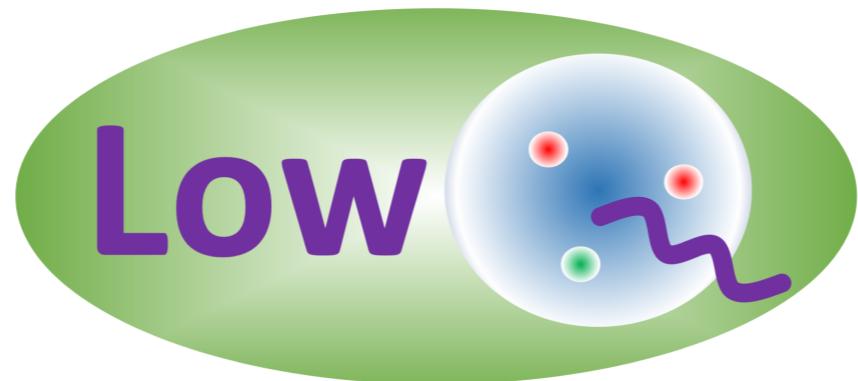


# Photon-photon fusion processes to pions and pion polarizability extraction

Igor Danilkin

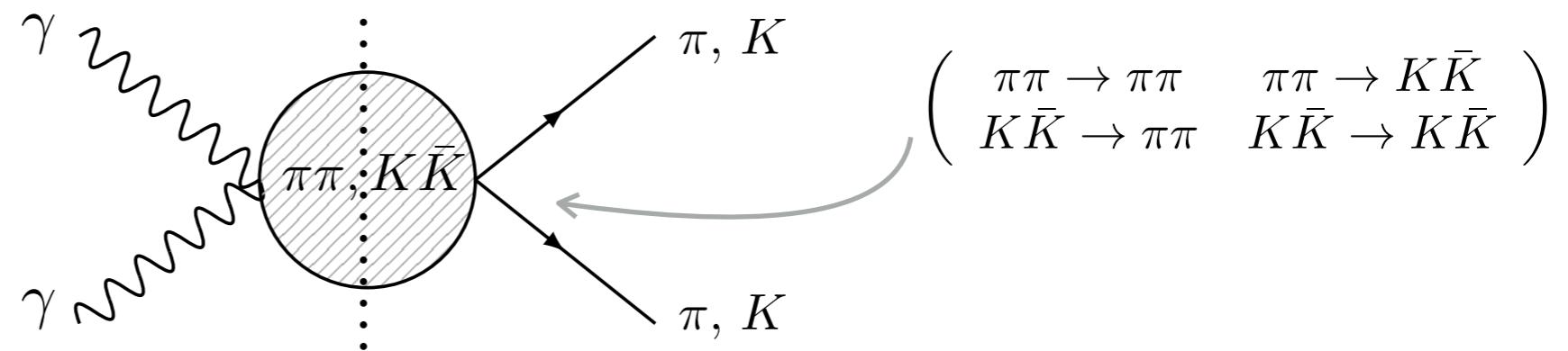


May 16, 2023

# Motivation

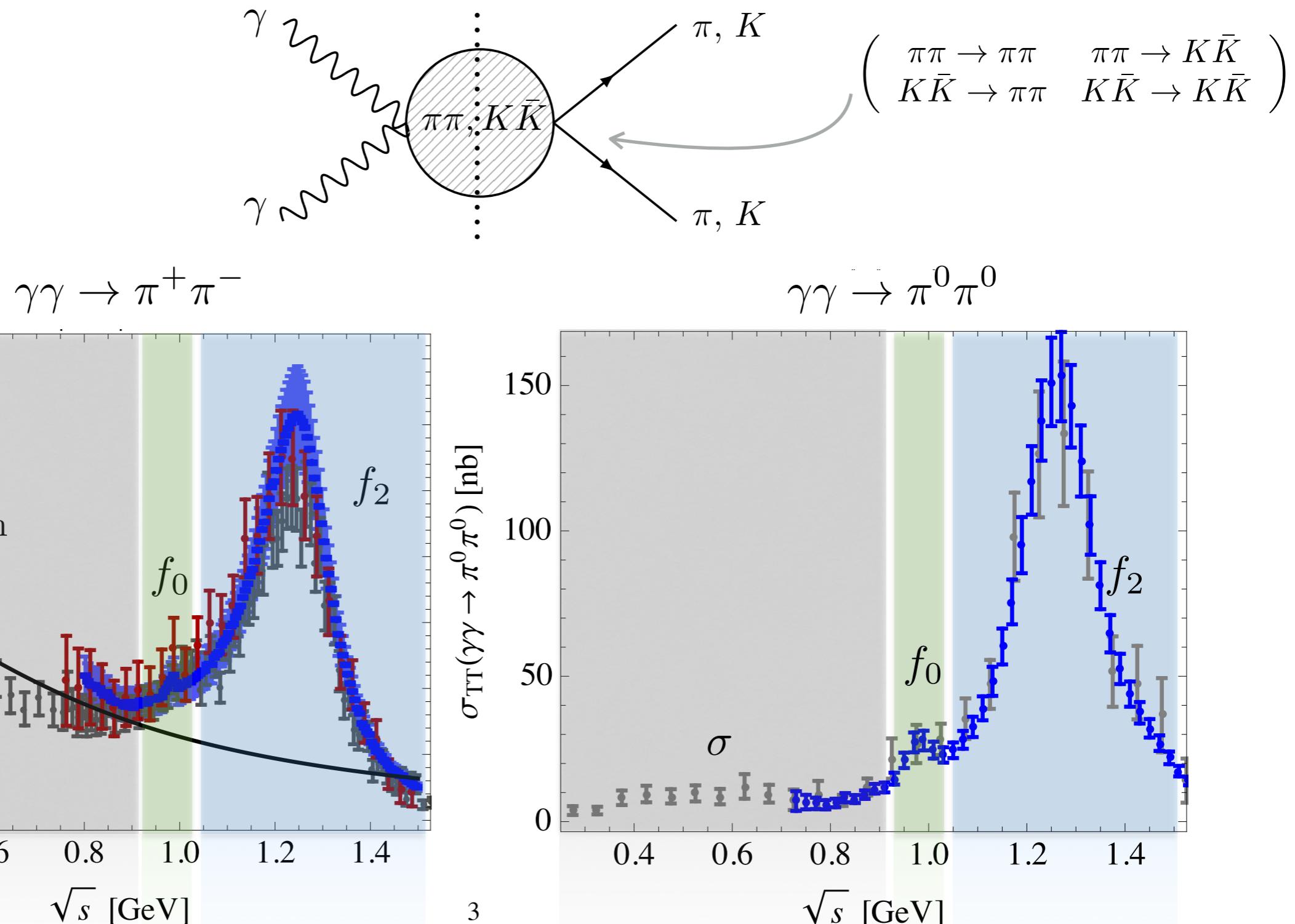
# Motivation

- The total cross sections for  $\gamma\gamma \rightarrow \pi\pi$  are very sensitive to **hadronic final state interaction** and therefore probe scalar resonances  $f_0(500)$  and  $f_0(980)$



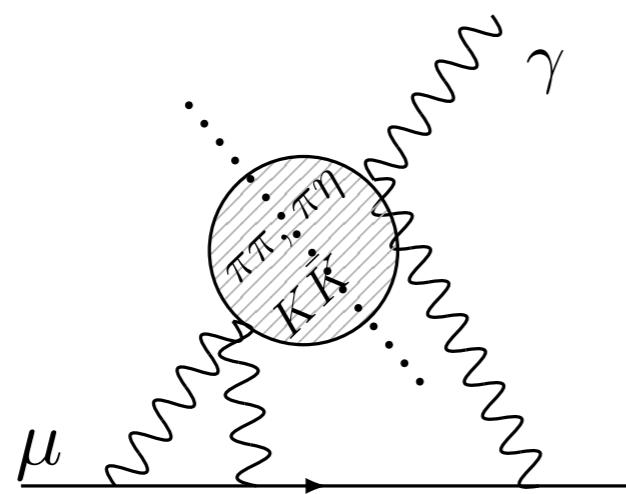
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- The double virtual photon fusion processes are important ingredients in the calculation of **two particle** intermediate states to **g-2** of the muon



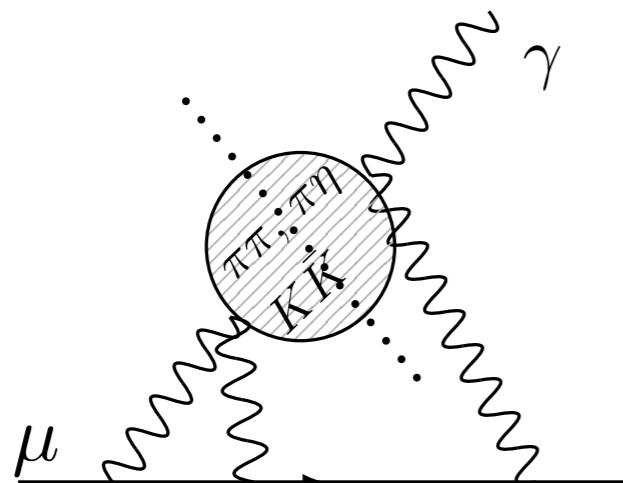
Required ingredients:

$$\gamma^* \gamma^* \rightarrow \pi\pi, K\bar{K}, \dots$$

$$q^2 = -Q^2 < 0 \quad \text{space-like } \gamma^*$$

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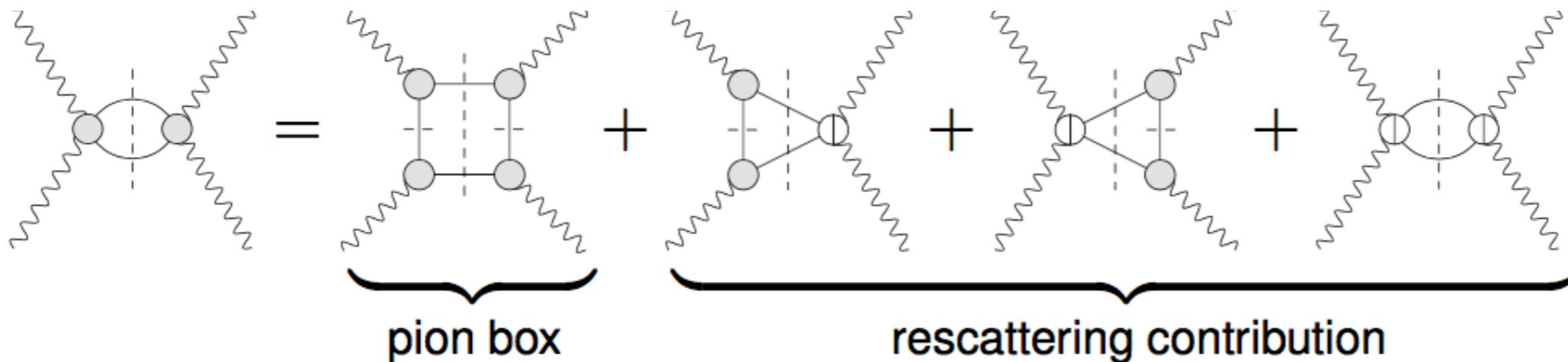


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$$q^2 = -Q^2 < 0 \quad \text{space-like } \gamma^*$$

$$a_\mu^{\text{HLbL}} = \frac{2\alpha^3}{3\pi^2} \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2} Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1, Q_2, Q_3) \bar{\Pi}_i(Q_1, Q_2, Q_3),$$



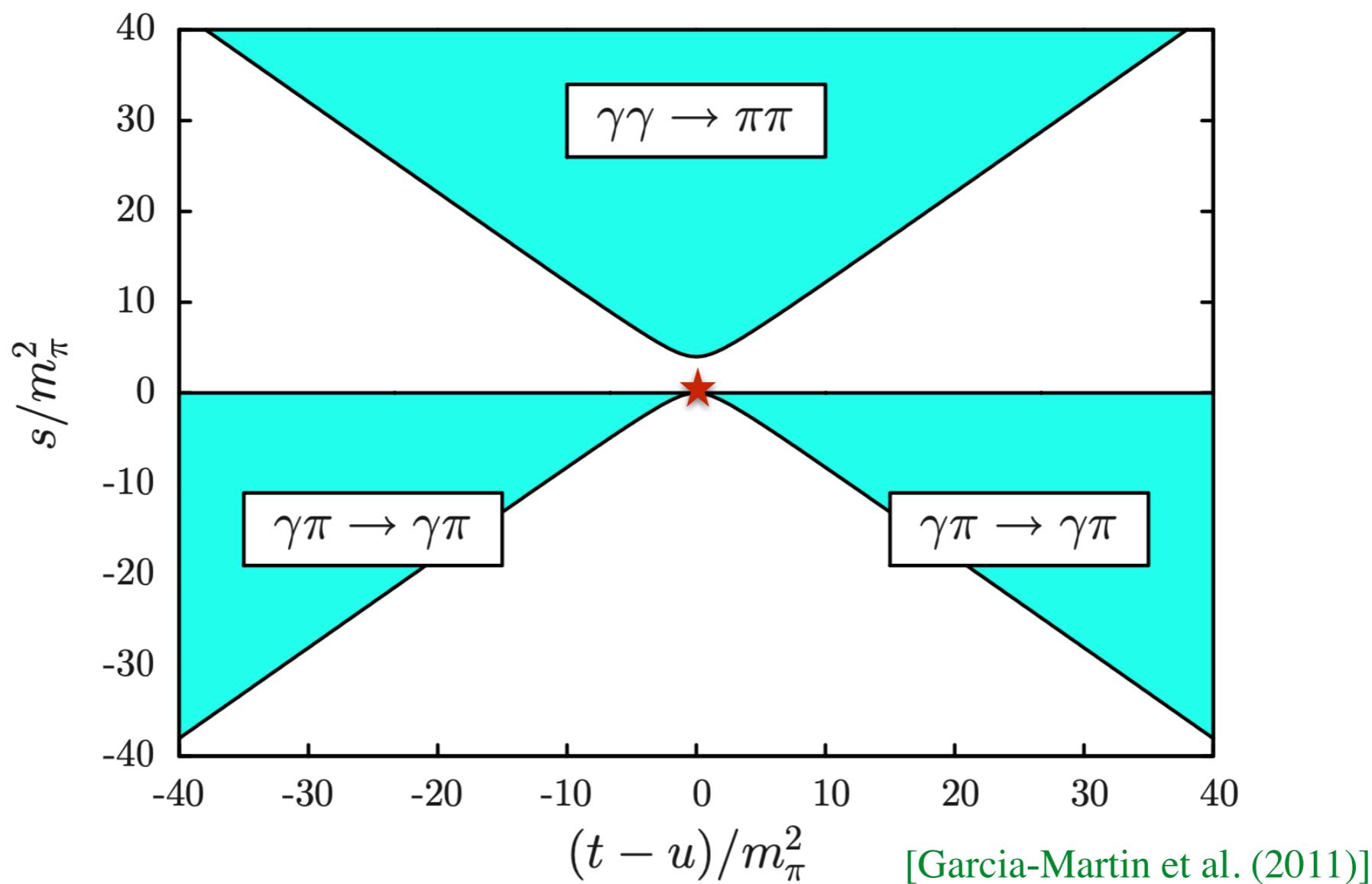
$$a_\mu^{\pi\pi, KK} [\text{box}] = -16.4(2) \times 10^{-11}$$

[Colangelo et al. (2014-2017)]

# Motivation

- Extraction of the pion polarizabilities

$$\frac{2\alpha}{m_\pi} \bar{H}_{++}(s, t = m_\pi^2) = s(\alpha_1 - \beta_1)_\pi + \frac{s^2}{12}(\alpha_2 - \beta_2)_\pi + \dots$$



# Formalism

# Formalism

- S-matrix theory:
  - Unitarity
  - Analyticity (causality)
  - Crossing symmetry

$$t_{ab}(s) = \int_{-\infty}^{s_L} \frac{ds'}{\pi} \frac{\text{Im } t_{ab}(s')}{s' - s} + \int_{s_{th}}^{\infty} \frac{ds'}{\pi} \frac{\text{Im } t_{ab}(s')}{s' - s} \quad (J=0 \text{ case})$$

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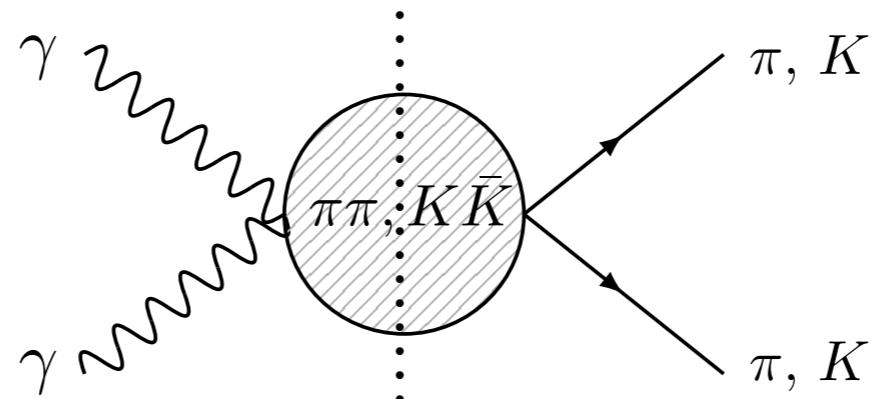
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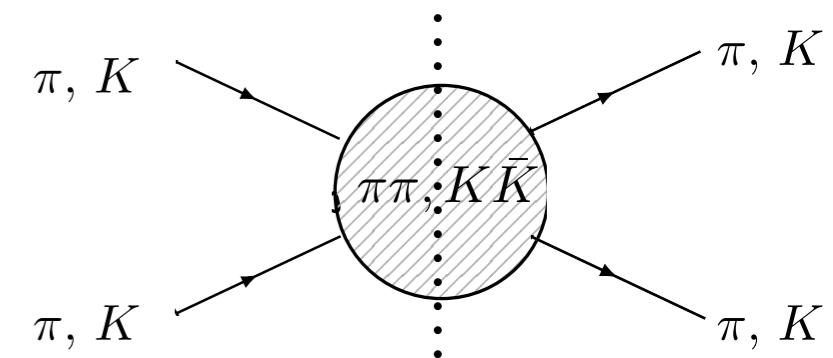
soft photon theorem  
 $t_{\gamma\gamma \rightarrow \pi\pi}(0) = \text{Born}$



