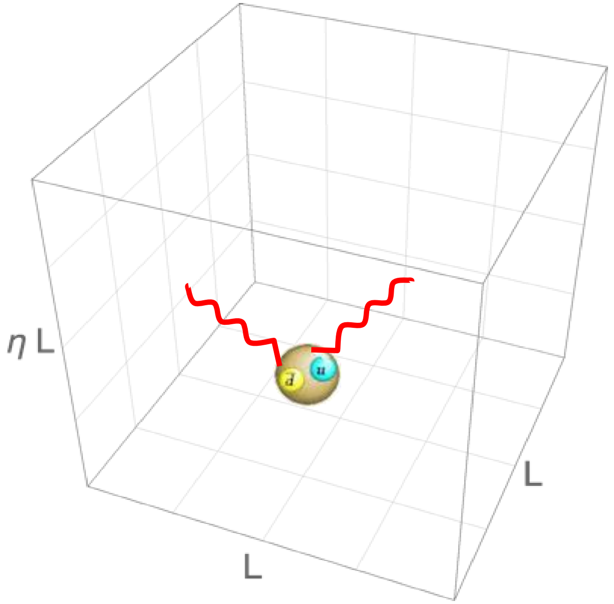
Polarizabilities from four-point functions in lattice QCD

Frank Lee (GW)
Collaborators: Andrei Alexandru (GW), Chris Culver (Liverpool), Walter Wilcox (Baylor)

Outline

1) Motivation
2) Background field method
3) Four-point function method
4) Lattice simulations and results
5) Conclusion



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Nucleon Structure at Low Q, 16 May 2023, Crete, Greece

Polarizability of hydrogen atom

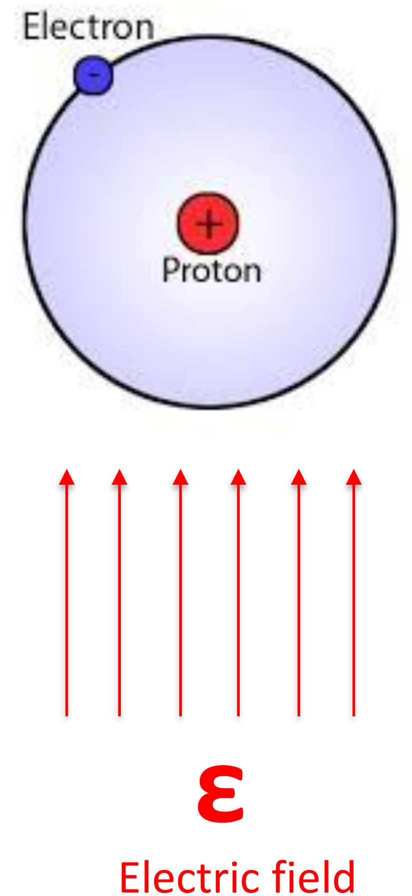
2nd order perturbation in quantum mechanics:

$$H = H_0 + (-e\mathcal{E}z)$$

$$\Delta H = \sum_{n\neq 1,l,m} \frac{|\langle nlm|H'|100\rangle|^2}{E_n - E_1} \equiv -\frac{1}{2}\alpha_E \mathcal{E}^2$$

$$E_n = -\frac{13.6}{n^2}\ (\mathrm{eV})$$

$$\alpha_E = 4.5a_0^3 \qquad a_0 = 0.529\ \mathring{A}$$



1) Polarizability is measured by volume of system.
2) Hydrogen atom is electrically soft.

Hadron polarizabilities (in units of 10^{-4} fm^3)

- Polarizabilities encode information on charge and current distributions inside hadrons at low energies.
- An active community in nuclear physics is engaged in the effort (experiment, theory, lattice QCD)

1) Hadrons are hard
2) QCD+QED

Charged pion ($\pi^{\pm}$) $\alpha_E = 2.0(6)(7) = -\beta_M$ (PDG)

$\alpha_E = 2.93(5)$, $\beta_M = -2.77(11)$ (ChPT)

Neutral pion (π^0) $\alpha_E = -0.69(7)(4) = -\beta_M$ (PDG)

$\alpha_E = -0.40(18)$, $\beta_M = 1.50(27)$ (ChPT)

Charged kaon ($K^{\pm}$) $\alpha_E = 0.58 = -\beta_M$ (ChPT)

IJMPA34 (2019), Moinester and Scherer

Proton $\alpha_{E1} = 11.2(0.4)$, $\beta_{M1} = 2.5(1.2)$ (PDG)

$\alpha_{E1} = 11.2(0.7)$, $\beta_{M1} = 3.9(0.7)$ (ChPT)

$\gamma_{E1E1} = -3.3(0.8)$, $\gamma_{M1M1} = 2.9(1.5)$,

$\gamma_{E1M2} = -0.2(0.2)$, $\gamma_{M1E2} = 1.1(0.3)$ (ChPT)

Neutron $\alpha_{E1} = 11.8(1.1)$, $\beta_{M1} = 3.7(1.2)$ (PDG)

$\alpha_{E1} = 13.7(3.1)$, $\beta_{M1} = 4.6(2.7)$ (ChPT)

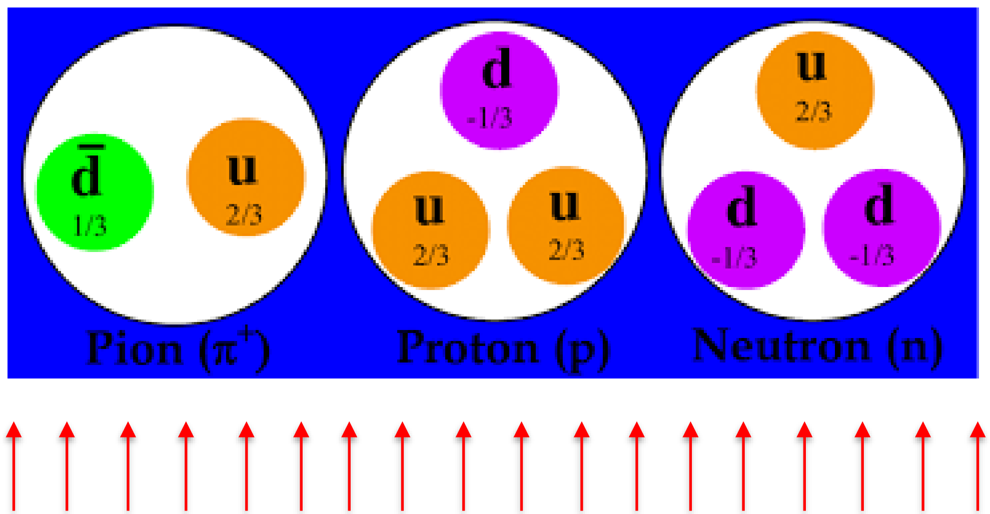
$\gamma_{E1E1} = -4.7(1.1)$, $\gamma_{M1M1} = 2.9(1.5)$,

$\gamma_{E1M2} = 0.2(0.2)$, $\gamma_{M1E2} = 1.6(0.4)$ (ChPT)

Eur. Phys. J. C75 (2015) Lensky, McGovern, Pascalutsa

Symmetry (2020), Hagelstein.

Background field method in QCD



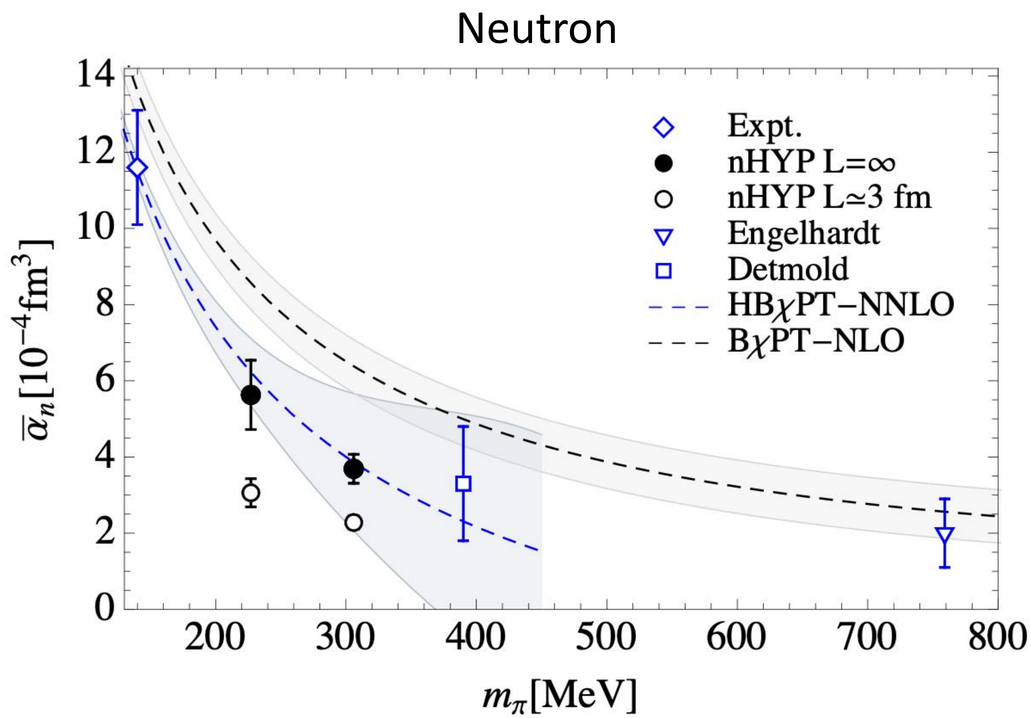
E or B field

Interaction Hamiltonian for weak fields:

$$
\begin{aligned}
H \;=\; & -\vec{p}\cdot\vec{E} - \vec{\mu}\cdot\vec{B} - \frac{1}{2}\alpha E^2 - \frac{1}{2}\beta B^2 \\
& -\frac{1}{2}\left(\gamma_{E1}\vec{\sigma}\cdot\vec{E}\times\dot{\vec{E}} + \gamma_{M1}\vec{\sigma}\cdot\vec{B}\times\dot{\vec{B}} - 2\gamma_{E2}E_{ij}\sigma_i B_j + 2\gamma_{M2}B_{ij}\sigma_i E_j\right) \\
& -\frac{1}{2}\left(\alpha_{E\nu}\dot{\vec{E}}^2 + \beta_{M\nu}\dot{\vec{B}}^2\right) - \frac{1}{12}4\pi\left(\alpha_{E2}E_{ij}^2 + \beta_{M2}B_{ij}^2\right) + \cdots .
\end{aligned}
$$

It works well for neutral hadrons (π^0, K^0, n)

Examples from background field method



π^0: $\alpha_E \simeq -0.5$
K^0: $\alpha_E = 0.356(74)$

π^0: $\alpha_E = -0.69(7)(4) = -\beta_M$ (PDG)
K^0: $\alpha_E = 0.58 = -\beta_M$ (ChPT)

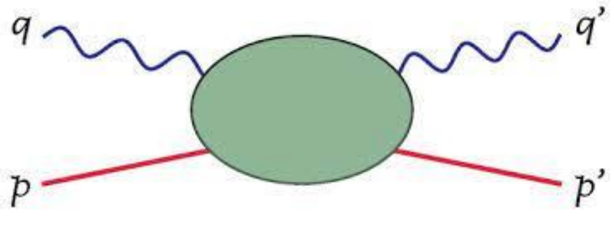
PRD94 (2016), Lujan, Alexandru, Freeman, Lee

New challenges arise for charged particles:

- Acceleration in electric fields
- Landau levels in magnetic field
- They come at leading order (polarizabilities at 2nd order)
- Their energies must be disentangled from the total to obtain the deformation energy on which polarizabilities are defined.

