L_∞ algebras and homotopy transfer

Victor Lekeu (Imperial College London)

[work in progress with Alex Arvanitakis, Olaf Hohm and Chris Hull]

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L_∞ algebras are an interesting algebraic structure that first appeared in closed string field theory [Zwiebach 92].

Actually, an L_∞ algebra can be associated to any (classical, perturbative) field theory !

[Stasheff 96, 97; Barnich, Fulp, Lada, Stasheff 98; Hohm, Zwiebach 17]

$\rightarrow$ What do the L_∞ math theorems mean for us?

$\rightarrow$ In this talk: Homotopy transfer = (tree-level) integrating out

Plan :

1. Review of (Lie and) L_∞ algebras
2. Field theory dictionary
3. Homotopy transfer theorem

1. Review of (Lie and) L_∞ algebras

Lie algebra = vector space X with an antisymmetric bracket

$$[\cdot\,,\cdot] : X \times X \to X$$

satisfying the Jacobi identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0 \quad \forall x, y, z \in X.$$

L_∞ algebra = graded vector space X, with some number (potentially infinite) of brackets of degree -1

$$b_n : X \times \cdots \times X \to X$$

satisfying (a lot of) *generalized Jacobi identities*.

If only $b_2 \neq 0$, back to the (super) Lie algebra case.

1. Review of (Lie and) L_∞ algebras

Generalized Jacobi identities?
One for each $n \geq 2$, involving all brackets b_i, b_j with $i + j = n$.

$n = 2$: $b_1(b_1(x)) = 0$, so $\partial \equiv b_1$ is a nilpotent operator

$$\partial^2 = 0.$$

$n = 3$: ∂ is a derivation of the 2-bracket $[\cdot\,,\,\cdot] \equiv b_2$,

$$\partial[x, y] + [\partial x, y] + (-1)^{|x|}[x, \partial y] = 0$$

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$$\partial[x, y] + [\partial x, y] + (-1)^{|x|}[x, \partial y] = 0$$

$n = 4$: Jacobi of b_2 is satisfied only up to terms involving the 3-bracket:

$$\begin{aligned} 0 = {} & [[x, y], z] + [[y, z], x] + [[z, x], y] \\ & + \partial(b_3(x, y, z)) + b_3(\partial x, y, z) \\ & + (-1)^x b_3(x, \partial y, z) + (-1)^{x+y} b_3(x, y, \partial z) \end{aligned}$$

$n = 5$: "$b_1 b_4 + b_2 b_3 + b_3 b_2 + b_4 b_1 = 0$"

$\vdots$

2. Field theory dictionary

Fields $\psi \in X_0$, gauge parameters $\lambda \in X_1$, reducibilities $\chi \in X_2$, ...
Define brackets by perturbative expansion.

Equations of motion:

$$0 = \partial\psi + \frac{1}{2}[\psi, \psi] + \frac{1}{3!}b_3(\psi, \psi, \psi) + \cdots \in X_{-1}$$

Similarly, gauge transformations:

$$\delta\psi = \partial\lambda + [\lambda, \psi] + \frac{1}{2}b_3(\lambda, \psi, \psi) + \cdots \in X_0$$

The L_∞ identities encode consistency (gauge-invariance of the EOM, Noether identities, ...)

2. Field theory dictionary

Example 1: 0D field theory

$$S[v] = \frac{1}{2} A_{ij} v^i v^j + \sum_{n=2}^{\infty} \frac{1}{n!} A_{i_1 \dots i_n} v^{i_1} \dots v^{i_n}$$

with $v \in \mathbb{R}^n$ gives L_∞ structure on $\mathbb{R}^n \oplus (\mathbb{R}^n)^*$, with brackets

$$(\partial v)_i = A_{ij} v^j \,, \quad b_n(v_1, \dots, v_n)_j = A_{j\, i_1 \dots i_n} v_1^{i_1} \dots v_n^{i_n}$$

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Example 2: Yang-Mills

A lot more brackets:

- From EOM: ∂A, $b_2(A_1, A_2)$, $b_3(A_1, A_2, A_3)$
- From gauge trsfs: $\partial\lambda$, $b_2(\lambda, A)$
- Gauge algebra: $b_2(\lambda_1, \lambda_2)$
- ...

2. Field theory dictionary

Aside: more rigorous definition in the BRST-BV field-antifield formalism [Batalin, Vilkovisky 81, 83]

Lie algebras

Introduce anticommuting coordinates θ^a and define

$$Q = \frac{1}{2} C^a_{bc} \theta^b \theta^c \frac{\partial}{\partial \theta^a}$$

Then, Jacobi identity if and only if $Q^2 = 0$

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L_∞ algebras

Similarly, introduce z^a's of opposite parities and define

$$Q = \sum_{n=1}^{\infty} \frac{1}{n!} C^a_{b_1 \dots b_n} z^{b_1} \dots z^{b_n} \frac{\partial}{\partial z^a}$$

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→ For any field theory, L_∞ from BRST-BV differential Q_{BV}
L_∞ relations $\Leftrightarrow Q^2_{\text{BV}} = 0 \Leftrightarrow$ master equation $(S, S) = 0$

3. Homotopy transfer

X some L_∞ algebra, with subspace $\bar{X}$. If the projection

$$\Pi : X \to \bar{X}$$

is *homotopic* to the identity, i.e.

$$\Pi = 1 + \partial \circ h + h \circ \partial$$

for some h, then there is a L_∞ algebra structure on $\bar{X}$.

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- Define $\bar{\partial} = \Pi\partial$:

$$\bar{\partial}^2 = \Pi\partial\,\Pi\partial = \Pi\,\partial(1 + \partial h + h\partial)\partial = 0$$

- Define $\bar{b}_2(x, y) = \Pi b_2(x, y)$: Jacobi will fail even if $b_3 = 0$
  Can be compensated by defining

$$\bar{b}_3(x, y, z) = \Pi\left([h([x, y]), z] \pm [[x, y], h(z)] \pm \text{perm.}\right)$$

- Generic formula at all orders

3. Homotopy transfer

0D field theory example: split $(v^i) = (v^\alpha, v^a)$ and take Π the projector on the v^α subspace. Then,

$$h^{ab} = -(A^{-1})^{ab}\,, \text{other components} = 0$$

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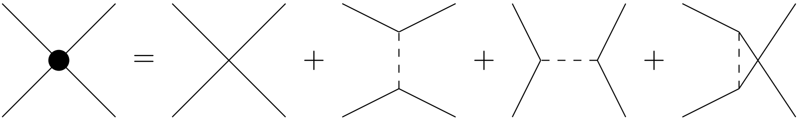
New $\bar{b}_3$ from homotopy transfer:

$$\bar{b}_3(v_1, v_2, v_3)_\alpha = \Big(A_{\alpha\beta\gamma\delta} - 3A_{a\alpha(\beta}(A^{-1})^{ab}A_{\gamma\delta)b}\Big)\, v_1^\beta v_2^\gamma v_3^\delta$$



→ Reproduces field-theory expectation

→ All orders result: general formula gives sum over trees

4. Conclusions and open questions

Recap

- L_∞ algebras are associated to (classical, perturbative) field theories
- New entry in this dictionary:
  homotopy transfer $=$ *integrating out*

Open questions

- Physical meaning of homotopy condition?
- Loops? Quantum BV equation $(S, S) + 2i\hbar\Delta S = 0$ defines "loop/quantum L_∞ algebras"
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  [Sen 2016]: String field theory, project onto massless states

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Thanks for your attention!