# partial wave dispersion relation (S wave)

- Hadronic part:  $a, b = \pi\pi, K\bar{K}$

$$t_{ab}(s) = t_{ab}(s_M) + \underbrace{\frac{s - s_M}{\pi} \int_{-\infty}^{s_L} \frac{ds'}{s' - s_M} \frac{\text{Im } t_{ab}(s')}{s' - s}}_{U_{ab}(s)} + \frac{s - s_M}{\pi} \sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s' - s_M} \frac{t_{ac}(s') \rho_c(s') t_{cb}^*(s')}{s' - s}$$



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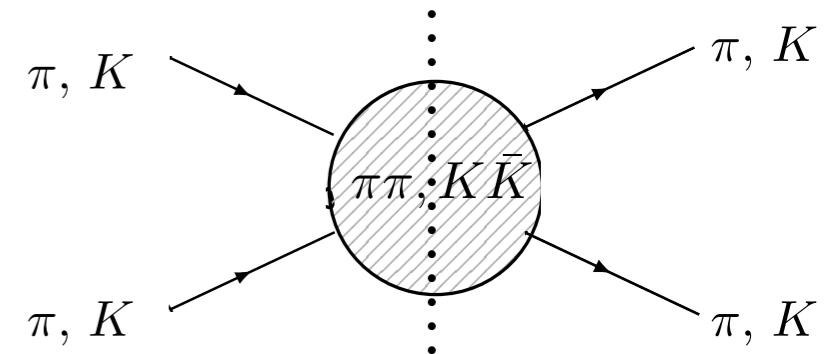
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can be solved using N/D method

$$t_{ab}(s) = \sum_c D_{ac}^{-1} N_{cb}(s)$$

$$N_{ab}(s) = U_{ab}(s) + \frac{s - s_M}{\pi} \sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s' - s_M} \frac{N_{ac}(s') \rho_c(s') (U_{cb}(s') - U_{cb}(s))}{s' - s}$$

$$D_{ab}(s) = \delta_{ab} - \frac{s - s_M}{\pi} \int_{s_{th}}^{\infty} \frac{ds'}{s' - s_M} \frac{N_{ab}(s') \rho_b(s')}{s' - s}$$



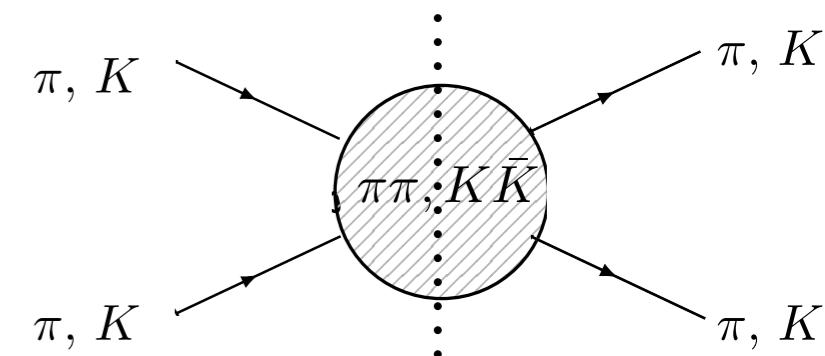
[Chew, Mandelstam (1960)]  
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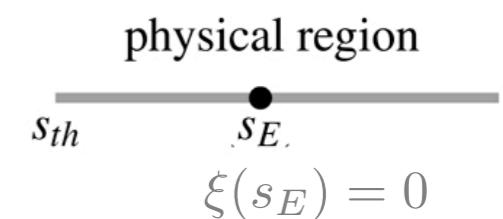
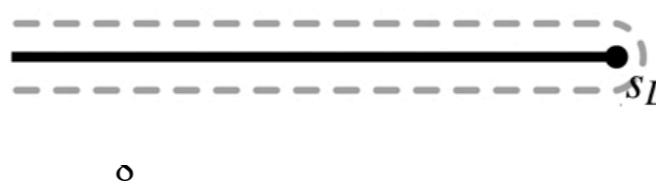
$$D_{ab}(s) = \delta_{ab} - \frac{s - s_M}{\pi} \int_{s_{th}}^{\infty} \frac{ds'}{s' - s_M} \frac{N_{ab}(s') \rho_b(s')}{s' - s}$$

[Chew, Mandelstam (1960)]  
 [Luming (1964)]  
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- Using the known analytical structure of left-hand cuts, one can approximate  $U_{ab}(s)$  as an expansion in a **conformal mapping variable**  $\xi(s)$   
[Gasparyan, Lutz (2010)]

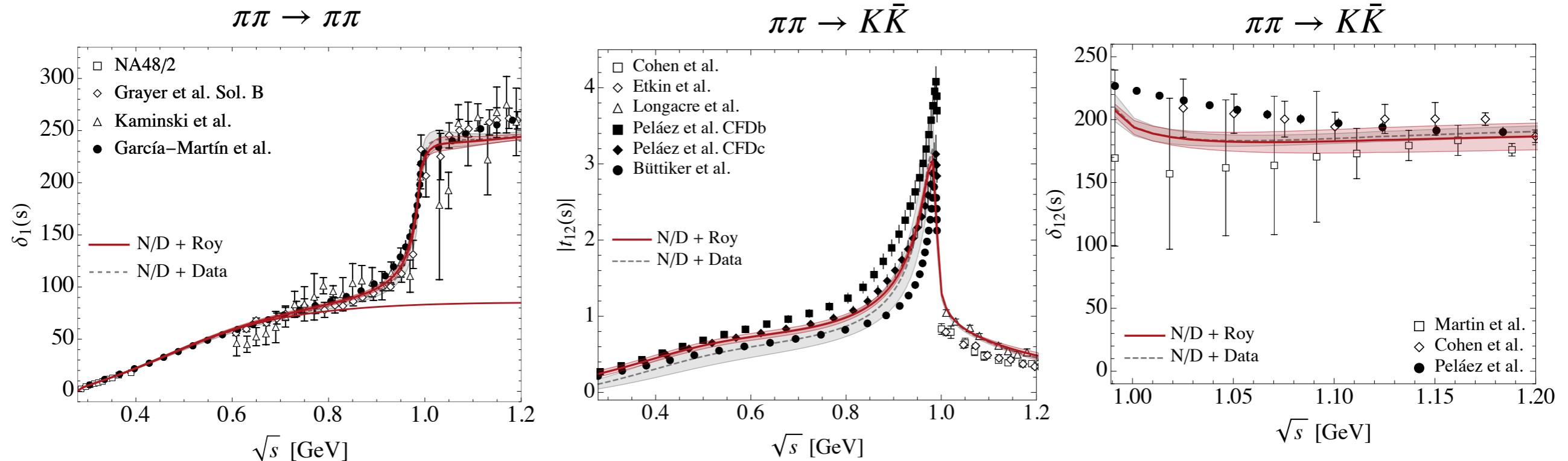
$$U_{ab}(s) = \sum_{n=0}^{\infty} C_{ab,n} (\xi_{ab}(s))^n$$

unknown coefficients fitted to data



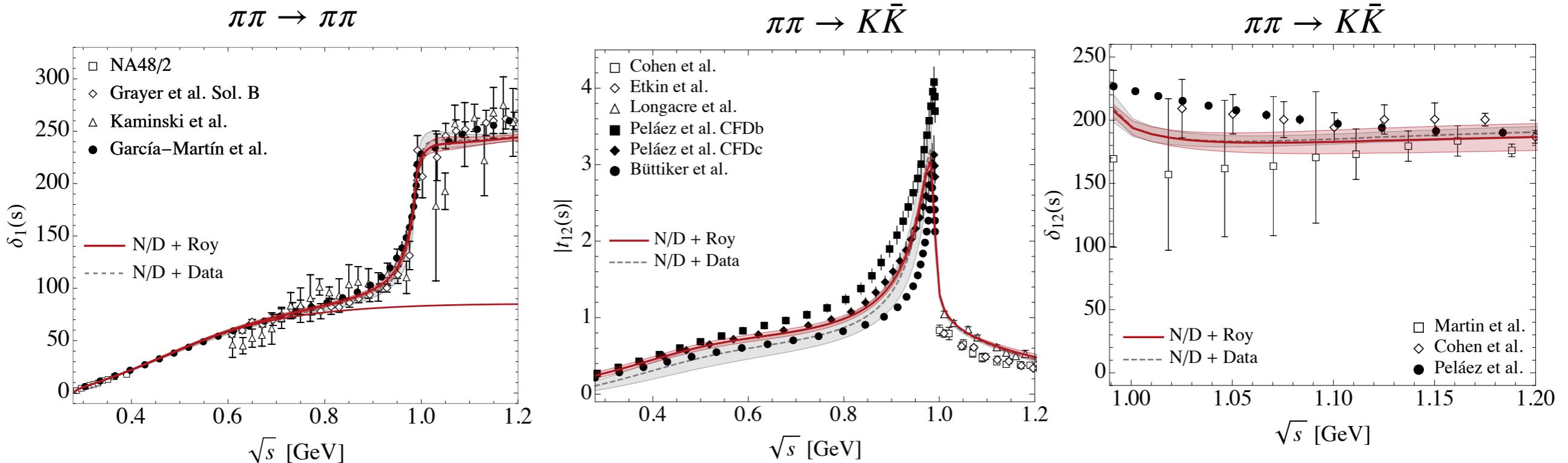
$$\xi(s_E) = 0$$

# Coupled-channel analysis $\{\pi\pi, K\bar{K}\}$



**Input:** experimental data/Roy analysis + threshold parameters NNLO (a, b) + Adler zero NLO

# Coupled-channel analysis $\{\pi\pi, K\bar{K}\}$



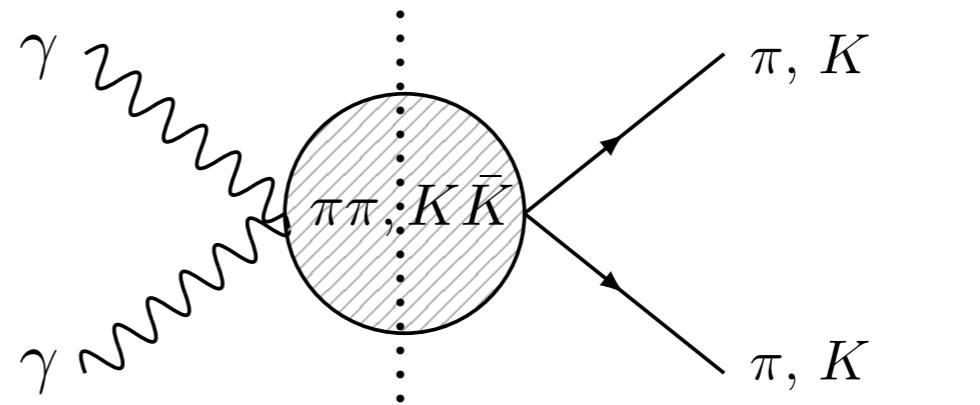
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	Our results		Roy-like analyses		[Caprini et al. (2006)] [Garcia-Martin et al. (2011)] [Moussallam (2011)]
	pole position, MeV	couplings, GeV	pole position, MeV	couplings, GeV	
$\sigma/f_0(500)$	$458(10)^{+7}_{-15} - i 256(9)^{+5}_{-8}$	$\pi\pi : 3.33(8)^{+0.12}_{-0.20}$ $K\bar{K} : 2.11(17)^{+0.27}_{-0.11}$	$449^{+22}_{-16} - i 275(15)$	$\pi\pi : 3.45^{+0.25}_{-0.29}$ $K\bar{K} : -$	
fit to Exp	$454(12)^{+6}_{-7} - i 262(12)^{+8}_{-12}$				
$f_0(980)$	$993(2)^{+2}_{-1} - i 21(3)^{+2}_{-4}$	$\pi\pi : 1.93(15)^{+0.07}_{-0.12}$ $K\bar{K} : 5.31(24)^{+0.04}_{-0.24}$	$996^{+7}_{-14} - i 25^{+11}_{-6}$	$\pi\pi : 2.3(2)$ $K\bar{K} : -$	
fit to Exp	$990(7)^{+2}_{-4} - i 17(7)^{+4}_{-1}$				

- Omnes function fulfills the unitarity relation on the right-hand cut and analytic everywhere else
- For the case of no bound states or CDD poles:  $\Omega_{ab}(s) = D_{ab}^{-1}(s)$

# Formalism

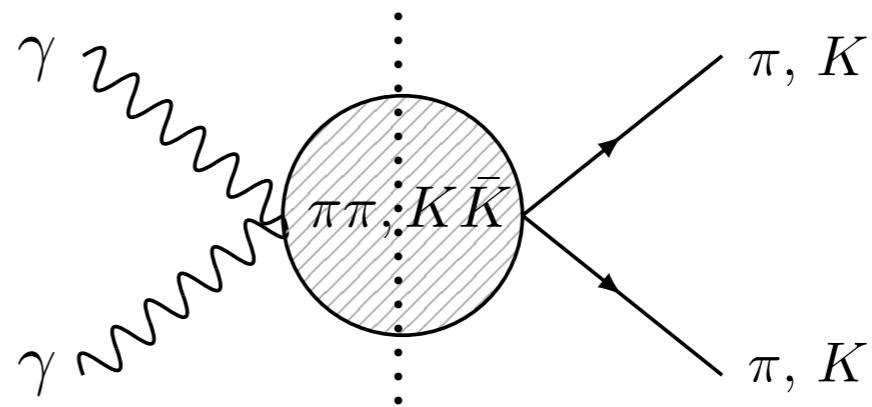
- **Photon-fusion part**  
(S-wave, fixed isospin,  
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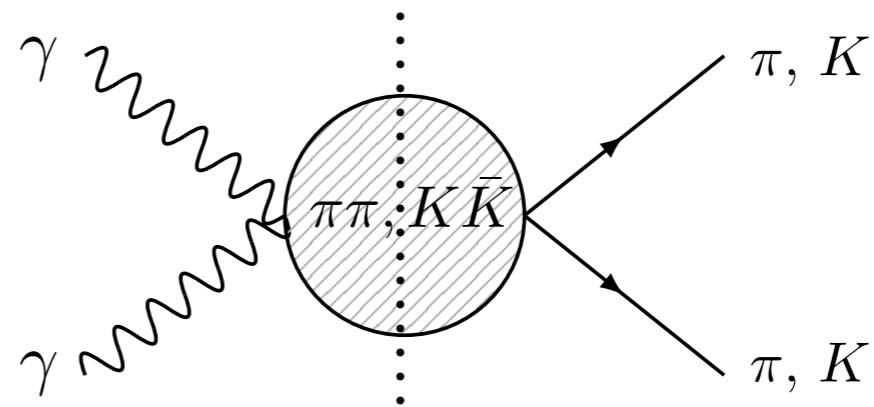
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- Can be solved using modified Muskhelishvili-Omnes formalism, i.e. by writing a dispersion relation for  $\Omega^{-1}(s)$  ( $t_{\gamma\gamma \rightarrow \pi\pi} - \text{Born}$ ) [Garcia-Martin et. al (2010)]

$$t_{\gamma\gamma \rightarrow \pi\pi}(s) = \text{Born} + s \Omega(s) \left( \frac{1}{\pi} \int_{-\infty}^{s_L} \frac{ds'}{s'} \frac{\Omega^{-1}(s') \text{Im } (t_{\gamma\gamma \rightarrow \pi\pi}(s') - \text{Born})}{s' - s} - \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'} \frac{\text{Im } \Omega(s') \text{Born}}{s' - s} \right)$$

