Update on the subtraction term in the Lamb shift in muonic hydrogen

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Work done in collaboration with Judith McGovern

Eur. Phys. J. A **48** (2012) 120

Phys. Rev. D **98** (2018) 038503

Lamb shift in μH 1

CREMA experiment at PSI: $2p_{\frac{3}{2}} \to 2s_{\frac{1}{2}}$ transitions to both hyperfine $2s$ states

Pohl et al, Nature **466** (2010) 213; Antognini et al, Science **339** (2013) 417

Eliminate hyperfine splitting to get

$$\Delta E_L^{\text{expt}} = E(2p_{\frac{1}{2}}) - E(2s_{\frac{1}{2}}) = 202.3706(23) \text{ meV}$$

Much larger than in electronic hydrogen, dominated by vacuum polarisation and much more sensitive to proton structure , in particular, its charge radius

Theory gives:

$$\Delta E_L^{\text{th}} = 206.0668(25) - 5.2275(10)\langle r_E^2 \rangle \text{ meV}$$

Results of many years of effort by Borie, Pachucki, Indelicato, Jentschura and others;

collated in Antognini et al, Ann. Phys. **331** (2013) 127

Current experimental and theoretical errors comparable: $\sim 2\mu$eV

But PSI group hope to reduce experimental error by ~ 5

Lamb shift in μH 2

Includes contribution from two-photon exchange

$$\Delta E^{2\gamma} = 33.2 \pm 2.0\ \mu\text{eV}$$

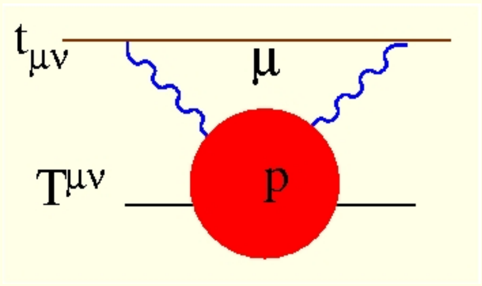
Sensitive to polarisabilities of proton by virtual photons

Largest single theoretical uncertainty

- important contribution to uncertainty in $\langle r_E^2 \rangle$
- and hence to the uncertainty in the Rydberg

Two-photon exchange

Contribution to Lamb shift:



Integral over $T^{\mu\nu}(\nu, q^2)$ – doubly-virtual Compton amplitude for proton

Spin-averaged, forward scattering $\rightarrow$ two independent tensor structures

Common choice:

$$T^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) T_1(\nu, Q^2) + \frac{1}{M^2}\left(p^\mu - \frac{p\cdot q}{q^2} q^\mu\right)\left(p^\nu - \frac{p\cdot q}{q^2} q^\nu\right) T_2(\nu, Q^2)$$

multiplied by scalar functions of $\nu = p\cdot q/M$ and $Q^2 = -q^2$

Doubly-virtual Compton scattering 1

Amplitude contains elastic (Born) and inelastic pieces

$$T^{\mu\nu} = T_B^{\mu\nu} + \overline{T}^{\mu\nu}$$

Elastic amplitude from Dirac nucleon with Dirac and Pauli form factors

K. Pachucki, Phys. Rev. A **60** (1999) 3593

$$\begin{aligned} T_1^B(\nu, Q^2) &= \frac{e^2}{M}\left[\frac{Q^4\left(F_D(Q^2)+F_P(Q^2)\right)^2}{Q^4-4M^2\nu^2} - F_D(Q^2)^2\right] \\ T_2^B(\nu, Q^2) &= \frac{4e^2MQ^2}{Q^4-4M^2\nu^2}\left[F_D(Q^2)^2 + \frac{Q^2}{4M^2}F_P(Q^2)^2\right] \end{aligned}$$

• need to remove terms already accounted for in Lamb shift (iterated Coulomb, leading dependence on $\langle r_E^2\rangle$)
$\to$ leaves "third Zemach moment" plus relativistic corrections

