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Perturbative verification of new sum rules and relations for Compton scattering

[arXiv:2305.08814]

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in collaboration with
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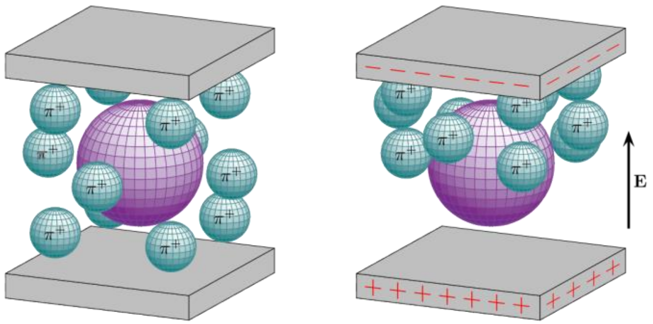
II low-Q Workshop, Kolymbari, Crete, Greece, 2023

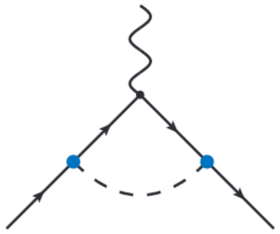
Outline

- Introduction: proton dipole polarizabilities and forward doubly-virtual Compton scattering
- Bernabéu-Tarrach (BT) sum rule: the data-driven way to access the nucleon electric dipole polarizability α_{E1}
  - Perturbative validation of BT sum rule in baryon ChPT
  - Evaluation of α_{E1} for the proton via BT sum rule with existing data
  - The data-driven approach for the subtraction function in the proton polarizability effect in muonic hydrogen
- The Schwinger sum rule validation in ultraviolet-complete theories
- Conclusions and outlook

[talks by M. Vanderhaeghen, C. Alexandrou and others]

$$\vec{P} = \alpha_{E1}\,\vec{E}$$





Proton dipole polarizabilities: theory vs. experiment

χEFT fit

[McGovern et al., EPJA (2013)]
[Griesshammer et al., EPJA (2016)]

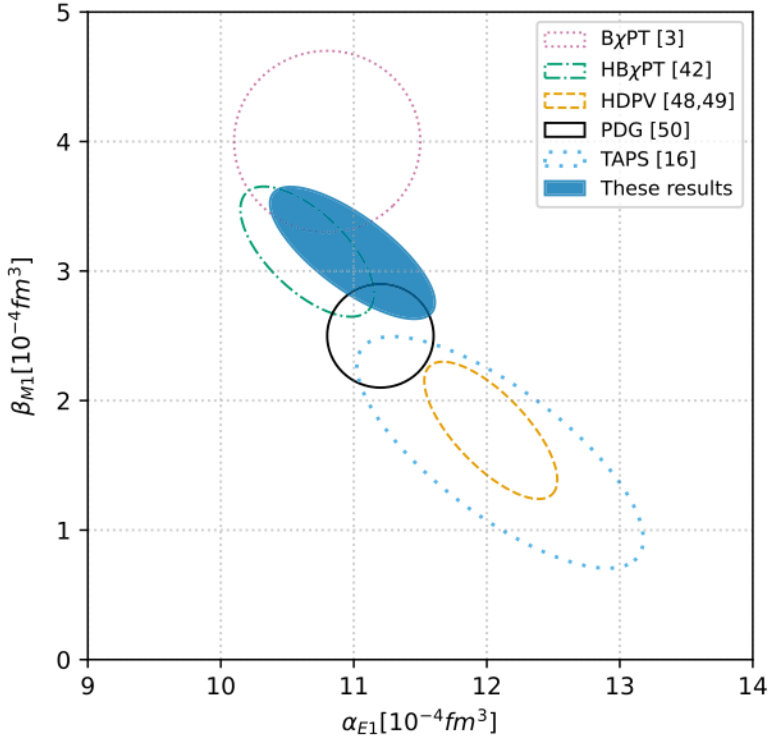
$$\alpha_{E1} = 10.75(47)$$
$$\beta_{M1} = 3.25(47)$$

χEFT prediction

[Lensky and Pascalutsa, EPJC (2010), (2015)]

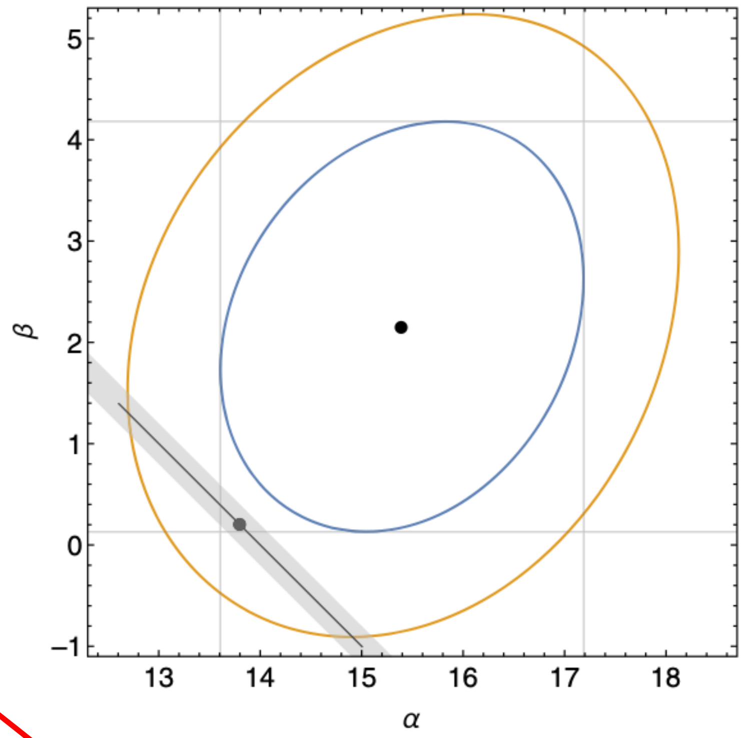
$$\alpha_{E1} = 10.8(0.7)$$
$$\beta_{M1} = 3.9(0.7)$$

[A2 at MAMI, PRL (2022)]



$$\alpha_{E1} = 10.99(63)$$
$$\beta_{M1} = 3.14(51)$$

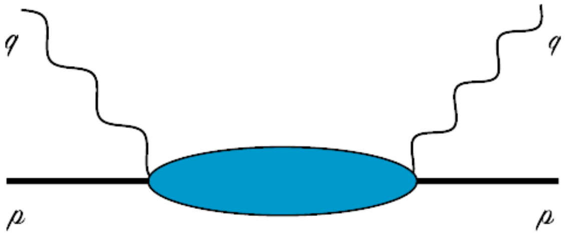
[HIγS, PRL (2022)]



$$\alpha_{E1} = 13.8(1.2)$$
$$\beta_{M1} = 0.2(1.2)$$

What is the reason of the tension between MAMI and HIγS results?

