# Lorentzian CFT correlators in momentum space

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**Iberian Strings - January 2020**

# Introduction

- Conformal invariance imposes strong constraints on the form of correlators, 2 & 3-point functions are fixed up to constants.

- Well known in **position** space. [Polyakov ’70, Osborn-Petkos ’94]

$$\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)\, \mathcal{O}_3(x_3) \rangle = \frac{c_{123}}{(x_{23})^{\Delta_t - 2\Delta_1}\,(x_{13})^{\Delta_t - 2\Delta_2}\,(x_{12})^{\Delta_t - 2\Delta_3}}$$

  - In **Euclidean** : symmetric under permutations, analytic at non-coincident points
  - In **Lorentzian** : Wick rotation and *i-epsilon* precription, causality relations

- In **momentum** space,
  - In **Euclidean** : known in general dimension [Giannotti et al. ’08, Armillis et al ’09,.. , Bzowski et al. ’13-‘18]
  - In **Lorentzian** : naive Wick rotation is not enough

# Motivation

- Interest & progress in Lorentzian CFTs :
  - Analytic bootstrap program [Komargosdki Zhiboedov'12, Fitzpatrick et al.'12, Caron-Huot '17, Costa Hansen Penedones '17, …]
  - ANEC: follows from causality [Hartmann et al. '16]

$$\langle \int dx^- T_{--} \rangle \geq 0$$

  - Implications of the ANEC : bounds on anomalies and conformal dimensions [Hofman Maldacena '08, Córdova Diab '18, …]
  - Positivity of non-local light-ray operators [Kravchuk Simmons-Duffin '16]
- Study of implications of causality may be more natural in momentum space
  - Calculation of ANEC expectation values on HM states
  - Monotonicity theorems : in general dimensions, new ones for boundary anomalies

    Good to study in momentum space, thanks to tensor decomposition and form factors [Cappelli '01]

# Outline

1. Scalar 2p function

2. Scalar 3p function

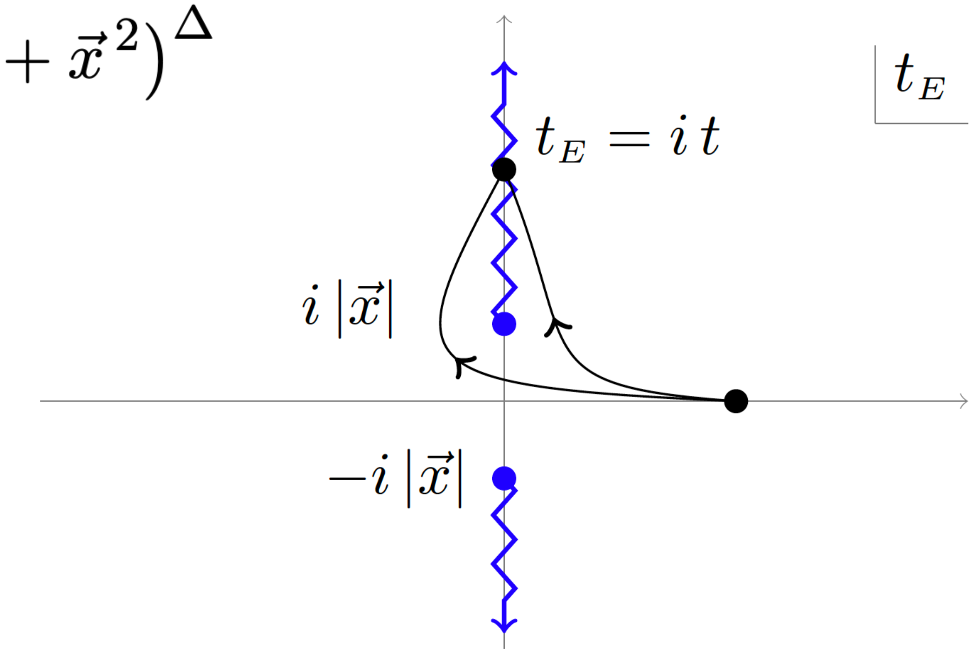
3. Tensorial 3p functions & ANEC expectation values