Alternative approach: four-point functions

- Mimics the Compton scattering process on the lattice
- Instead of background field, electromagnetic currents couple to quarks
- All photon, gluon, and quark interactions are included
- Charged and neutral hadrons are on equal footing



Compton scattering amplitude

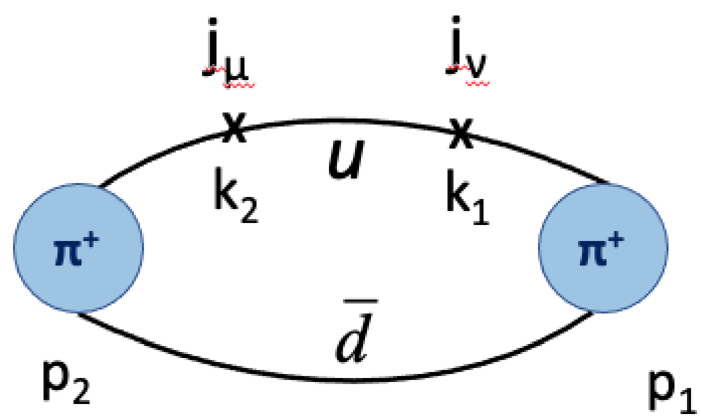
$$\mathcal{T} = \alpha\, \epsilon_2^{\mu *} T_{\mu\nu} \epsilon_1^{\nu} \qquad \frac{d\sigma}{d\Omega} \propto |\mathcal{T}|^2$$

Four-point tensor

$$T_{\mu\nu} = i \int d^4x e^{ik_2\cdot x} (\pi(p_2)|T j_\mu(x) j_\nu(0)|\pi(p_1))$$

Kinematics,

$$p_2 + k_2 = k_1 + p_1$$



Low-energy parametrization,

$$\alpha \epsilon_2^{\mu *} T_{\mu\nu} \epsilon_1^{\nu} =$$

$$\hat{\epsilon}_1 \cdot \hat{\epsilon}_2^* \left[-\frac{\alpha}{m} \left(1 + \frac{\langle r^2 \rangle}{6} (k_1^2 + k_2^2) \right) + \alpha_E \omega_1 \omega_2 \right]$$

$$+ \beta_M (\hat{\epsilon}_1 \times \vec{k}_1) \cdot (\hat{\epsilon}_2^* \times \vec{k}_2)$$

Compton tensor

- Lorentz invariant
- Gauge invariant
- Crossing symmetry

$$\sqrt{2E_1 2E_2}\, T_{\mu\nu} =$$

$$- \frac{T_\mu(p_1+k_1,p_1)T_\nu(p_2,p_2+k_2)}{(p_1+k_1)^2-m^2}$$
$$- \frac{T_\mu(p_2,p_2-k_1)T_\nu(p_1-k_2,p_1)}{(p_1-k_2)^2-m^2} + 2g_{\mu\nu}$$
$$+ A(k_1^2 g_{\mu\nu} - k_{1\mu}k_{1\nu} + k_2^2 g_{\mu\nu} - k_{2\mu}k_{2\nu})$$

Born
(or elastic)

$$+ B(k_1 \cdot k_2 g_{\mu\nu} - k_{2\mu}k_{1\nu})$$
$$+ C(k_1 \cdot k_2 Q_\mu Q_\nu + Q \cdot k_1 Q \cdot k_2 g_{\mu\nu}$$
$$- Q \cdot k_2 Q_\mu k_{1\nu} - Q \cdot k_1 Q_\nu k_{2\mu}),$$

Non-Born
(or inelastic)

$$Q = p_1 + p_2$$

form factor:

$$T_\mu(p',p) = (p'_\mu + p_\mu)F_\pi(q^2) + q_\mu \frac{p'^2 - p^2}{q^2}(1 - F_\pi(q^2))$$

$$q = p' - p$$

small q expansion: $$F_\pi(q^2) = 1 + \frac{\langle r^2 \rangle}{6} q^2 + \frac{\langle r^4 \rangle}{120} q^4$$

Charged pion polarizability formulas

$$\alpha_E = \frac{\alpha\langle r_E^2\rangle}{3m_\pi} + \frac{2\alpha}{\boldsymbol{q}^2}\int_0^\infty dt \left[Q_{44}(\boldsymbol{q},t) - Q_{44}^{elas}(\boldsymbol{q},t)\right]$$

$$\beta_M = -\frac{\alpha\langle r_E^2\rangle}{3m_\pi} + \frac{2\alpha}{\boldsymbol{q}^2}\int_0^\infty dt \left[Q_{11}^{inel}(\boldsymbol{q},t) - Q_{11}^{inel}(\boldsymbol{0},t)\right]$$

Charge radius can be extracted from elastic part of the same Q_{44} ,

$$Q_{44}^{elas}(\boldsymbol{q},t) = \frac{(E_\pi + m_\pi)^2}{4E_\pi m_\pi} F_\pi^2(\boldsymbol{q}^2) e^{-a(E_\pi(\boldsymbol{q})-m_\pi)t}$$

PRD104 (2021), Wilcox, Lee

Proton Compton tensor

$$B = \frac{2m\beta_M}{\alpha}$$

$$C = -\frac{\alpha_E + \beta_M}{2m\alpha}$$

$$\sqrt{2E_1 2E_2}\, T_{\mu\nu} = T_{\mu\nu}^{Born} + B(k_1 \cdot k_2 g_{\mu\nu} - k_{2\mu} k_{1\nu})$$

$$+ C(k_1 \cdot k_2 Q_\mu Q_\nu + Q \cdot k_1 Q \cdot k_2 g_{\mu\nu} - Q \cdot k_2 Q_\mu k_{1\nu} - Q \cdot k_1 Q_\nu k_{2\mu})$$

$$T_{\mu\nu}^{Born} = \frac{\bar{u}(p_2, s_2)\Gamma_\mu(-k_2)(\not{p}_1 + \not{k}_1 + m_p)\Gamma_\nu(k_1)u(p_1, s_1)}{m_p^2 - s} + \frac{\bar{u}(p_1, s_2)\Gamma_\mu(k_1)(\not{p}_2 - \not{k}_2 + m_p)\Gamma_\nu(-k_2)u(p_2, s_1)}{m_p^2 - u}$$

(Gasser, Leutwyler, arXiv:1506.06747)

form factors: $$\Gamma_\mu(q) \equiv \gamma_\mu F_1(q) + \frac{iF_2(q)}{2m_p}\sigma_{\mu\lambda}q^\lambda, \quad q = p' - p$$

$$F_1 = \frac{G_E + \tau G_M}{1+\tau}, \quad F_2 = \frac{G_M - G_E}{1+\tau}, \quad \tau \equiv \frac{-q^2}{4m_p^2}$$

small q expansion: $$G_E(q) = 1 + \frac{\langle r_E^2 \rangle}{6} q^2 + \frac{\langle r_E^4 \rangle}{120} q^4 + \cdots$$

$$G_M(q) = (1+\kappa)\left(1 + \frac{\langle r_M^2 \rangle}{6} q^2 + \frac{\langle r_M^4 \rangle}{120} q^4 + \cdots\right)$$

Proton formulas

$$\alpha_E = \frac{\alpha\langle r_E^2\rangle}{3m_p} + \frac{\alpha(1+\kappa^2)}{4m_p^3} + \frac{2\alpha}{\boldsymbol{q}^2}\int_0^\infty dt\,\left[Q_{44}(\boldsymbol{q},t) - Q_{44}^{elas}(\boldsymbol{q},t)\right]$$

$$\beta_M = -\frac{\alpha\langle r_E^2\rangle}{3m_p} - \frac{\alpha(1+\kappa+\kappa^2)}{2m_p^3} + \frac{2\alpha}{\boldsymbol{q}^2}\int_0^\infty dt\,\left[Q_{11}(\boldsymbol{q},t) - Q_{11}^{elas}(\boldsymbol{q},t) - Q_{11}(\boldsymbol{0},t)\right]$$

$$Q_{44}^{elas}(\boldsymbol{q},t) \xrightarrow[t\gg 1]{} \left[1 - \boldsymbol{q}^2\left(\frac{1}{4m_p^2} + \frac{\langle r_E^2\rangle}{3}\right)\right]e^{-(E_p-m_p)t}$$

$$Q_{11}^{elas}(\boldsymbol{q},t) \xrightarrow[t\gg 1]{} \frac{(1+\kappa)^2}{4m_p^2}\boldsymbol{q}^2 e^{-(E_p-m_p)t}$$

PRD104 (2021), Wilcox, Lee

Neutron formulas

$$\alpha_E = \frac{\alpha\kappa^2}{4m_p^3} + \frac{2\alpha}{\boldsymbol{q}^2}\int_0^\infty dt\,\left[Q_{44}(\boldsymbol{q},t) - Q_{44}^{elas}(\boldsymbol{q},t)\right]$$

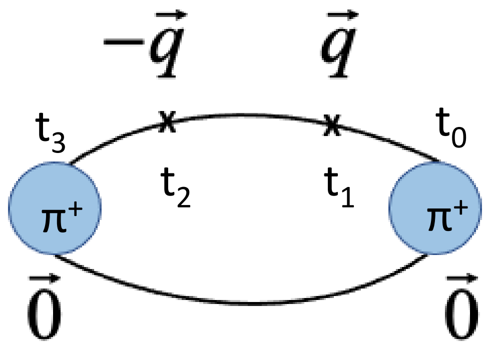
$$\beta_M = -\frac{\alpha\kappa^2}{2m_p^3} + \frac{2\alpha}{\boldsymbol{q}^2}\int_0^\infty dt\,\left[Q_{11}(\boldsymbol{q},t) - Q_{11}^{elas}(\boldsymbol{q},t) - Q_{11}(\boldsymbol{0},t)\right]$$

$$Q_{11}^{elas}(\boldsymbol{q},t) \xrightarrow[t\gg 1]{} \frac{\kappa^2}{4m_p^2}\boldsymbol{q}^2 e^{-(E_p-m_p)t}$$

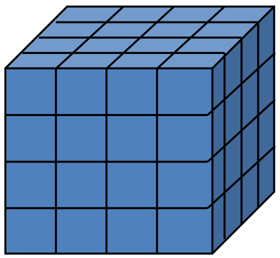
Four-point function in lattice QCD

$$\frac{\sum_{\boldsymbol{x}_3,\boldsymbol{x}_2,\boldsymbol{x}_1,\boldsymbol{x}_0} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}_2} e^{i\boldsymbol{q}\cdot\boldsymbol{x}_1} \langle\Omega|\psi^\dagger(x_3) : j^L_\mu(x_2) j^L_\nu(x_1) : \psi(x_0)|\Omega\rangle}{\sum_{\boldsymbol{x}_3,\boldsymbol{x}_0} \langle\Omega|\psi^\dagger(x_3)\psi(x_0)|\Omega\rangle}$$
$$\equiv Q_{\mu\nu}(\boldsymbol{q}, t_3, t_2, t_1, t_0)$$

Kinematics
(zero-momentum Breit frame)



Path integrals in Euclidean spacetime



Proof-of-concept simulation:

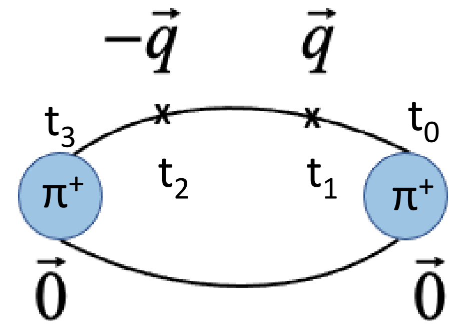
- Quenched Wilson action on 24^3x48 lattice with spacing a=0.085 fm.
- Dirichlet boundary condition in time, periodic in space.
- Quark mass parameter κ=0.1520, 0.1543, 0.1555, 0.1565 corresponding to pion mass m_π=1100, 800, 600, 370 MeV. Analyzed 1000 configurations for each mass.
- 5 momenta **q**={0,0,0}, {0,0,1}, {0,1,1}, {1,1,1}, {0,0,2} per mass

Operators

$$\boxed{\frac{\sum_{\boldsymbol{x}_3,\boldsymbol{x}_2,\boldsymbol{x}_1,\boldsymbol{x}_0} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}_2} e^{i\boldsymbol{q}\cdot\boldsymbol{x}_1} \langle\Omega|\psi^\dagger(x_3) : j^L_\mu(x_2) j^L_\nu(x_1) : \psi(x_0)|\Omega\rangle}{\sum_{\boldsymbol{x}_3,\boldsymbol{x}_0} \langle\Omega|\psi^\dagger(x_3)\psi(x_0)|\Omega\rangle} \equiv Q_{\mu\nu}(\boldsymbol{q}, t_3, t_2, t_1, t_0)}$$

