Stochastic Multiple Fields Inflation and Primordial Black Holes Formation

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Outline:

- The Inflationary Theory
- δN formalism
- Stochastic Process
- Stochastic δN formalism
- **Part 1: Inflation with Stochastic Boundaries**
- **Part 2 : Stochastic Multiple Fields Inflation**
- Summary and Conclusions

Based on the following works:

- A. Nassiri-Rad, **K. A.**, H. Firouzjahi, PRD 106, 123528 (2022)
- **K. A.**, A. Nassiri-Rad, H. Firouzjahi, arXiv:2304.00577

A short review of the Big-Bang model

Success and shortcomings

Inflationary Theory:

A key characteristic of inflation is that all physical quantities are slowly varying, despite the fact that the **space is expanding rapidly**.

The simplest models of inflation implement the time-dependent dynamics during inflation in terms of the evolution of **a scalar field, $\phi(t, x)$,** called the inflaton.

Associated with each field value is a **potential energy density V (ϕ)**. When the field is dynamical then it also carries a kinetic energy density.

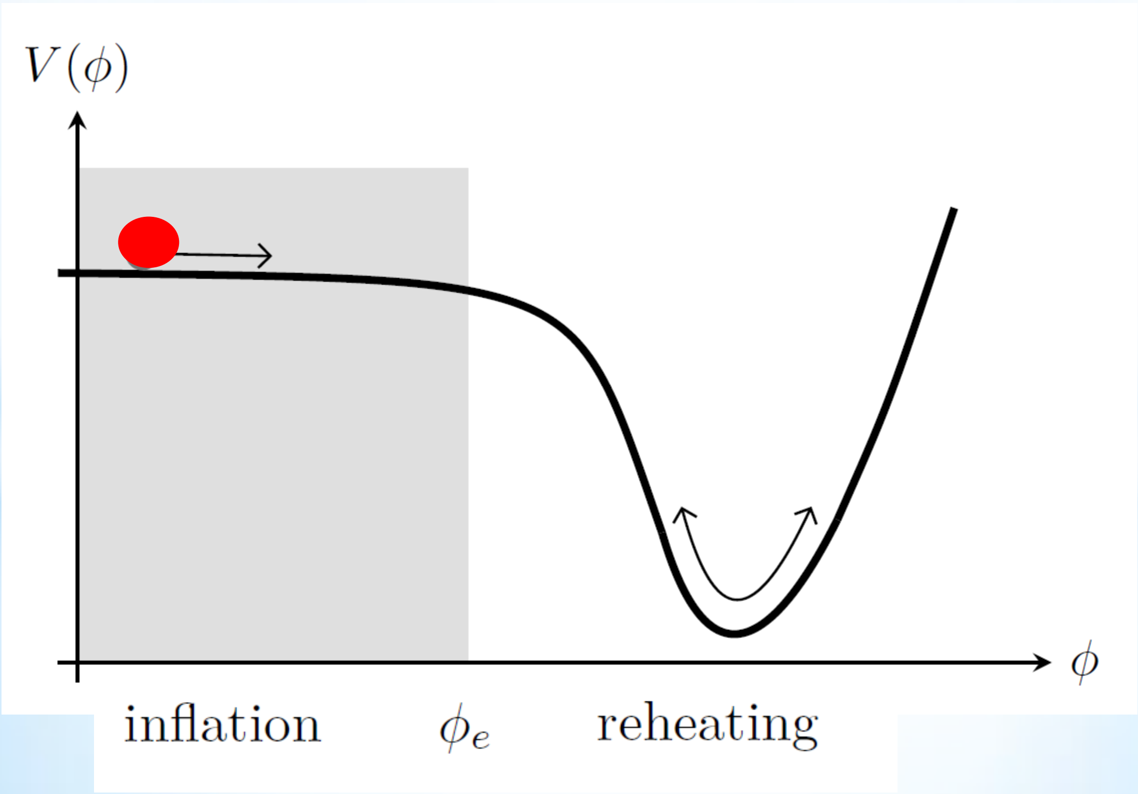
Inflationary Theory:

The dynamics during inflation is then determined by a combination of the Friedmann and Klein-Gordon equations

$$H^2 = \frac{1}{3M_{\rm Pl}^2}\left[\frac{1}{2}\dot{\phi}^2 + V\right]$$

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi}$$

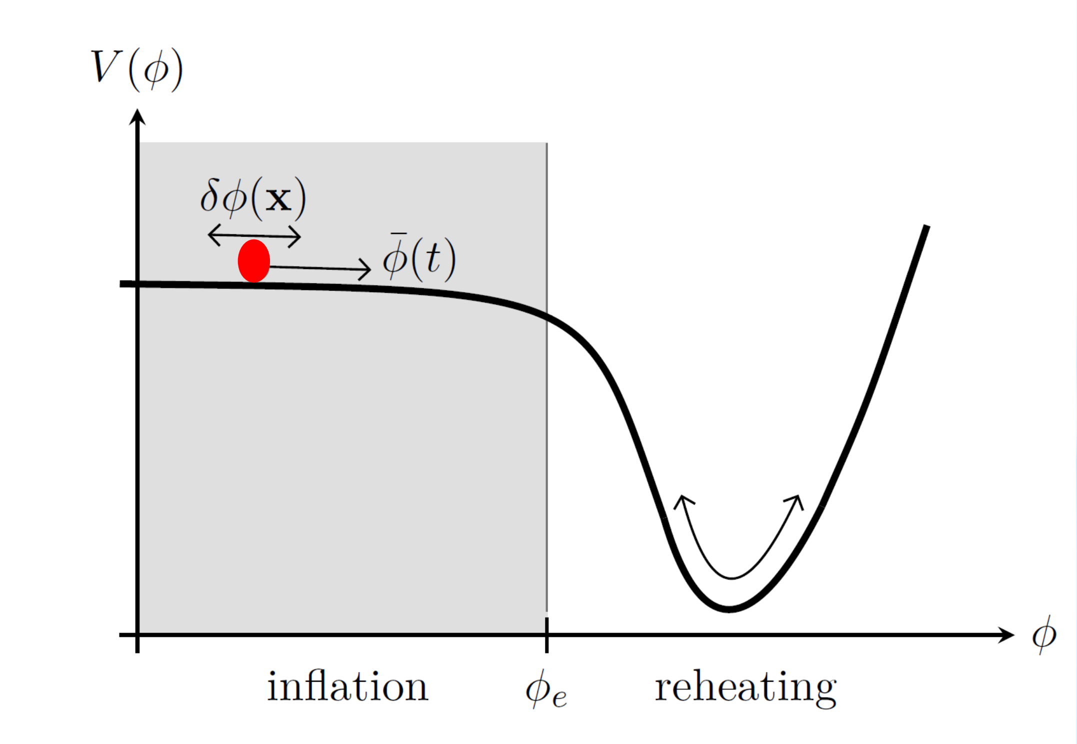
Slow-Roll Inflation



$$\dot{\phi}^2 \ll V$$

Inflationary Perturbation:

Quantum fluctuations $\delta\phi(t, \mathrm{x})$ around the classical background evolution $\phi(t)$ affect the duration of inflation. Regions acquiring negative fluctuations $\delta\phi$ remain potential-dominated longer, and hence inflate longer, than regions with positive $\delta\phi$. **Different parts of the universe therefore undergo slightly different evolutions, which induces variations in the density after inflation, $\delta\rho(t, \mathrm{x})$.**

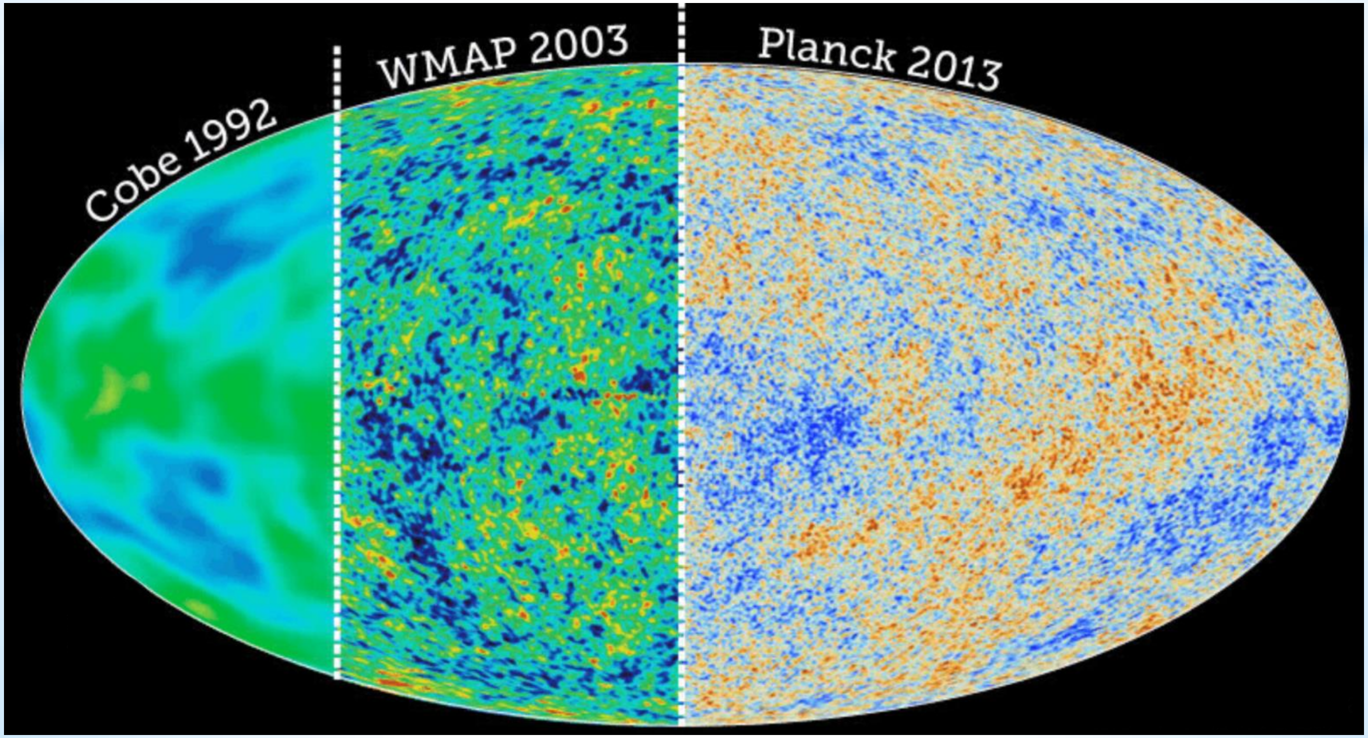


Inflation and Observation:

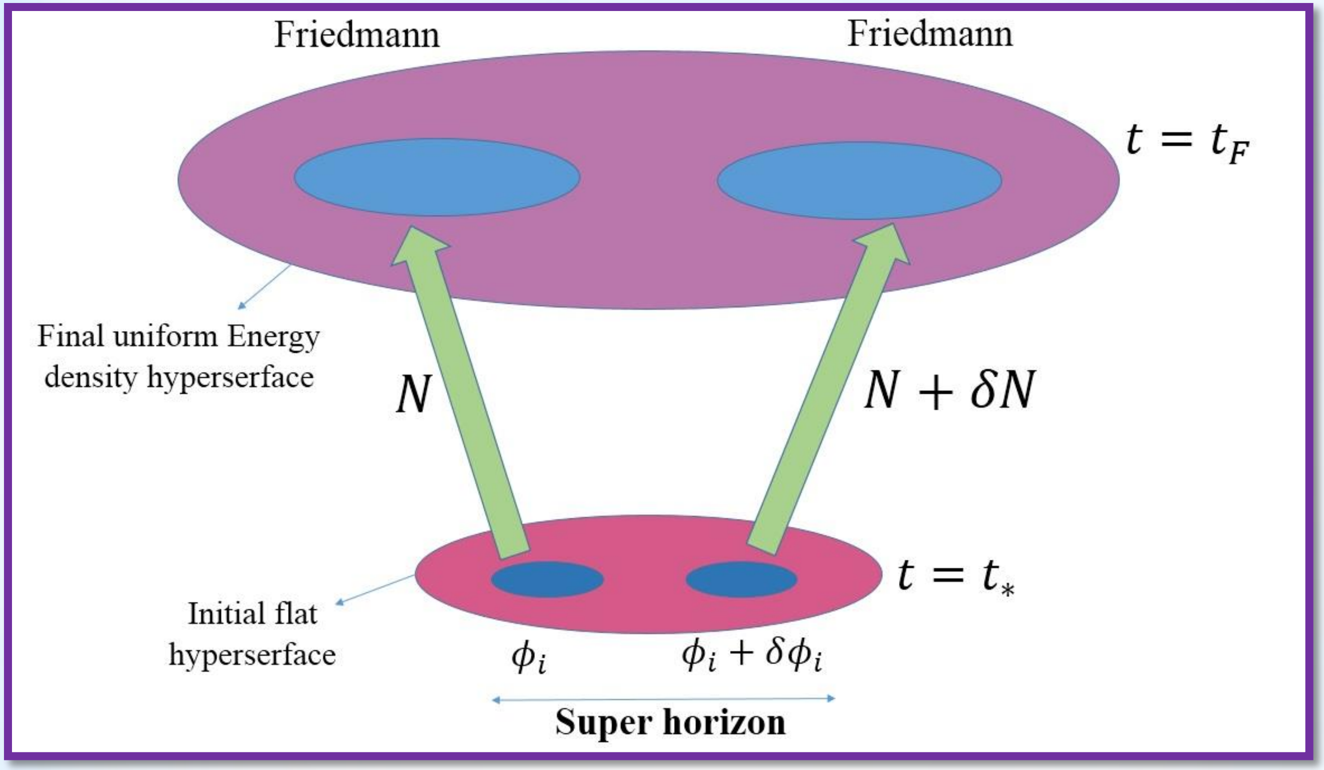
All Observations strongly support inflation.

