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Higgs and Beyond at Future Colliders

INFIERI 2021 | **Homework** | **Date: 05.08.2021**

Feel free to send me your solutions and I will send you feedback.

Exercise 1: Order of magnitude estimates

a) How long does it take for a photon to travel from your feet to your brain? Can you guess when is AI going to take over humans?
b) Estimate the energy of the cosmic rays given that the lifetime of a muon is about $1\,\mu$s.

c) A $1\,\mathrm{cm}^3$ piece of ice melts in about 40 minutes under the sun. Compute the volume of oil to burn 1 cm-thick ice cap surrounding the sun at a distance of 150 million kilometres (the effects of the atmosphere will be neglected and you'll look on wikipedia to find the latent heat of the ice and the energy realised by burning one liter of oil). What do you conclude concerning the origin of the energy radiated by the Sun?

Exercise 2: Natural units

a) Show that $[\hbar] = M \cdot L^2 \cdot T^{-1}$ and $[c] = L \cdot T^{-1}$.
b) Check the consistency of the classical/quantum correspondence at the dimensional level:

$$E \to i\hbar\frac{\partial}{\partial t} \quad \& \quad p \to i\hbar\frac{\partial}{\partial x}$$

c) Show that

$$1\,\mathrm{s} = 1.52\cdot 10^{27}\,\hbar/\mathrm{TeV}, \quad 1\,\mathrm{m} = 5.1\cdot 10^{18}\,\hbar c/\mathrm{TeV}, \quad 1\,\mathrm{kg} = 5.61\cdot 10^{23}\,\mathrm{TeV}/c^2$$

d) Using the Newton constant, $\hbar$ and c, construct a mass scale, a length scale and a time scale. They are defining the Planck scales. Compute the matter density of the universe today (10^{-29} g/cm^3) in Planck units.
e) The Schwarzschild radius of an object of mass m is the measure of its mass in Planck units. The Compton wavelength is defined as $\hbar/(mc)$. Compute the Schwarzschild radius of the Earth, the Sun, a neutron star, a stellar black-hole, a super-massive BH, a micro-BH (you'll check on Wikipedia the characteristic mass of these objects). What do you conclude? Compute the Schwarzschild radius of a micro-BH assuming that the Planck scale has been reduced to 1 TeV. What do you conclude?
f) Using e, m_e and c, construct a length scale. This is the classical radius of the electron.

Using e, m_e and $\hbar$, construct a length scale. This is the Bohr radius of the electron.
g)The pion Compton wavelength in natural units is $\lambda_\pi = \hbar/(M_\pi c) \rightarrow 1/M_\pi \simeq (140 \text{ MeV})^{-1}$. Convert this to conventional units by multiplying with a combination of $\hbar$ and c to get a distance unit.
h) A typical hadronic cross section is of order $\sigma \simeq \lambda_\pi^2 \simeq 1/M_\pi^2 \simeq 1/(140)^2\text{MeV}^{-2}$. Express this quantity in units of barns (1 barn = $10^{-28}\,\text{m}^2$).

Exercise 3: Value of e in HEP units

The electromagnetic fine-structure constant was defined by A. Sommerfeld in 1916. It is given by

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c},$$

where $e = 1.6\times10^{-19}\,\text{C}$ is the unit electric charge, $\epsilon_0 = 8.8\times10^{-12}\,\text{F}\cdot\text{m}^{-1}$ is the vacuum permittivity.
a) Compute the value of α. Check that it is a dimensionless quantity (we remind that $1\,\text{F} = 1\,\text{C}^2.\text{J}^{-1}$)
b) Deduce the value of the electric charge e in the HEP units ($\hbar = c = \epsilon_0 = 1$).

Exercise 4: Wave equations

a) Derive the Schrödinger equation from the classical expression of the energy.
b) Derive the Klein–Gordon equation from the relativistic energy expression.

Exercise 5: Schwarzschild radius

Our present understanding of the Universe and its evolution implies the existence of black holes, bodies whose masses are packed in such small volumes that not even light can escape. From a theoretical point of view, black holes are a direct consequence of the fact that we must use General Relativity to describe the late stages of gravitational collapse. The physical magnitude used to evaluate this singularity point is known as Schwarzschild radius. The aim of this exercise is to determine this important quantity for a set of different objects:
a) the entire Milky Way, with a mass of 250 billion suns ($\text{M}_\text{Sun} = 2\times10^{30}$ kg)
b) our moon with a mass of 7.35×10^{22} kg.
c) a black hole with a mass of an average human (M = 60 kg).
(Express the answers to two significant figures)

Exercise 6: Stephan Boltzmann Law

a) The Stefan-Boltzmann law states that a black hole loses energy at a rate $P = \sigma A T^4$. The total energy is given by the Einstein's equation $E = mc^2$. Use these to estimate the lifetime of a black hole in terms of its mass.
b) Take the Sun to be a blackbody with a surface temperature of 6'000 K. The Sun's radius is 7.0×10^5 km. Calculate the Sun's luminosity, in watts (Joules/second).
c) Mars is 1.5 AU from the Sun. Calculate the brightness of the Sun at Mars' distance (i.e., the solar flux on Mars' surface) expressed in watts per square meter.