Charged pion: $\psi_{\pi^+}(x) = \bar{d}(x)\gamma_5 u(x)$

Local current: $j_\mu^{(PC)} = Z_V \left(q_u \bar{u}\gamma_\mu u + q_d \bar{d}\gamma_\mu d\right)$



Conserved current (Z_V=1):

$$\begin{aligned} j_\mu^{(PS)}(x) = q_u\kappa[&-\bar{u}(x)(1-\gamma_\mu)U_\mu(x)u(x+a\hat{\mu}) + \bar{u}(x+a\hat{\mu})(1+\gamma_\mu)U^\dagger_\mu(x)u(x)] \\ + q_d\kappa[&-\bar{d}(x)(1-\gamma_\mu)U_\mu(x)d(x+a\hat{\mu}) + \bar{d}(x+a\hat{\mu})(1+\gamma_\mu)U^\dagger_\mu(x)d(x)] \end{aligned}$$

Current conservation leads to following property for Q_{44} (**q**=0),

$$\frac{\sum\limits_{\boldsymbol{x}_3,\boldsymbol{x}_2,\boldsymbol{x}_1,\boldsymbol{x}_0} \langle\Omega|\psi(x_3) j^L_4(x_2) j^L_4(x_1)\psi^\dagger(x_0)|\Omega\rangle}{\sum\limits_{\boldsymbol{x}_3,\boldsymbol{x}_0} \langle\Omega|\psi(x_3)\psi^\dagger(x_0)|\Omega\rangle} = q_1 q_2$$

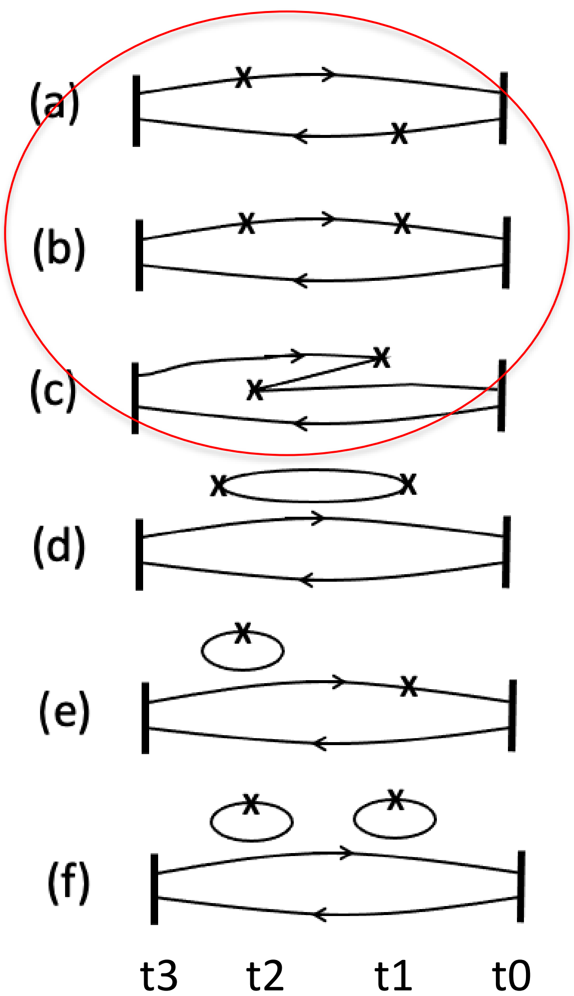
(used for numerical validation of the diagrams)

Wick contractions

$$\frac{\sum_{\boldsymbol{x}_3,\boldsymbol{x}_2,\boldsymbol{x}_1,\boldsymbol{x}_0} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}_2} e^{i\boldsymbol{q}\cdot\boldsymbol{x}_1} \langle\Omega|\psi^\dagger(x_3) : j_\mu^L(x_2) j_\nu^L(x_1) : \psi(x_0)|\Omega\rangle}{\sum_{\boldsymbol{x}_3,\boldsymbol{x}_0} \langle\Omega|\psi^\dagger(x_3)\psi(x_0)|\Omega\rangle}$$
$$\equiv Q_{\mu\nu}(\boldsymbol{q}, t_3, t_2, t_1, t_0)$$

$$\bar{u}d \qquad \boxed{u\bar{u} + d\bar{d}} \qquad \boxed{u\bar{u} + d\bar{d}} \qquad u\bar{d}$$

Connected contributions

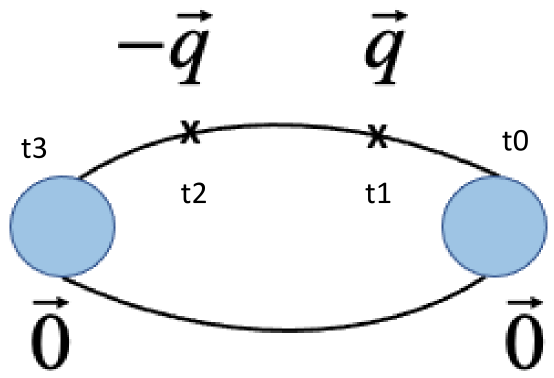


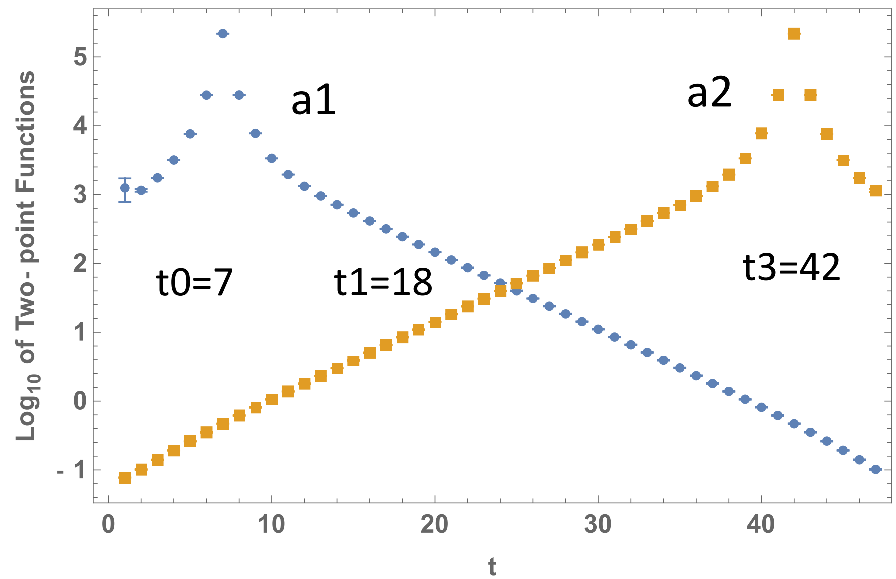
$$d_4^{\mathrm{A}} = -2\,\mathrm{tr}\left[S(t_1,t_3)\gamma_5 S(t_3,t_2)\gamma_\mu e^{-i\mathbf{q}} S(t_2,t_0)\gamma_5 S(t_0,t_1)\gamma_\nu e^{i\mathbf{q}}\right]$$
$$d_2^{\mathrm{A\text{-}bwd}} = -2\,\mathrm{tr}\left[S(t_2,t_3)\gamma_5 S(t_3,t_1)\gamma_\nu e^{i\mathbf{q}} S(t_1,t_0)\gamma_5 S(t_0,t_2)\gamma_\mu e^{-i\mathbf{q}}\right]$$
$$d_1^{\mathrm{B}} = 4\,\mathrm{tr}\left[S(t_2,t_3)\gamma_5 S(t_3,t_0)\gamma_5 S(t_0,t_1)\gamma_\nu e^{i\mathbf{q}} S(t_1,t_2)\gamma_\mu e^{-i\mathbf{q}}\right]$$
$$d_7^{\mathrm{B\text{-}bwd}} = 1\,\mathrm{tr}\left[S(t_0,t_3)\gamma_5 S(t_3,t_2)\gamma_\mu e^{-i\mathbf{q}} S(t_2,t_1)\gamma_\nu e^{i\mathbf{q}} S(t_1,t_0)\gamma_5\right]$$
$$d_0^{\mathrm{C}} = 4\,\mathrm{tr}\left[S(t_1,t_3)\gamma_5 S(t_3,t_0)\gamma_5 S(t_0,t_2)\gamma_\mu e^{-i\mathbf{q}} S(t_2,t_1)\gamma_\nu e^{i\mathbf{q}}\right]$$
$$d_9^{\mathrm{C\text{-}bwd}} = 1\,\mathrm{tr}\left[S(t_0,t_3)\gamma_5 S(t_3,t_1)\gamma_\nu e^{i\mathbf{q}} S(t_1,t_2)\gamma_\mu e^{-i\mathbf{q}} S(t_2,t_0)\gamma_5\right]$$
$$d_{10}^{\mathrm{D}} = -5\,\mathrm{tr}\left[S(t_0,t_3)\gamma_5 S(t_3,t_0)\gamma_5\right]\mathrm{tr}\left[S(t_1,t_2)\gamma_\mu e^{-i\mathbf{q}} S(t_2,t_1)\gamma_\nu e^{i\mathbf{q}}\right]$$
$$d_5^{\mathrm{El}} = -2\,\mathrm{tr}\left[S(t_1,t_3)\gamma_5 S(t_3,t_0)\gamma_5 S(t_0,t_1)\gamma_\nu e^{i\mathbf{q}}\right]\mathrm{tr}\left[S(t_2,t_2)\gamma_\mu e^{-i\mathbf{q}}\right]$$
$$d_6^{\mathrm{El\text{-}bwd}} = 1\,\mathrm{tr}\left[S(t_0,t_3)\gamma_5 S(t_3,t_1)\gamma_\nu e^{i\mathbf{q}} S(t_1,t_0)\gamma_5\right]\mathrm{tr}\left[S(t_2,t_2)\gamma_\mu e^{-i\mathbf{q}}\right]$$
$$d_3^{\mathrm{Er}} = -2\,\mathrm{tr}\left[S(t_2,t_3)\gamma_5 S(t_3,t_0)\gamma_5 S(t_0,t_2)\gamma_\mu e^{-i\mathbf{q}}\right]\mathrm{tr}\left[S(t_1,t_1)\gamma_\nu e^{i\mathbf{q}}\right]$$
$$d_8^{\mathrm{Er\text{-}bwd}} = 1\,\mathrm{tr}\left[S(t_0,t_3)\gamma_5 S(t_3,t_2)\gamma_\mu e^{-i\mathbf{q}} S(t_2,t_0)\gamma_5\right]\mathrm{tr}\left[S(t_1,t_1)\gamma_\nu e^{i\mathbf{q}}\right]$$
$$d_{11}^{\mathrm{F}} = 1\,\mathrm{tr}\left[S(t_0,t_3)\gamma_5 S(t_3,t_0)\gamma_5\right]\mathrm{tr}\left[S(t_2,t_2)\gamma_\mu e^{-i\mathbf{q}}\right]\mathrm{tr}\left[S(t_1,t_1)\gamma_\nu e^{i\mathbf{q}}\right]$$