The basic predictions of inflationary scenarios are that the primordial perturbations which are nearly scale invariant, nearly adiabatic and nearly Gaussian.

Quantum vacuum fluctuations are observed in CMB perturbations



δN Formalism:



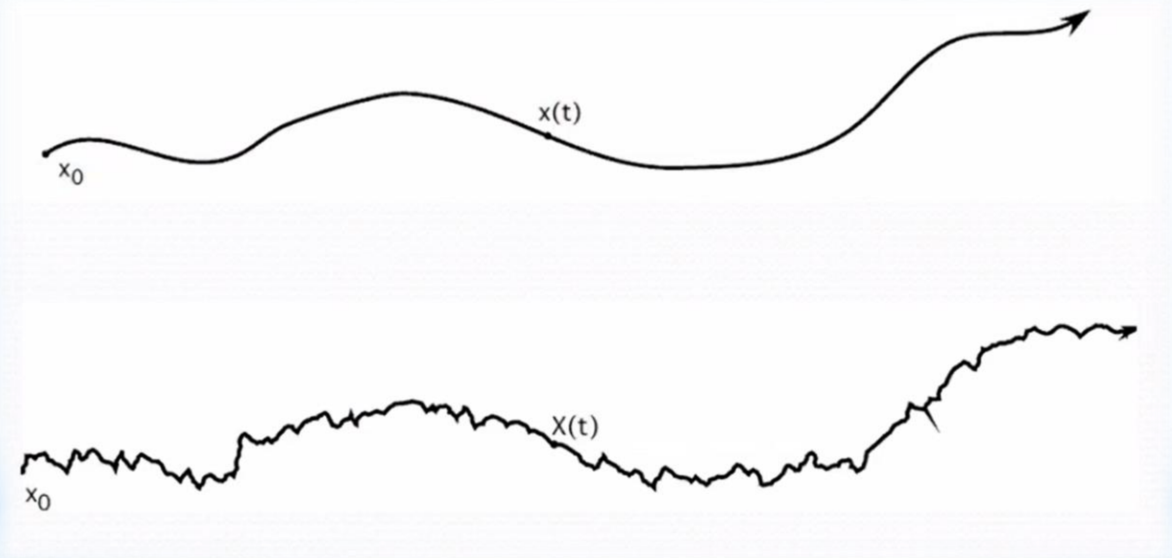
$$\zeta(t_F) \approx \delta N(t_F, t_*, x)\,.$$

The power spectrum of curvature perturbation is given by:

$$\mathcal{P}_\zeta(k) = \mathcal{P}_{\delta N}(k) = \frac{d}{d\langle N\rangle}\langle \delta N^2\rangle$$

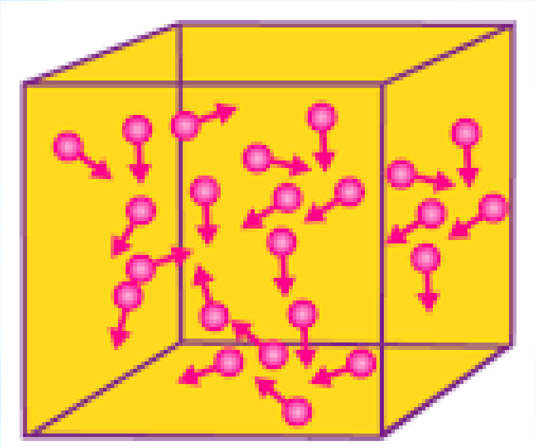
Stochastic Process:

A stochastic process is a set of time dependent random variables through which the evolution of the system is non-deterministic



Stochastic Process:

A stochastic process is a set of time dependent random variables through which the evolution of the system is non-deterministic



The most two important stochastic processes are **the Poisson process** and **the Wiener process** .

Stochastic Inflation

Langevin Equation:

a **Langevin equation** is a stochastic differential equation describing how a system evolves when subjected to a combination of deterministic and fluctuating forces.

$$\frac{d\phi(N)}{dN} = -\frac{V_{,\phi}}{3H^2} + \frac{H}{2\pi}\xi(N)$$

deterministic

Stochastic

- **Drift dominated regime**
- **Diffusion dominated regime**

Fokker Planck Equation:

the **Fokker–Planck equation** is a partial differential equation that describes the time evolution of the probability density function of the velocity of a particle under the influence of drift and diffusion terms, as in Brownian motion.

$$\frac{\partial}{\partial N} P(x, N) = -\frac{\partial}{\partial x}\left(A(x, N)P(x, N)\right) + \frac{\partial^2}{\partial x^2}\left(B(x, N)P(x, N)\right)$$

Stochastic $\delta\mathcal{N}$ Formalism:

The time it takes the field to reach the end of inflation is itself a stochastic variable $\mathcal{N}$

