Weyl metrics and Wiener-Hopf factorization

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Introduction

Consider the field equations of theories of gravity

in $D \geq 4$ spacetime dimensions, $\Lambda = 0$, at two-derivative level.

Focus on the solution subspace with $D - 2$ commuting isometries:

- dimensional reduction to $d = 2$ dimensions
- non-linear field equations in $d = 2$ form an integrable system:

  solvability conditions for a certain Lax pair,

  Breitenlohner-Maison (BM) linear system (1987);
- BM linear system: associated Riemann-Hilbert (RH) problem for a monodromy matrix $\mathcal{M}$

  canonical Wiener-Hopf matrix factorization problem.

With a few natural assumptions: arXiv:1910.10632

Theorem: If $\mathcal{M}$ possesses canonical Wiener-Hopf matrix factorization, get exact solution to the non-linear PDE's for the spacetime metric g.

Wiener-Hopf matrix factorization uses complex analytic tools.

WH matrix factorization as a solution generating technique

Katsimpouri, Kleinschmidt, Virmani, arXiv:1211.3044; Câmara, C, Mohaupt, Nampuri, arXiv:1703.10366; C, Serra, arXiv:1711.01113

How to pick a monodromy matrix $\mathcal{M}$?

In general: Taylor made factorization techniques needed.

Rational monodromy matrices: factorization by means of (generalized) Liouville's theorem in complex analysis.

Crucial role: spectral curve.

Dimensionally reduced theory in $d = 2$ dimensions

$d = 3$:

$$L_3 \sim R_3 - g^{mn} \mathrm{Tr} \left(A_m A_n \right)$$

where

$$A = M^{-1} dM \quad , \quad M \in G/H$$

Symmetric space G/H: Lie group involution $\natural$, $M^{\natural} = M$.

In $d = 2$:

- $(\rho, v) \in \mathbb{R}^2$, $\rho > 0$. Weyl upper-half plane.
- Field equations take the form

$$d \left(\rho \star A \right) = 0 \quad , \quad A = M^{-1} dM$$

Example: $D = 4$

- Space-time metric in Weyl-Lewis-Papapetrou form

$$ds_4^2 = -\sigma\Delta(dy + B\,d\phi)^2 + \Delta^{-1}\left(e^{\psi}\,ds_2^2 + \rho^2 d\phi^2\right) \ ,$$

  - $\sigma = \pm 1$

$$ds_2^2 = \sigma\,d\rho^2 + dv^2 \quad , \quad ds_2^2 = d\rho^2 + \sigma\,dv^2$$

  - $\Delta > 0$,
  - Δ, B and ψ are functions of the Weyl coordinates (ρ, v) only.

- $M \in G/H$:
  - $SL(2,\mathbb{R})/SO(2)$ or $SL(2,\mathbb{R})/SO(1,1)$: $D = 4$ Einstein gravity,

    $M^{\natural} = M^T$, $\qquad M_{2\times 2}$
  - $SL(3,\mathbb{R})/SO(2,1)$ or $SL(3,\mathbb{R})/SO(3)$: $D = 5$ Einstein gravity
  - $SU(2,1)/(SL(2,\mathbb{R}) \times U(1))$ or $SU(2,1)/(SU(2) \times U(1))$:

    $D = 4$ Einstein + Maxwell theory

Breitenlohner-Maison linear system

How to determine a solution $M(\rho, v)$? 3 Steps:

Step 1: field equations $d\,(\rho \star A) = 0\;,\; A = M^{-1} dM$ form an integrable system:

Breitenlohner-Maison (BM) linear system (1987):

$$\varphi(\rho, v)\left[\, dX(\rho, v) + A(\rho, v) X(\rho, v)\,\right] = \star\, dX(\rho, v)$$

- Input:
  - $A = M^{-1} dM$, $n \times n$ matrix M
  - function φ, $\tilde{\varphi} \equiv -\sigma/\varphi$

$$v + \sigma\, \frac{\rho}{2} \left(\frac{\sigma - \varphi_\omega^2(\rho, v)}{\varphi_\omega(\rho, v)} \right) = \omega, \quad \omega \in \mathbb{C}$$

- Output: $n \times n$ matrix X

Breitenlohner-Maison linear system

Theorem

If $\varphi\Big(dX + A\,X\Big) = \star\, dX$ *is satisfied, then* $A = M^{-1} dM$ *solves*

$$d(\rho \star A) = 0 \; .$$

Step 2:

Theorem

Given $\varphi, \widetilde{\varphi} = -\sigma/\varphi$, *let* X *and* $\widetilde{X}$ *denote solutions to the corresponding BM linear system based on* $A = M^{-1} dM$. *Then, the* $n \times n$ *matrix*

$$\mathcal{M}(\rho, v) = \widetilde{X}^{\natural}(\rho, v)\, M(\rho, v)\, X(\rho, v) = \mathcal{M}^{\natural}(\rho, v)$$

satisfies

$$d\mathcal{M}(\rho, v) = 0.$$

$\mathcal{M}$ is independent of (ρ, v)!