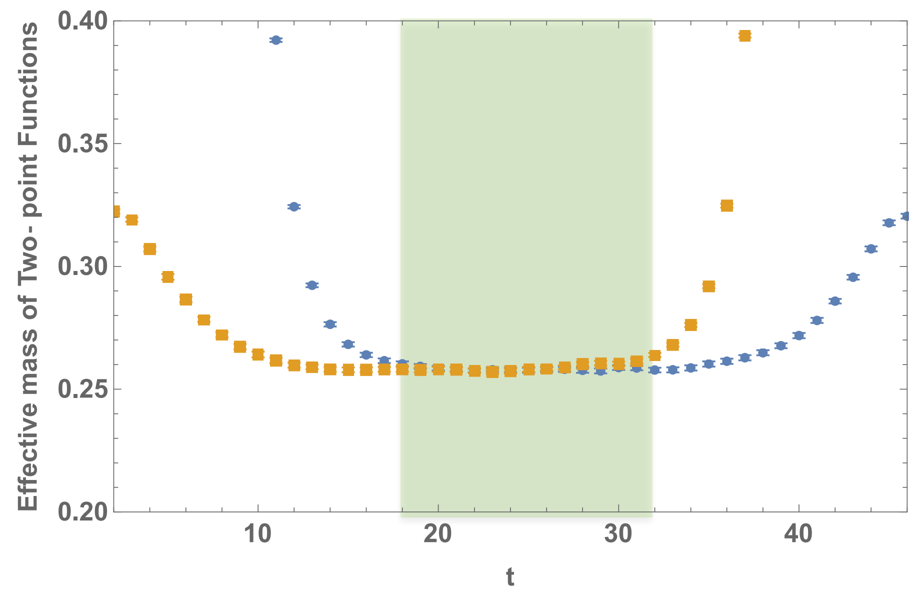
Quark propagator $S_q(t_2, t_1) \equiv \langle q\bar{q}\rangle = \dfrac{1}{\not{D} + m_q}$

Two-point functions

Pion wall-to-point correlators at m_π=600 MeV

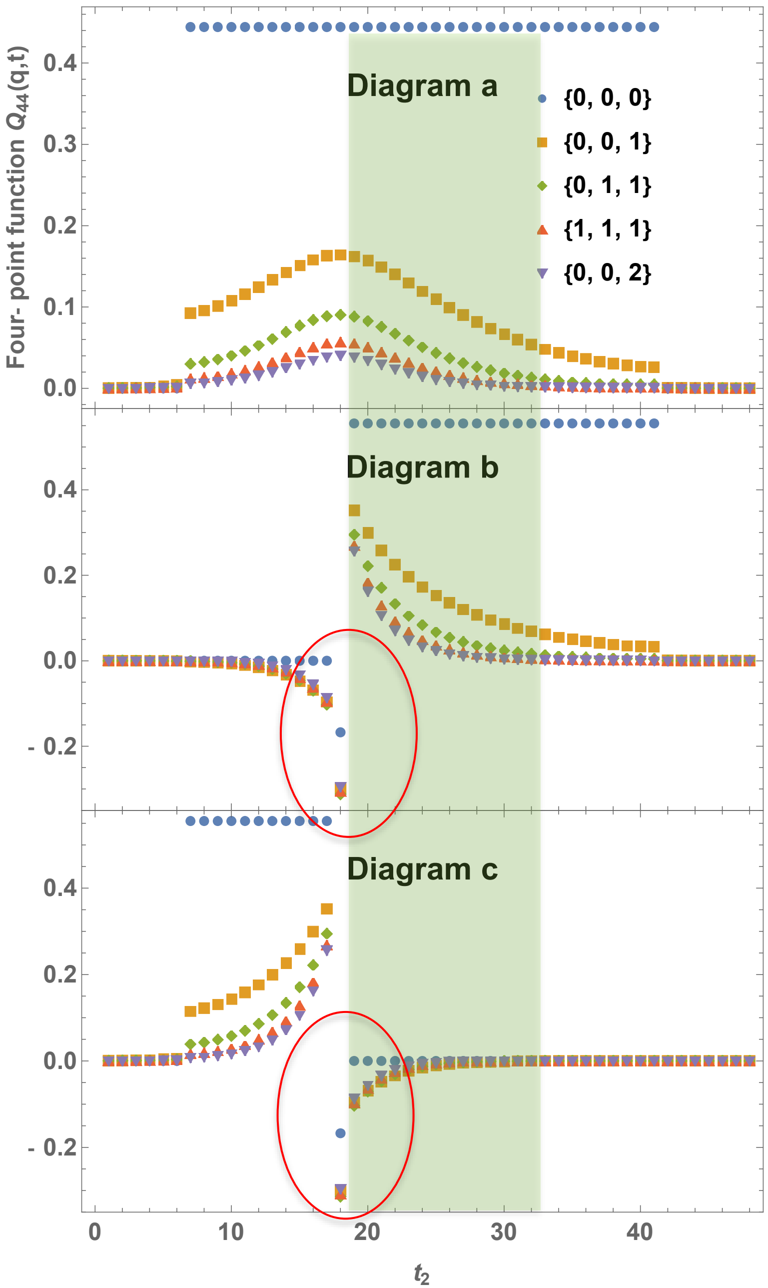






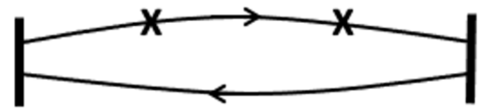
- Measured m_π and m_ρ from a1 correlators.
- Current 1 fixed at where ground state dominates.
- Limited `window of opportunity' for four-point functions.

Four-point functions Q_{44} for α_E



$4/9 = 2q_u q_{\bar{d}}$ (current conservation at **q**=0)

$5/9 = q_u q_u + q_{\bar{d}} q_{\bar{d}}$ (current conservation at **q**=0)



$5/9 = q_u q_u + q_{\bar{d}} q_{\bar{d}}$ (current conservation at **q**=0)

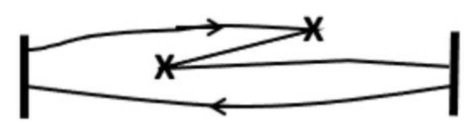
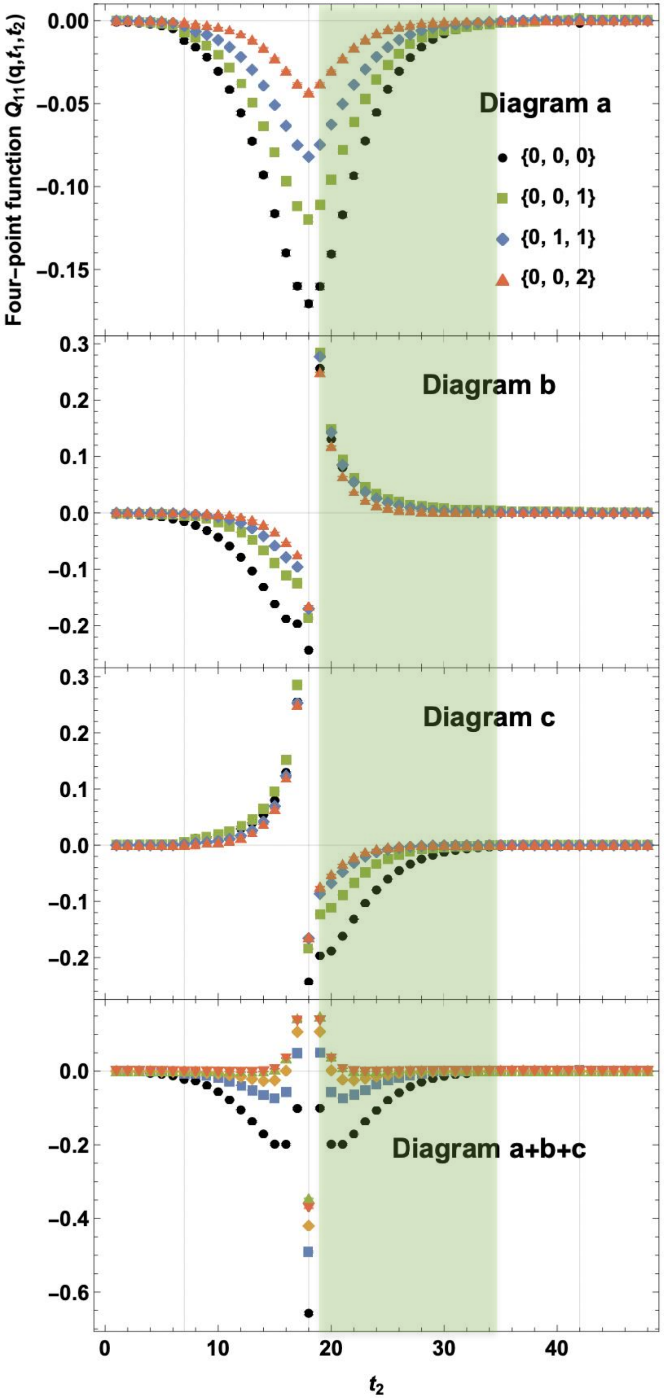
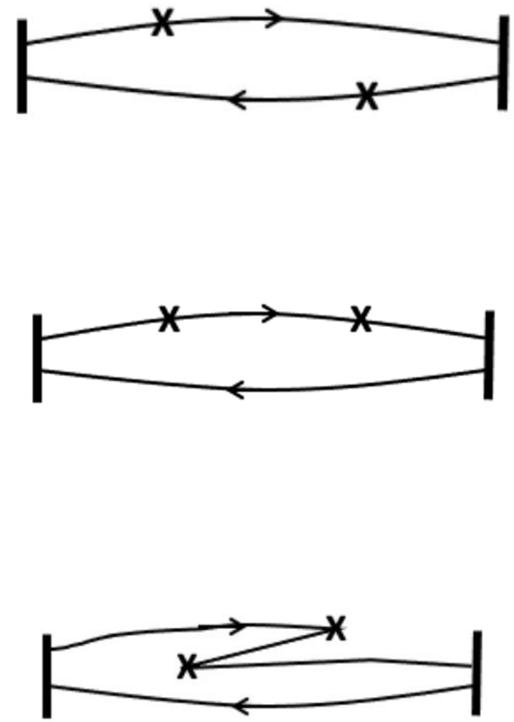


Diagram b and c have unphysical contact interactions (we avoid t_1=t_2)

Four-point functions Q_{11} for β_M



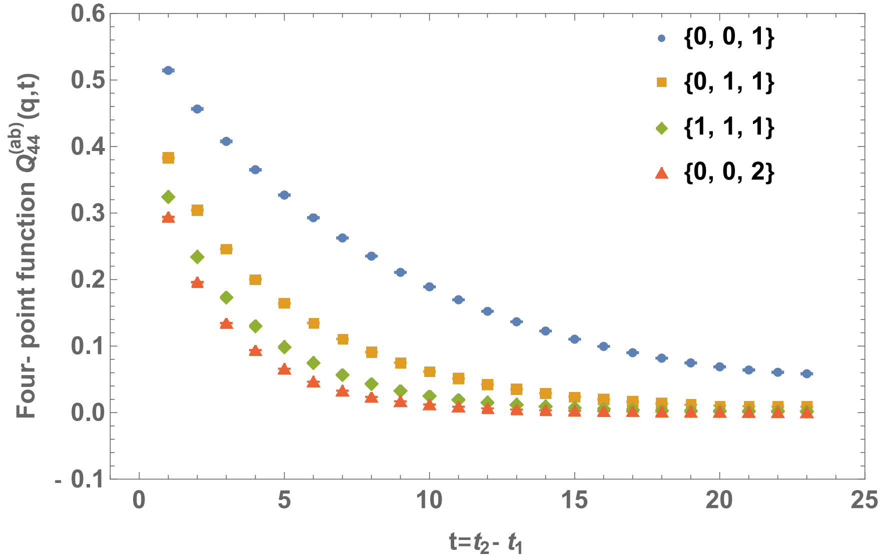


Extracting pion form factor

m_π=600 MeV

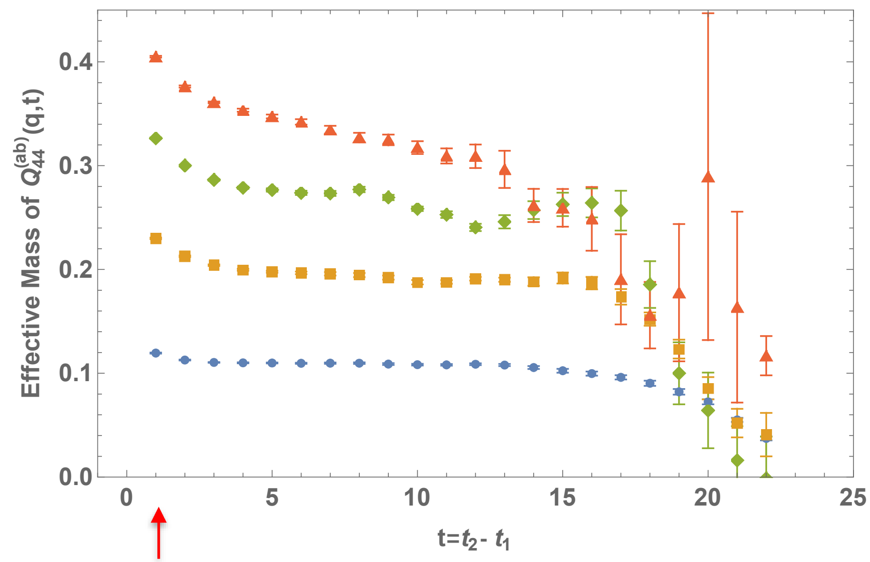
$$Q_{44}^{elas}(\mathbf{q},t) = \frac{(E_\pi + m_\pi)^2}{4E_\pi m_\pi} F_\pi^2(\mathbf{q}^2)\, e^{-a(E_\pi - m_\pi)t}$$

(switch x-axis to $t=t_2-t_1$)



Horizontal lines are continuum dispersion relation

$$E_\pi = \sqrt{\mathbf{q}^2 + m_\pi^2}$$



Fit $Q^{(ab)}_{44}$ data to $Q^{(elas)}_{44}$ function treating both F_π and E_π as free parameters.