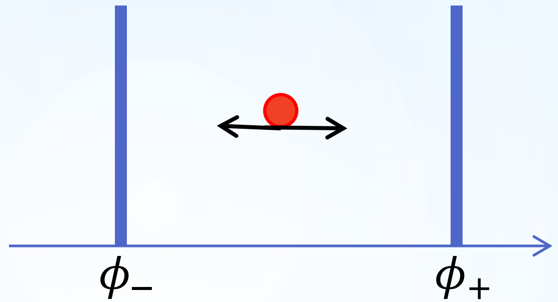
$$\mathcal{P}_{\mathcal{R}} = \mathcal{P}_{\delta\mathcal{N}} = \frac{\mathrm{d}\left\langle \delta\mathcal{N}^2 \right\rangle}{\mathrm{d}\left\langle \mathcal{N} \right\rangle}$$

$$\left\langle \delta\mathcal{N}^2 \right\rangle \equiv \left\langle \left(\mathcal{N} - \left\langle \mathcal{N} \right\rangle \right)^2 \right\rangle = \left\langle \mathcal{N}^2 \right\rangle - \left\langle \mathcal{N} \right\rangle^2$$

Brownian Motion During Inflation:

Consider the simple case where there is no initial velocity, $\dot{\phi}_0 = 0$. The quantum kicks will govern the dynamics of ϕ and we have a Brownian motion.

$$\phi\left(\mathcal{N}\right) = \frac{H}{2\pi} W\left(\mathcal{N}\right)$$



$$W\left(N\right) \equiv \int_0^N \xi\left(N\right) dN$$

$$\langle W(\mathcal{N})^2 \rangle = \langle \mathcal{N} \rangle$$

The first boundary crossing probabilities, $p_\pm$ can be obtained as:

$$\langle \phi(\mathcal{N}) \rangle = 0$$

$$\langle \phi(\mathcal{N}) \rangle = p_+ \phi_+ + p_- \phi_-$$

$$p_+ = \frac{-\phi_-}{\phi_+ - \phi_-}, \qquad p_- = \frac{\phi_+}{\phi_+ - \phi_-}.$$

$\langle \mathcal{N} \rangle$ can also be obtained as:

$$\langle \phi(\mathcal{N})^2 \rangle = \left(\frac{H}{2\pi}\right)^2 \langle \mathcal{N} \rangle$$

$$\langle \mathcal{N} \rangle = \frac{-\phi_- \phi_+}{\left(\frac{H}{2\pi}\right)^2} = \left(\frac{\phi_+}{\frac{H}{2\pi}}\right)\left(\frac{-\phi_-}{\frac{H}{2\pi}}\right)$$

Part 1: Inflation with Stochastic Boundaries

A. Nassiri-Rad, K. A., H. Firouzjahi, PRD 106, 123528 (2022)

Now consider a situation where the boundaries undergo a Brownian motion

What is the motivation?

consider the model containing two fields: the inflaton ϕ and the spectator field σ. The field σ is massless, and it does not contribute to potential and the background expansion. However, it affects the surface of the end of inflation.

Inflation terminates on a surface $f(\phi , \sigma) = 0$.

As the field σ modulates the surface of the end of inflation, its perturbations generate an additional contribution into the curvature perturbations:

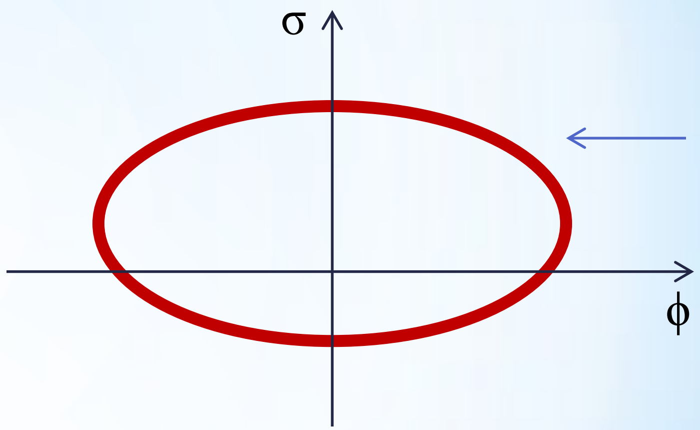
$$\mathcal{R} = -\frac{H}{\dot{\phi}}\delta\phi + \mathcal{R}_e$$

represents the curvature perturbations induced from the surface of the end of inflation.

An example:

consider a model in which the surface of the end of inflation is determined by the ellipse:

$$\lambda_\phi \phi^2 + \lambda_\sigma \sigma^2 = M^2$$



in the presence of the spectator field, there can be additional perturbations generated at the surface of the end of inflation via:

$$\delta\phi_e = -\frac{\lambda_\sigma \sigma_e}{\lambda_\phi \phi_e}\delta\sigma.$$

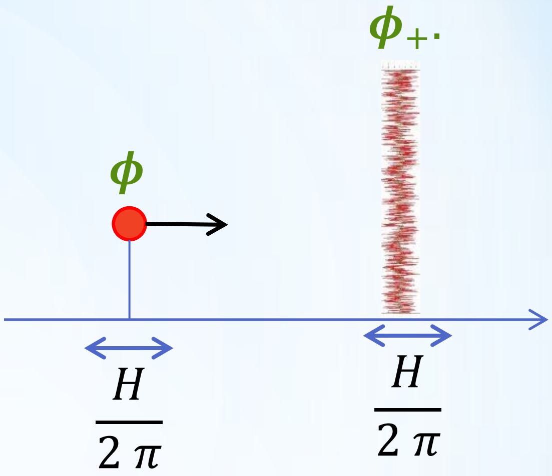
the total curvature perturbation : $$\mathcal{R} = -\frac{H}{\dot\phi}\left(\delta\phi - \frac{\lambda_\sigma \sigma_e}{\lambda_\phi \phi_e}\delta\sigma\right)$$

And its power spectrum : $$\mathcal{P}_\mathcal{R} = \frac{H^2}{8\pi^2 M_P^2 \epsilon_e}\left[1 + \left(\frac{\lambda_\sigma \sigma_e}{\lambda_\phi \phi_e}\right)^2\right]$$

A. Nassiri-Rad, K. A., H. Firouzjahi, arXiv:2208.08229

Stochastic Boundary in Drift-Dominated Regime:

Now consider an slow-roll inflationary model which is drift-dominated. The position of the stochastic boundary is denoted by $\phi_+(N)$.