Doubly-virtual Compton scattering 2

On-shell intermediate nucleon states $\rightarrow$ poles at $\nu = \pm Q^2/2M$

- residues given unambiguously by elastic form factors

Final term in T_1: no pole corresponding to on-shell intermediate nucleon

But leading terms required by low-energy theorems

- Thomson limit at $O(1)$, Dirac radius at $O(q^2)$

$$F_D(Q^2)^2 = 1 - \left[\frac{1}{3}\langle r_E^2\rangle - \frac{\kappa}{2M^2}\right] Q^2 + O(Q^4)$$

$\rightarrow$ choose to keep all of it as part of Born amplitude

Others include it in inelastic part: Carlson and Vanderhaeghen, Phys. Rev. A **84** (2011) 020102

Low-energy theorems

VVCS not directly measurable, but inelastic part is constrained by LETs

Expand in tensor basis without kinematic singularities $(1/q^2)$

Tarrach, Nuov Cim **28A** (1975) 409

$\rightarrow$ two independent tensors of order q^2: correspond to polarisabilities $\alpha+\beta$ and β from real Compton scattering

$$\begin{aligned}\overline{T}_1(\omega,Q^2) &= 4\pi Q^2\beta + 4\pi\omega^2(\alpha+\beta) + O(q^4)\\ \overline{T}_2(\omega,Q^2) &= 4\pi Q^2(\alpha+\beta) + O(q^4)\end{aligned}$$

- electric polarisability: α
- magnetic polarisability: β

HBChPT	3.15 ± 0.50	McGovern *et al*, Eur Phys J A 49 (2013) 12
BChPT	3.9 ± 0.7	Lensky *et al*, Eur Phys J C 75 (2015) 604
3 methods	3.14 ± 0.51	A2: Mornacchi *et al*, Phys Rev Lett 128 (2022) 132503
DR	2.4 ± 0.6	Mornacchi *et al*, Phys Rev Lett 129 (2022) 102501

Dispersion relations

Get information on forward VVCS away from $q=0$ from structure functions $F_{1,2}(\nu,Q^2)$ via dispersion relations

$$\overline{T}_2(\nu,Q^2)=\int_{\nu_{th}^2}^{\infty}\mathrm{d}\nu'^2\,\frac{F_2(\nu',Q^2)}{\nu'(\nu'^2-\nu^2)}$$

– integral converges since $F_2\sim 1/\nu^{0.9}$ at high energies

But $F_1\sim\nu^{0.5}$ so need to use subtracted dispersion relation

$$\overline{T}_1(\nu,Q^2)=\overline{T}_1(0,Q^2)+\frac{\nu^2}{M}\int_{\nu_{th}^2}^{\infty}\frac{\mathrm{d}\nu'^2}{\nu'^2}\,\frac{F_1(\nu',Q^2)}{\nu'^2-\nu^2}$$

$F_{1,2}(\nu,Q^2)$ well determined from electroproduction experiments at JLab

Subtraction function $\overline{T}_1(0,Q^2)$ not experimentally accessible
Maybe via second subtraction at an unphysical point Biloshytskyi *et al*, arXiv:2305.0881
but only way to avoid a subtracted DR is to extract the Regge behaviour for large ν and handle it separately Gasser *et al*, Eur Phys J C 80 (2020) 1121

Subtraction term 1

Satisfies LET: $\overline{T}_1(0,Q^2)/Q^2 \to 4\pi\beta$ as $Q^2 \to 0$

But Lamb shift requires integral over all Q^2

Define form factor

$$\overline{T}_1(0,Q^2) = 4\pi\beta\, Q^2\, F_\beta(Q^2)$$

Large Q^2: operator-product expansion (OPE) gives $Q^2 F_\beta(Q^2) \propto Q^{-2}$