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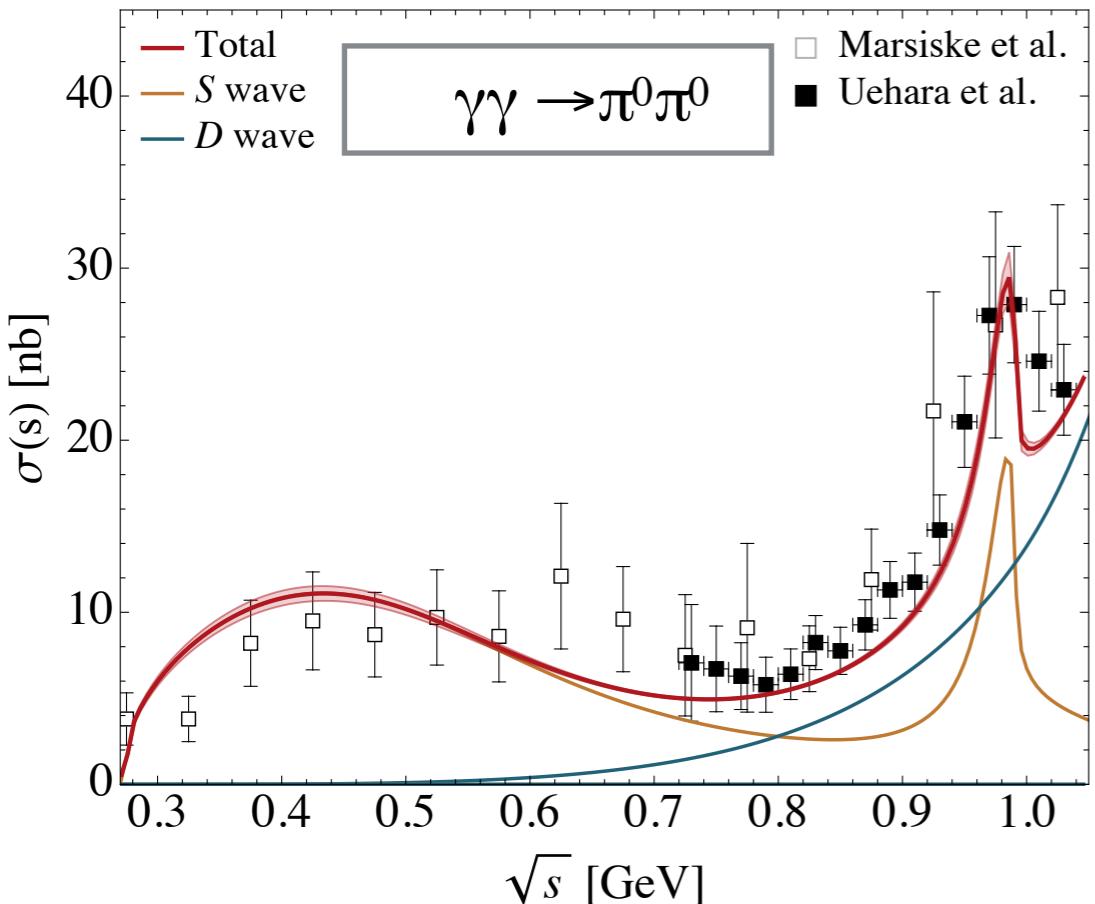
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- Similar equations can be written for **higher partial waves**, but one needs to take into account the correct threshold behaviour

$$t_{\gamma\gamma \rightarrow \pi\pi}^{(J)}(s) - \text{Born} \sim s (p(s) q(s))^J, \quad \{\lambda_1, \lambda_2\} = + +$$

$$t_{\gamma\gamma \rightarrow \pi\pi}^{(J)}(s) - \text{Born} \sim (p(s) q(s))^J, \quad \{\lambda_1, \lambda_2\} = + -$$

# $\gamma\gamma \rightarrow \pi\pi$ (postdiction)



$$\Gamma_{\gamma\gamma}[f_0(500)] = 1.37(13)^{+0.09}_{-0.06} \text{ keV}$$

$$\Gamma_{\gamma\gamma}[f_0(980)] = 0.33(16)^{+0.04}_{-0.16} \text{ keV}$$

[I.D, Deineka,  
Vanderhaeghen,(2021)]

consistent with

$$\Gamma_{\gamma\gamma}^{\text{Roy-Steiner}}[f_0(500)] = 1.7(4) \text{ keV}$$

$$\Gamma^{\text{MO}}[f_0(980)] = 0.29(21)^{+0.02}_{-0.07} \text{ keV}$$

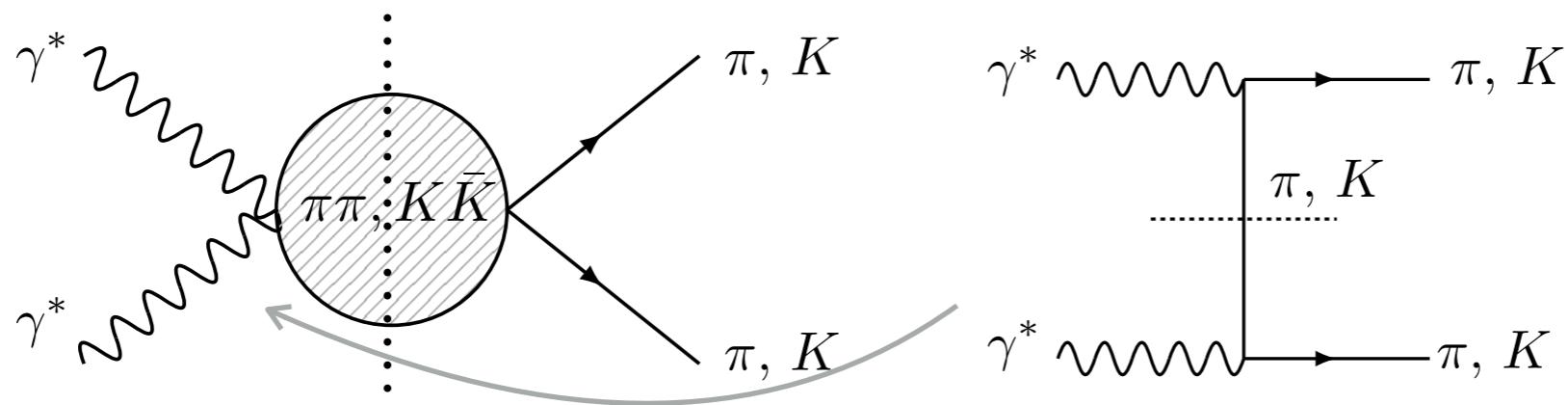
$$\Gamma^{\text{Ampl. analys.}}[f_0(980)] = 0.32(5) \text{ keV}$$

[Hoferichter et. al. (2011)]  
[Moussallam (2011)]  
[Dai et al. (2014)]

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*take into account only  
Born LHC*

# Left-hand cuts (pion/kaon pole)

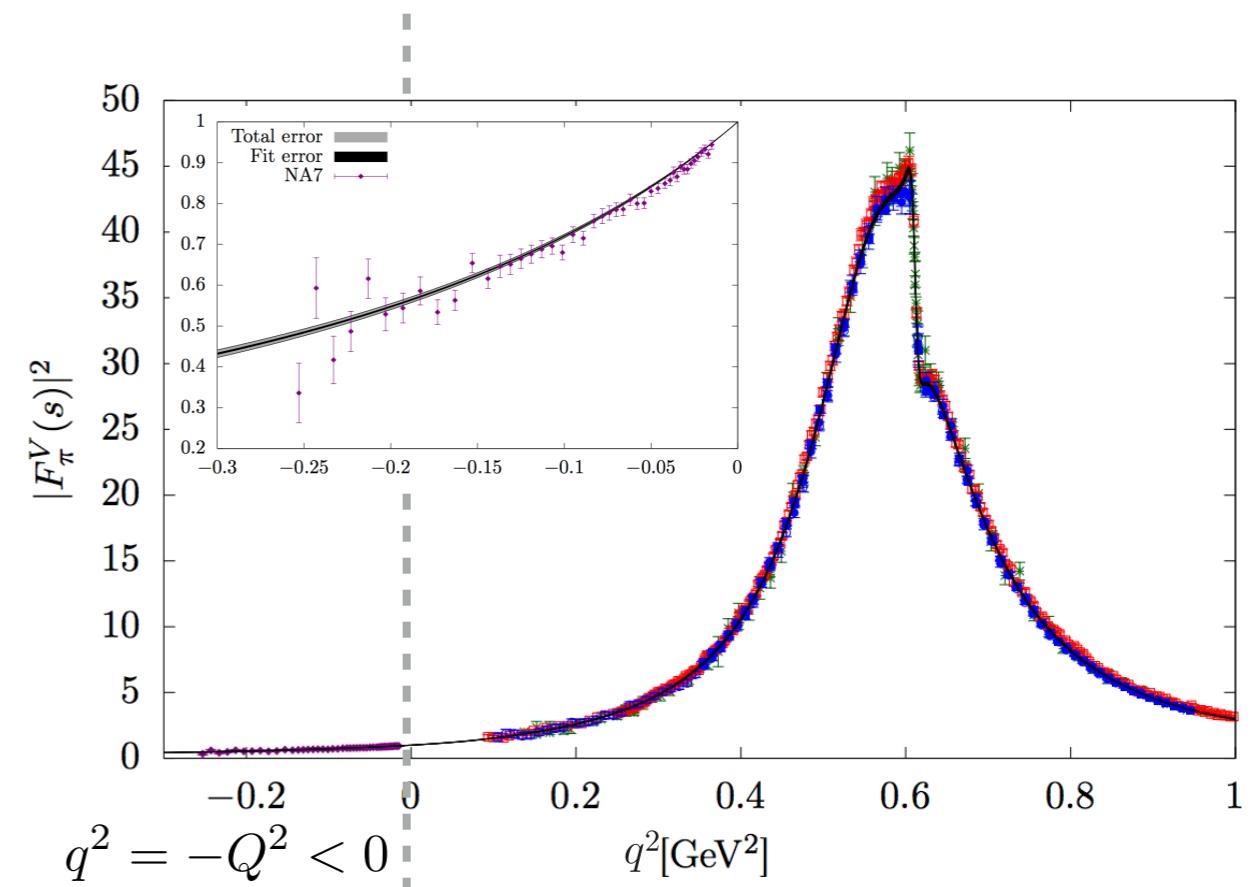


- Left-hand cuts requires knowledge from  $\gamma^*\pi\pi, \gamma^*KK$  form factors

$$\text{Disc} \left[ \begin{array}{c} \gamma^* \\ \hline \pi & \pi \end{array} \right] = \gamma^* \begin{array}{c} \pi \\ \hline \text{had} \\ \pi \end{array}$$

- p.w. helicity amplitudes suffer from kinematic constraints. For S-wave ( $\bar{t} \equiv t - \text{Born}$ )

$$\bar{t}_{\gamma\gamma \rightarrow \pi\pi,++} \pm \bar{t}_{\gamma\gamma \rightarrow \pi\pi,00} \sim (s + (Q_1 \pm Q_2)^2)$$



# Contribution to (g-2)

- Using S-wave elastic helicity amplitudes on  $\gamma^*\gamma^*\rightarrow\pi\pi$ ,  $f_0(500)$  contribution was calculated previously

$$a_\mu^{\text{HLbL}}[\text{S-wave}, I = 0]_{\text{rescattering}} = -9.3(1) \times 10^{-11} \quad [\text{Colangelo et al. (2014-2017)}]$$

- Extending to KK channel allowed us to access energies up to  $\sim 1.2$  GeV ( $f_0(500) + f_0(980)$  contributions)

$$a_\mu^{\text{HLbL}}[\text{S-wave}, I = 0]_{\text{rescattering}} = -9.8(1) \times 10^{-11} \quad [\text{I.D. Hoferichter, Stoffer (2021)}]$$

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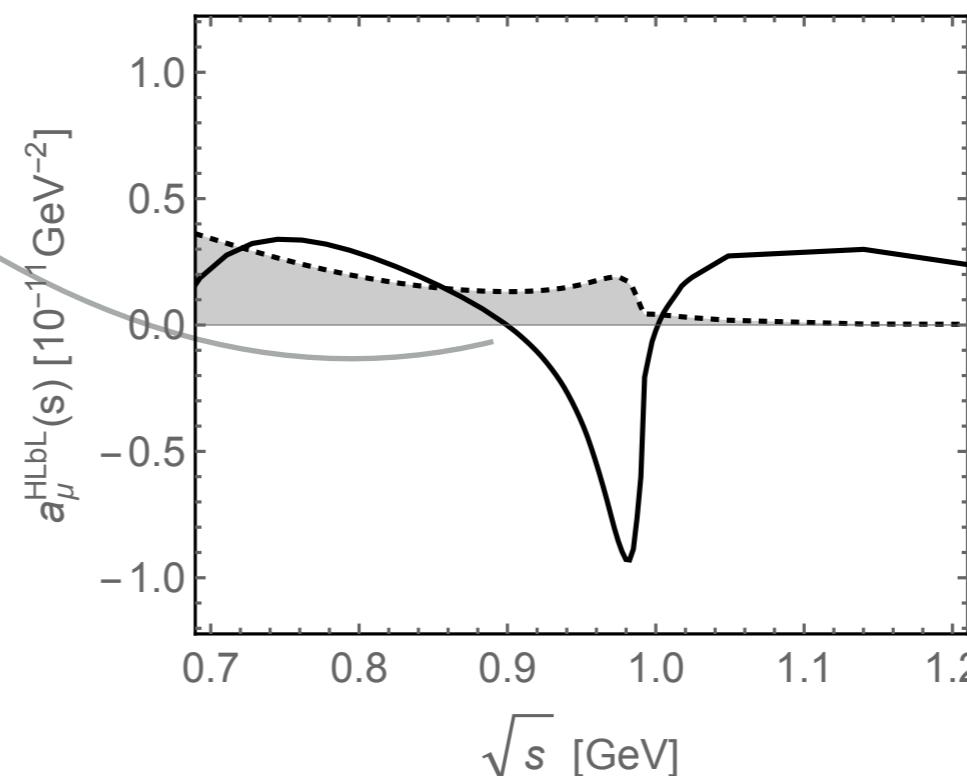
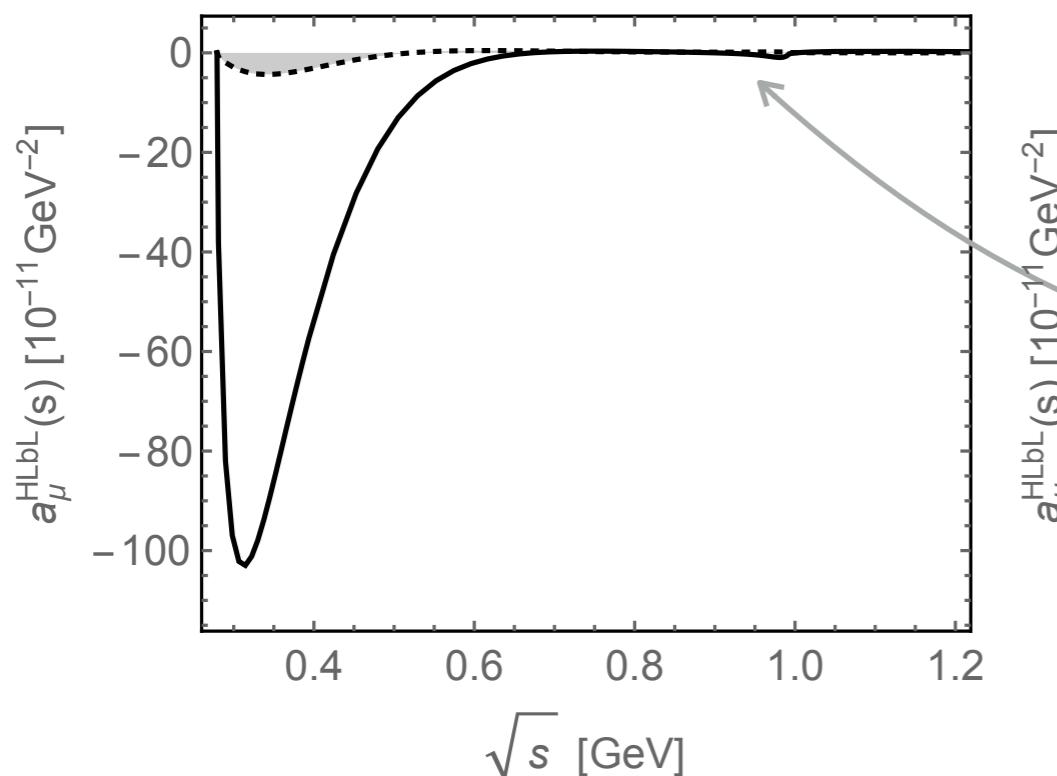
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[I.D, Hoferichter, Stoffer (2021)]

- The contribution just from  $f_0(980)$  we defined as an integral over the deficit in shape

$$a_\mu^{\text{HLbL}} = \int_{4m_\pi^2}^\infty ds' \frac{da_\mu^{\text{HLbL}}(s')}{ds'}$$

$$a_\mu^{\text{HLbL}}[f_0(980)]_{\text{rescattering}} = -0.2(1) \times 10^{-11}$$



**shaded area** is a  
sum rule violation  
→  
result is largely  
basis independent

# Pion polarizabilities

- **Polarizabilities** are defined as

$$\frac{2\alpha}{m_\pi} \bar{t}_{\gamma\gamma \rightarrow \pi\pi,++}(s) = s (\alpha_1 - \beta_1)_\pi + \dots$$

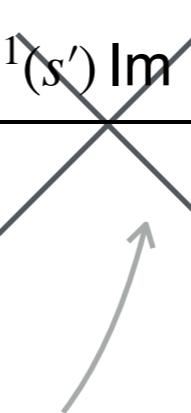

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	Dispersive	ChPT	Experiment
	$\pi$ pole LHC	NLO	NNLO
	COMPASS		
$(\alpha_1 - \beta_1)_{\pi^\pm} [10^{-4}\text{fm}^3]$	5.5 <sub>SC</sub> , 6.1 <sub>CC</sub>	6.0	5.7(1.0)
$(\alpha_1 - \beta_1)_{\pi^0} [10^{-4}\text{fm}^3]$	8.9 <sub>SC</sub> , 9.5 <sub>CC</sub>	-1.0	-1.9(2)