The stochastic equation of motion for:

The field: $$\phi(N) = \phi_0 + CN + AW(N),$$

The boundary: $$\phi_+(N) = \phi_+^{(0)} + BW_+(N)$$

We rewrite it for the case the boundary is fixed and the stochastic nature of the boundary is transferred to the inflaton field:

$$\phi(N) = \phi_0 + CN + \mathrm{D}W(N),$$

A. Nassiri-Rad, K. A., H. Firouzjahi, arXiv:2208.08229

Stochastic Boundary in Drift-Dominated Regime:

Slow-Roll limit:

$$\langle \mathcal{N} \rangle = \frac{\phi_+^{(0)} - \phi_0}{C},$$

$$\langle \mathcal{N}^2 \rangle = \langle \mathcal{N} \rangle^2 + \frac{A^2 + B^2}{C^2} \langle \mathcal{N} \rangle,$$

After obtaining the moments we obtain the power spectrum using the stochastic $\delta\mathcal{N}$ formalism:

$$\mathcal{P}_{\mathcal{R}} = \frac{d\delta\mathcal{N}^2}{d\langle \mathcal{N} \rangle}, \longrightarrow \mathcal{P}_{\mathcal{R}} = \frac{A^2 + B^2}{C^2} = \frac{H^2}{8\pi^2 M_P^2 \epsilon} \left(1 + \frac{B^2}{A^2}\right)$$

Similar to the studied example in two-fields inflationary model

$$\mathcal{P}_{\mathcal{R}} = \frac{H^2}{8\pi^2 M_P^2 \epsilon_e} \left[1 + \left(\frac{\lambda_\sigma \sigma_e}{\lambda_\phi \phi_e}\right)^2\right]$$

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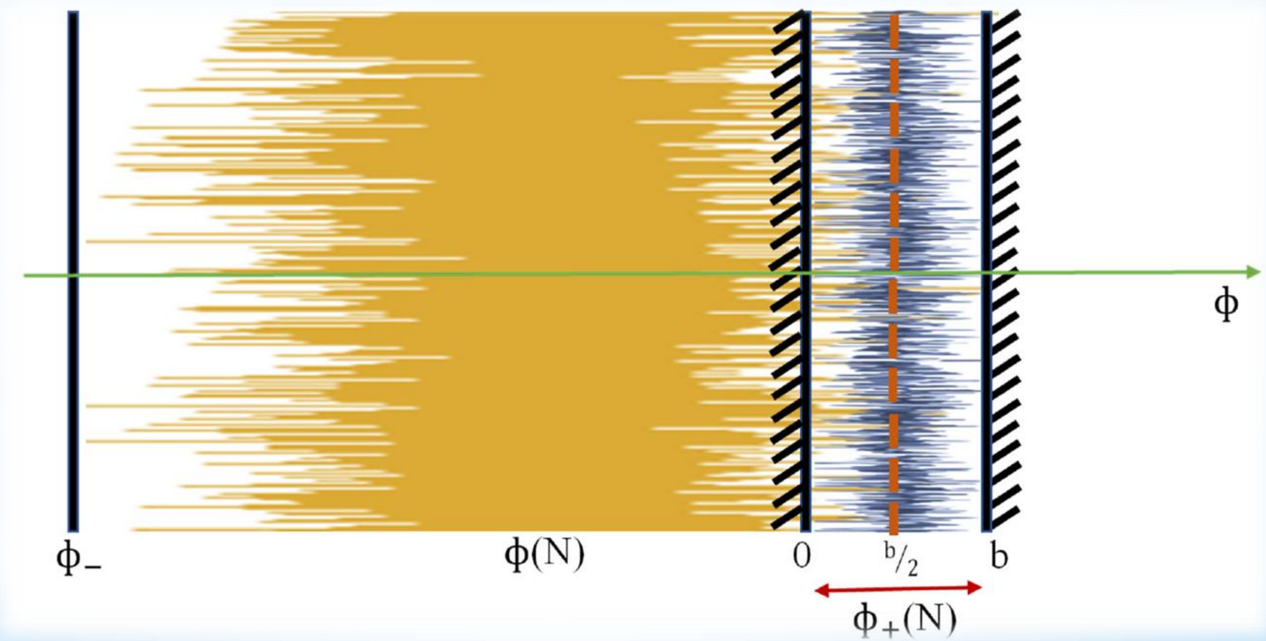
Diffusion-Dominated Regime:

A schematic view of the setup in the case of the diffusion-dominated regime.

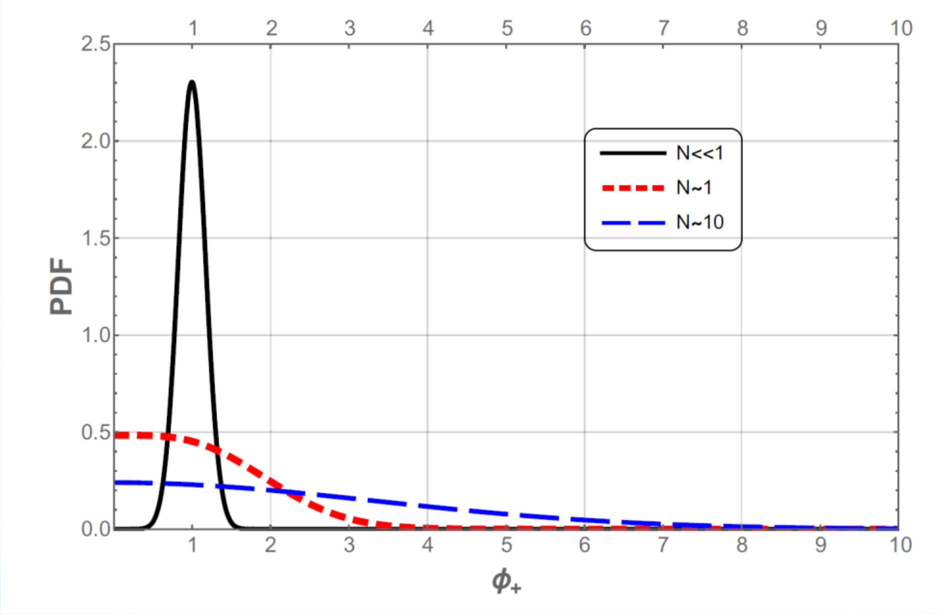


The stochastic behavior of the field is shown by an orange noise, while that of the right boundary is denoted by a blue noise.

A. Nassiri-Rad, K. A., H. Firouzjahi, arXiv:2208.08229

Diffusion-Dominated Regime

The probability density function for a Brownian boundary without a right barrier ($b \longrightarrow \infty$).