# 1. Scalar 2p function

# Position space

- Euclidean 2-point function of two scalar operators with dimension $\Delta$

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle_E = \frac{1}{x^{2\Delta}} = \frac{1}{\left({t_E}^2 + \vec{x}^{\,2}\right)^{\Delta}}$$




- For Lorentzian, do Wick rotation

  Ambiguity for $t \geq |\vec{x}|$

  ⟶ two possible rotations $t_E = i\,(t \pm i\epsilon)$

  ⟶ two possible Wightman functions

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$$G(x) \equiv \langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{1}{\left(-(t - i\,\epsilon)^2 + \vec{x}^{\,2}\right)^{\Delta}} \qquad \langle \mathcal{O}(0)\mathcal{O}(x)\rangle = \frac{1}{\left(-(t + i\,\epsilon)^2 + \vec{x}^{\,2}\right)^{\Delta}} = G(-x)$$

- $i\epsilon$ prescription : Lorentzian time of the operator to the left gets a more negative imaginary part

# Momentum space

$$\langle \mathcal{O}(p)\mathcal{O}(q)\rangle = (2\pi)^d\, \delta^{(d)}(p+q)\, G(p)$$

- Euclidean 2-point function follows from Fourier transform :

$$G_E(p) = \frac{\pi^{d/2}\,\Gamma(d/2-\Delta)}{2^{2\Delta-d}\,\Gamma(\Delta)}\ (p_E^2+\vec{p}^{\,2})^{\Delta-d/2}$$

- In Lorentzian, Fourier transform becomes complicated

⟶ Obtain from Euclidean space. But naive Wick rotation $p_E = ip^0$ is not enough.

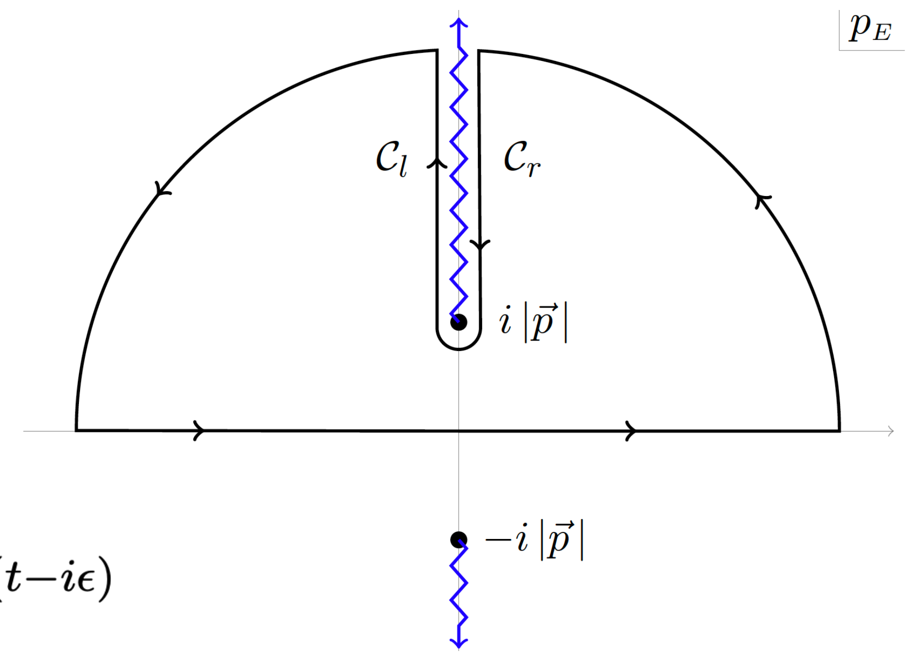
- Wick rotation within the Fourier transform :

$$G(x) = G_E(i(t-i\epsilon),\vec{x}) = \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}}\, e^{i\vec{p}\cdot\vec{x}} \int_{-\infty}^{\infty} \frac{dp_E}{2\pi}\, e^{-p_E(t-i\epsilon)}\, G_E(p_E,\vec{p})$$

To recast as a Lorentzian Fourier transform, we Wick-rotate $p_E$ to the imaginary axis taking into account the analytic properties of $G_E(p_E,\vec{p})$