Collins, Nucl Phys B 149 (1979) 90; Hill and Paz, Phys. Rev. D 95 (2017) 094017

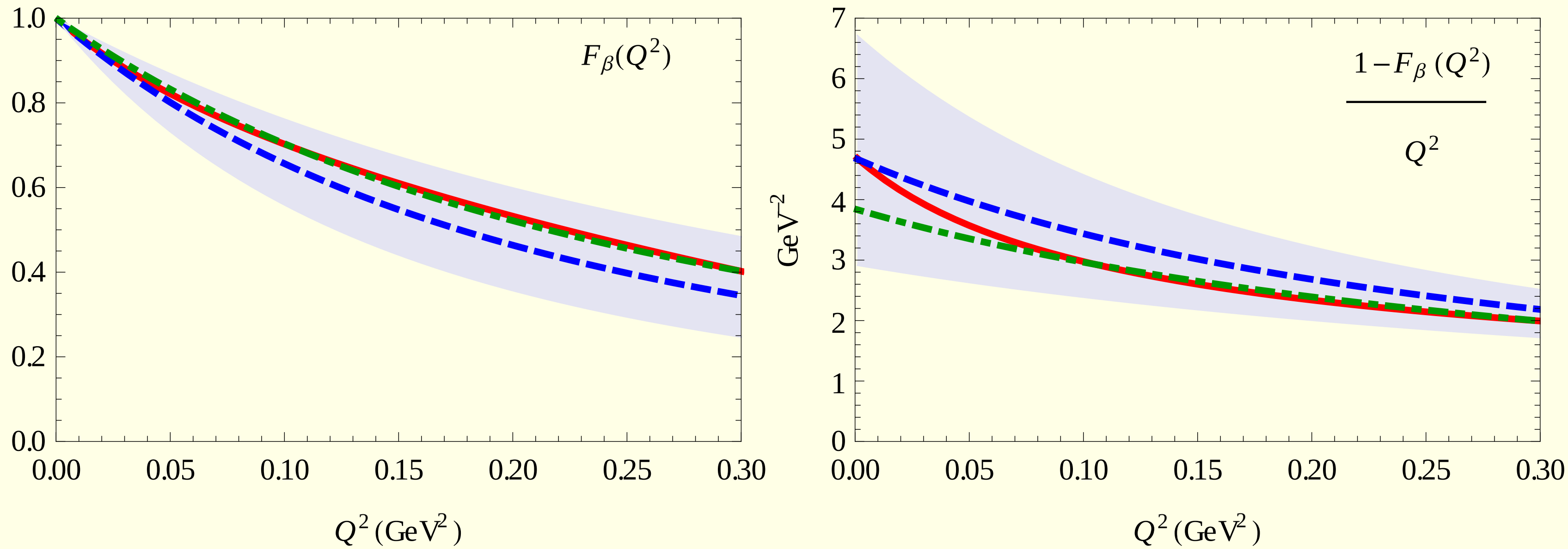
Small Q^2: use chiral effective field theories to calculate $F_\beta(Q^2)$

- HBChPT at 4th order, plus leading effect of $\gamma N\Delta$ form factor
- same diagrams as for real Compton scattering

McGovern et al, Eur. Phys. J. A 49 (2013) 12

- subtract elastic contribution calculated to this order (pole + nonpole)

Form factor 1



EFT calculation

Dipole matched at $Q^2 = 0 \to M_\beta = 462$ MeV; at $Q^2 \sim m_\pi^2 \to M_\beta = 510$ MeV

Form-factor mass

$$M_\beta = 485 \pm 100 \pm 40 \pm 25 \text{ MeV}$$

Uncertainties from:

- higher-order effects and uncertainties in input (shaded)
- $\beta = (3.1 \pm 0.5) \times 10^{-4}$ fm^3 Griesshammer *et al*, Prog Part Nucl Phys **67** (2012) 841
- matching uncertainty

Form factor 2

Extended and corrected OPE calculation gives coefficient of Q^{-2} for large Q^2

Hill and Paz, Phys. Rev. D 95 (2017) 094017

$$\frac{Q^2 T_1(0,Q^2)}{4\pi\,\alpha_{\rm EM} M} \sim 0.27 - 0.37$$

Our extrapolation: 0.2–23

Our central value too high by factor of 3 to 4

But wide uncertainty band covers OPE result

And Lamb shift integral is heavily weighted to small Q^2

$\rightarrow$ interpolation from EFT to OPE will not shift result outside our error band

Muonic H energy shift 1

$$\Delta E^{2\gamma}_{\rm sub}(2p-2s) = \frac{\alpha_{\rm EM}\,\phi(0)^2}{4\pi m}\int_0^\infty {\rm d}Q^2\,\frac{\overline{T}_1(0,Q^2)}{Q^2}\left[1+\left(1-\frac{Q^2}{2m^2}\right)\left(\sqrt{\frac{4m^2}{Q^2}+1}-1\right)\right]$$