$\phi_- = 0$,

$\phi_+^{(0)} = 1$

It can be seen that, at very early time $N \ll 1$, the field has a Gaussian distribution with a maximum around ϕ_+.
However, as time passes, the maximum of distribution is shifted toward ϕ_-, which is a consequence of ϕ_- being a reflective boundary.

A. Nassiri-Rad, K. A., H. Firouzjahi, arXiv:2208.08229

Diffusion-Dominated Regime:

the evolution of the field and the right boundary is given as follows

$$\phi_+(N) = \phi_+^{(0)} + D\tilde{W}_+(N),$$

$$\phi(N) = \phi_0 + W(N),$$

the time-dependent probability density function (PDF) associated to the Brownian movement of the right boundary, f_+, is described by the Fokker-Planck equation as follows

$$\frac{\partial f_+(\phi_+, N)}{\partial N} = \frac{D^2}{2}\frac{\partial^2 f_+(\phi_+, N)}{\partial \phi_+^2}$$

As ϕ_+ is limited in the interval [0,b] with reflective barriers, we have the following Neumann boundary conditions:

$$\left.\frac{\partial f_+(\phi_+, N)}{\partial \phi_+}\right|_{\phi_+=0} = \left.\frac{\partial f_+(\phi_+, N)}{\partial \phi_+}\right|_{\phi_+=b} = 0,$$

And the initial condition as:

$$f_+(\phi_+, N = N_0) = \delta(\phi_+ - \phi_+^{(0)})$$

The solution for f_+ gives:

$$f_+(\phi_+, N) = \frac{1}{b} + \frac{2}{b}\sum_{m=1}^{\infty} \cos\left(\frac{m\pi}{b}\phi_+^{(0)}\right) \cos\left(\frac{m\pi}{b}\phi_+\right) e^{-\frac{m^2\pi^2 D^2}{2b^2}N}$$

A. Nassiri-Rad, K. A., H. Firouzjahi, arXiv:2208.08229

Diffusion-Dominated Regime:

the Fokker-Planck equation governing the stochastic dynamics of the field is given by

$$\frac{\partial f(\phi, N|\phi_0, N_0)}{\partial N} = \frac{1}{2}\frac{\partial^2 f(\phi, N|\phi_0, N_0)}{\partial \phi^2}$$

Whose solution is

$$f(\phi, N|\phi_0, N_0) = \frac{1}{\sqrt{2\pi(N-N_0)}}\exp\left(-\frac{(\phi-\phi_0)^2}{2(N-N_0)}\right)$$

A. Nassiri-Rad, K. A., H. Firouzjahi, arXiv:2208.08229

Diffusion-Dominated Regime

First Crossing Probability:

$$
\begin{aligned}
p_+ &= 1 - p_- \\
&= \frac{2(\phi_0 - \phi_-)}{b - 2\phi_-} - 4b \sum_{m=1}^{\infty} \left[\frac{(-1)^m - 1}{m^2 \pi^2 (b - 2\phi_-)} \right] \\
&\quad \times \Gamma^+ \left(\frac{m^2 \pi^2 D^2}{2b^2} \right) \cos\left(\frac{m\pi}{b} \phi_+^{(0)} \right).
\end{aligned}
$$

LO

$$
p_+^{\mathrm{LO}} = \frac{2(\phi_0 - \phi_-)}{b - 2\phi_-}
$$

NLO

$$
p_+^{\mathrm{NLO}} = p_+^{\mathrm{LO}} + \frac{8b}{\pi^2 (b - 2\phi_-)} \cos\left(\frac{\pi \phi_+^{(0)}}{b} \right) Y
$$

A. Nassiri-Rad, K. A., H. Firouzjahi, arXiv:2208.08229

Diffusion-Dominated Regime

Mean number of e-folds:

We note that while the clock N is a deterministic variable, $\mathcal{N}$, the number of e-folds hitting either of the boundaries, is a stochastic variable.

$$\langle\mathcal{N}\rangle^{\rm LO} = \frac{(\phi_0 - \phi_-)}{3(b-2\phi_-)}\left[2b^2 - 3b(\phi_- + \phi_0) + 6\phi_-\phi_0\right].$$

$$\langle\mathcal{N}\rangle^{\rm NLO} = \langle\mathcal{N}\rangle^{\rm LO} - \frac{4bY}{3\pi^2(b-2\phi_-)}\left[b^2 - 6b\phi_- + 6\phi_-^2\right] \cos\left(\frac{\pi\phi_+^{(0)}}{b}\right)$$

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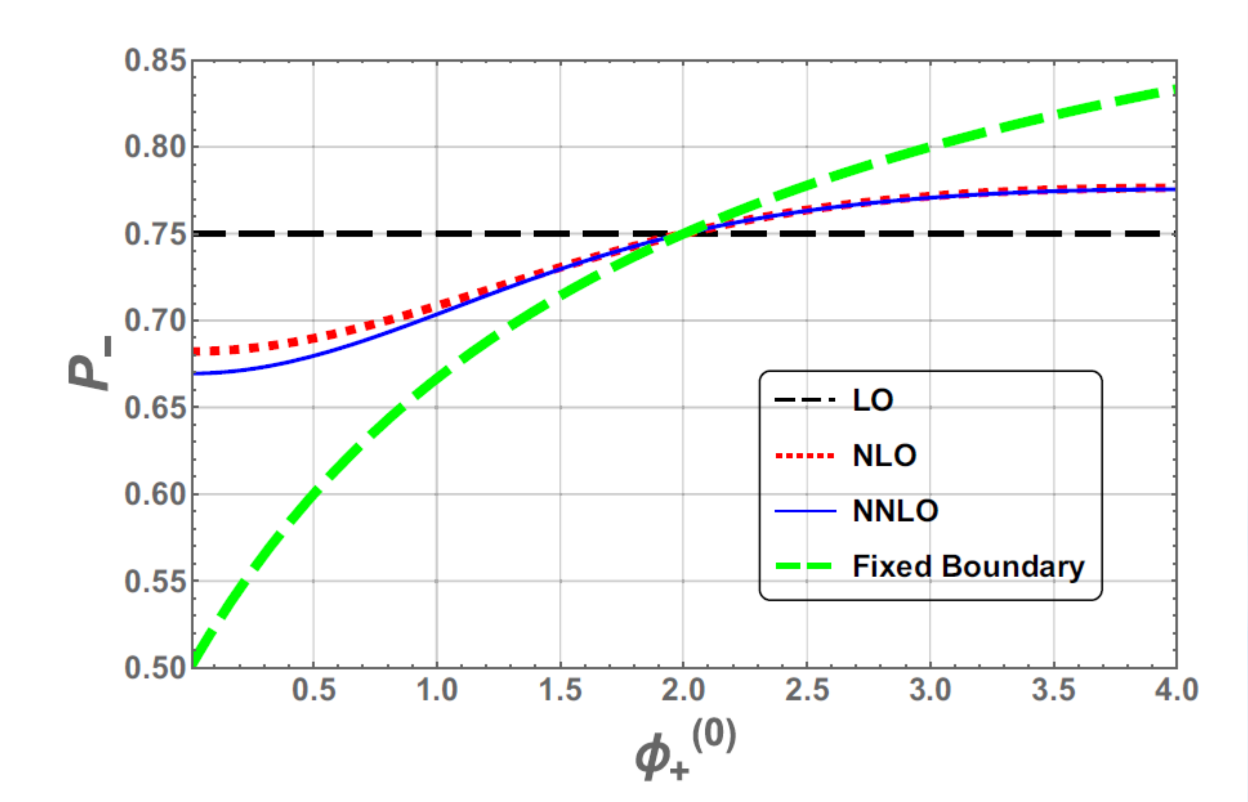
Diffusion-Dominated Regime

