV}) \quad 1 + 3.07 \left(\frac{\alpha_s}{\pi}\right) + 13.3 \left(\frac{\alpha_s}{\pi}\right)^2 + 62.7 \left(\frac{\alpha_s}{\pi}\right)^3$$

$$m_b^{\rm kin}(1\text{ GeV}) : \overline{m}_c(2\text{ GeV}) \quad 1 \quad 1.24 \left(\frac{\alpha_s}{\pi}\right) \quad 3.65 \left(\frac{\alpha_s}{\pi}\right)^2 \quad 1.0 \left(\frac{\alpha_s}{\pi}\right)^3$$

Improvement in $|V_{cb}|$

- Fit BR and moments from B factories
- Global fit strategy from 2014
  Gambino, Schwanda, Phys.Rev.D 89 (2014) 014022
  Alberti, Gambino, Healey, Nandi, Phys.Rev.Lett. 114 (2015) 6, 061802
- $O(\alpha_s^3)$ semileptonic width
- $O(\alpha_s^3)$ relation between $\overline{m}_b \quad m_b^{\mathrm{kin}}$
  MF, Schönwald, Seinhauser, Phys.Rev.Lett. 125 (2020) 052003;
  Phys.Rev.D 103 (2021) 1, 014005

- Precise input from lattice

$$\overline{m}_c(3\text{ GeV}) = 0.988\,(7)\text{ GeV}$$
$$\overline{m}_b(\overline{m}_b) = 4.198\,(12)\text{ GeV}$$
$$!\;\; m_b^{\mathrm{kin}} = 4.56\,(19)\text{ GeV}$$

FLAG2019

$$jV_{cb}j = 42.16\,(30)_{\mathrm{th}}(32)_{\mathrm{exp}}(25)_{\Gamma} \quad 10\quad{}^{3}$$
$$= 42.16\,(50) \quad 10\quad{}^{3}$$

Bordone, Capdevila, Gambino, hep-ph/2107.00604
error improvement of 34% compared to 2014.

Conclusions

- QED $O(\alpha^3)$ to the muon lifetime
- QCD α_s^3 corrections to $\Gamma(b \to X_c \ell \nu)$.
- Heavy daughter limit: small parameter $\delta = 1 - m_e/m_\mu$ and $\delta = 1 - m_c/m_b$.
- $\Delta q^{(3)}$ with relative 15% uncertainty. Theory error on Δq reduced to 0.03 ppm.
- 1% theory uncertainty in semileptonic decays width.
- Improved extraction of $|V_{cb}|$ inclusive by about 34%.