Forward doubly-virtual Compton scattering: spin-independent part



$$T^{\mu\nu}(q,p) = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) T_1(\nu,Q^2) + \frac{1}{M^2}\left(p^\mu - \frac{p\cdot q}{q^2}\, q^\mu\right)\left(p^\nu - \frac{p\cdot q}{q^2}\, q^\nu\right) T_2(\nu,Q^2)$$

spin-independent

$$-\frac{1}{M}\gamma^{\mu\nu\alpha} q_\alpha\, S_1(\nu,Q^2) - \frac{1}{M^2}\left(\gamma^{\mu\nu} q^2 + q^\mu \gamma^{\nu\alpha} q_\alpha - q^\nu \gamma^{\mu\alpha} q_\alpha\right) S_2(\nu,Q^2),$$

spin-dependent

- Unitarity:

$$\mathrm{Im} T_1(\nu,Q^2) = \nu\, \sigma_T(\nu,Q^2)$$

$$\mathrm{Im} T_2(\nu,Q^2) = \frac{Q^2 \nu}{\nu^2 + Q^2}\left[\sigma_T + \sigma_L\right](\nu,Q^2)$$

- Analyticity, crossing symmetry:

$$T_i(\nu,Q^2) = \frac{2}{\pi}\int_{\nu_{\mathrm{el}}}^{\infty} d\nu' \,\frac{\nu' \mathrm{Im} T_i(\nu',Q^2)}{\nu'^2 - \nu^2 - i0^+}$$

- Low-energy theorems:

$$\frac{1}{4\pi}\left[T_1 - T_1^{\mathrm{pole}}\right](\nu,Q^2) = -\frac{Z^2\alpha_{\mathrm{em}}}{M} + \left(\frac{Z\alpha_{\mathrm{em}}}{3M}\langle r^2\rangle_1 + \beta_{M1}\right) Q^2 + (\alpha_{E1} + \beta_{M1})\,\nu^2 + \dots$$

$$\frac{1}{4\pi}\left[T_2 - T_2^{\mathrm{pole}}\right](\nu,Q^2) = (\alpha_{E1} + \beta_{M1})\, Q^2 + \dots$$

Sum rules for spin-independent polarizabilities

The dispersion relation for the T_1 amplitude: $$T_1(\nu, Q^2) = T(0, Q^2) + \frac{2\nu^2}{\pi}\int_{\nu_0}^{\infty} \mathrm{d}\nu' \, \frac{\sigma_T(\nu', Q^2)}{\nu'^2 - \nu^2}$$

The low-energy expansion of the T_1 amplitude:

$$\frac{1}{4\pi}\left[T_1 - T_1^{\mathrm{pole}}\right](\nu, Q^2) = -\frac{Z^2\alpha_{\mathrm{em}}}{M} + \left(\frac{Z\alpha_{\mathrm{em}}}{3M}\langle r^2\rangle_1 + \beta_{M1}\right)Q^2 + (\alpha_{E1} + \beta_{M1})\,\nu^2 + \ldots$$

Thomson term

[Sucher, (1972)]
[Bernabéu and Tarrach, (1975)]
[L'vov, (1998)]

It is impossible to write the unsubtracted sum rule for this term: dispersion relation requires a subtraction at each value of Q^2.

Baldin sum rule: [Baldin, (1960)]

$$\alpha_{E1} + \beta_{M1} = \frac{1}{2\pi^2}\int_0^{\infty} \mathrm{d}\nu \frac{\sigma_T(\nu)}{\nu^2}$$

- a powerful tool for data-driven evaluation of the sum of the electric and magnetic nucleon polarizabilities.

Don't we have the sum rule for each separate polarizability?

Bernabéu-Tarrach sum rule

The Compton helicity amplitude with two longitudinally polarized photons:

$$T_L(\nu, Q^2) = \left(1 - \frac{\nu^2}{Q^2}\right) T_2(\nu, Q^2) - T_1(\nu, Q^2) = \frac{2}{\pi} \int_{\nu_0}^{\infty} d\nu' \, \nu'^2 \, \frac{\sigma_L(\nu', Q^2)}{\nu'^2 - \nu^2}$$

$$T_L(\text{non-pole Born}) = -\frac{\pi \alpha_{\text{em}} Q^2}{M^3} F_2^2(Q^2) \qquad T_L(\text{non-Born}) = 4\pi Q^2 \alpha_{E1} + O(\nu^2, Q^4)$$

Bernabéu-Tarrach sum rule:

$$\boxed{\alpha_{E1} - \frac{\alpha_{\text{em}} \varkappa^2}{4M^3} = \frac{1}{2\pi^2} \int_0^{\infty} \mathrm{d}\nu \left[\frac{\sigma_L(\nu, Q^2)}{Q^2}\right]_{Q^2 \to 0}}$$

[Bernabéu and Tarrach, PLB (1975)]

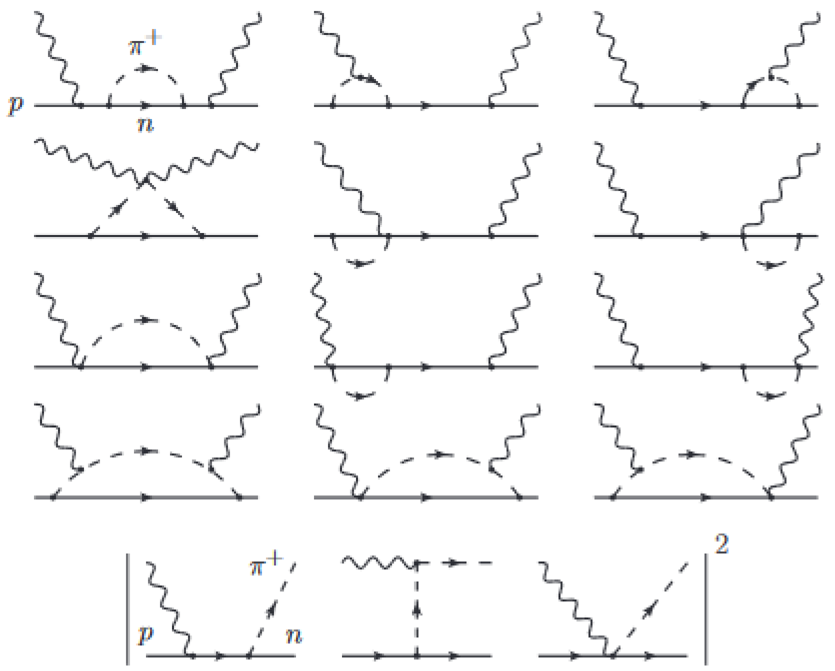
$\varkappa$ is the anomalous magnetic moment of the nucleon.

Bernabéu-Tarrach sum rule: validation

- [Llanta and Tarrach, PLB (1978)]: the sum rule, albeit convergent, does not hold in QED. The r.h.s differs from l.h.s. by a constant.