Starts at t=1

Charge radius from pion form factor

1) Monopole (vector meson dominance)

$$F_\pi(\mathbf{q}^2) = \frac{1}{1 + \frac{\mathbf{q}^2}{m_V^2}}$$

2) z-expansion

$$F_\pi(\boldsymbol{q}^2) = 1 + \sum_{k=1}^{k_{max}} a_k\, z^k,$$

$$\text{where } z \equiv \frac{\sqrt{t_{cut} - t} - \sqrt{t_{cut} - t_0}}{\sqrt{t_{cut} - t} + \sqrt{t_{cut} - t_0}}$$

$$\text{and } t = -\boldsymbol{q}^2,\ t_{cut} = 4m_\pi^2,$$

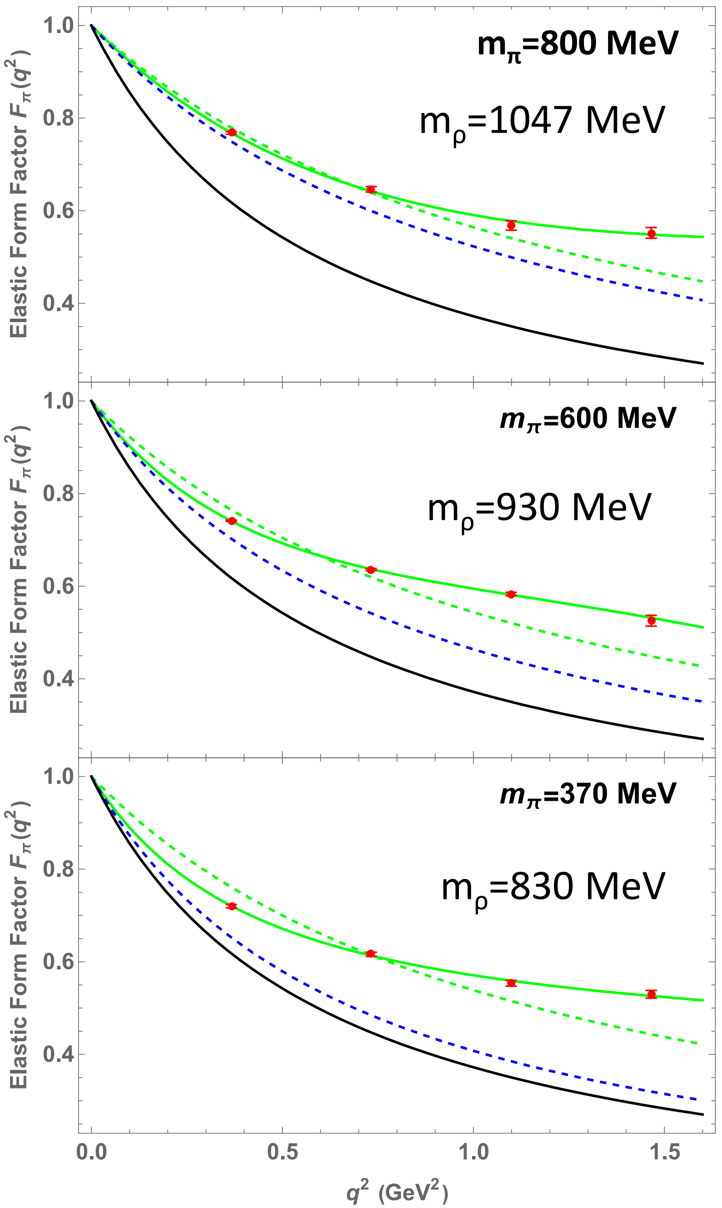
Solid green = z-expansion fit with k_{max}=3

Dashed green = monopole fit

Dashed blue = monopole with measured m_ρ

Solid black = monopole with physical m_ρ

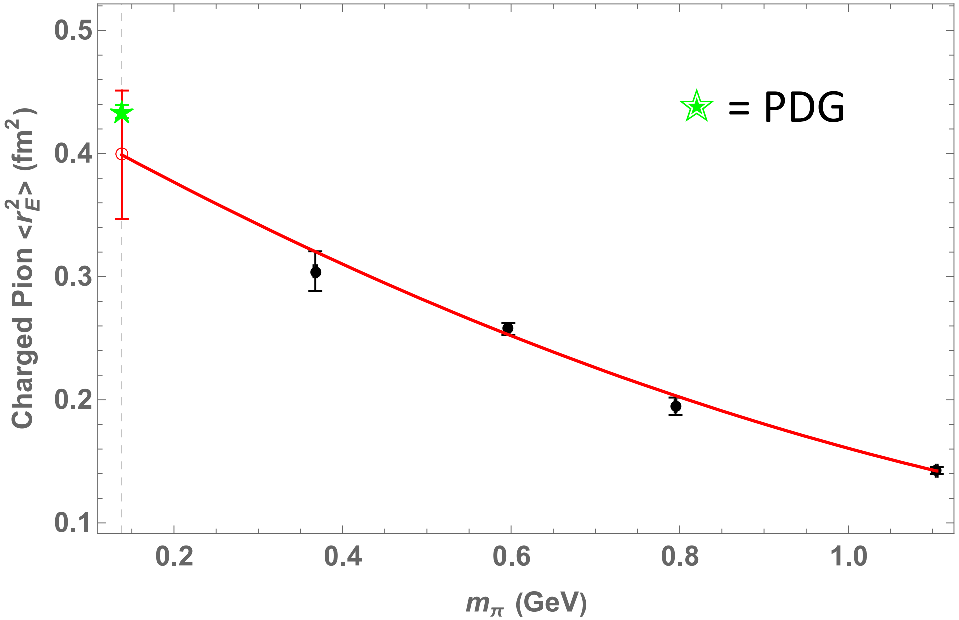
$$\langle r_E^2 \rangle = -6 \frac{dF_\pi(q^2)}{dq^2}\Big|_{q^2 \to 0}$$



Chiral extrapolation of charge radius

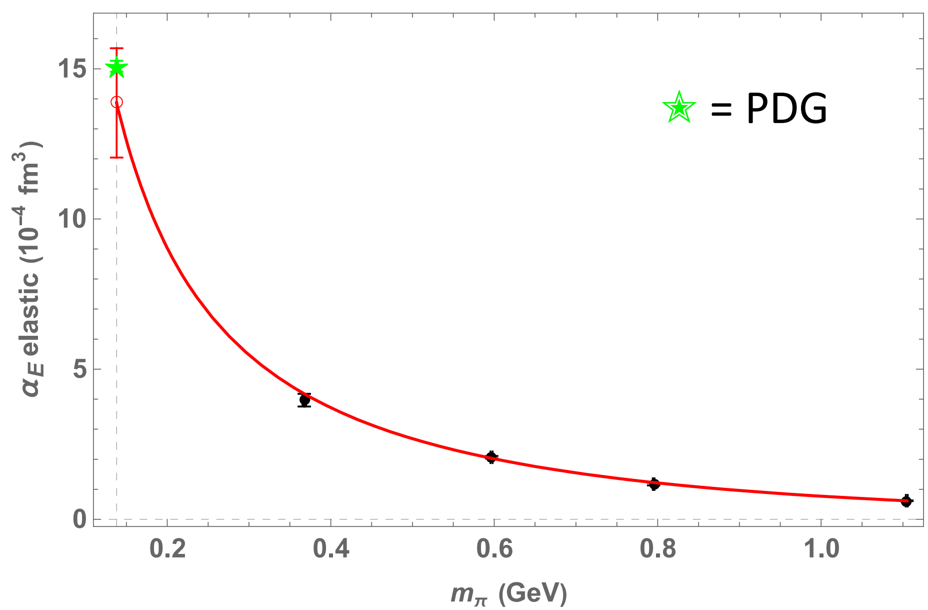
$$\alpha_E^\pi = \frac{\alpha \langle r_E^2 \rangle}{3m_\pi} + \frac{2\alpha a}{\boldsymbol{q}^2} \int_0^\infty dt \left[Q_{44}(\boldsymbol{q},t) - Q_{44}^{elas}(\boldsymbol{q},t) \right]$$

$$a + b m_\pi + c\, m_\pi^2$$



Elastic contribution:

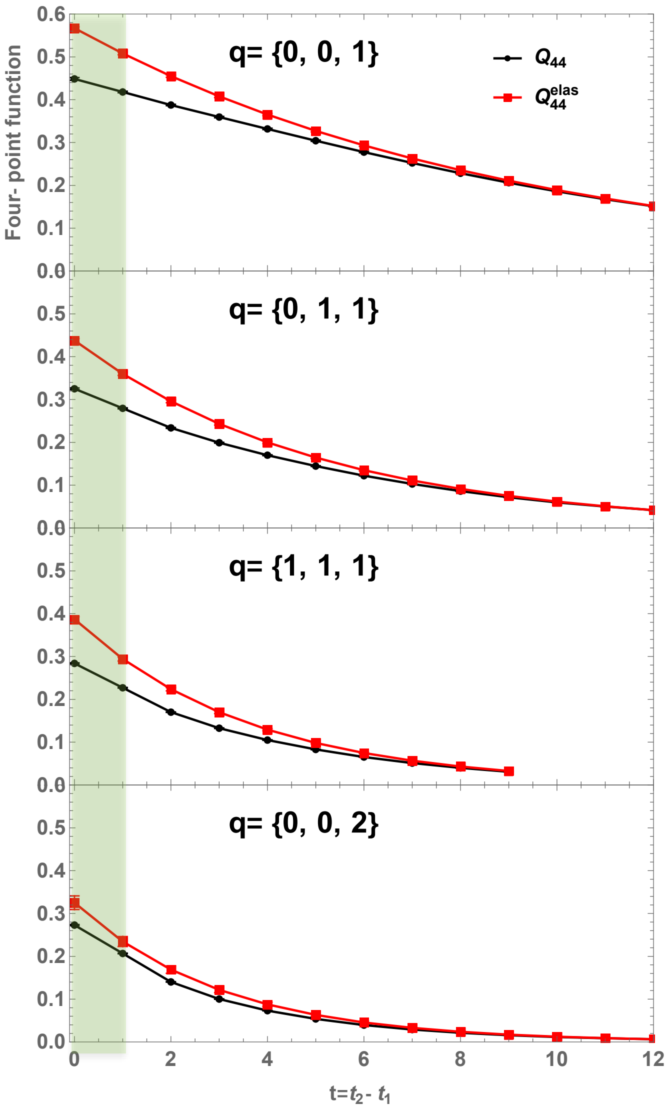
$$\frac{a}{m_\pi} + b + c\, m_\pi$$



Time integrals

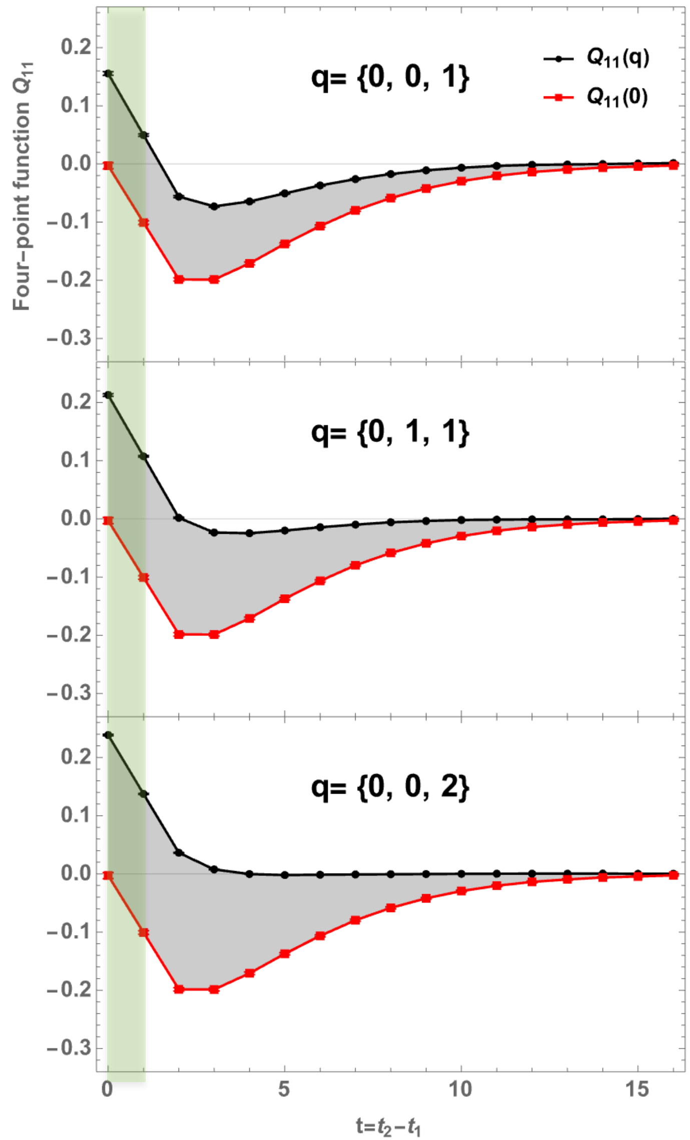
$$\alpha_E^\pi = \frac{\alpha\langle r_E^2\rangle}{3m_\pi} + \frac{2\alpha a}{\boldsymbol{q}^2}\int_0^\infty dt\,[Q_{44}(\boldsymbol{q},t) - Q_{44}^{elas}(\boldsymbol{q},t)]$$

$$\beta_E^\pi = -\frac{\alpha\langle r_E^2\rangle}{3m_\pi} + \frac{2\alpha a}{\boldsymbol{q}^2}\int_0^\infty dt\,[Q_{11}(\boldsymbol{q},t) - Q_{11}(\boldsymbol{0},t)]$$



Signal is negative of shaded area

Extrapolation to t=0

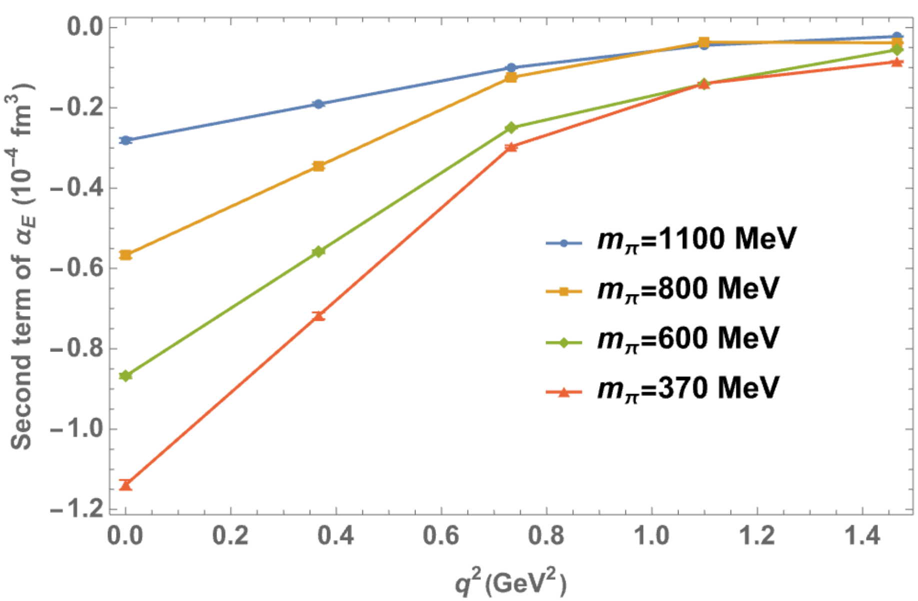


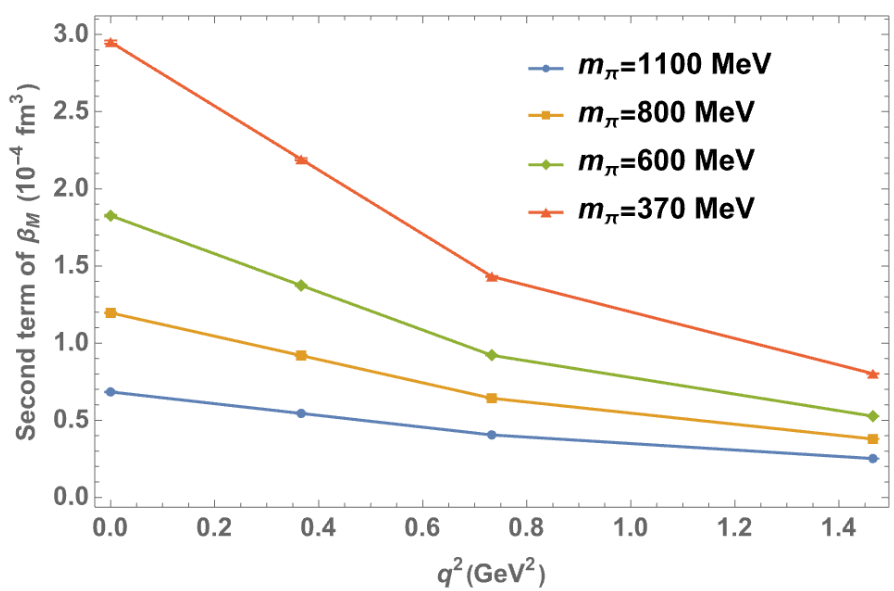
Signal is positive shaded area

Extrapolation to $\mathbf{q}^2$=0

$$\alpha_E^{\pi} = \frac{\alpha \langle r_E^2 \rangle}{3m_\pi} + \frac{2\alpha a}{\boldsymbol{q}^2} \int_0^\infty dt \left[Q_{44}(\boldsymbol{q},t) - Q_{44}^{elas}(\boldsymbol{q},t) \right]$$

$$\beta_M = -\frac{\alpha \langle r_E^2 \rangle}{3m_\pi} + \frac{2\alpha}{\boldsymbol{q}^2} \int_0^\infty dt \left[Q_{11}^{inel}(\boldsymbol{q},t) - Q_{11}^{inel}(\boldsymbol{0},t) \right]$$

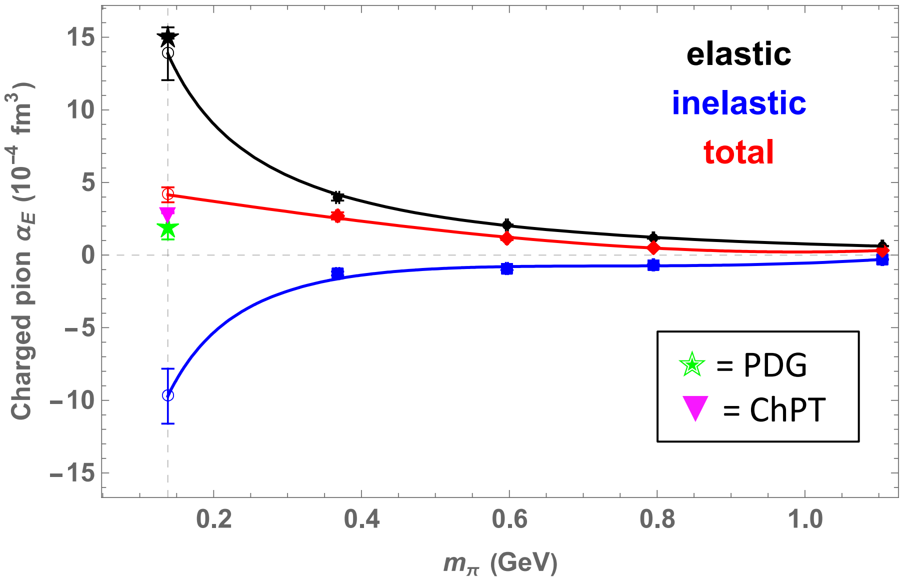


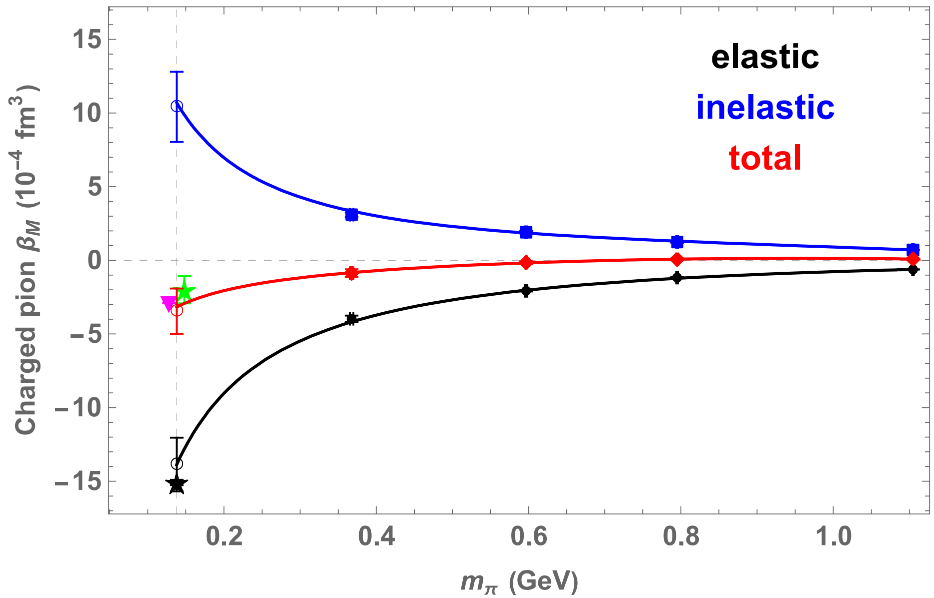


Chiral extrapolation

$$\alpha_E^\pi = \frac{\alpha\langle r_E^2\rangle}{3m_\pi} + \frac{2\alpha a}{\boldsymbol{q}^2}\int_0^\infty dt\left[Q_{44}(\boldsymbol{q},t) - Q_{44}^{elas}(\boldsymbol{q},t)\right]$$

$$\beta_M = -\frac{\alpha\langle r_E^2\rangle}{3m_\pi} + \frac{2\alpha}{\boldsymbol{q}^2}\int_0^\infty dt\left[Q_{11}^{inel}(\boldsymbol{q},t) - Q_{11}^{inel}(\boldsymbol{0},t)\right]$$





arXiv:2301.05200,
Lee, Alexandru, Culver, Wilcox

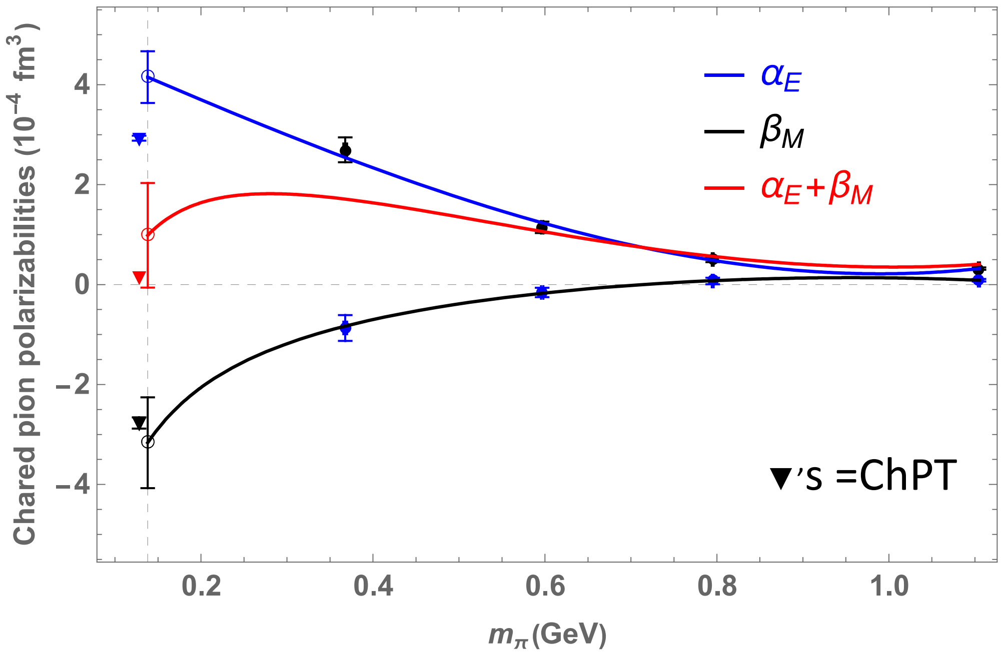
$$\frac{a}{m_\pi} + b\,m_\pi + c\,m_\pi^3$$