# Wick rotation

$$\int_{-\infty}^{\infty} \frac{dp_E}{2\pi}\, e^{-p_E(t-i\epsilon)}\, G_E(p_E, \vec{p})$$




1. Change of variable : $p_E = ip^0$
2. Integral along the branch cut : $\int_{|\vec{p}\,|}^{\infty} dp^0\, e^{-ip^0(t-i\epsilon)}$
3. Phase difference between each side of the cut : $\sin\left(\pi(\Delta - d/2)\right)$

Obtain Lorentzian Fourier transform, $G(x) = \int \frac{d^dp}{(2\pi)^d}\, e^{ip\cdot x}\, G(p)$ from which we read off

$$G(p) = \frac{\pi^{d/2+1}}{2^{2\Delta-d-1}\,\Gamma(\Delta - d/2+1)\,\Gamma(\Delta)}\,\frac{\theta(p^0 - |\vec{p}\,|)}{|p^2|^{d/2-\Delta}}$$

- The other Wightman function $G(-p) \sim \theta(-p^0 - |\vec{p}\,|)$
- Coefficient does not diverge : no renormalisation in Lorentzian signature

# 2. Scalar 3p function

# As a triple-Bessel integral

- In position space : $\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)\, \mathcal{O}_3(x_3)\rangle = \dfrac{c_{123}}{(x_{23})^{\Delta_t-2\Delta_1}\,(x_{13})^{\Delta_t-2\Delta_2}\,(x_{12})^{\Delta_t-2\Delta_3}}$

  For Lorentzian : $t_j^E = i\,(t_j - i\epsilon_j)$ with $\epsilon_1 > \epsilon_2 > \epsilon_3$

- In momentum space : $\langle \mathcal{O}_1(p_1)\, \mathcal{O}_2(p_2)\, \mathcal{O}_3(p_3)\rangle = (2\pi)^d\, \delta^{(d)}(p_1+p_2+p_3)\langle\!\langle \mathcal{O}_1\, \mathcal{O}_2\, \mathcal{O}_3\rangle\!\rangle$

  In Euclidean : given by the triple-K integral [Barnes et al. '10]

$$\langle\!\langle \mathcal{O}_1\, \mathcal{O}_2\, \mathcal{O}_3\rangle\!\rangle_E = c_E(\Delta_j) \int_0^\infty dt\, t^{d/2-1} \prod_{j=1}^{3} p_j^{\nu_j}\, K_{\nu_j}(p_j\, t)$$

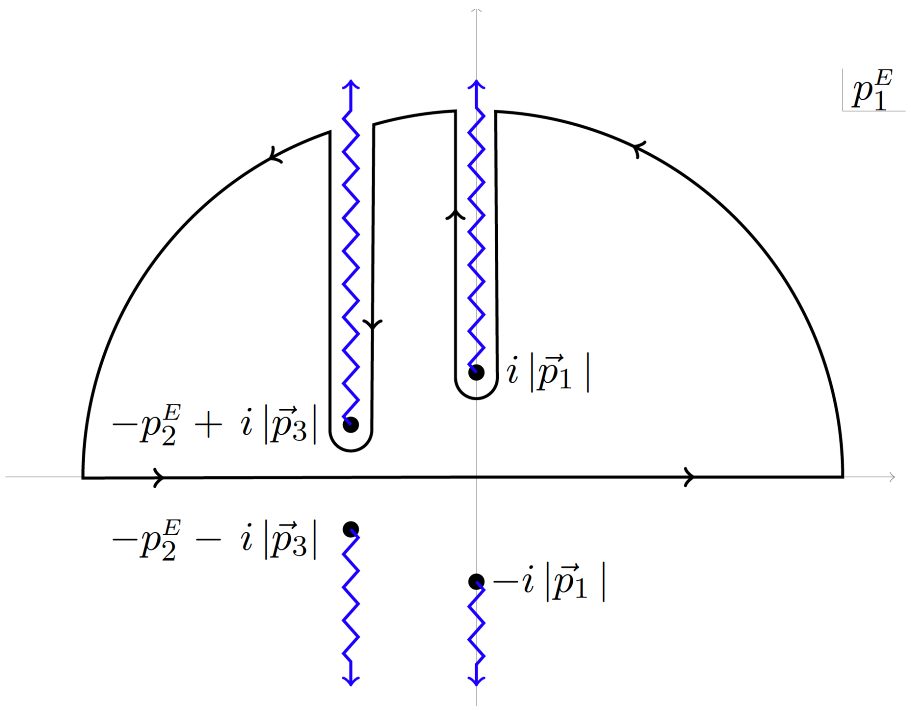
$$c_E(\Delta_j) = c_{123}\, \frac{2^{4+3d/2-\Delta_t}\,\pi^d}{\Gamma(\frac{\Delta_t-d}{2})\, \prod_{j=1}^{3} \Gamma(\frac{\Delta_t}{2}-\Delta_j)} \qquad \nu_j = \Delta_j - \frac{d}{2}$$