- [L'vov, NPA (1998)]: the sum rule is in general invalid and should not converge without the subtraction; it is violated for the (negative) pion electric polarizability in pure σ-model.

We have found the case when this sum rule holds exactly!

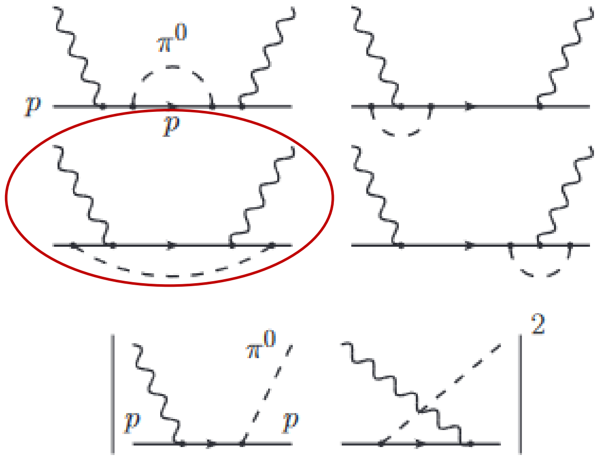


The sum rule is validated in the manifestly covariant baryon χPT for the $O(p^3)$ contribution to the proton electric polarizability that comes from the <u>charged</u> pion loops.

The results of [Bernard et al., PRL (1991), NPB (1992)], were reproduced.

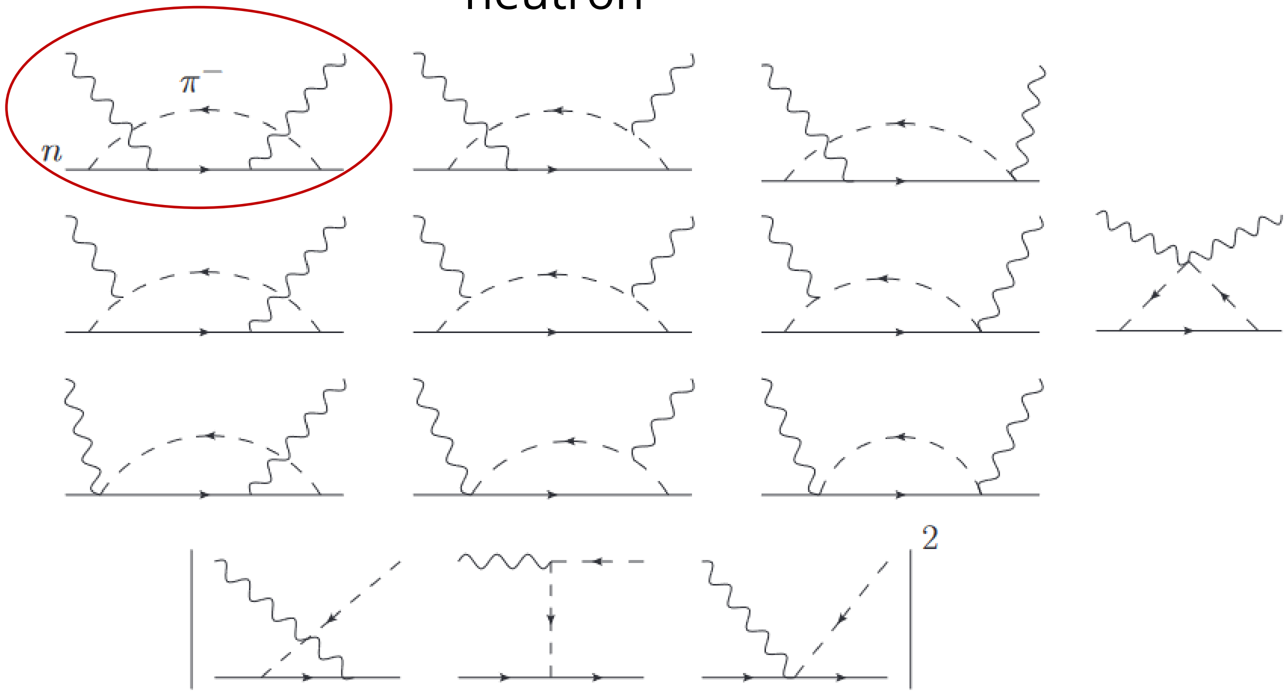
Bernabéu-Tarrach sum rule: (in)validation?

Neutral pion contribution to the Compton scattering off the proton



$$T_L^{\pi^0 p\text{-loops}}(\infty, Q^2) = -\frac{\alpha_{\rm em}}{12\pi}\frac{g_{\pi N}^2}{M^3}Q^2 + O(Q^4)$$

The Compton scattering off the neutron



$$T_L^{\pi^- n\text{-loops}}(\infty, Q^2) = -\frac{\alpha_{\rm em}}{6\pi}\frac{g_{\pi N}^2}{M^3}Q^2 + O(Q^4)$$

In these cases the dispersion relation for T_L must be modified as follows:
[Sugawara and Kanazawa, PhysRev (1961)]

$$T_L(\nu, Q^2) - T_L(\infty, Q^2) = \frac{2}{\pi}\int_{\nu_0}^{\infty} d\nu' \, \nu'^2 \, \frac{\sigma_L(\nu', Q^2)}{\nu'^2 - \nu^2}$$