[Colangelo et al. 2017]  
 [I.D., Vanderhaeghen (2018)]

$$t_{\gamma\gamma \rightarrow \pi\pi}(s) = \text{Born} + s \Omega(s) \left( \frac{1}{\pi} \int_{-\infty}^{s_L} \frac{ds'}{s'} \frac{\Omega^{-1}(s') \text{Im} (t_{\gamma\gamma \rightarrow \pi\pi}(s') - \text{Born})}{s' - s} - \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'} \frac{\text{Im} \Omega(s') \text{Born}}{s' - s} \right)$$


*take into account only  
Born LHC*

# Pion polarizabilities

- Polarizabilities are defined as

$$\frac{2\alpha}{m_\pi} \bar{t}_{\gamma\gamma \rightarrow \pi\pi,++}(s) = s (\alpha_1 - \beta_1)_\pi + \dots$$


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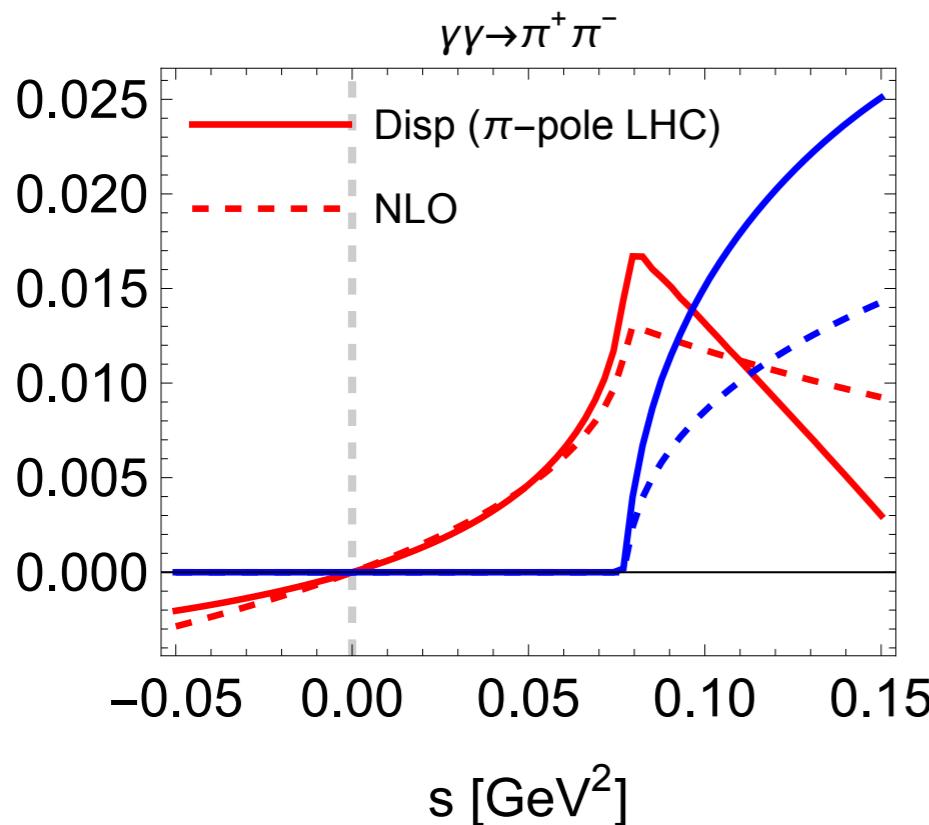


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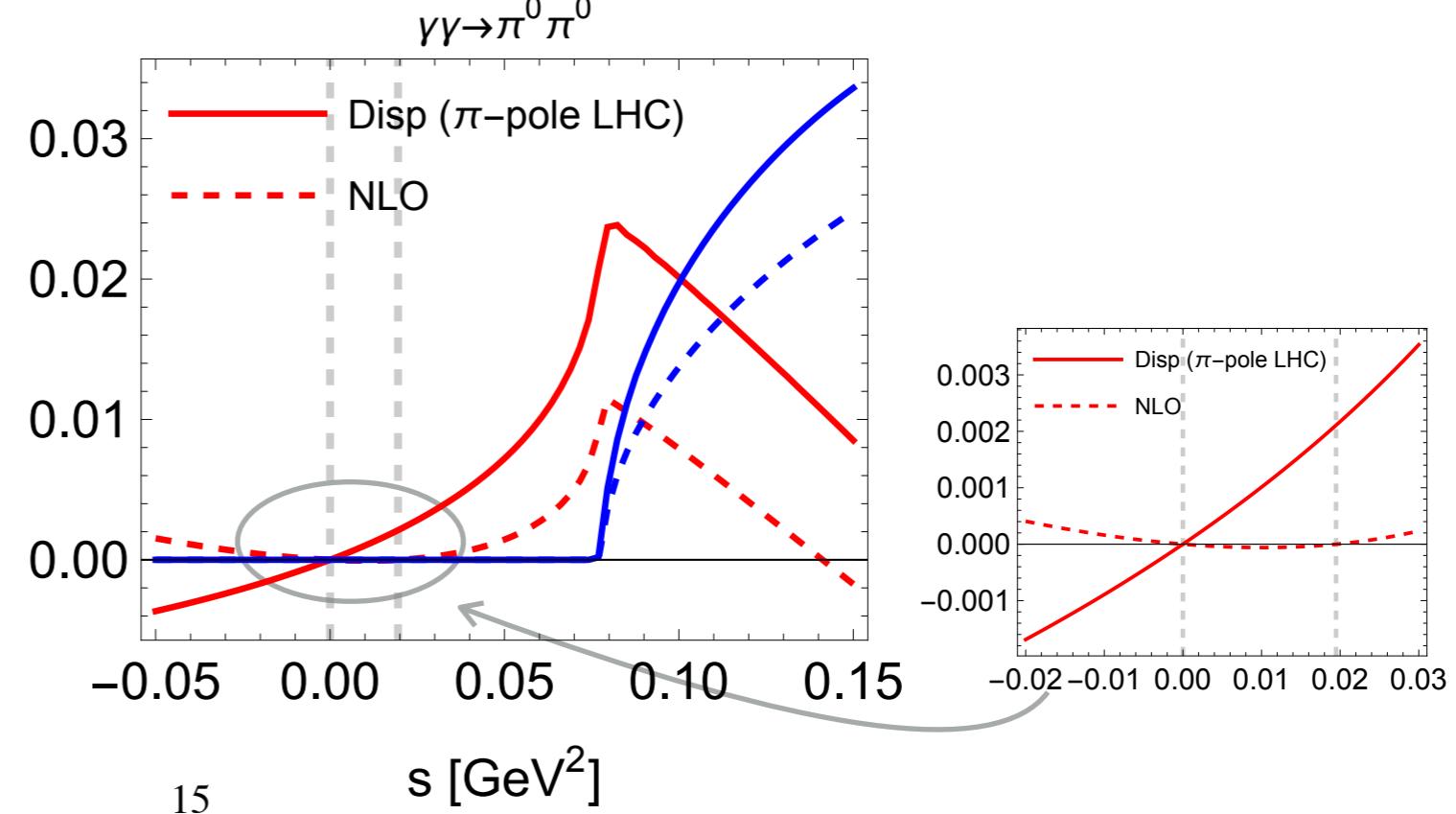
	Dispersive	ChPT	Experiment	
	$\pi$ pole LHC	NLO	NNLO	COMPASS
$(\alpha_1 - \beta_1)_{\pi^\pm} [10^{-4}\text{fm}^3]$	5.5 <sub>SC</sub> , 6.1 <sub>CC</sub>	6.0	5.7(1.0)	4.0(1.2) <sub>stat</sub> (1.4) <sub>syst</sub>
$(\alpha_1 - \beta_1)_{\pi^0} [10^{-4}\text{fm}^3]$	8.9 <sub>SC</sub> , 9.5 <sub>CC</sub>	-1.0	-1.9(2)	-

[Colangelo et al. 2017]  
 [I.D., Vanderhaeghen (2018)]

$\pi^\pm$  pol: agreement with ChPT/COMPASS



$\pi^0$  pol: no Adler zero in  $\gamma\gamma \rightarrow \pi^0 \pi^0$  ( $s_A \sim m_\pi^2$ )



# Pion polarizabilities

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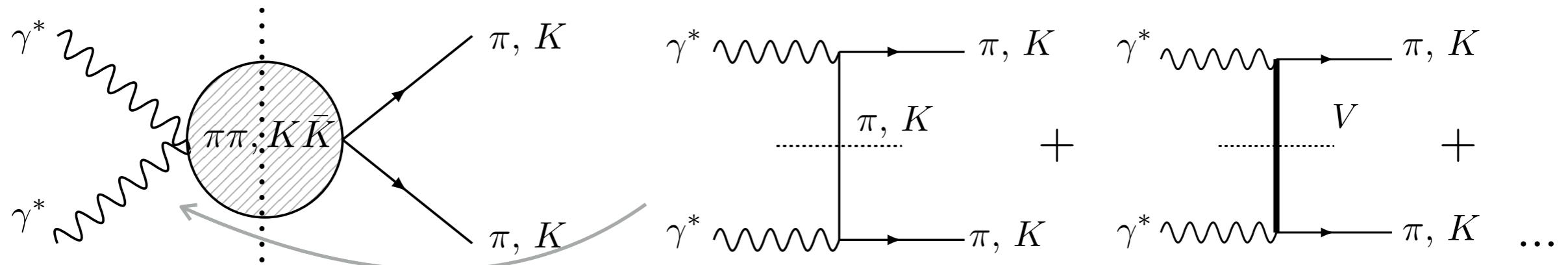


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	Dispersive	ChPT	Experiment
	$\pi$ pole LHC	NLO	NNLO
	COMPASS		
$(\alpha_1 - \beta_1)_{\pi^\pm} [10^{-4}\text{fm}^3]$	5.5 <sub>SC</sub> , 6.1 <sub>CC</sub>	6.0	5.7(1.0)
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- $\pi^0$  pol. gets large corrections when vector-meson left-hand cuts are added (without spoiling  $\pi^\pm$  pol.)

$$\frac{\Gamma_{\omega \rightarrow \pi^0 \gamma}}{\Gamma_{\rho^{\pm,0} \rightarrow \pi^{\pm,0} \gamma}} \sim 10$$



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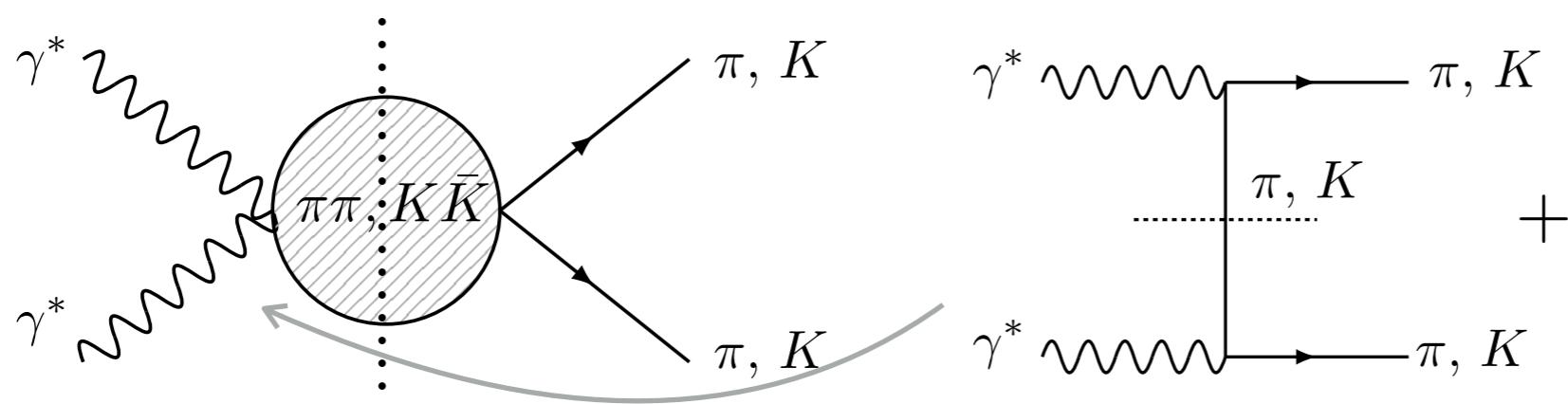
$$\frac{\Gamma_{\omega \rightarrow \pi^0 \gamma}}{\Gamma_{\rho^{\pm,0} \rightarrow \pi^{\pm,0} \gamma}} \sim 10$$

- In **modified Muskhelishvili-Omnes** formalism heavier LHC enter as **pole contributions**

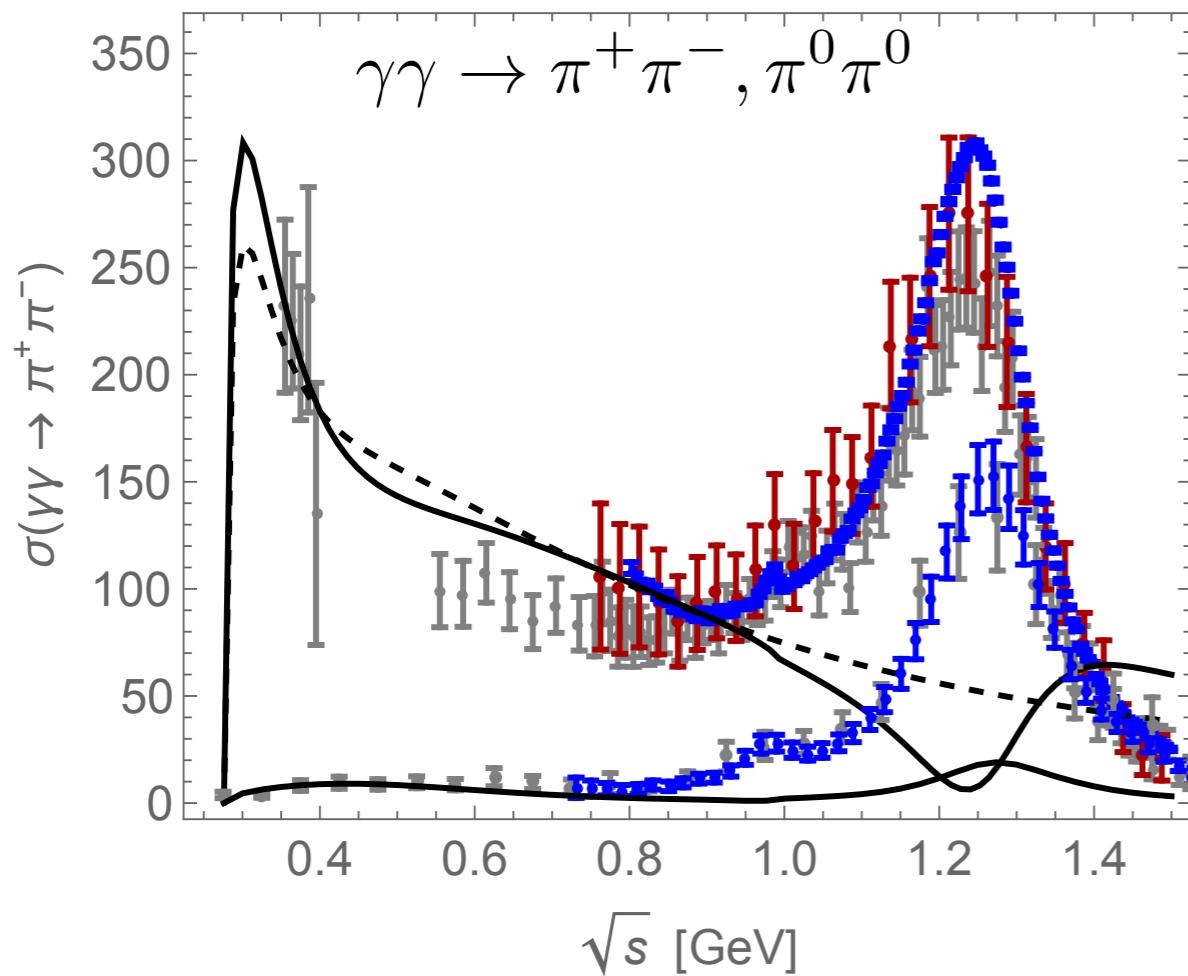
$$t_{\gamma\gamma \rightarrow \pi\pi}(s) = \text{Born} + s \Omega(s) \left( \frac{1}{\pi} \int_{-\infty}^{s_L} \frac{ds'}{s'} \frac{\Omega^{-1}(s') \text{Im} (t_{\gamma\gamma \rightarrow \pi\pi}(s') - \text{Born})}{s' - s} - \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'} \frac{\text{Im} \Omega(s') \text{Born}}{s' - s} \right)$$

- Need to introduce at least **one-subtraction** in **S-wave** to improve the convergence under dispersive integrals