First Crossing Probability:

b = 4



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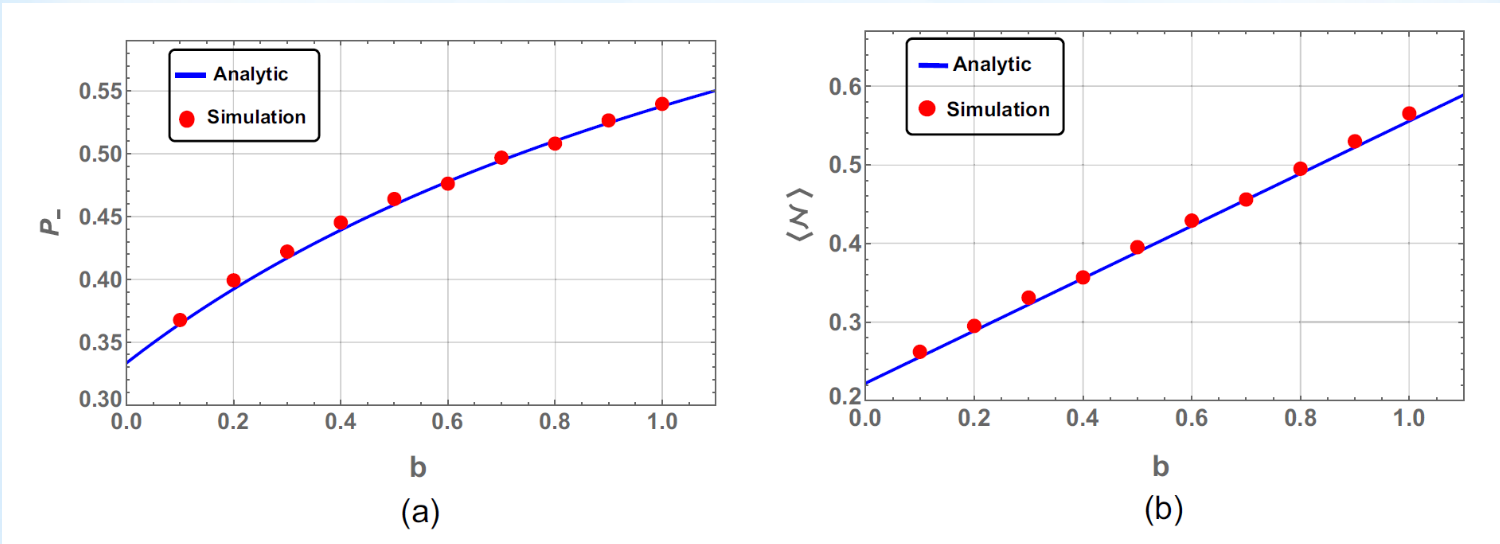
Boundary with Uniform Distribution:

$$\phi_- = -1 \ \text{and} \ \phi_0 = -\frac{1}{3}.$$



$$
\begin{aligned}
p_+ &= \int_0^b p(\phi \,\text{hits}\, \phi_+ \,\text{first} | \phi_+ = y)\, \frac{dy}{b} \\
&= \int_0^b \left(\frac{\phi_0 - \phi_-}{y - \phi_-} \right) \frac{dy}{b} \\
&= \left(\frac{\phi_0 - \phi_-}{b} \right) \ln \left(\frac{\phi_- - b}{\phi_-} \right).
\end{aligned}
$$

$$
\begin{aligned}
\langle \mathcal{N} \rangle &= \int_0^b (\phi_0 - \phi_-)(y - \phi_0)\, \frac{dy}{b} \\
&= (\phi_0 - \phi_-) \left(\frac{b}{2} - \phi_0 \right).
\end{aligned}
$$

A. Nassiri-Rad, K. A., H. Firouzjahi, arXiv:2208.08229

Summary of Part 1:

- Within the context of stochastic inflation, we have studied the Brownian motion of a field which is restricted to move between two boundaries one of them is fixed at a constant value, while the other one undergoes a Brownian motion.

- We have presented the Langevin equation in various related examples and have calculated the mean number of e-folds and the first hitting probabilities for the field to hit either of the boundaries.

Motivation: surface of the end of inflation is modulated by a light spectator field

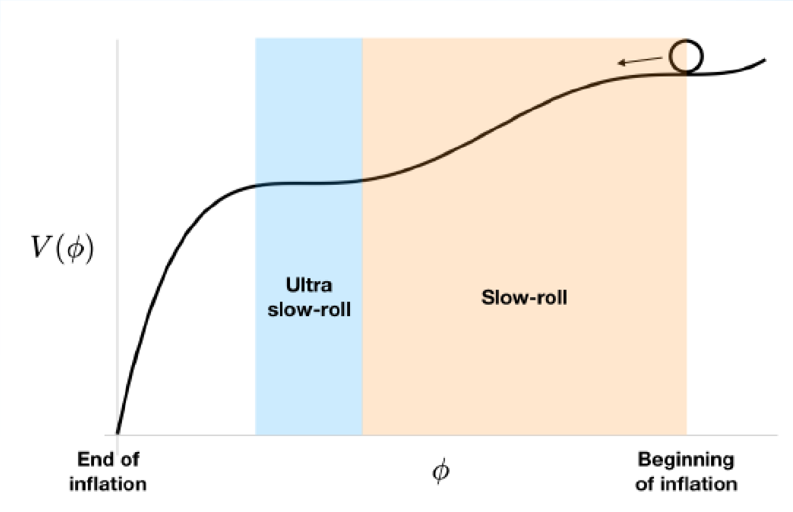
we studied: → classical drift- dominanted regime

→ diffusion dominated

Stochastic Multiple Fields Inflation: Diffusion Dominated Regime

We study multiple fields inflation in diffusion dominated regime using stochastic δN formalism.