Step 3: the reverse question:

Given an $n \times n$ matrix $\mathcal{M} = \mathcal{M}^\natural$ that is independent of (ρ, v), can we,

by matrix factorization,

obtain an $M(\rho, v)$ from it that solves the field equations?

This is where Riemann-Hilbert problems come into play:

- RH problem: matrix factorization problem for $\mathcal{M}$, called canonical Wiener-Hopf matrix factorization problem.

What is a Riemann-Hilbert problem?

A boundary value problem involving sectionally analytic functions.

Canonical Wiener-Hopf factorization of $\mathcal{M}(\tau)$

Several ingredients: 1) Spectral curve:

- Recall $v + \sigma \, \frac{\rho}{2} \left(\frac{\sigma - \varphi_\omega^2(\rho, v)}{\varphi_\omega(\rho, v)} \right) = \omega, \quad \omega \in \mathbb{C} \qquad (*)$.

  Accordingly, introduce

$$v + \sigma \, \frac{\rho}{2} \left(\frac{\sigma - \tau^2}{\tau} \right) = u, \quad u \in \mathbb{C} \quad , \quad \tau \in \mathbb{C} \backslash \{0\}$$

  Invariant under $\tau \mapsto -\sigma/\tau$.

  Fixpoints: $\sigma = 1 : \; \pm i \qquad \sigma = -1 : \; \pm 1$

  For each fixed (ρ, v): algebraic curve in $(u, \tau) \in \mathbb{C}^2$.

  Spectral curve. Note difference with $(*)$. Change of viewpoint.

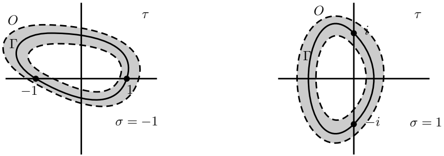
- Let $\mathcal{M}(u) = \mathcal{M}^\natural(u)$. By composition:

$$\mathcal{M} \left(v + \sigma \, \frac{\rho}{2} \frac{\sigma - \tau^2}{\tau} \right) = \mathcal{M}(\tau) \quad , \quad \tau \in \mathbb{C} \backslash \{0\} \; .$$

Canonical Wiener-Hopf factorization of $\mathcal{M}(\tau)$

2) Pick any simple closed curve Γ in the τ-plane,

that encircles the origin, and passes through the fixpoints.



If $\mathcal{M}(\tau)$ admits a canonical Wiener-Hopf factorization with respect to Γ,

1. $$\mathcal{M}(\tau) = M^-(\tau)\, M^+(\tau) \quad \text{on} \quad \Gamma \subset O\,,$$
   where $M^+(\tau)$ (respectively $M^-(\tau)$) and its inverse are analytic and bounded in the open set $D^+ \cup O$ (respectively $D^- \cup O$).
2. $M^+(0) = \mathbb{I}$
3. unique!

Canonical Wiener-Hopf factorization of $\mathcal{M}(\tau)$

Then, defining

$$M(\rho, v) \equiv \lim_{\tau \to \infty} M^-(\tau) \, ,$$

obtain

Theorem

$M(\rho, v)$ is a solution of the field equations $\quad d(\rho \star M^{-1} dM) = 0.$

Proof uses:

- spectral curve relation $u = v + \sigma \, \frac{\rho}{2} \left(\frac{\sigma - \tau^2}{\tau} \right)$
- normalization condition $M^+(0) = \mathbb{I}$
- contour Γ passes through fixpoints of involution $\tau \mapsto -\sigma/\tau$

WH factorization as a solution generating technique

1) $\mathcal{M}(\tau)$ has singularities in τ-plane. Several choices of contours $\Gamma \to$ several factorizations $\to$ several ST metrics from a single $\mathcal{M}(\tau)$!

2) New gravitational solutions by means of deformation of $\mathcal{M}(\tau)$:

`with João Serra, arXiv:1711.01113`

$$\mathcal{M}(u) = \left(\frac{H_2}{H_1}\right)^{1/3} \begin{pmatrix} H_1 H_2 & \sqrt{2} H_1 & -1 \\ -\sqrt{2} H_1 & -H_1/H_2 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

where

$$H_1(u) = h_1 + \frac{Q}{u} \quad , \quad H_2(u) = h_2 + \frac{P}{u} \, .$$

Deformation of rational matrix: deformation parameters $h_1, h_2 \in \mathbb{R}$

New solution of EDM system in $D = 4$: power series in $J = h_1 P - h_2 Q$, interpolates between a stationary geometry with a NUT parameter J and a Killing horizon with area QP.

Thanks!