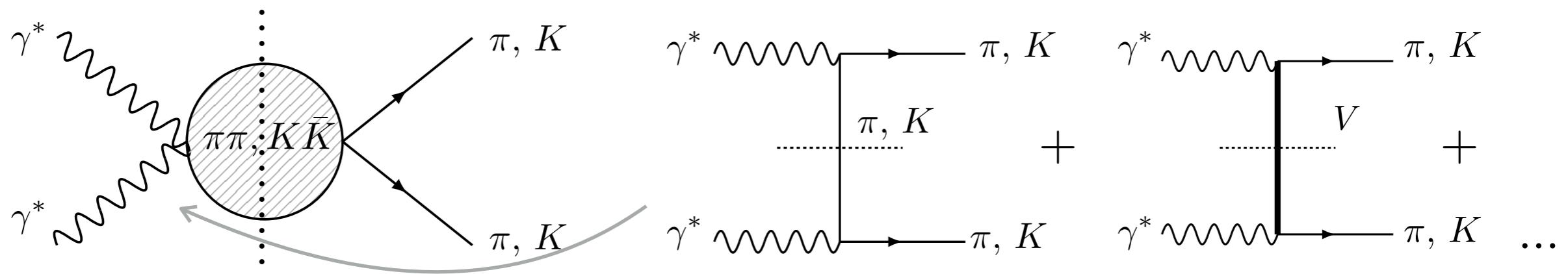
# $f_2(1270)$ region



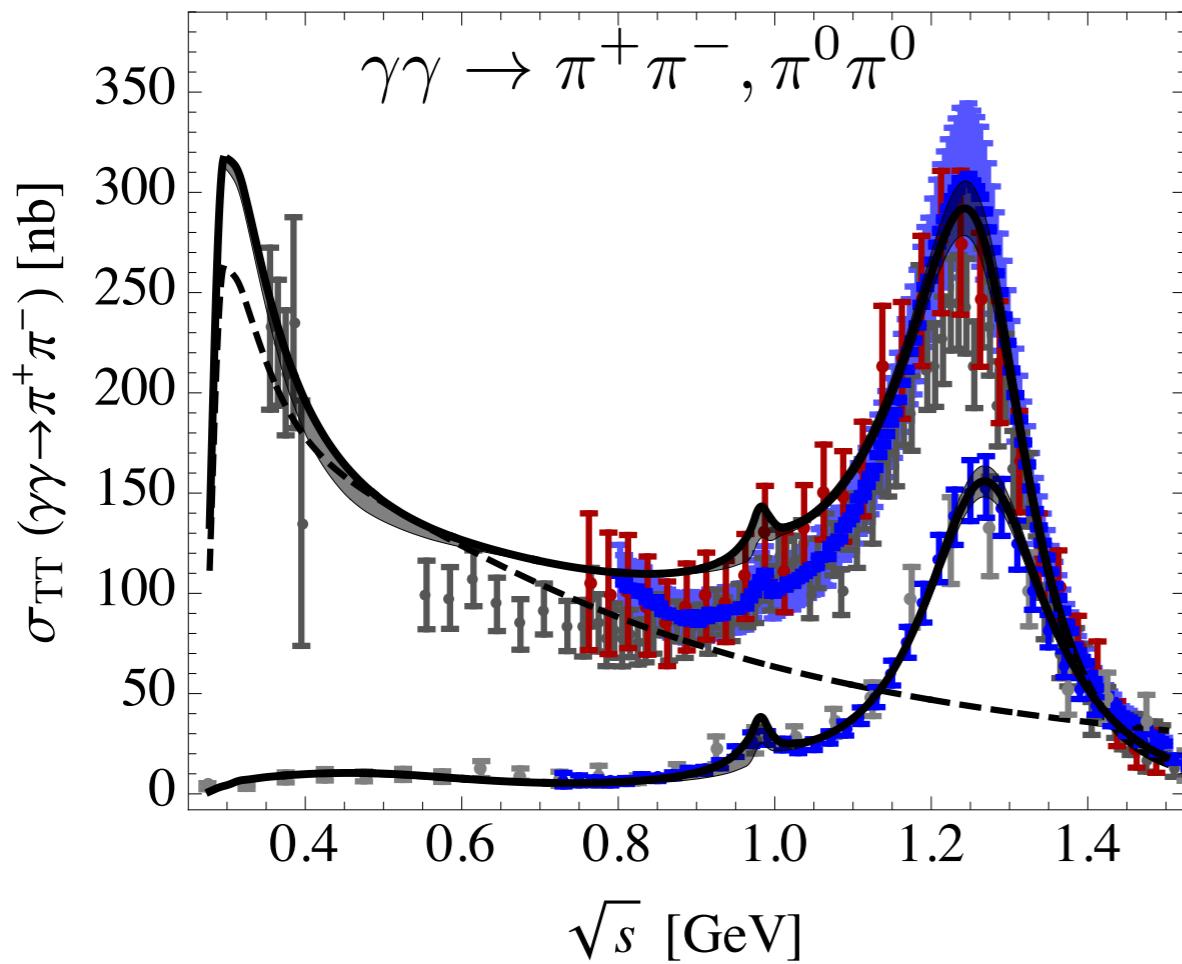
- Heavier LHCs are absolutely necessary to describe  $f_2(1270)$  resonance dispersively  
[Garcia-Martin et. al (2010)]



# $f_2(1270)$ region



- Heavier LHCs are absolutely necessary to describe  $f_2(1270)$  resonance dispersively  
[Garcia-Martin et. al (2010)]



- Fitted parameter is the coupling:

$$g_{V \rightarrow \pi\gamma} \simeq C_{\rho^{\pm,0} \rightarrow \pi^{\pm,0}\gamma} \simeq \frac{1}{3} C_{\omega \rightarrow \pi^0\gamma}^{\text{PDG}} = 0.37(2) \text{ GeV}^{-1}$$

$$g_{V \rightarrow \pi\gamma} = 0.33 \text{ GeV}^{-1}$$

[I.D., Vanderhaeghen (2018)]

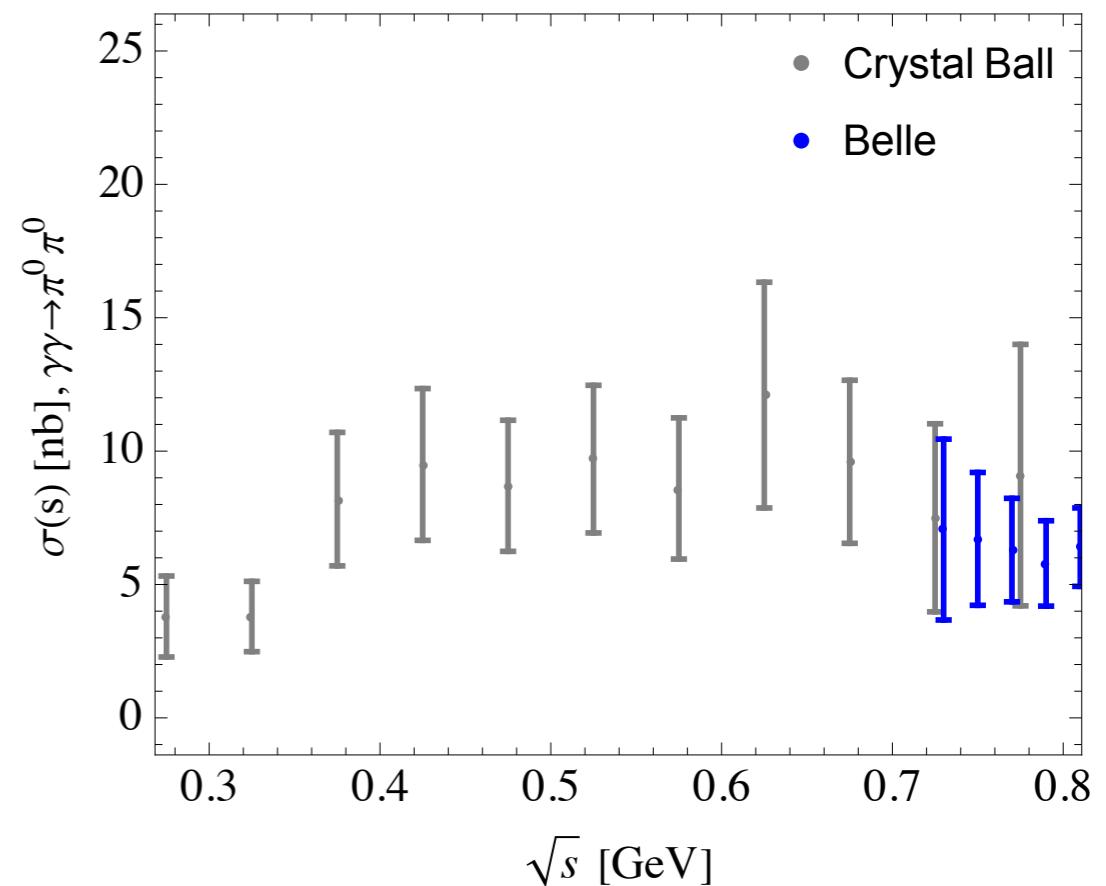
# Pion polarizabilities

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$$\frac{2\alpha}{m_\pi} \bar{t}_{\gamma\gamma \rightarrow \pi\pi,++}(s) = s (\alpha_1 - \beta_1)_\pi + \dots$$

	Dispersive	ChPT	Experiment			
Present work	Garcia-Martin et al 2010					
	$\pi$ pole	$\pi, V$ poles	$\pi, V, A, T$ poles	NLO	NNLO	COMPASS
$(\alpha_1 - \beta_1)_\pi^\pm [10^{-4}\text{fm}^3]$	5.5 <sub>SC</sub>	4.3(4)(2)	4.7	6.0	5.7(1.0)	4.0(1.2) <sub>stat</sub> (1.4) <sub>sys</sub>
$(\alpha_1 - \beta_1)_{\pi^0} [10^{-4}\text{fm}^3]$	8.9 <sub>SC</sub>	-1.5(0)(3)	-1.25(17)	-1.0	-1.9(2)	-

fixed **Adler zero**  $t_{\gamma\gamma \rightarrow \pi^0\pi^0}(s_A) = 0$  and  
fitted one subtraction to  $\sigma_{\gamma\gamma \rightarrow \pi^0\pi^0}$  data in the  
elastic region (preliminary results)



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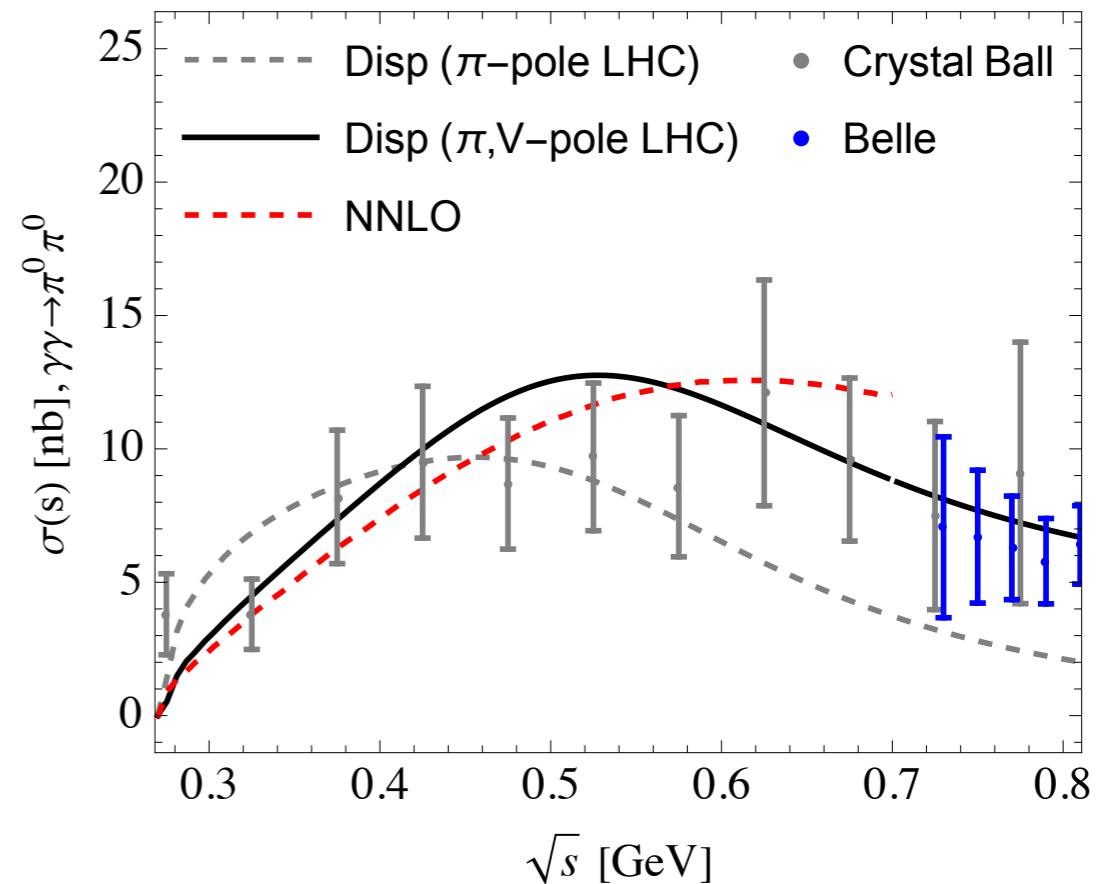
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elastic region (preliminary results)

$$\chi^2/\text{d.o.f}(\pi\text{-pole LHC}) = 2.4$$

$$\chi^2/\text{d.o.f}(\pi, V\text{-pole LHC}) = 0.9$$

more precise data on  $\sigma_{\gamma\gamma \rightarrow \pi^0\pi^0}$  is highly anticipated (Jlab, Hall D)

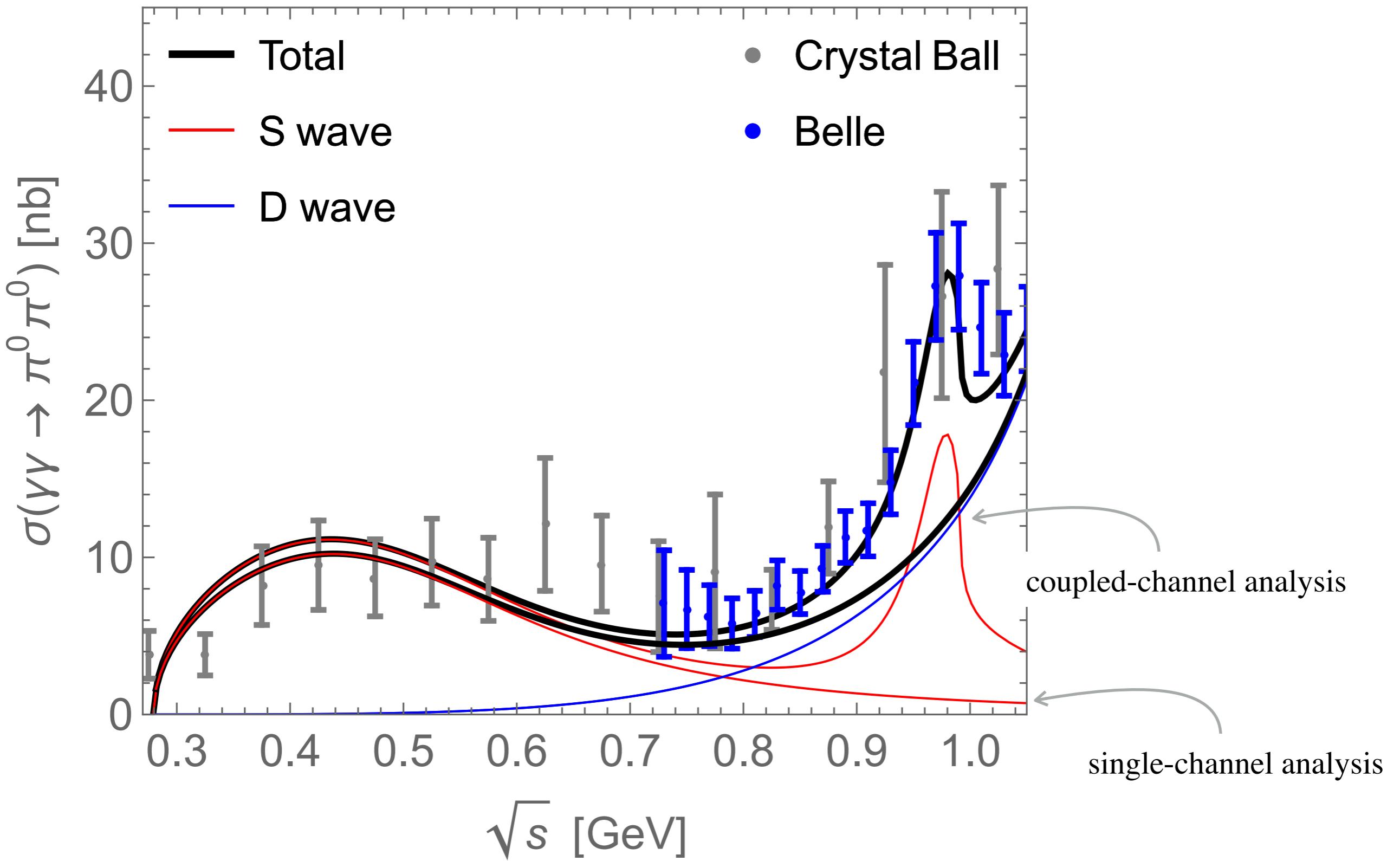


# Summary and outlook

- We presented a data driven analysis of  $\{\pi\pi, K\bar{K}\}$  reactions using **p.w. dispersion relation**
  - for  $f_0(500)$ ,  $f_0(980)$  resonances we obtained consistent results with Roy-like analyses, therefore one can apply it for processes, where no Roy analysis is available
- Obtained coupled-channel  $\{\pi\pi, KK\}$  Omnes matrix has already been implemented in the analysis of  $\gamma^*\gamma^* \rightarrow \{\pi\pi, K\bar{K}\}$  and consequently the contribution of  $f_0(500)$ ,  $f_0(980)$  to  $(g-2)_\mu$  was calculated (re-calculated) based on pion-pole (kaon-pole) LHC approximation
  - The extraction of pion polarizabilities requires adding at least **vector left-hand cuts** and fitting unknown subtraction constants to the experimental data (Adler zero constraint helps to reduce the number of unknown parameters).
  - For single-virtual case, one can fix subtraction constants by fitting future BESIII data on  $\gamma\gamma^* \rightarrow \{\pi^+\pi^-, \pi^0\pi^0\}$  in the range  $0.2 < Q^2 < 2.2 \text{ GeV}^2$

Thank you!