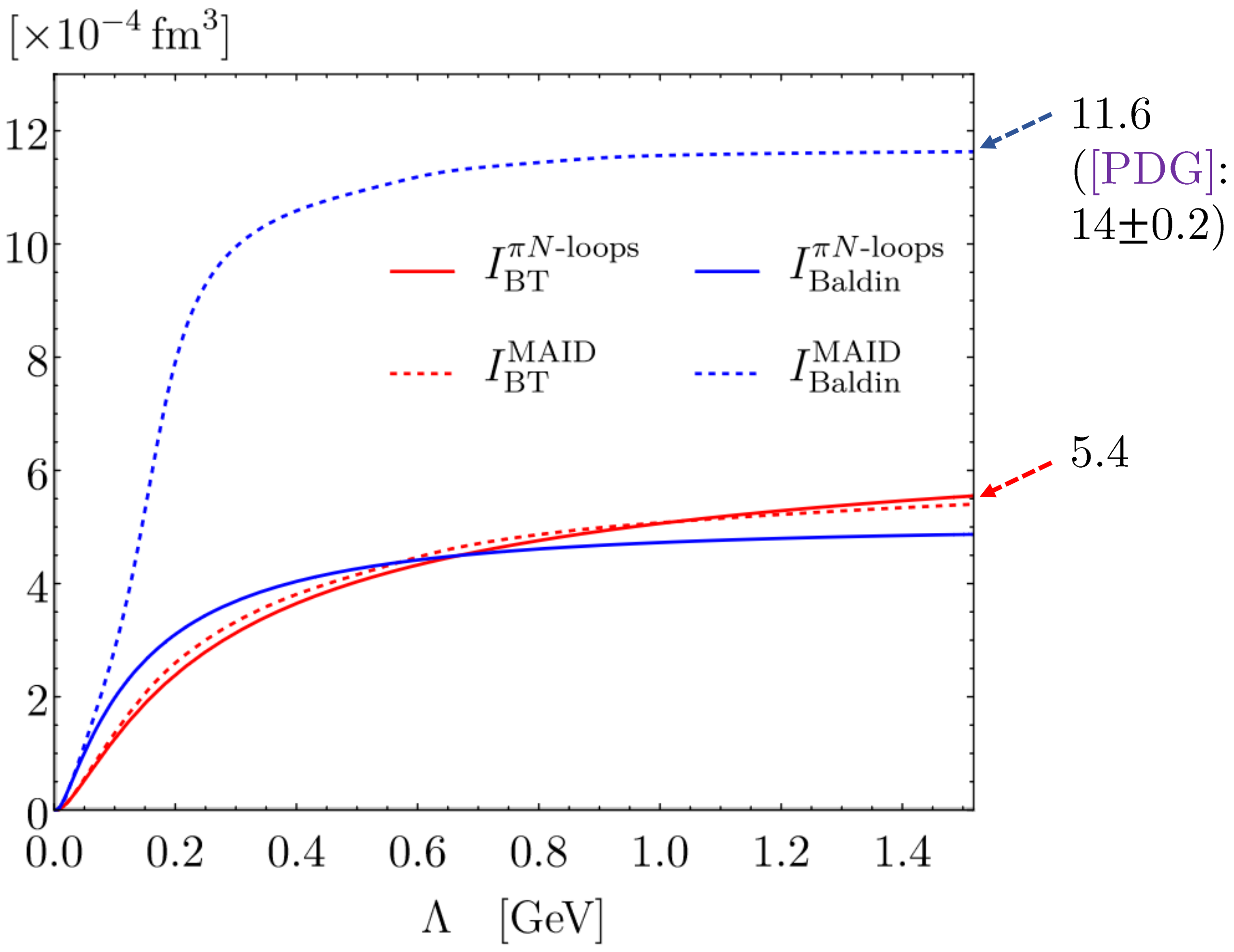
How to deal with asymptotic constants?

Our point: the sum rule is valid if convergent.

- The low-energy physics should not depend on the behavior at very high energies (i.e. physics at the Plank scale)
- The asymptotic constants are the artifacts of the low-energy theory, which is not valid at high energies.
- With proper ultraviolet completion, the theory does not produce the asymptotic constants in the sum rules.

In conclusion, we ought to treat the Bernabéu-Tarrach sum rule as the valid sum rule

Saturation of the sum rule integral



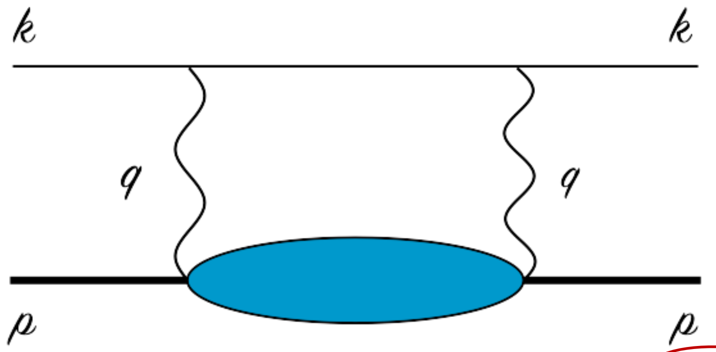
$$I_{\mathrm{BT}}(\Lambda) = \frac{1}{2\pi^2}\int_{\nu_0}^{\Lambda} \mathrm{d}\nu \left[\frac{\sigma_L(\nu,Q^2)}{Q^2}\right]_{Q^2\to 0}$$

$$I_{\mathrm{Baldin}}(\Lambda) = \frac{1}{2\pi^2}\int_{\nu_0}^{\Lambda} \mathrm{d}\nu \frac{\sigma_T(\nu)}{\nu^2}.$$

source	α_{E1} [$\times 10^{-4}$ fm^3]
I_{BT}(MAID) extrapolated	5.4 ≃7
Kappa term	0.5
resonances*	0.5-1*
total (w/o Regge region)	8-8.5*
[PDG]	11.2±0.4

*Currently, we have no parametrization of the existing data, which has a stable behavior within the limit $Q^2 \to 0$

Proton polarizability contribution to the Lamb shift



The two-photon exchange contribution to the Lamb shift in a hydrogen-like atom:

$$\Delta E(nS) = 8\pi\alpha m\,\phi_n^2\,\frac{1}{i}\int_{-\infty}^{\infty}\frac{\mathrm{d}\nu}{2\pi}\int\frac{\mathrm{d}\mathbf{q}}{(2\pi)^3}\,\frac{(Q^2-2\nu^2)\,T_1(\nu,Q^2)-(Q^2+\nu^2)\,T_2(\nu,Q^2)}{Q^4(Q^4-4m^2\nu^2)}$$

$$T_1(\nu,Q^2) = \boxed{T_1(0,Q^2)} + \frac{32\pi Z^2\alpha M\nu^2}{Q^4}\int_0^1 \mathrm{d}x\,\frac{x f_1(x,Q^2)}{1-x^2(\nu/\nu_{\mathrm{el}})^2-i0^+}$$

$$T_2(\nu,Q^2) = \frac{16\pi Z^2\alpha M}{Q^2}\int_0^1 \mathrm{d}x\,\frac{f_2(x,Q^2)}{1-x^2(\nu/\nu_{\mathrm{el}})^2-i0^+}$$

Unknown subtraction function!

- Can be extracted from the dilepton electroproduction on the nucleon
[Pauk, Carlson, Vanderhaeghen, PRC (2020)]

- Can be calculated on a lattice at the subtraction point $\nu = iQ$
[Hagelstein and Pascalutsa, NPA (2021)]

- Can be obtained via another sum rule

Sum rule for the subtraction function

Dispersion relation for T_L:

$$T_L(\nu, Q^2) = \frac{2}{\pi} \int_{\nu_0}^{\infty} \mathrm{d}\nu' \, \nu'^2 \, \frac{\sigma_L(\nu', Q^2)}{\nu'^2 - \nu^2}$$

Siegert theorem:

$$T_1(iQ, Q^2) = -T_L(iQ, Q^2).$$

Dispersion relation for T_1:

$$T_1(\nu, Q^2) = T(0, Q^2) + \frac{2\nu^2}{\pi} \int_{\nu_0}^{\infty} \mathrm{d}\nu' \, \frac{\sigma_T(\nu', Q^2)}{\nu'^2 - \nu^2}$$

$$\boxed{T_1(0, Q^2) = \frac{2}{\pi} Q^2 \int_{\nu_0}^{\infty} \frac{\mathrm{d}\nu}{\nu^2 + Q^2} \left[\sigma_T - \frac{\nu^2}{Q^2} \sigma_L\right](\nu, Q^2)}$$