[Bzowski et al. '13-'16]

# As a triple-Bessel integral

Obtain the Lorentzian $C(p_1, p_2)$ by Wick rotation inside the Fourier transform.

$$\int dp_1^E\, dp_2^E\, \langle\!\langle \mathcal{O}_1\, \mathcal{O}_2\, \mathcal{O}_3 \rangle\!\rangle_E\, e^{-p_1^E(t_{13}-i\,\epsilon_{13})-p_2^E(t_{23}-i\,\epsilon_{23})}$$




$$\langle\!\langle \mathcal{O}_1\, \mathcal{O}_2\, \mathcal{O}_3 \rangle\!\rangle = \frac{\pi^2}{2}\, c_E(\Delta_j)\, \theta(p_1^0 - |\vec{p}_1|)\, \theta(-p_3^0 - |\vec{p}_3|) \int_0^\infty dt\, t^{d/2-1} \prod_{j=1}^{3} p_j^{\nu_j}$$
$$\times \Big\{ 2\,\theta(p_2^0 + |\vec{p}_2|)\,\theta(-p_2^0 + |\vec{p}_2|)\, J_{\nu_1}(p_1\,t)\, K_{\nu_2}(p_2\,t)\, J_{\nu_3}(p_3\,t)$$
$$-\pi\,\theta(p_2^0 - |\vec{p}_2|)\, J_{\nu_1}(p_1\,t)\, \left[J_{\nu_2}(p_2\,t)\, Y_{\nu_3}(p_3\,t) + Y_{\nu_2}(p_2\,t)\, J_{\nu_3}(p_3\,t)\right]$$
$$-\pi\,\theta(-p_2^0 - |\vec{p}_2|)\, \left[J_{\nu_1}(p_1\,t)\, Y_{\nu_2}(p_2\,t) + Y_{\nu_1}(p_1\,t)\, J_{\nu_2}(p_2\,t)\right]\, J_{\nu_3}(p_3\,t) \Big\}$$

No renormalisation required.

# As a momentum integral

Position : $\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)\, \mathcal{O}_3(x_3)\rangle = \dfrac{c_{123}}{(x_{23}^2)^{\beta_1}\,(x_{13}^2)^{\beta_2}\,(x_{12}^2)^{\beta_3}}$ $\qquad \beta_j = \dfrac{\Delta_t}{2} - \Delta_j$

Momentum: from the Fourier Transform we can obtain

$$\langle\langle \mathcal{O}_1\, \mathcal{O}_2\, \mathcal{O}_3\rangle\rangle = c_{123} \int \frac{d^d k}{(2\pi)^d}\, G^{\beta_1}(p_2+k)\, G^{\beta_2}(p_1-k)\, G^{\beta_3}(k)$$

Product of three 2-point functions. So for general dimensions

$$\langle\langle \mathcal{O}_1\, \mathcal{O}_2\, \mathcal{O}_3\rangle\rangle = c(\beta_j) \int \frac{d^d k}{(2\pi)^d}\, \frac{\theta(k^0 - |\vec{k}|)}{|k|^{d-2\beta_3}}\, \frac{\theta(p_2^0 + k^0 - |\vec{p}_2 + \vec{k}|)}{|p_2+k|^{d-2\beta_1}}\, \frac{\theta(p_1^0 - k^0 - |\vec{p}_1 - \vec{k}|)}{|p_1-k|^{d-2\beta_2}}$$

Equivalence between the two expressions is not easy to check in Lorentzian.