# Spares



Omnès function:  $\text{Disc } \Omega(s) = t(s)\rho(s)\Omega^*(s), \quad s \geq s_{\text{thr}}$

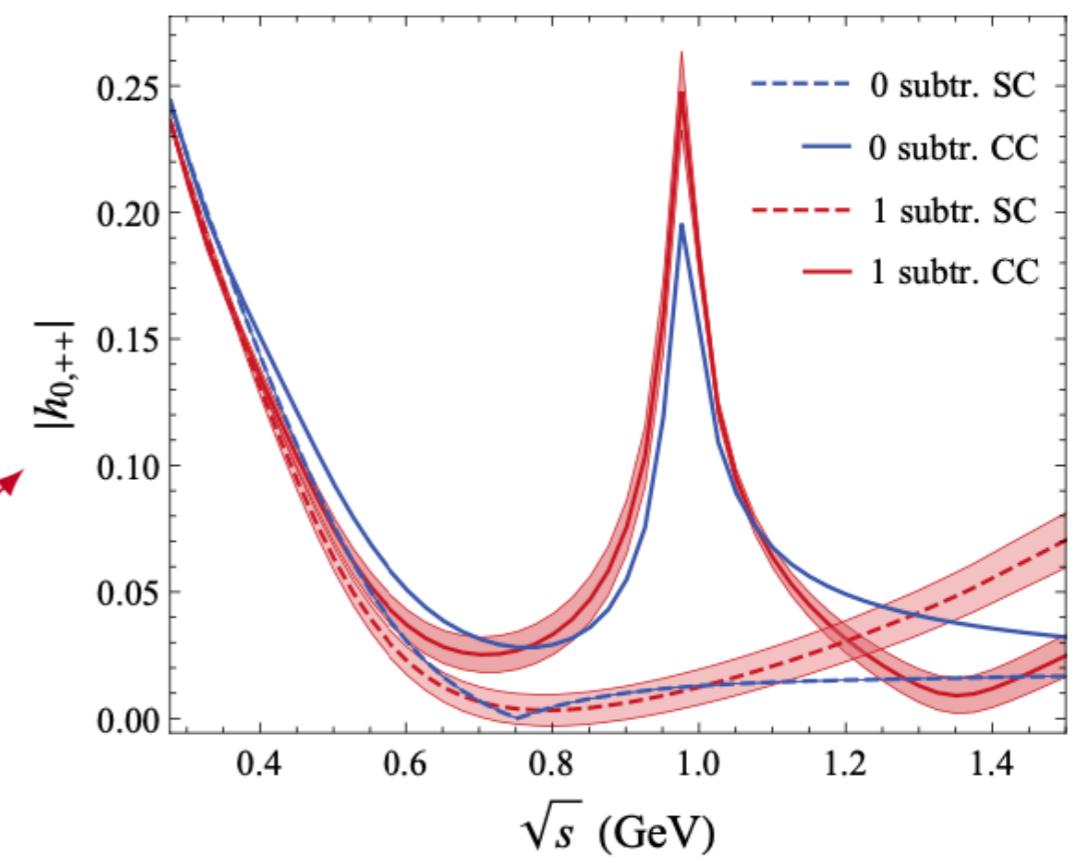
$$\Omega_J(s) = \begin{pmatrix} \Omega_J(s)_{\pi\pi \rightarrow \pi\pi} & \Omega_J(s)_{\pi\pi \rightarrow K\bar{K}} \\ \Omega_J(s)_{K\bar{K} \rightarrow \pi\pi} & \Omega_J(s)_{K\bar{K} \rightarrow K\bar{K}} \end{pmatrix}$$

**Single channel:**  $\gamma\gamma \rightarrow \pi\pi, \quad I = 0, 2, \text{ D-wave}$

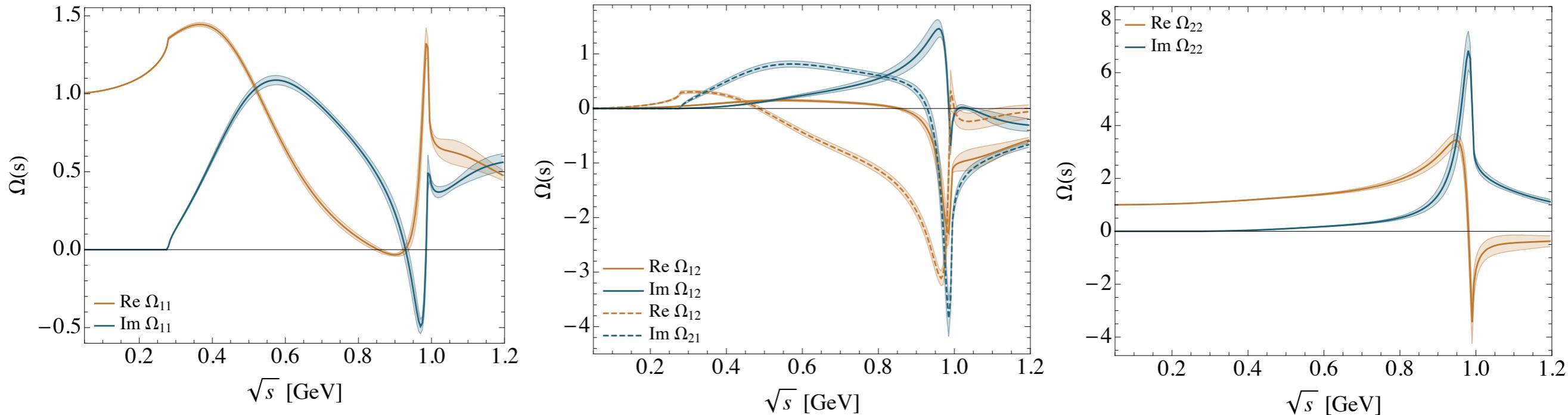
$\gamma\gamma \rightarrow \pi\pi, \quad I = 2, \text{ S-wave}$

**Coupled channel:**  $\gamma\gamma \rightarrow \pi\pi, \quad I = 0, \text{ S-wave}$

$\gamma\gamma \rightarrow \pi\eta, \quad I = 1, \text{ S-wave}$



# Omnes matrix $\{\pi\pi, KK\}$



Omnes function fulfils the unitarity relation on the right-hand cut and analytic everywhere else.  
For the case of no bound states or CDD poles:

$$\Omega_{ab}(s) = D_{ab}^{-1}(s)$$

which automatically satisfies a once-subtracted dispersion relation (i.e.  $\Omega(s)$  is asymptotically bounded)

$$\Omega_{ab}(s) = \delta_{ab} + \frac{s}{\pi} \sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'} \frac{t_{ac}^*(s') \rho_c(s') \Omega_{cb}(s')}{s' - s}$$

different from  
 [Donoghue et al. (1990)]  
 [Moussallam (2000)]

# Kinematic constraints

- p.w. helicity amplitudes suffer from kinematic constraints

$$h_{\lambda_1 \lambda_2}^{(J)} = \int \frac{d \cos \theta}{2} d_{\lambda_1 - \lambda_2, 0}^J(\theta) H_{\lambda_1 \lambda_2}$$

- Helicity amplitudes

$$H_{\lambda_1, \lambda_2} = \epsilon_\mu(\lambda_1) \epsilon_\nu(\lambda_2) \sum_{n=1}^5 F_n(s, t) \underbrace{L_n^{\mu\nu}}_{H^{\mu\nu}}$$

Bardeen et al. (1968), Tarrach (1975)  
 Metz et al. (1998), Colangelo et al. (2015)

- Unconstrained basis for Born subtracted p.w. amplitudes  $\bar{h}_i^{(J)} \equiv h_i^{(J)} - h_i^{(J), \text{Born}}$   
 For S-wave

$$\bar{h}_{++}^{(0)} \pm \bar{h}_{00}^{(0)} \sim (s - s_{\text{kin}}^{(\mp)}), \quad s_{\text{kin}}^{(\pm)} \equiv -(Q_1 \pm Q_2)^2$$

$$\bar{h}_{i=1,2}^{(0)}(s) = \frac{\bar{h}_{++}^{(0)}(s) \pm \bar{h}_{00}^{(0)}(s)}{s - s_{\text{kin}}^{(\mp)}}$$

Colangelo et al. (2017)  
 Hoferichter, Stoffer (2019)  
 I.D., Deineka, Vanderhaeghen (2019)

For D-wave

$$\bar{h}_i^{(J)} = K_{ij} \bar{h}_j^{(J)} \quad j \equiv \lambda_1 \lambda_2 = \{++, +- , +0, 0+, 00\}$$

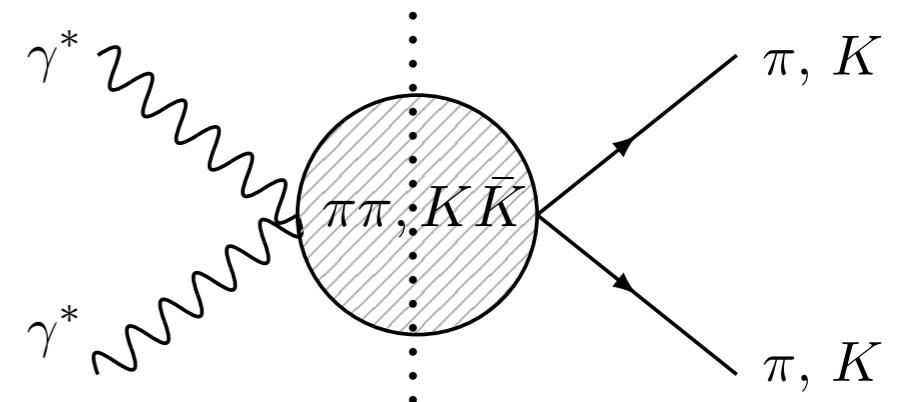
$K_{ij}$  is  $5 \times 5$  matrix

# Dispersion relation

- Unsubtracted dispersion relation for kinematically unconstrained p.w. amplitudes

$$\bar{h}_i^{(J)} = \int_{-\infty}^0 \frac{ds'}{\pi} \frac{\text{Disc } \bar{h}_i^{(J)}(s')}{s' - s} + \int_{4m_\pi^2}^\infty \frac{ds'}{\pi} \frac{\text{Disc } \bar{h}_i^{(J)}(s')}{s' - s}$$

$$\bar{h}_i^{(J)} \equiv h_i^{(J)} - h_i^{(J),\text{Born}}$$



which can be solved using modified MO method, i.e. by writing a dispersion relation for  $\Omega^{(J)}(s)^{-1} \bar{h}_i^{(J)}(s)$

- For S-wave, I=0

$$\begin{pmatrix} \bar{h}_i^{(0)}(s) \\ \bar{k}_i^{(0)}(s) \end{pmatrix} = \Omega^{(0)}(s) \left[ \int_{-\infty}^0 \frac{ds'}{\pi} \frac{\Omega^{(0)}(s')^{-1}}{s' - s} \begin{pmatrix} \text{Disc } \bar{h}_i^{(0)}(s') \\ \text{Disc } \bar{k}_i^{(0)}(s') \end{pmatrix} - \int_{4m_\pi^2}^\infty \frac{ds'}{\pi} \frac{\text{Disc } \Omega^{(0)}(s')^{-1}}{s' - s} \begin{pmatrix} h_i^{(0),\text{Born}}(s') \\ k_i^{(0),\text{Born}}(s') \end{pmatrix} \right]$$

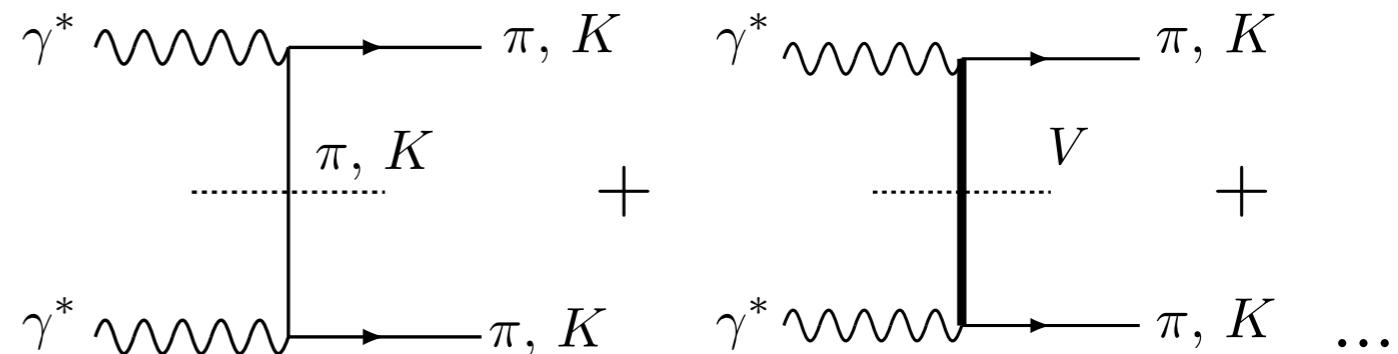
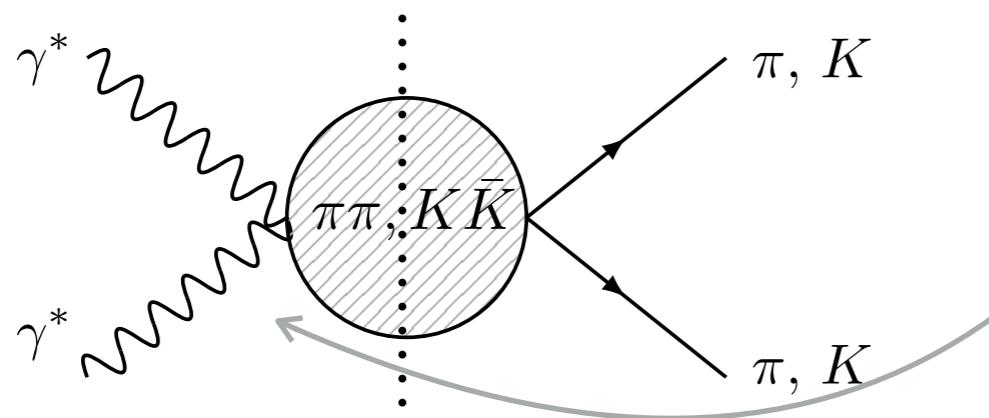
$\gamma^*\gamma^*\rightarrow\pi\pi$

$\gamma^*\gamma^*\rightarrow KK$

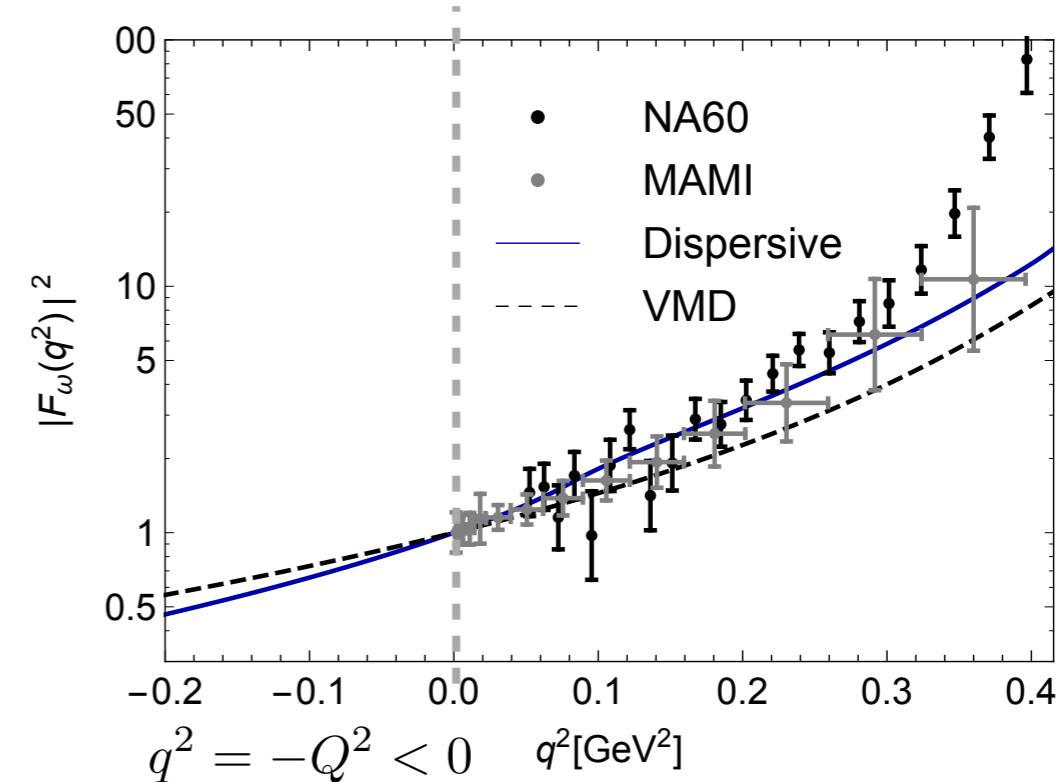
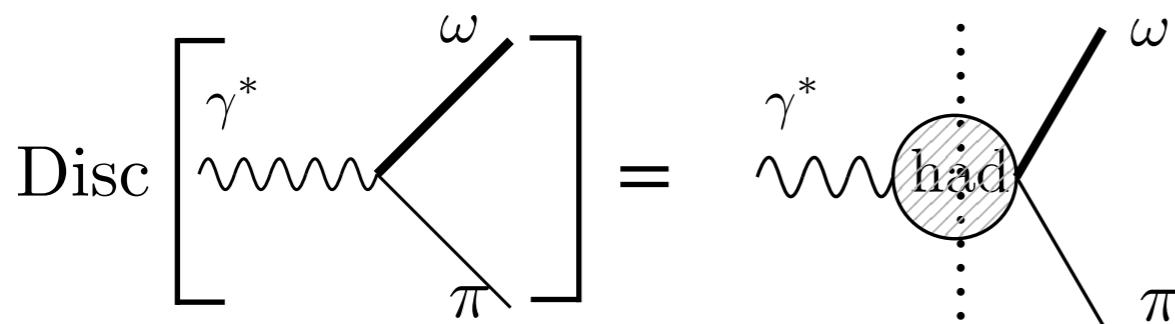
V-exch