Exercise 7: Star Wars Death Star and Hawking Black Hole radiation

In the movie, we learn that the Death Star has a radius of about one-tenth of the Endor planet. Endor is very comparable to the Earth (life develops, humans experience gravitational potential as on Earth, there is a breathable atmosphere).
a) Estimate the size and the mass of the Death Star.
b) The Death Star is a weapon that can destroy planets similar to the Earth, like Alderande. Using dimensional arguments, estimate the energy needed to destroy Alderande, i.e. compute the gravitational potential energy of Alderande. For comparison, the total amount of energy produced on Earth in one year is of the order of 10^{20} J. Given that the Death Star needs about 3 days to produce/store this energy, compute the power of the source of energy.
c) In practise, the Death Star is doing more than destroying Alderande: each little fragment of the planet is expelled with a velocity of about 10^4 km/s. Compute the energy needed by the Death Star to achieve such a destructive action. What is the power of the source of energy? Assuming that this energy is produced by burning oil, compute the volume of oil needed. How many power-plants are needed to reach the same power?
d) Assuming that the energy of the Death Star is produced by the annihilation of matter and antimatter into energy, what would be the amount of anti-matter needed?
e) S. Hawking understood that the laws of quantum mechanics imply that a BH is actually radiating particles, hence energy. Based on dimensional arguments, find the Hawking temperature of a BH of mass M. There is a priori a 1D infinite family of solution, you'll retain the solution that scales with a single power of $\hbar$. The exact formula derived by Hawking is smaller by a factor $1/(8\pi)$ compared to the naive estimate.
f) Assuming that a BH is a perfect black-body, use the Stefan–Boltzmann law ($P \propto T^4$) to derive the luminosity of a BH of mass M. Numerically, compute the power of a BH of solar mass (we recall that the Stefan–Boltzmann constant is equal to $\pi^2 k_B^4/(60\hbar^3 c^2)$).
g) Using the conservation of energy, derive the differential equation controlling the time evolution of the BH mass. Integrate this equation to obtain the BH life time.
h) What is the lower bound on the mass of a BH to be as old as the Universe?

Exercise 8: Solar neutrino flux

The main source of energy of the Sun is the thermonuclear reaction:

$$4\,{}^{1}_{1}H + 4e^- \to {}^{4}_{2}He + 2\,e^- + 2\,\nu_e$$

a) The mass of Helium is 4.002602 atomic unit, the mass the proton is 1.007276466879 atomic unit (or 938.2720813 MeV) and the mass of the neutron is 1.00866491588 atomic unit. Compute the amount of energy emitted by the reaction above.
b) From the luminosity of the Sun computed in the previous exercise, compute the amount of matter lost and transformed into energy every second in the Sun. Is it now compatible with the age of the Sun?
c) From the value of the luminosity of the Sun, estimate the number of neutrinos produced by the Sun every second.
d) Compute the flux of neutrinos emitted by the Sun and received on Earth.

Exercise 9: LHC as a Higgs discovery machine

a) The scattering cross section of a neutron on ^{235}U is about 1 b (=10^{-28} m^2). Assuming a simple scaling of the volume and area of a nucleus with the number of nucleons, obtain an estimate of the proton-proton cross section. Deduce the size of the proton (assuming that the proton is a hard sphere).
b) Estimate the cross-section $\sigma(pp \to W)$. Argue that $\sigma(pp \to h)$ should be about 3 orders of magnitude smaller.
c) One way to identify the Higgs boson is to look at its decay mode into 2 photons. About 1 out of 100 such events can be reconstructed and seen in the detector. Compute the collision rate (i.e., the number of proton-proton collisions) needed to observe 100 Higgs bosons in a year.
d) On disk, the recorded data associated to one proton-proton collision occupies about 1 MB. What is the fraction of collisions that one can afford to record? This is called the trigger rate.
e) Protons are grouped into bunches, each containing about 10^{11} protons. What should be the separation between two bunches to be able to observe 100 Higgs boson a year? What should be the transverse size of the bunches? And the minimum longitudinal size of the bunches?

Exercise 10: Higgs production at e^+e^- colliders

At low energy e^+e^- colliders (E<500 GeV), the dominant production mode of the Higgs boson is the so-called Higgs-strahlung mode $e^+e^- \to Zh$.

a) Draw the tree-level Feynman diagram for this production mode.

b) In the center of mass frame, compute the energy of the Z boson as a function of m_Z, m_h and $\sqrt{s}$ (the total energy in the center-of-mass frame).

c) Can you comment of the advantage of this process over the different Higgs production modes at a hadron collider?

Exercise 11: Electroweak phase transition in presence of $|H|^6$ interaction

In the SM, the EW phase transition is first order only for a Higgs mass below 50 GeV or so. In this exercise, you will see that this phase transition can be first order for $m_h = 125$ GeV provided that there is an additional contribution to the Higgs potential. More precisely, we considered the following Higgs potential where a $|H|^6$ interactions in added to the SM potential:

$$V(H) = \lambda\left(|H|^2 - \frac{v^2}{2}\right)^2 + \frac{1}{\Lambda^2}\left(|H|^2 - \frac{v^2}{2}\right)^3.$$

a) Find the condition on Λ, λ and v^2 to ensure that the potential at $T = 0$ breaks the EW symmetry.

b) By approximating the finite temperature correction with a simple thermal mass (computed in the high temperature expansion), compute the Higgs thermal mass, cT^2 in

$$V(h,T) = \frac{1}{2}cT^2h^2 + V(h,T=0).$$

(For an introduction to quantum field theory at finite temperature, see hep-ph/9901312. In particular Eqs. (212) and (213) give expression of the thermal potential and its expansion at high temperature).

c) Compute the critical temperature and the value of the Higgs vacuum expectation value at the critical temperature.

d) Find the condition on Λ to ensure that $v_c/T_c > 1$.

e) Find the deviation of the Higgs cubic self-coupling at zero temperature in the region of parameter space that leads to a first order phase transition for $m_h = 125\,\mathrm{GeV}$.