# 3. Tensorial 3p functions

# &

# ANEC expectation values

# Tensorial 3-point functions

In position space, 3-p. functions of conserved currents were fully worked out in any dimension [Osborn, Petkos '94]

$$\frac{x_{ij}^{\mu}\, x_{kl}^{\nu}\, \ldots}{(x_{23}^2)^{\alpha_1}\,(x_{13}^2)^{\alpha_2}\,(x_{12}^2)^{\alpha_3}}$$

In the Fourier transform, numerator traded by momentum derivatives :

$$\int \prod_i d^d x_i\, e^{-ip_i\cdot x_i}\, \frac{x_{12\,\mu}\;x_{12\,\nu}}{(x_{23}^2)^{\alpha_1}\,(x_{13}^2)^{\alpha_2}\,(x_{12}^2)^{\alpha_3}} = -(2\pi)^d\,\delta(p_1+p_2+p_3)\left(\frac{\partial}{\partial p_1^{\mu}}-\frac{\partial}{\partial p_2^{\mu}}\right)\left(\frac{\partial}{\partial p_1^{\nu}}-\frac{\partial}{\partial p_2^{\nu}}\right)\langle\!\langle \mathcal{O}_1\,\mathcal{O}_2\,\mathcal{O}_3\rangle\!\rangle$$

⟶ Tensorial correlators calculated by differentiating scalar correlators.

For example, $\langle\!\langle \mathcal{O}\,T_{\mu\nu}\,\mathcal{O}\rangle\!\rangle =$

$$c(\Delta)\int \frac{d^dk}{(2\pi)^d}\,\frac{\delta(k^0-|\vec{k}|)}{|\vec{k}|}\;\frac{\delta(p_2^0+k^0-|\vec{p}_2+\vec{k}|)}{|\vec{p}_2+\vec{k}|}\;\frac{\theta(p_1^0-k^0-|\vec{p}_1-\vec{k}|)}{|(p_1-k)^2|^{d-1-\Delta}}\left[\frac{8(d-1)}{d-2}\;\left(k_\mu k_\nu+\,k_{(\mu}\,p_{2\nu)}\right)+2\,p_{2\mu}\,p_{2\nu}\right]+$$

$$+\ \text{terms} \sim \eta_{\mu\nu}$$

# ANEC expectation values

ANEC : $\langle \mathcal{E} \rangle = \langle \int dx^- \, T_{--} \rangle \geq 0$

In a CFT, it was used to put bounds on anomalies [Hofman & Maldacena]

These bounds seem to be optimal : ANEC op. at null infinity is translation invariant

$$\langle \mathcal{E} \rangle = \lim_{x^+ \to \infty} (x^+)^{d-2} \, \langle \mathcal{O}(q)^\dagger \int_{-\infty}^{\infty} dx^- \, T_{--}(x^+, x^-) \, \mathcal{O}(q) \rangle \geq 0$$

⟶ The calculation is more natural in momentum space

$$\langle \mathcal{E} \rangle = \lim_{x^+ \to \infty} (x^+)^{d-2} \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} \, e^{2ip^1 x^+} \langle\!\langle \mathcal{O}(-q, \vec{0}\,) \, T_{--}(-p^1, \vec{p}\,) \, \mathcal{O}(p^1 + q, -\vec{p}\,) \rangle\!\rangle$$

$$\sim \langle\!\langle \mathcal{O}(-q) \, T_{--}(0) \, \mathcal{O}(q) \rangle\!\rangle$$

This is an expectation value

The calculation simplifies a lot

## Conclusions

- A way to compute Lorentzian 3-point functions from Euclidean ones by Wick rotation inside the Fourier transform
- Important is the analytic properties of the Euclidean correlator in the momenta plane
- Calculation of ANEC expectation values is more natural and simplifies in momentum space

## Outlook

- Relation to Appell $F_4$ function [Gillioz '19]
- Obtain tensorial correlators from tensorial decomposition and triple-Bessel expression for scalar form factors
- Implications of the ANEC away from fixed point