$\alpha_E + \beta_M$

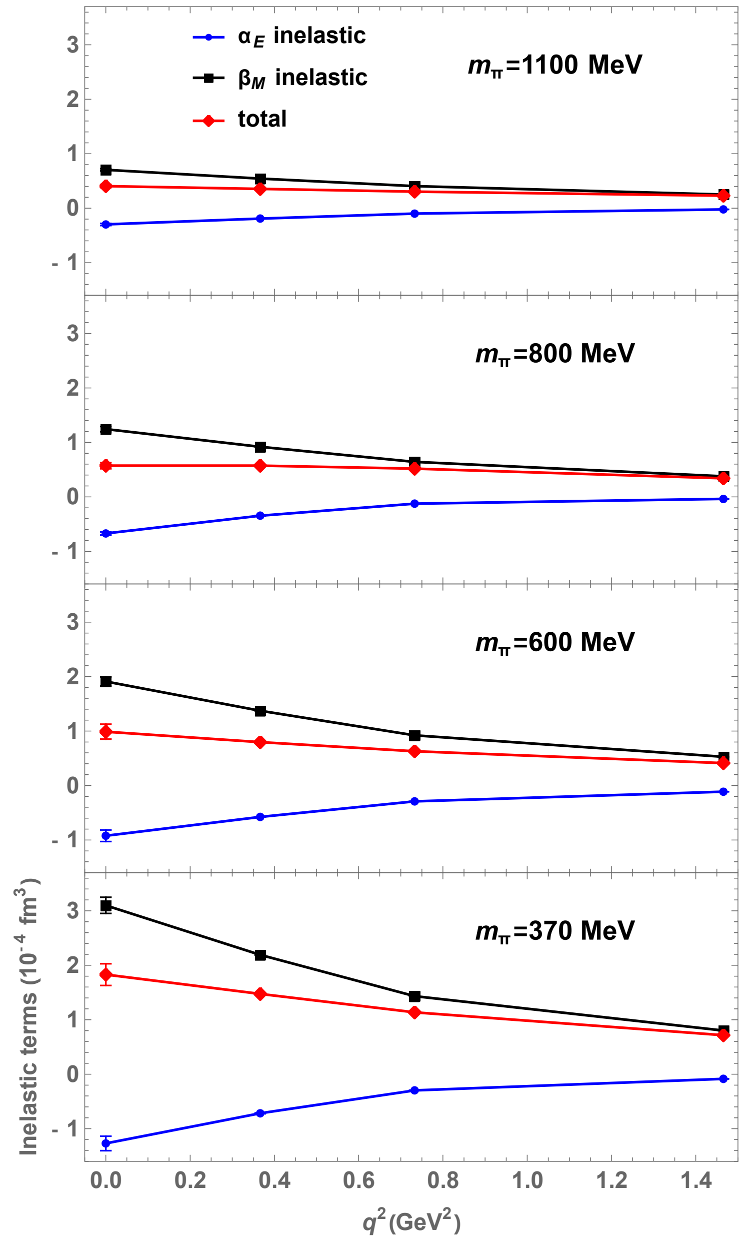
$$\alpha_E^\pi = \frac{\alpha\langle r_E^2\rangle}{3m_\pi} + \frac{2\alpha a}{\boldsymbol{q}^2}\int_0^\infty dt\,[Q_{44}(\boldsymbol{q},t) - Q_{44}^{elas}(\boldsymbol{q},t)]$$

$$\beta_E^\pi = -\frac{\alpha\langle r_E^2\rangle}{3m_\pi} + \frac{2\alpha a}{\boldsymbol{q}^2}\int_0^\infty dt\,[Q_{11}(\boldsymbol{q},t) - Q_{11}(\boldsymbol{0},t)]$$

Pion mass dependence



Momentum dependence



Summary table for charged pion electric and magnetic polarizabilities from four-point functions in lattice QCD

TABLE I. Summary of results in physical units from two-point and four-point functions. Results for charge radius and α_E are taken from previous work [37]. Elastic β_M and total β_M are chirally extrapolated to the physical point. Inelastic β_M at the physical point is taken as the difference of the two. Known values from ChPT and PDG are listed for reference. All polarizabilities are in units of 10^{-4} fm^3.

	κ=0.1520	κ=0.1543	κ=0.1555	κ=0.1565	physical point	known value
m_π (MeV)	1104.7 ± 1.2	795.0 ± 1.1	596.8 ± 1.4	367.7 ± 2.2	138	138
m_ρ (MeV)	1273.1 ± 2.5	1047.3 ± 3.4	$930. \pm 7.$	$830. \pm 17.$	770	770
$\langle r_E^2 \rangle$ (fm^2)	0.1424 ± 0.0029	0.195 ± 0.007	0.257 ± 0.005	0.304 ± 0.016	0.40 ± 0.05	0.434 ± 0.005 (PDG)
α_E elastic	0.618 ± 0.012	1.17 ± 0.04	2.07 ± 0.04	3.97 ± 0.21	13.9 ± 1.8	15.08 ± 0.13 (PDG)
α_E inelastic	-0.299 ± 0.019	-0.672 ± 0.030	-0.92 ± 0.11	-1.27 ± 0.13	-9.7 ± 1.9	
α_E total	0.319 ± 0.023	0.50 ± 0.05	1.15 ± 0.11	2.70 ± 0.25	4.2 ± 0.5	2.93 ± 0.05 (ChPT)
						$2.0 \pm 0.6 \pm 0.7$ (PDG)
β_M elastic	-0.618 ± 0.012	-1.17 ± 0.04	-2.07 ± 0.04	-3.97 ± 0.21	-13.9 ± 1.8	-15.08 ± 0.13 (PDG)
β_M inelastic	0.705 ± 0.021	1.24 ± 0.05	1.91 ± 0.09	3.10 ± 0.15	10.7 ± 2.0	
β_M total	0.087 ± 0.024	0.07 ± 0.06	-0.16 ± 0.09	-0.87 ± 0.26	-3.2 ± 0.9	-2.77 ± 0.11 (ChPT)
						$-2.0 \pm 0.6 \pm 0.7$ (PDG)

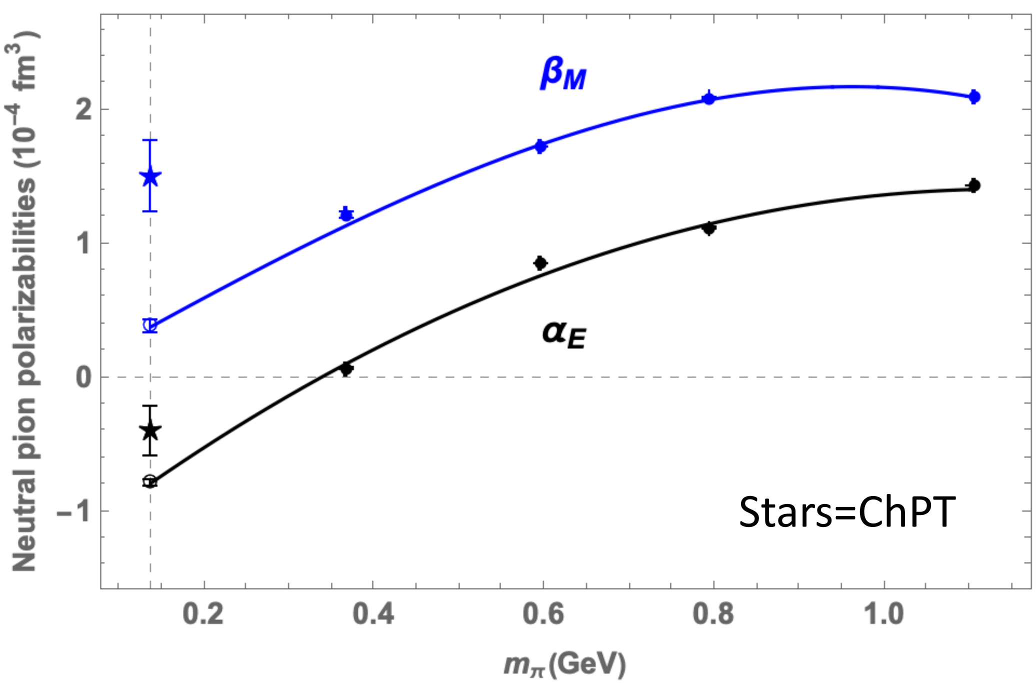
Neutral pion

$$\alpha_E^\pi = \cancel{\frac{\alpha\langle r_E^2\rangle}{3m_\pi}} + \frac{2\alpha a}{\boldsymbol{q}^2}\int_0^\infty dt\,\left[Q_{44}(\boldsymbol{q},t) - \cancel{Q_{44}^{elas}}(\boldsymbol{q},t)\right]$$