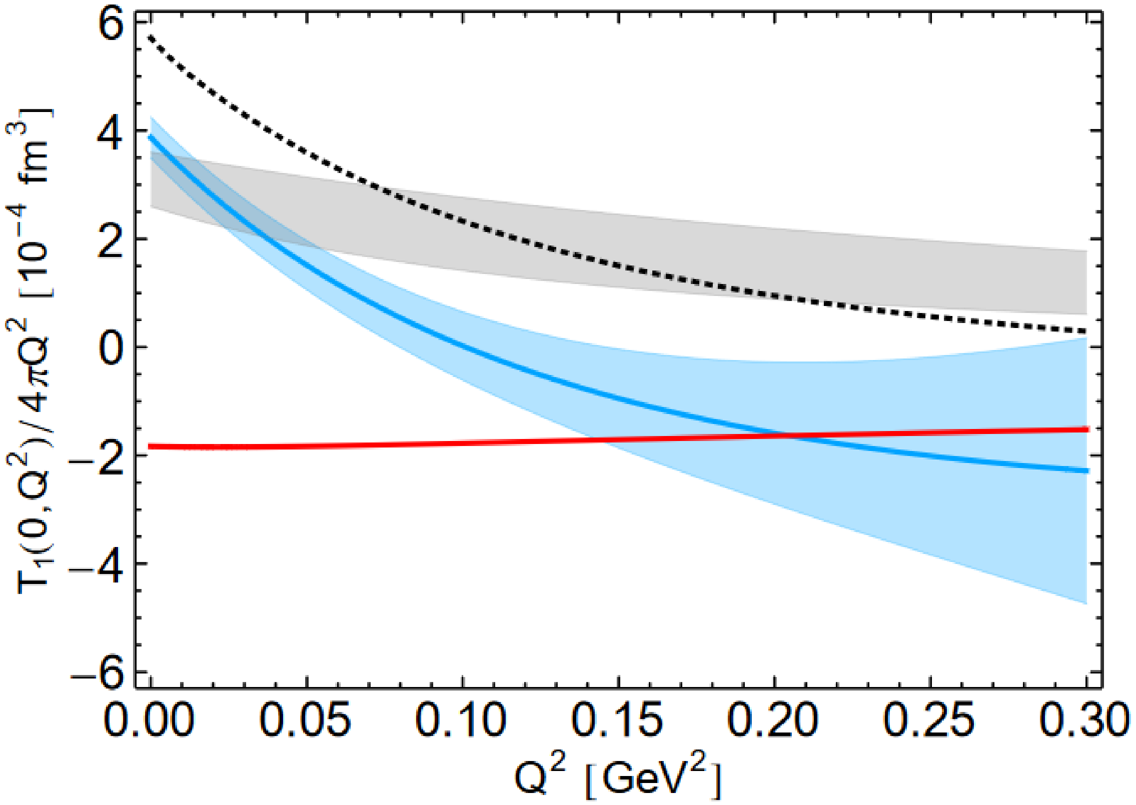
This sum rule is also validated in the manifestly covariant baryon χPT for the $O(p^3)$ contribution to the proton electric polarizability that comes from the charged pion loops.

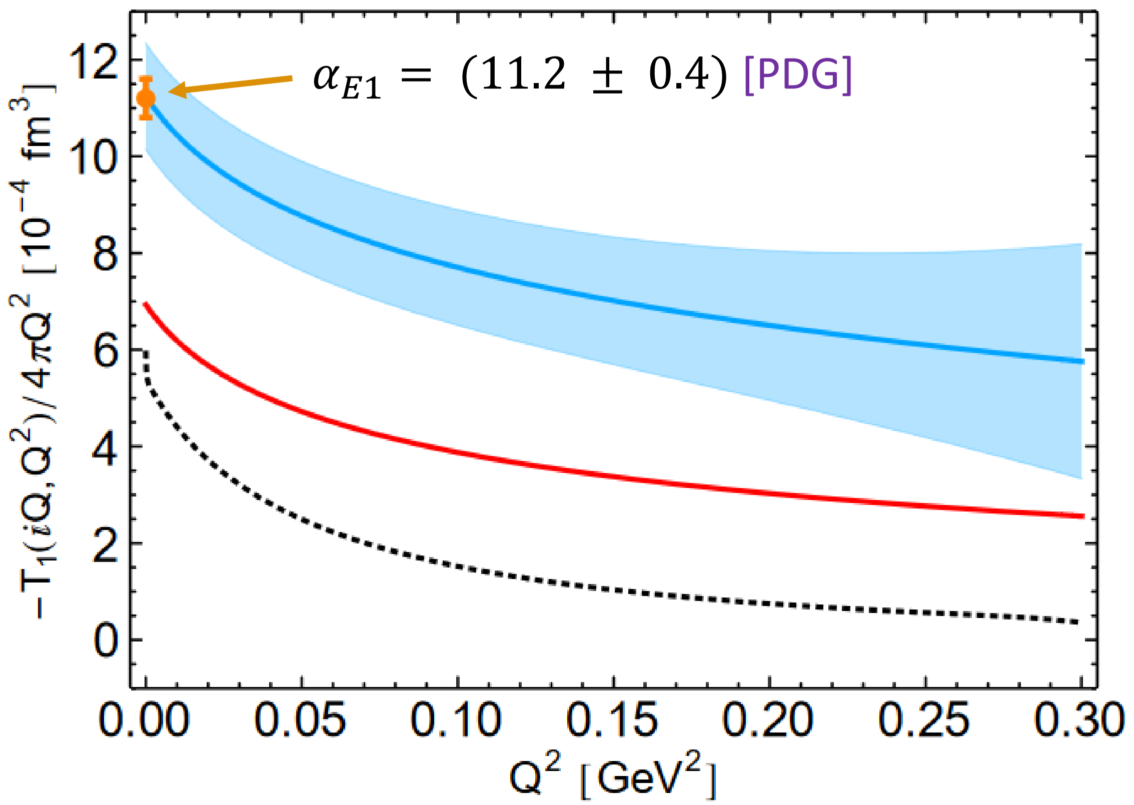
Note that at this order we only verify the polarizability contribution (no contributions from the possible non-pole Born terms)

Non-Born part of the subtraction functions

$$T_1(0,Q^2) = \frac{2}{\pi} Q^2 \int_{\nu_0}^{\infty} \frac{d\nu}{\nu^2 + Q^2} \left[\sigma_T - \frac{\nu^2}{Q^2} \sigma_L \right] (\nu, Q^2)$$

$$T_L(iQ,Q^2) = \frac{2}{\pi} \int_{\nu_0}^{\infty} d\nu' \, \nu'^2 \frac{\sigma_L(\nu', Q^2)}{\nu'^2 + Q^2}$$