- with dipole form, 90% comes from $Q^2 < 0.3$ GeV^2
- rather insensitive to extrapolation and value of M_β

Result:

$$\Delta E^{2\gamma}_{\rm sub} = -4.2 \pm 1.0\,\mu\text{eV}$$

Comparable to previous, model-based results Pachucki, Phys. Rev. A **60** (1999) 3593; Carlson and Vanderhaeghen, Phys. Rev. A **84** (2011) 020102

But with errors under much better control

Muonic H energy shift 2

Combine our result

- $\Delta E^{2\gamma}_{\rm sub} = -4.2 \pm 1.0\ \mu$eV

with those of Carlson and Vanderhaeghen

Carlson and Vanderhaeghen, Phys. Rev. A **84** (2011) 020102

- elastic (with nonpole term reinstated): $\Delta E^{2\gamma}_{\rm el} = 24.7 \pm 1.3\ \mu$eV
- inelastic (dispersive): $\Delta E^{2\gamma}_{\rm inel} = 12.7 \pm 0.5\ \mu$eV

$\rightarrow$ total: $\Delta E^{2\gamma} = 33.2 \pm 2.0\ \mu$eV Antognini *et al*

Main sources of uncertainty:

- magnetic polarisability β in subtraction term
- form factors in elastic contribution

(better measurement of β $\rightarrow$ better determination of Rydberg)

Additional slides

Subtraction term 2

3rd order EFTs give $F_\beta(Q^2)$ that can be integrated to give Lamb shift
But do not reproduce observed β
(and hence have incorrect slope for subtraction term at $Q^2 = 0$)
And single order gives no way to estimate convergence of chiral expansion
Alarcón et al, Eur Phys J C 74 (2014) 2852; Peset and Pineda, Eur Phys J A 51 (2015) 32

4th order EFTs contain LEC needed to reproduce experimental β
(and one to satisfy Dirac radius LET)
Difference between 3rd and 4th orders can be used to estimate errors
But give a form factor $F_\beta(Q^2)$ that cannot be integrated for large Q^2
Could be renormalised by μp contact interaction, fit to Lamb shift

Here: estimate of uncertainty from difference between 3rd and 4th orders
with allowance for possible slower convergence of Δ contributions
And extrapolate to higher Q^2 by matching EFT onto dipole form from OPE

$$F_\beta(Q^2) \sim \frac{1}{(1 + Q^2/2M_\beta^2)^2}$$

Born subtraction: pole?

Alternative dispersion relation for full amplitude including Born terms

Hill and Paz, Phys. Rev. D 95 (2017) 094017

Subtraction term for $T_1(\nu, Q^2)$ has slope for $Q^2 \to 0$

$$\frac{T_1(0,Q^2) - T_1(0,0)}{Q^2} = -\frac{4\pi\alpha_{\rm EM}}{3M}(1+\kappa)^2\langle r_M^2\rangle + \frac{4\pi\alpha_{\rm EM}}{3M}\langle r_E^2\rangle - \frac{2\pi\alpha_{\rm EM}}{M^3}\kappa + 4\pi\beta$$

- first term: Born pole, -3.93 ± 0.39 GeV^{-3}
- second and third terms: Born nonpole, 0.54 ± 0.01 GeV^{-3}
- final term: polarisability, 0.41 ± 0.06 GeV^{-3}

Born pole gives large slope with large uncertainty (from magnetic radius r_M)

Subtraction term with this slope multiplying poorly-known form factor $F_\beta(Q^2)$

$\to$ unnecessarily inflated error

Pole: well-defined structure, Q^2 dependence of residue given by elastic form factors

- can be extracted unambiguously from amplitude, DR applied to remainder

Born subtraction: nonpole?

Nonpole Born term different

- analytic in ν (in standard tensor basis)
- follows from Lorentz invariance (eg by "sticking form factors" into Dirac equation)
- but only terms up to order Q^2 fixed by LETs

(at higher orders: new LECs in V^{2}CS)

We choose to extract it from the subtraction term
and evaluate it using empirical form factors

- terms beyond order Q^2 contain contributions beyond order of our EFT
- effects of this choice should fall within our error estimate