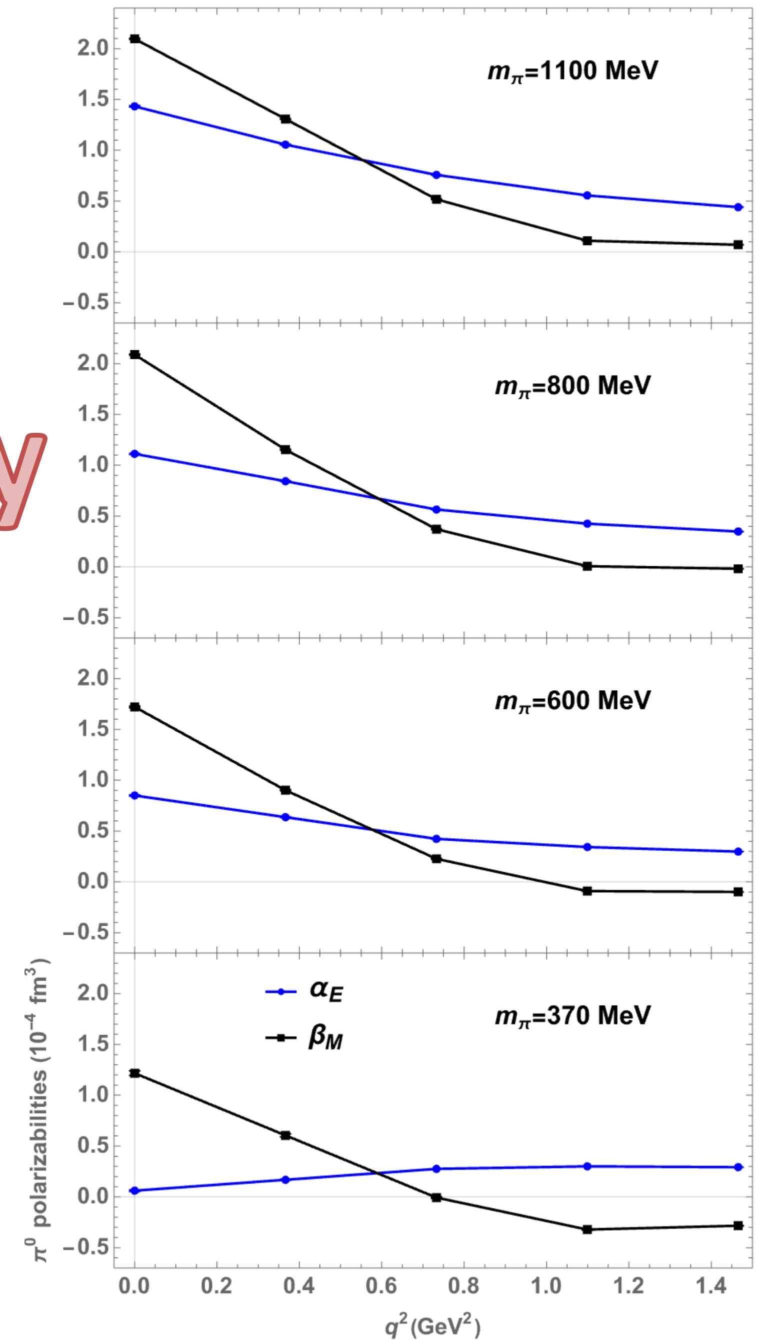
$$\beta_E^\pi = -\cancel{\frac{\alpha\langle r_E^2\rangle}{3m_\pi}} + \frac{2\alpha a}{\boldsymbol{q}^2}\int_0^\infty dt\,\left[Q_{11}(\boldsymbol{q},t) - Q_{11}(\boldsymbol{0},t)\right]$$

Preliminary

Pion mass dependence



Momentum dependence



“Pion electric polarizabilities from lattice QCD”

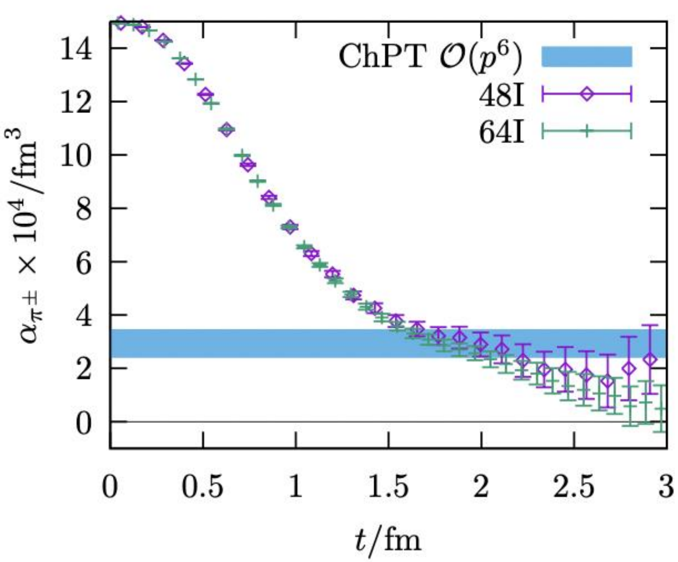
X. Feng, T. Izubuchi, L. Jin, M. Golterman

arXiv:2201.01396 (Lattice 2021)

Domain-wall ensembles
at physical pion mass

	Volume	a^{-1} (GeV)	L (fm)	M_π (MeV)	t_{sep} (a)
48I	$48^3 \times 96$	1.730(4)	5.5	135	12
64I	$64^3 \times 128$	2.359(7)	5.4	135	18
24D	$24^3 \times 64$	1.0158(40)	4.7	142	8
32D	$32^3 \times 64$	1.0158(40)	6.2	142	8

$$\alpha_\pi(t) = -\int_{-t<t_x<t} \int_{\vec{x}} \frac{t_x^2}{24\pi} \frac{1}{2M_\pi} \langle \pi | T \vec{J}(t_x,\vec{x}) \cdot \vec{J}(0,\vec{0}) | \pi \rangle - \alpha_\pi^{\mathrm{Born}}$$



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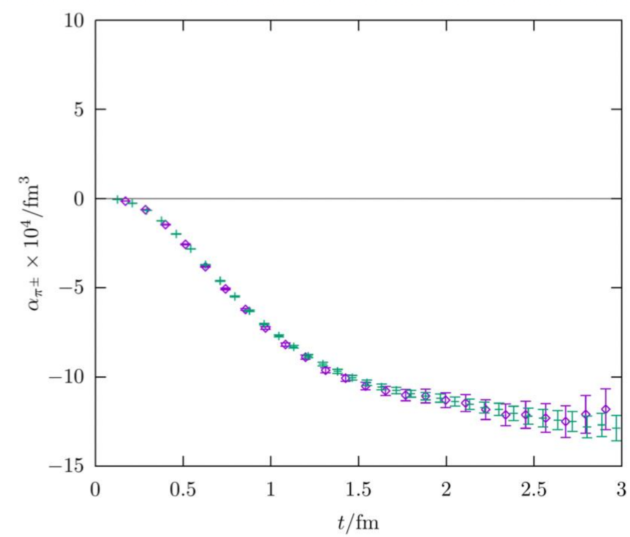
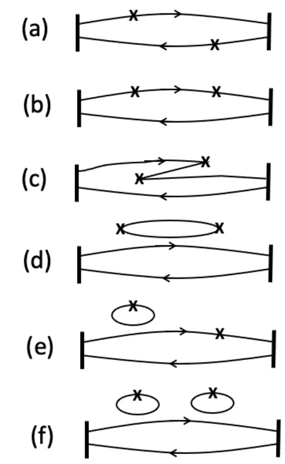


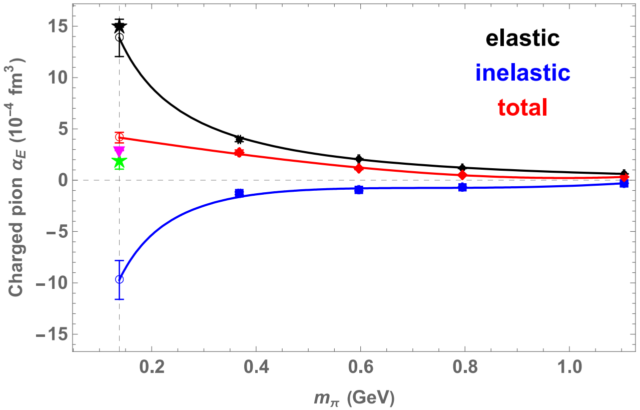
$$+ \frac{\alpha \langle r_E^2 \rangle}{3m_\pi}$$

$$\alpha_E^\pi = \frac{\alpha \langle r_E^2 \rangle}{3m_\pi} + \frac{2\alpha a}{\boldsymbol{q}^2} \int_0^\infty dt \left[Q_{44}(\boldsymbol{q},t) - Q_{44}^{elas}(\boldsymbol{q},t) \right]$$

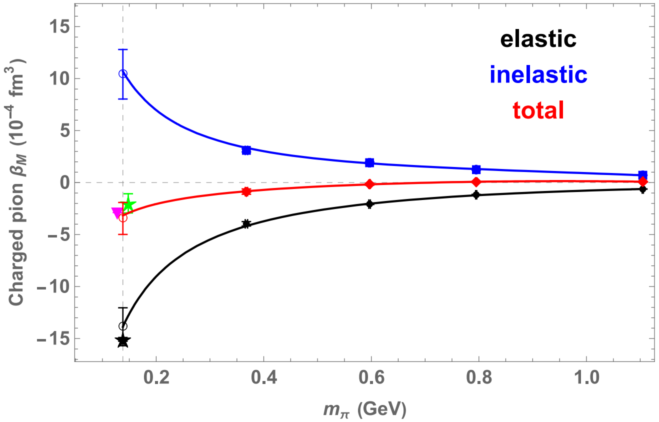
Conclusion



- Proof-of-concept simulations for charged pion show promise of four-point function methodology.
  - Physics payouts: form factors, polarizabilities, etc.
  - Clear pictures for α_E and β_M
  - Requires 2pt and 4pt (but not 3pt) functions



- Open issues
  - Fitting form factors (monopole vs z-expansion)
  - Extrapolation to t=0 (contact term)
  - Extrapolation to $\mathbf{q}^2$=0 (static limit)
  - Chiral extrapolation
  - Quenched approximation
  - Only connected contributions so far



- Outlook
  - Dynamical ensembles (two-flavor nHYP-clover, 315 and 227 MeV, elongated geometries for volume study and smaller Q^2)
  - Disconnected contributions
  - Next target: proton and neutron

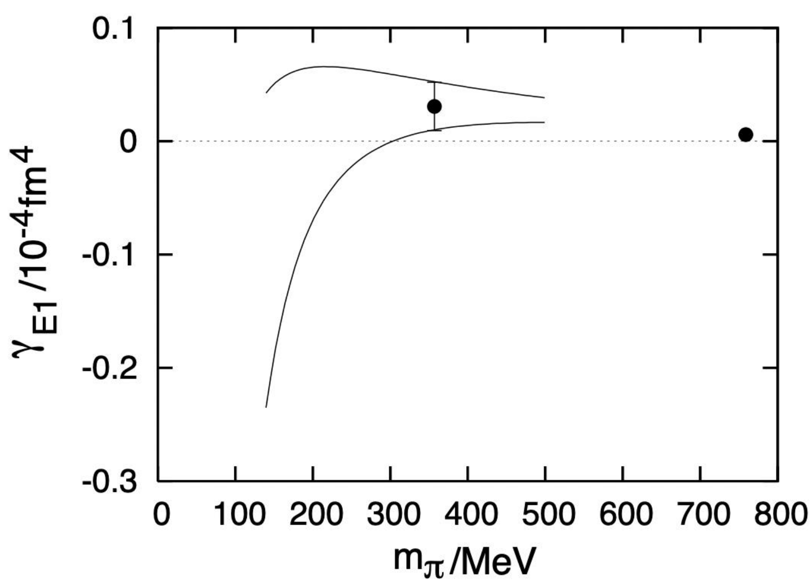
Reserve

Background field + 4pt function method

Perturbative expansion in the background field at the action level leads to the same diagrammatic structure in 4pt method.

Neutron electric polarizability: $\alpha_E = -\ 2.0(0.9)$ PRD76 (2007), Engelhardt

Neutron spin polarizability



arXiv1111.2686 (Lattice2011), Engelhardt

From action to answers: how to calculate observables in QCD?

Correlation functions: vacuum expectation values via path integrals,

$$\langle\Omega|O_2(t)O_1(0)|\Omega\rangle = \frac{\int DqD\bar{q}DG\, O_2[q,\bar{q},G]O_1[q,\bar{q},G]\, e^{-S_{QCD}}}{\int DqD\bar{q}DG\, e^{-S_{QCD}}}$$

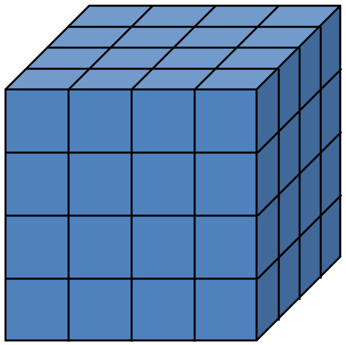
Quark fields anti-commute. They can be integrated out using Grassmann algebra,

$$S_{QCD} = S_G + \bar{q}(\not{D} + m_q)q$$

$$\langle O_2(t)\bar{O}_1(0)\rangle \equiv \frac{\int D[G]\, f(M^{-1})\det(M)e^{-S_G}}{\int D[G]\det(M)e^{-S_G}}$$

$$M = \not{D} + m_q$$

It resembles a statistical system with a probability distribution. Can be evaluated numerically on a spacetime lattice using Monte Carlo importance sampling methods.



$$\langle O\rangle \approx \frac{1}{N}\sum_{i=1}^{N} O[G_i]$$