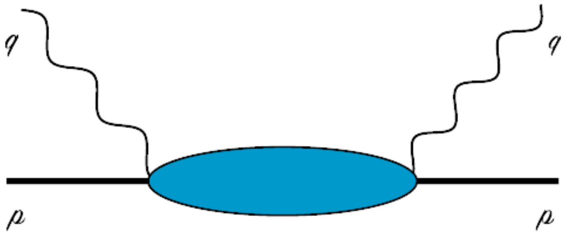
MAID

LO χPT: πN-loops

NLO χPT [Lensky et al., PRC (2014)] [Alarcón et al., PRD (2020)]

HBχPT [Birse and McGovern, EPJA, (2012)]

Schwinger sum rule



$$T^{\mu\nu}(q,p) = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) T_1(\nu,Q^2) + \frac{1}{M^2}\left(p^\mu - \frac{p\cdot q}{q^2} q^\mu\right)\left(p^\nu - \frac{p\cdot q}{q^2} q^\nu\right) T_2(\nu,Q^2)$$
$$\boxed{-\frac{1}{M}\gamma^{\mu\nu\alpha}q_\alpha S_1(\nu,Q^2) - \frac{1}{M^2}\left(\gamma^{\mu\nu}q^2 + q^\mu\gamma^{\nu\alpha}q_\alpha - q^\nu\gamma^{\mu\alpha}q_\alpha\right)S_2(\nu,Q^2),}$$

spin-dependent

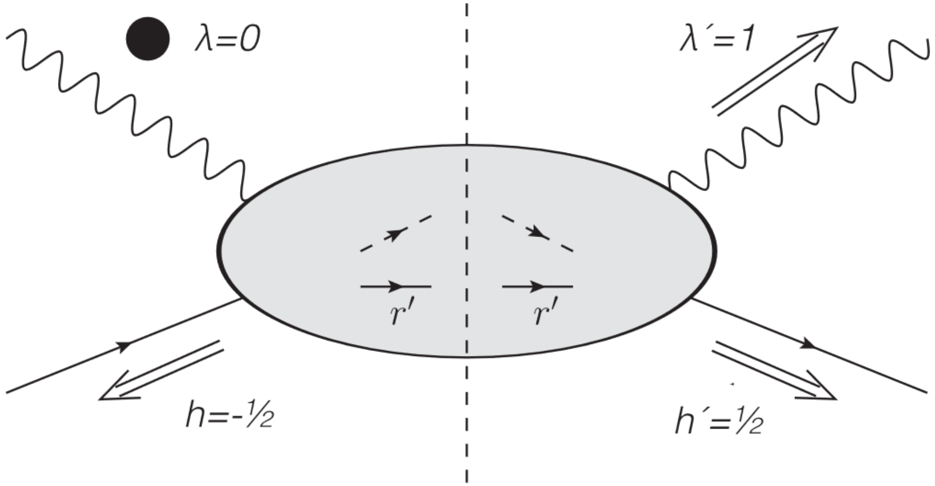
The LT-polarized Compton helicity amplitude:

$$S_{\mathrm{LT}}(\nu,Q^2) = \frac{m}{2\pi\alpha}\left[S_1(\nu,Q^2) + \frac{\nu}{m}S_2(\nu,Q^2)\right]$$

The Schwinger sum rule:

$$\varkappa = \frac{m^2}{\pi^2\alpha}\int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}(\nu,Q^2)}{Q}\right]_{Q^2\to 0}$$

[Schwinger, PNAS (1975)]



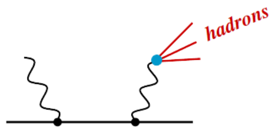
[Alarcón et al., PRD (2020)]

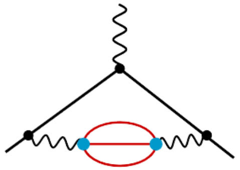
Unlike the Gerasimov-Drell-Hearn sum rule, the Schwinger sum rule contains the anomalous magnetic moment in a power of 1.

Schwinger sum rule for hadronic contributions to $(g-2)_\mu$

[Hagelstein and Pascalutsa, PRL, (2018)]
"Dissecting the Hadronic Contributions to $(g-2)_\mu$ by Schwinger's Sum Rule"

- With the timelike LT-polarized cross section, the Schwinger sum rule can reproduce the famous formula for HVP contribution to $(g-2)_\mu$

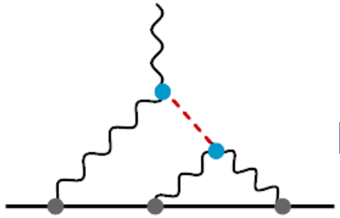


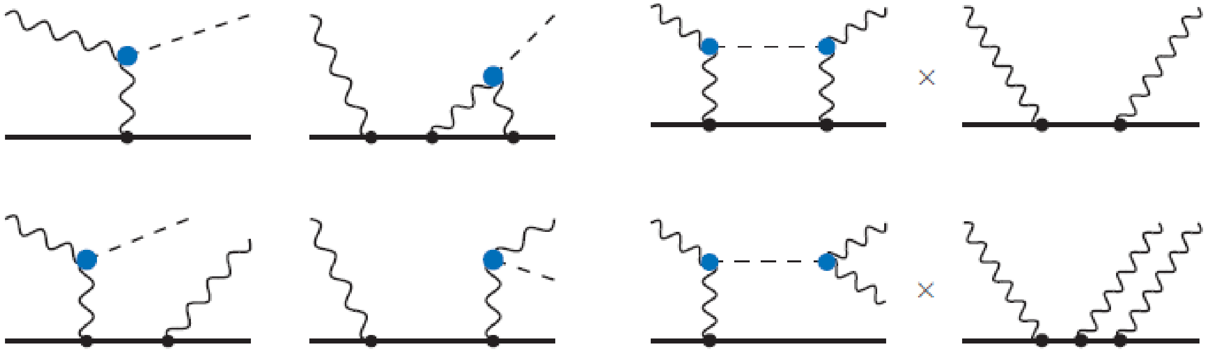


$$\varkappa = \frac{m^2}{\pi^2\alpha}\int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}(\nu,Q^2)}{Q}\right]_{Q^2\to 0} \longrightarrow \varkappa^{\mathrm{HVP}} = \frac{\alpha}{\pi^2}\int_{4m_\pi^2}^{\infty} ds\, K(s/m^2)\, \frac{\mathrm{Im}\,\Pi^{\mathrm{had}}(s)}{s}$$

- The procedure of the evaluation of HLbL contribution to $(g-2)_\mu$ via the Schwinger sum rule was formulated