Omnès (1958)  
Muskhelishvili (1953)  
Garcia-Martin et. al (2010)  
Hoferichter et. al. (2011,19)  
Dai et al. (2014)  
Moussallam (2013)

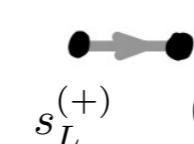
# Left-hand cuts (vector poles)



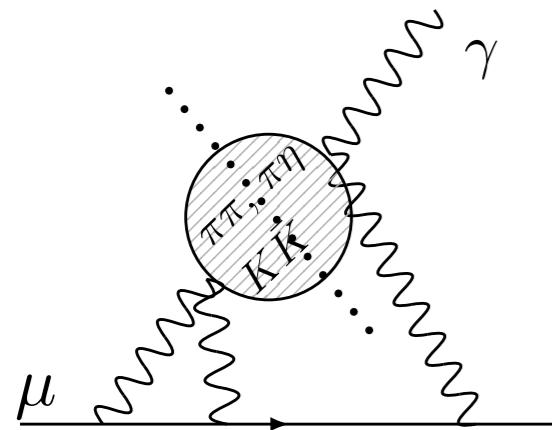
- Left-hand cuts requires knowledge from  $\gamma^*\pi\pi, \gamma^*KK; \gamma^*\pi V, \gamma^*KV$  form factors



- Left-hand cuts: “anomalous thresholds” for large virtualities  $Q_1^2 Q_2^2 > (M_V^2 - m_\pi^2)^2$



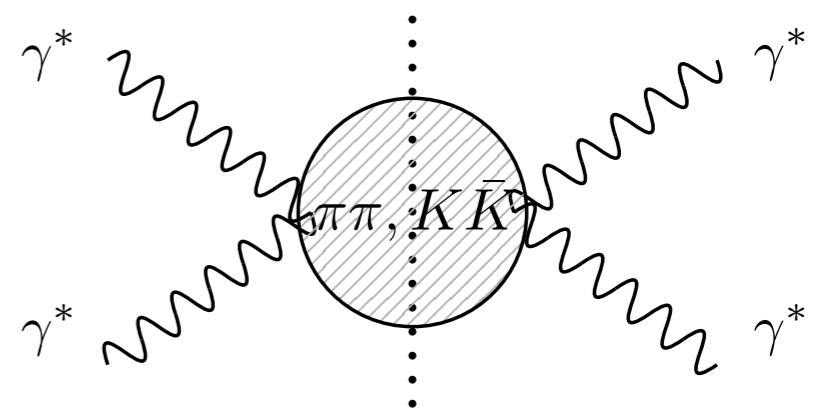
# Contribution to (g-2)



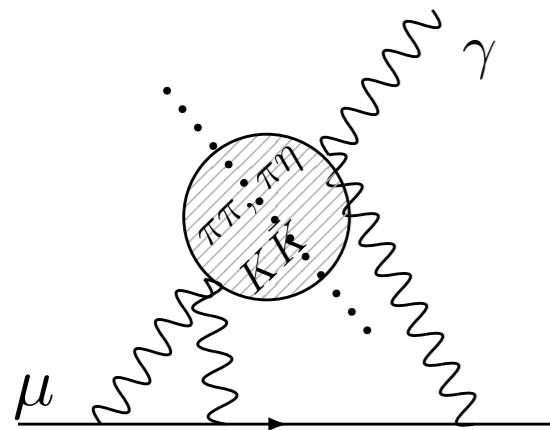
Important ingredients:

$$\gamma^* \gamma^* \rightarrow \pi\pi, K\bar{K}, \dots$$

$$q^2 = -Q^2 < 0 \quad \text{space-like } \gamma^*$$



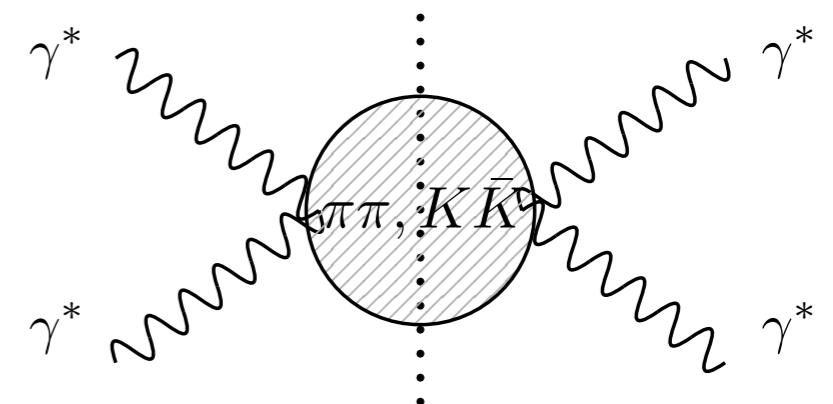
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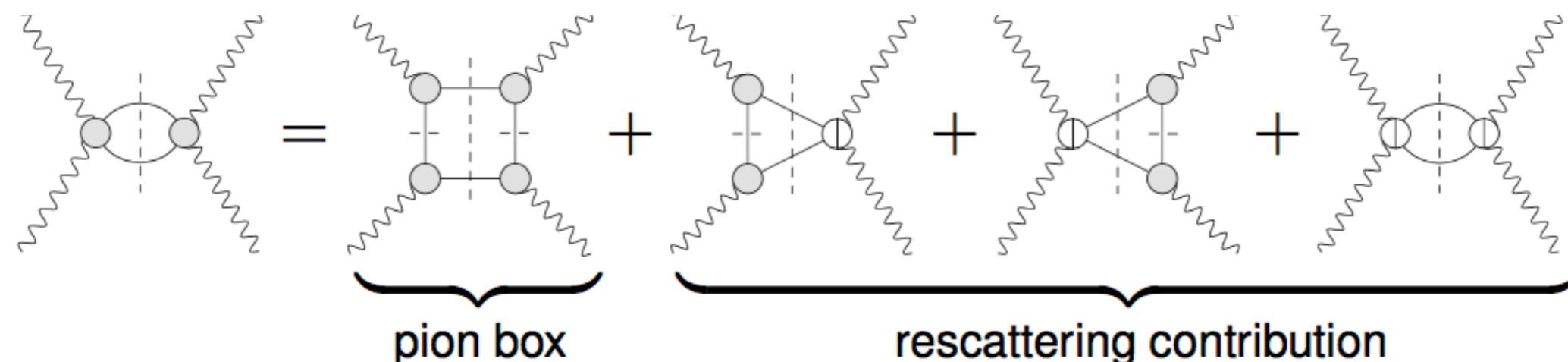
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$$a_\mu^{\text{HLbL}} = \frac{2\alpha^3}{3\pi^2} \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2} Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1, Q_2, Q_3) \bar{\Pi}_i(Q_1, Q_2, Q_3),$$

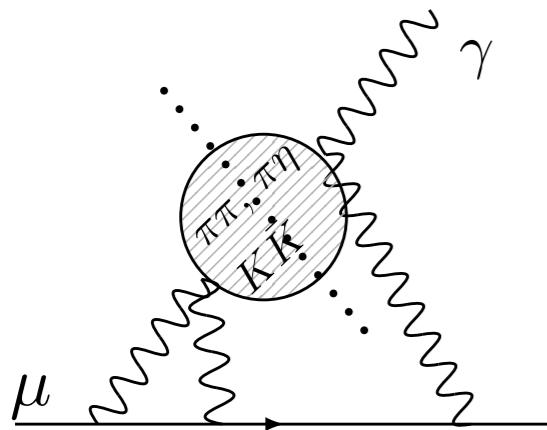
[Colangelo et al. (2014-2017)]

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i \quad \bar{\Pi}_i \text{ linear combination of } \Pi_i$$



$$a_\mu^{\pi\pi, KK} [\text{box}] = -16.4(2) \times 10^{-11}$$

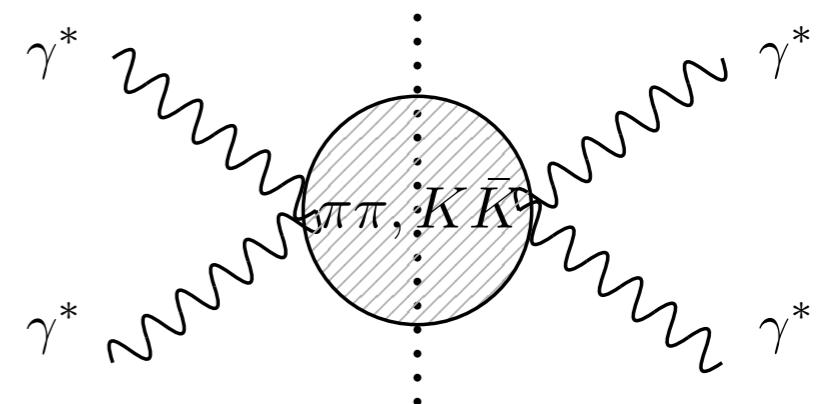
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Rescattering contribution ( $\bar{h} \equiv h - h^{\text{Born}}$ ) in the S-wave

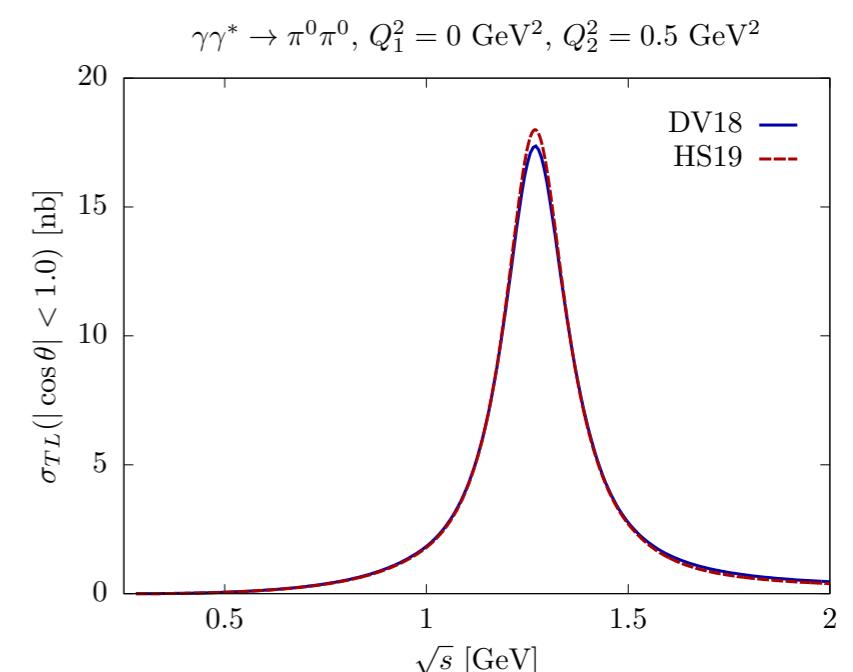
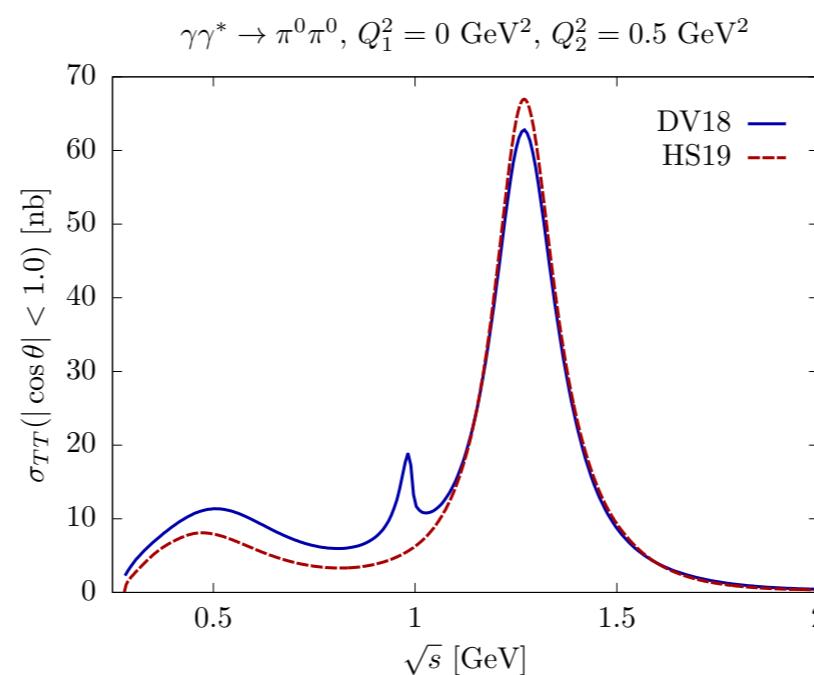
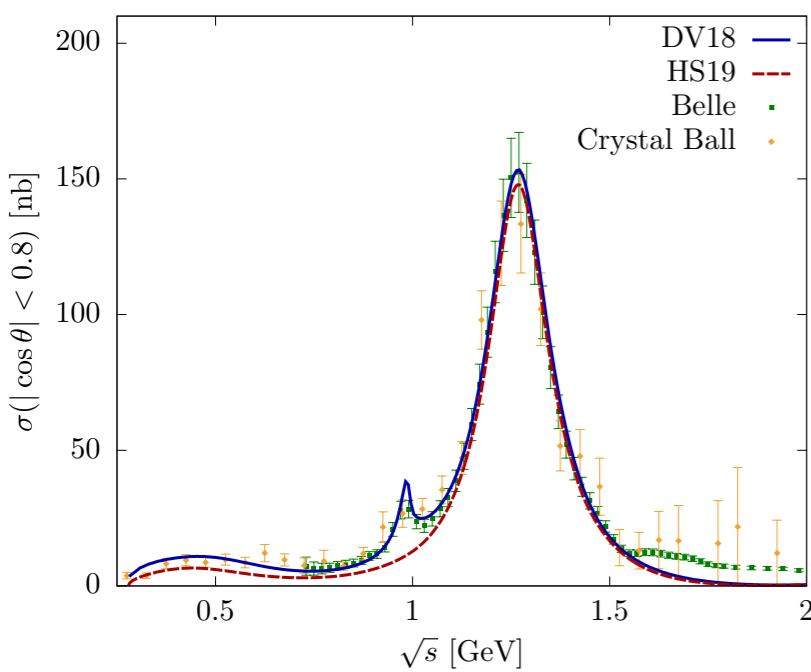
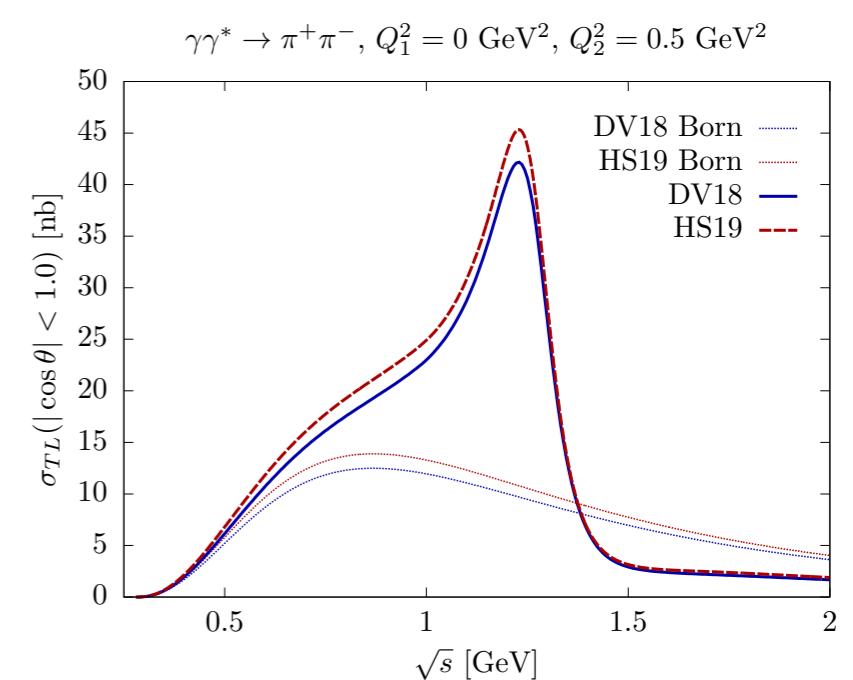
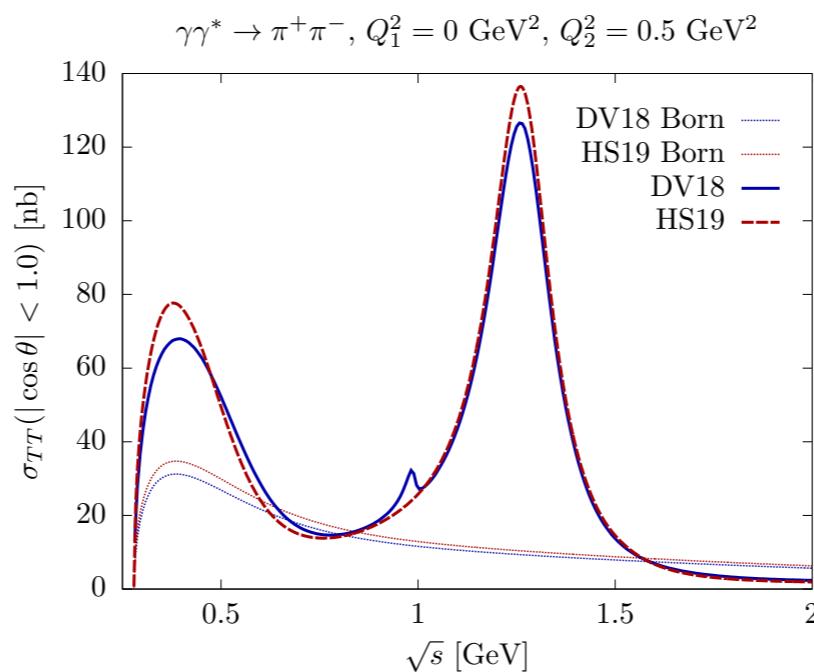
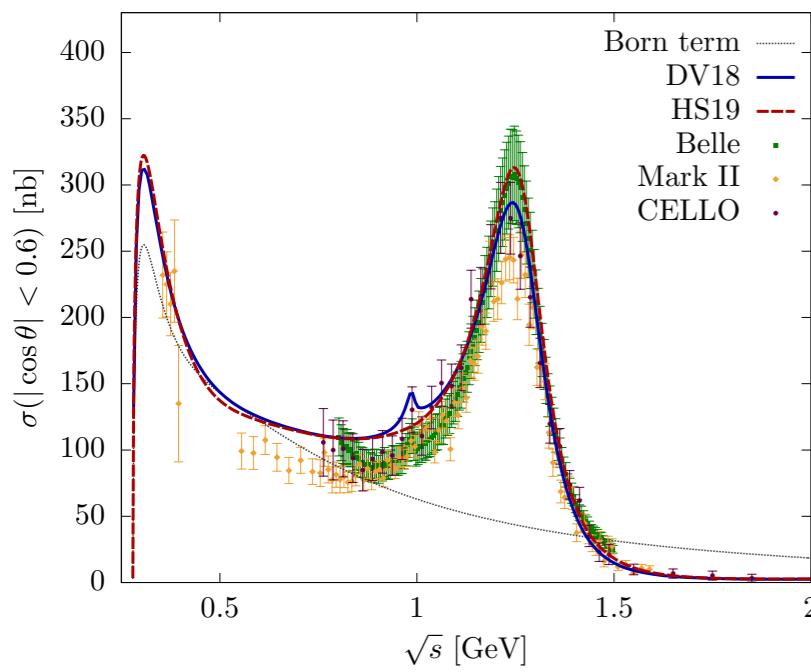
$$\begin{aligned} \bar{\Pi}_3^{J=0} &= \frac{1}{\pi} \int_{4m_\pi^2}^\infty ds' \frac{-2}{\lambda_{12}(s')(s'+Q_3^2)^2} \left( 4s' \text{Im} \bar{h}_{++,++}^{(0)}(s') - (s' - Q_1^2 + Q_2^2)(s' + Q_1^2 - Q_2^2) \text{Im} \bar{h}_{00,++}^{(0)}(s') \right) \\ \bar{\Pi}_9^{J=0} &= \frac{1}{\pi} \int_{4m_\pi^2}^\infty ds' \frac{4}{\lambda_{12}(s')(s'+Q_3^2)^2} \left( 2 \text{Im} \bar{h}_{++,++}^{(0)}(s') - (s' + Q_1^2 + Q_2^2) \text{Im} \bar{h}_{00,++}^{(0)}(s') \right) \end{aligned}$$