The leading-order contribution to HLbL: pion-pole contribution

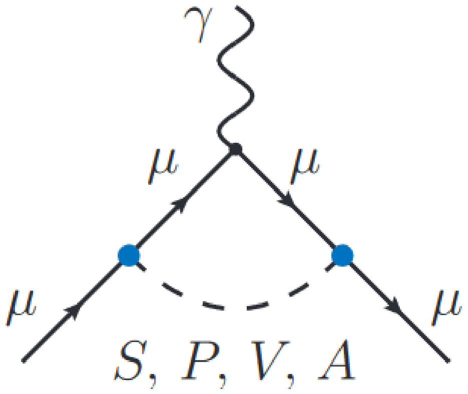




(a) Hadron photo-production channels (b) Electromagnetic channels

Schwinger sum rule: new physics and asymptotic values

The various contributions to $(g-2)_\mu$ at one loop:



$$
\begin{aligned}
\mathcal{L}^{\mathrm{S}}_{\mathrm{int}} &= C_{\mathrm{S}}\overline{\psi}(x)\psi(x)\phi(x) \\
\mathcal{L}^{\mathrm{P}}_{\mathrm{int}} &= C_{\mathrm{P}}\overline{\psi}(x)i\gamma_5\psi(x)\phi(x) \\
\mathcal{L}^{\mathrm{V}}_{\mathrm{int}} &= C_{\mathrm{V}}\overline{\psi}(x)\gamma^\rho\psi(x)\mathcal{V}_\rho(x) \\
\mathcal{L}^{\mathrm{A}}_{\mathrm{int}} &= C_{\mathrm{A}}\overline{\psi}(x)\gamma^\rho\gamma_5\psi(x)\mathcal{A}_\rho(x).
\end{aligned}
$$

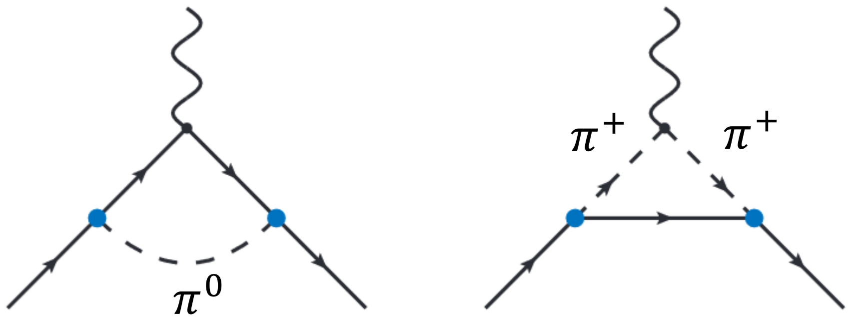
The Schwinger sum rule with the asymptotic constant:

$$
\boxed{\kappa = \lim_{\nu\to\infty} S_{LT}(\nu, Q^2=0) + \frac{m^2}{\pi^2\alpha}\int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}(\nu,Q^2)}{Q}\right]_{Q^2\to 0}}
$$

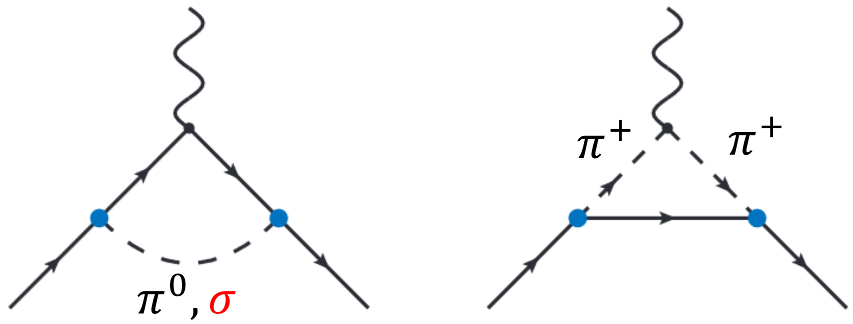
$$
\begin{aligned}
\lim_{\nu\to\infty} S^{\mathrm{S}}_{\mathrm{LT}}(\nu, Q^2=0) &= \frac{C_S^2}{8\pi^2}, \\
\lim_{\nu\to\infty} S^{\mathrm{P}}_{\mathrm{LT}}(\nu, Q^2=0) &= -\frac{C_P^2}{8\pi^2}, \\
\lim_{\nu\to\infty} S^{\mathrm{V}}_{\mathrm{LT}}(\nu, Q^2=0) &= 0, \\
\lim_{\nu\to\infty} S^{\mathrm{A}}_{\mathrm{LT}}(\nu, Q^2=0) &= -\frac{C_A^2}{8\pi^2}\left(\frac{2m}{M}\right)^2.
\end{aligned}
$$

Schwinger sum rule for the proton in baryon χPT and linear σ-model

$O(p^3)$ BχPT



Linear sigma-model

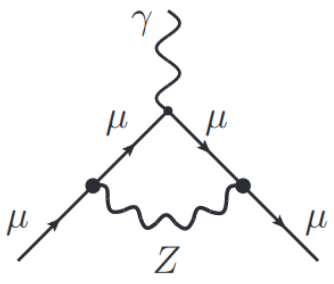


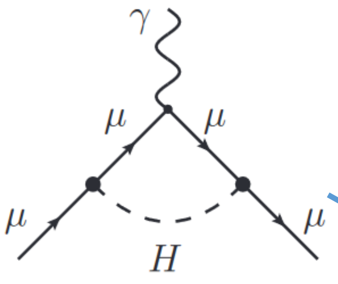
- The Schwinger sum rule holds for the charged pion contribution to $(g-2)_p$
- However the sum rule has the asymptotic constants for neutral pion contribution to $(g-2)_p$ →

$$\lim_{\nu\to\infty} S_{\mathrm{LT}}^{\mathrm{S}}(\nu, Q^2=0) = \frac{C_S^2}{8\pi^2},$$
$$\lim_{\nu\to\infty} S_{\mathrm{LT}}^{\mathrm{P}}(\nu, Q^2=0) = -\frac{C_P^2}{8\pi^2},$$

→ Due to the cancellation of the asymptotic constants, the Schwinger sum rule for the proton holds exactly in the linear sigma-model

Schwinger sum rule in SM: Z+H contribution





$$\lim_{\nu\to\infty} S^{\mathrm{A}}_{\mathrm{LT}}(\nu, Q^2=0) \;=\; -\frac{C_A^2}{8\pi^2}\left(\frac{2m}{M}\right)^2$$

$$\lim_{\nu\to\infty} S^{\mathrm{S}}_{\mathrm{LT}}(\nu, Q^2=0) \;=\; \frac{C_S^2}{8\pi^2}$$

The sum rule holds only for the total contribution of H and Z. Otherwise, it has the nonzero asymptotic constants from H and axial part of Z.

Conclusions and outlook

- The BT sum rule seems to be as valid as the Baldin sum rule . Then the dipole polarizabilities can be determined separately within the fully data-driven approach.