+crossed

Unitarity  $\gamma^* \gamma^* \rightarrow \gamma^* \gamma^*$

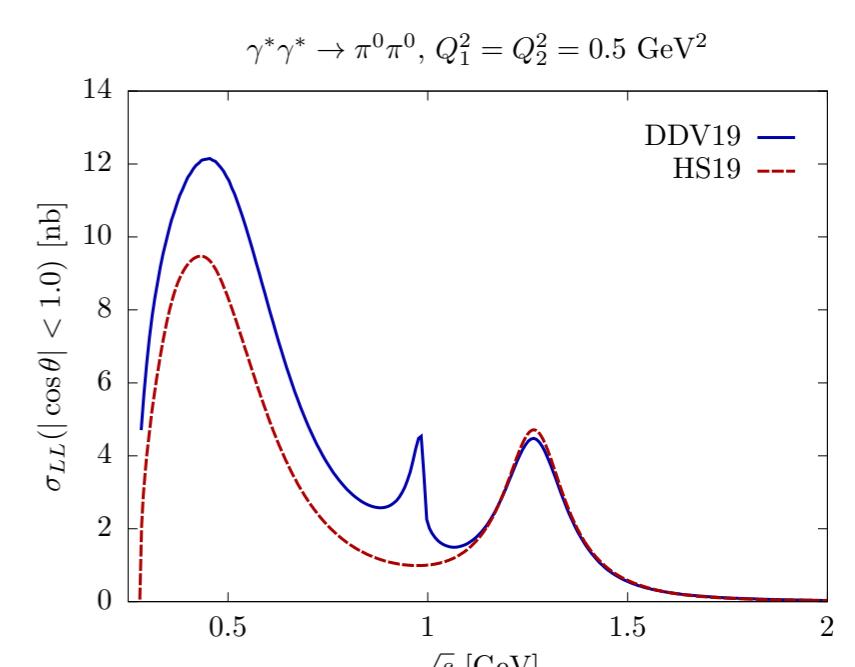
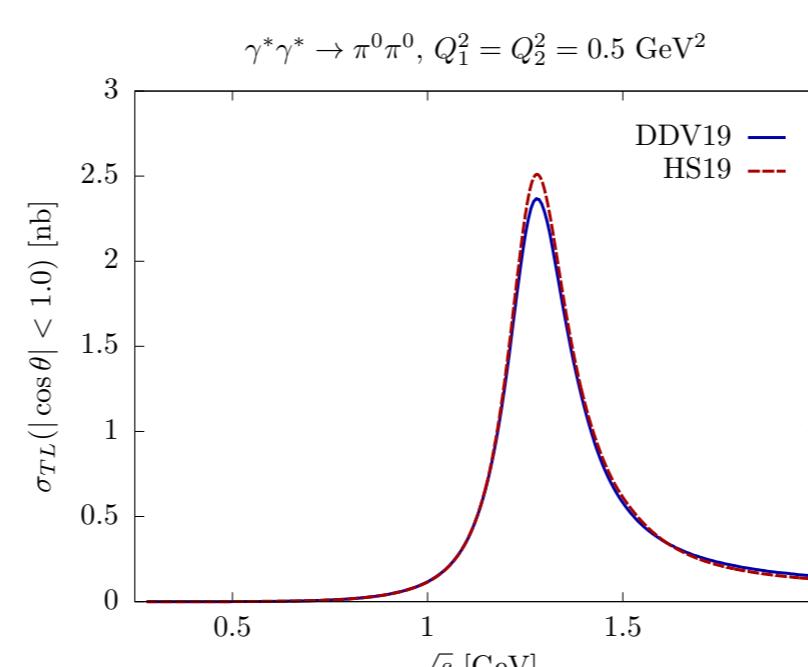
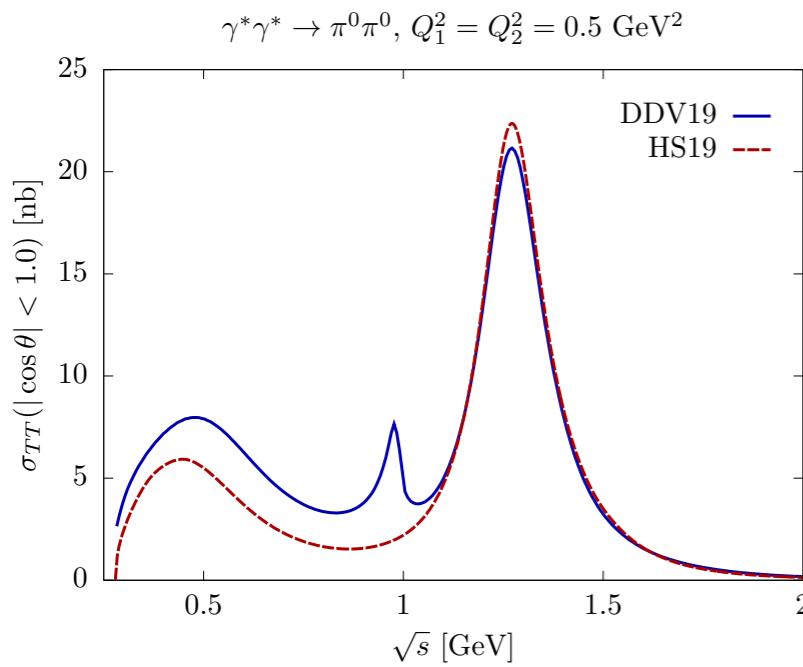
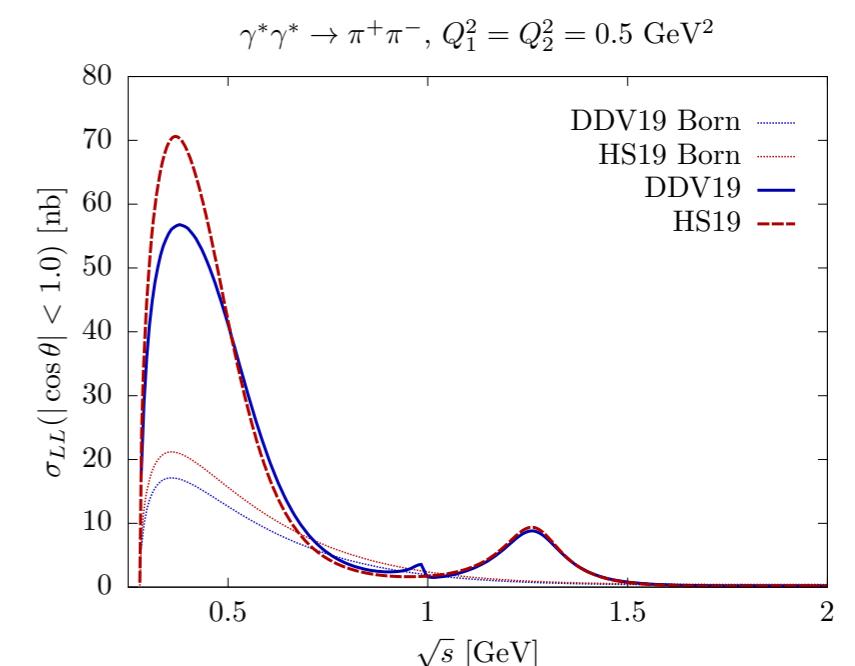
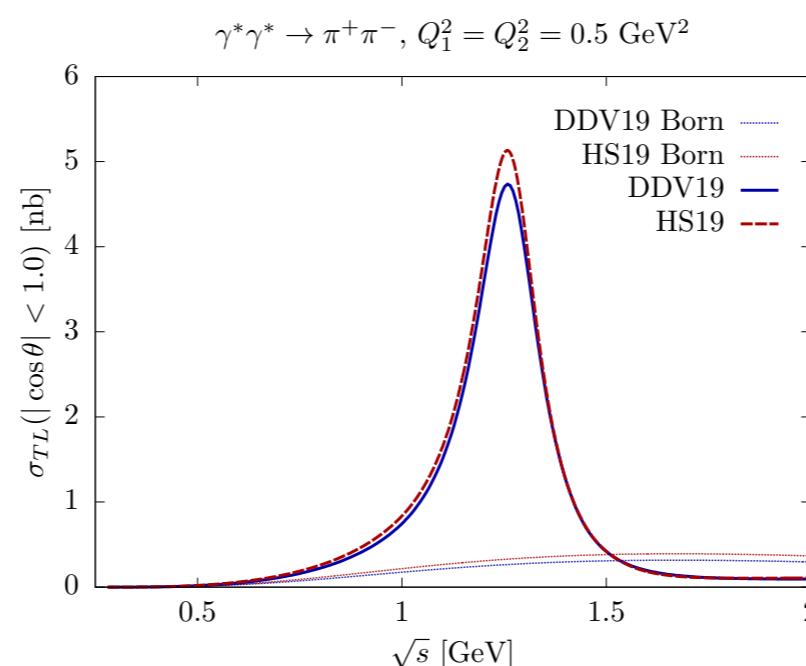
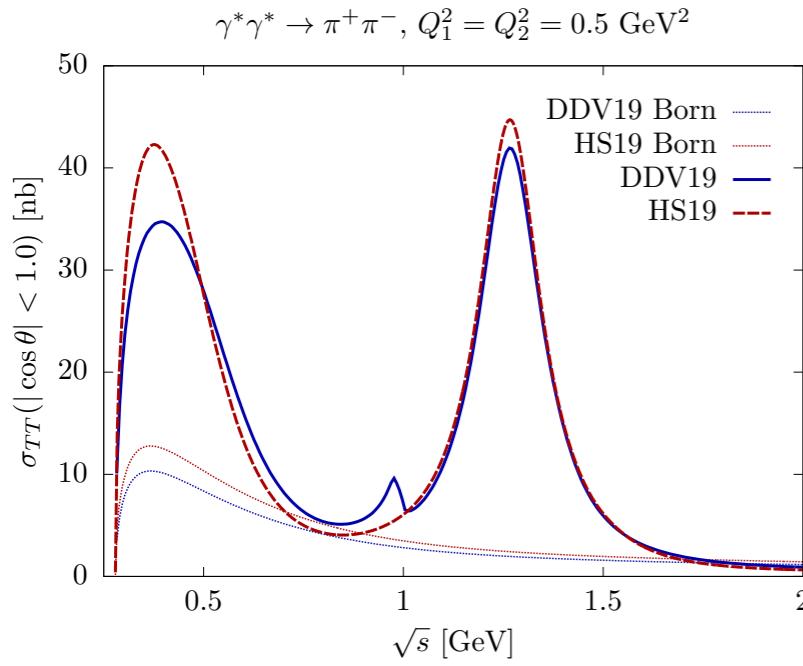
$$\text{Im} \bar{h}_{\lambda_1 \lambda_2, \lambda_3 \lambda_4}^{(0)}(s) = \frac{1}{2} \bar{h}_{\lambda_1 \lambda_2}^{(0)}(s) \rho_\pi(s) \bar{h}_{\lambda_3 \lambda_4}^{(0)*}(s) + \frac{1}{2} \bar{k}_{\lambda_1 \lambda_2}^{(0)}(s) \rho_K(s) \bar{k}_{\lambda_3 \lambda_4}^{(0)*}(s)$$

# Results for $\pi\pi$



I.D., Vanderhaeghen (2018)  
Hoferichter, Stoffer (2019)

# Results for $\pi\pi$



$$\frac{d\sigma_{TT}}{d\cos\theta} \sim |H_{++}|^2 + |H_{+-}|^2, \quad \frac{d\sigma_{TL}}{d\cos\theta} \sim |H_{+0}|^2, \quad \frac{d\sigma_{LL}}{d\cos\theta} \sim |H_{00}|^2$$

Hoferichter, Stoffer (2019)  
I.D., Deineka, Vanderhaeghen (2019)

# (g-2)

Contribution	PdRV(09) [471]	N/JN(09) [472, 573]	J(17) [27]	Our estimate
$\pi^0, \eta, \eta'$ -poles	114(13)	99(16)	95.45(12.40)	93.8(4.0)
$\pi, K$ -loops/boxes	-19(19)	-19(13)	-20(5)	-16.4(2)
$S$ -wave $\pi\pi$ rescattering	-7(7)	-7(2)	-5.98(1.20)	-8(1)
subtotal	88(24)	73(21)	69.5(13.4)	69.4(4.1)
scalars	—	—	—	— 1(3)
tensors	—	—	1.1(1)	
axial vectors	15(10)	22(5)	7.55(2.71)	6(6)
$u, d, s$ -loops / short-distance	—	21(3)	20(4)	15(10)
$c$ -loop	2.3	—	2.3(2)	3(1)
total	105(26)	116(39)	100.4(28.2)	92(19)

Table 15: Comparison of two frequently used compilations for HLbL in units of  $10^{-11}$  from 2009 and a recent update with our estimate. Legend: PdRV = Prades, de Rafael, Vainshtein (“Glasgow consensus”); N/JN = Nyffeler / Jegerlehner, Nyffeler; J = Jegerlehner.