- Consequently, the data-driven determination of the subtraction-function part of the proton polarizability contribution to the Lamb shift of hydrogen-like atoms is also possible.

- The BT sum rule, as well as the sum rule for the subtraction function, works properly for $O(p^3)$ BχPT contribution to the proton electric polarizability that comes from the charged pion loops.

- The Schwinger sum rule is verified perturbatively in some examples of the ultraviolet-complete theories.

➢ The high-quality parametrization of the current data on σ_L with the correct limit $Q^2 \to 0$ is highly needed!

Thank you for attention!

BACKUP: resonance contribution to BT sum rule for the proton

Existing fits:

- [Christy and Bosted, PRC 2010]
- [Hiller Blin et al., PRC 2019]

Both have the issues at low-Q limit.

The resonance contribution to the sum rule integral, which was obtained from Christy and Bosted fit, was ~1

Estimation done by Marc Vanderhaeghen:

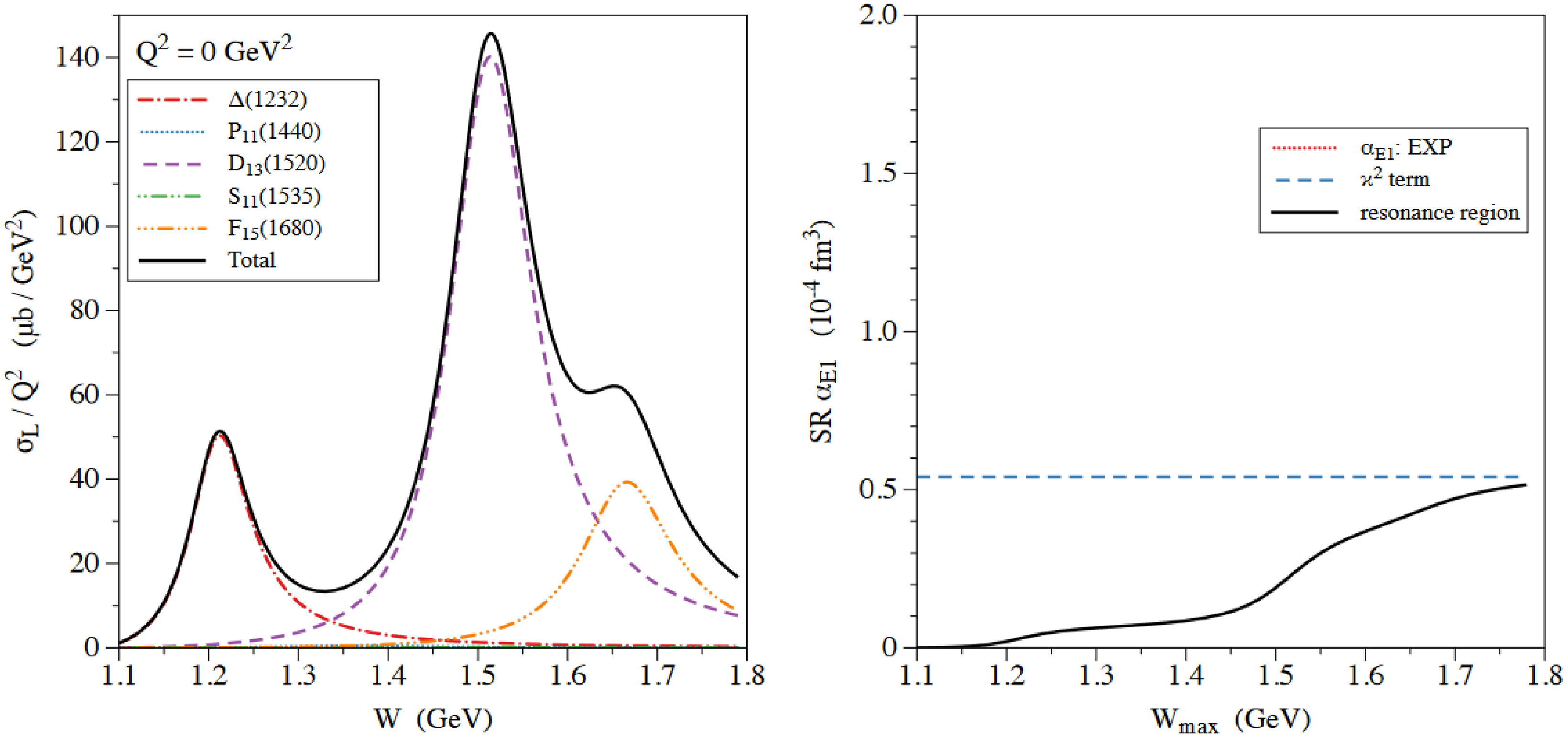


FIG. 2: Left panel: inclusive cross section σ_L/Q^2 in the limit for $Q^2 \to 0$. Right panel: the black solid curve gives the resonance contribution to the sum rule Eq. (12) for α_{E1}, as function of the upper integration limit W_{max} in the dispersion integral. The blue dashed curve indicates the contribution of the term proportional to κ_p^2 in Eq. (12).

BACKUP: Sugawara-Kanazawa theorem

The essence of the theorem:

If the amplitude tends to a constant value at the infinite real energy, then it tends to the same value at every point of the upper (lower) infinite semicircle part of the contour.
Therefore, the contribution of the latter to the dispersive integral can be obtained via the following formula

If one has the function $f(z)$ that is

1. analytic everywhere in the complex z-plane except for two cuts and poles on the real axis,
2. has the divergence at $|z| = \infty$, not stronger than a large but finite power of $|z|$,
3. has finite limits $f(\infty \pm i\epsilon)$ as $z \to \infty \pm i\epsilon$,

then the limits of $f(z)$ when z approaches infinity in any other direction are

$$\begin{aligned} \lim_{|z|\to\infty} &= f(\infty + i\epsilon) \quad \text{in the upper half-plane,} \\ &= f(\infty - i\epsilon) \quad \text{in the lower half-plane,} \end{aligned}$$

provided that $f(z)$ approaches definite (not necessarily finite) limits at $-\infty$. Thus the dispersion relation for $f(z)$ becomes

$$f(z) = \sum_i \frac{R_i}{z - x_i} + \frac{1}{\pi}\left(\int_{c_1}^{\infty} + \int_{-\infty}^{-c_2}\right)\frac{\Delta f(x)dx}{x - z} + \tilde{f}(\infty),$$

where

$$\begin{aligned} \Delta f(x) &= \frac{1}{2i}[f(x + i\epsilon) - f(x - i\epsilon)], \\ \tilde{f}(x) &= \frac{1}{2}[f(x + i\epsilon) + f(x - i\epsilon)], \end{aligned}$$

are respectively, the absorptive and dispersive parts of $f(z)$ when z approaches real x in the upper half plane and R_i is the residue at the pole at x_i.