This setup can be realized towards the final stages of the ultra slow-roll setup where the classical drifts fall off exponentially and the perturbations are driven by quantum kicks.



$\dot{\phi} \propto a^{-3}$ $\epsilon \propto a(t)^{-6}$ $\mathcal{R} \propto a^{3}$

We consider both symmetric and asymmetric boundaries with absorbing and reflective boundary conditions and calculate the average number of e-folds, the first crossing probabilities and the power spectrum.

We study the primordial black holes (PBHs) formation in this setup and calculate the mass fraction and the contribution of PBHs in dark matter energy density for various higher dimensional field spaces.

Primordial Black Holes Formation:

Quantum fluctuations generated during inflation are usually described by curvature perturbations. In the range of scales accessible to CMB observations, these fluctuations are constrained to be at the order $\mathcal{R} \sim 10^{-5}$

On the other hand, at smaller scales it can grow by few orders of magnitude, say $\mathcal{R} \sim 10^{-1}$ to seed the primordial black holes (PBHs) formation as a candidate for dark matter.

Models of ultra slow-roll (USR) inflation are well studied in recent years as a setup to generate PBHs. This is because in USR setups the **potential is very flat** and curvature perturbations can grow on super-horizon scales.

As the potential is flat, one expects that the quantum diffusion effects to play important roles during USR setup.

The limit of diffusion domination takes place at the final stage of USR setup where the fields' velocities are exponentially damped and the corresponding Langevin equations are dominated by the quantum diffusion terms.

Symmetric Boundaries

- **2-D Circular Boundary**
  - **Absorbing**
  - **Mixed**

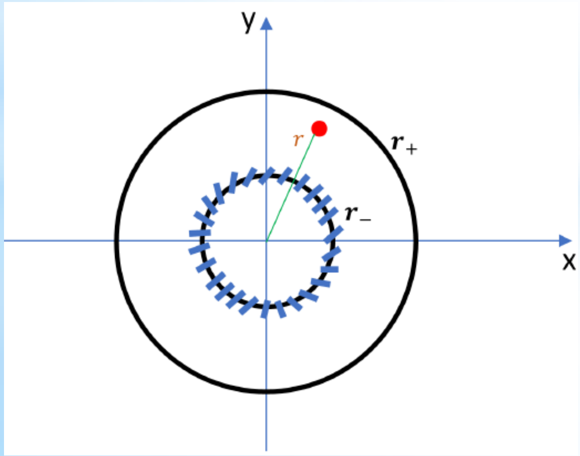
    Power Spectrum

    $$\mathcal{P}_{\mathcal{R}} = \frac{d\langle\delta\mathcal{N}^2\rangle}{d\langle\mathcal{N}\rangle}$$

    $$\mathcal{P}_{\mathcal{R}} = \frac{2\pi^2}{H^2}\left[\frac{4r_+^4}{r_+^2 - r^2}\ln\left(\frac{r_+}{r}\right) - \left(3r_+^2 - r^2\right)\right]$$

- **Generalization to n-D**
  - **Absorbing**
  - **Mixed**

    $$\mathcal{P}_{\mathcal{R}} = \frac{16\pi^2\left[r_+^n\left(n^2 r_+^2 - (n^2-4)\,r^2\right) + r^2\Big((n-2)r^n - (n+2)r^{-n}r_+^{2n}\Big)\right]}{n\,(n^2-4)\left(r^n - r_+^n\right)H^2}$$



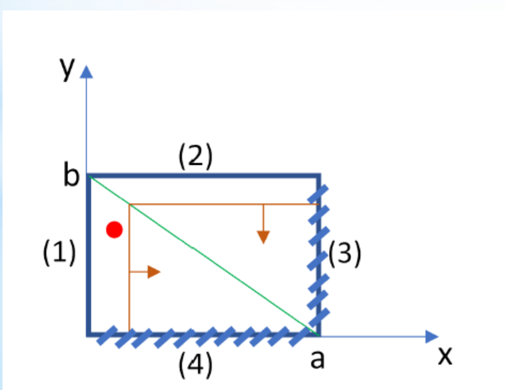
K. A. , A. Nassiri-Rad, H. Firouzjahi, arXiv:2304.00577

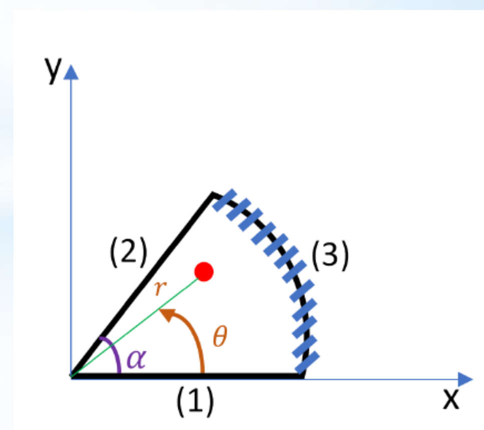
- Asymmetric Boundaries
  - Rectangle
    - all absorbing
    - Mixed
  - Sector
    - Absorbing

$$\langle \mathcal{N} \rangle = -4\alpha^2 \sum_{k=1}^{\infty} \left[\frac{k\pi r^2 - 2\alpha R^2 \left(\frac{r}{R}\right)^{\frac{k\pi}{\alpha}}}{\pi^2 k^2 \left(k^2\pi^2 - 4\alpha^2\right)} \right] \left(\cos(k\pi) - 1\right) \sin\left(\frac{k\pi}{\alpha}\theta\right)$$

$$p_1 = -\frac{\theta}{\alpha} + 1$$

    - Mixed





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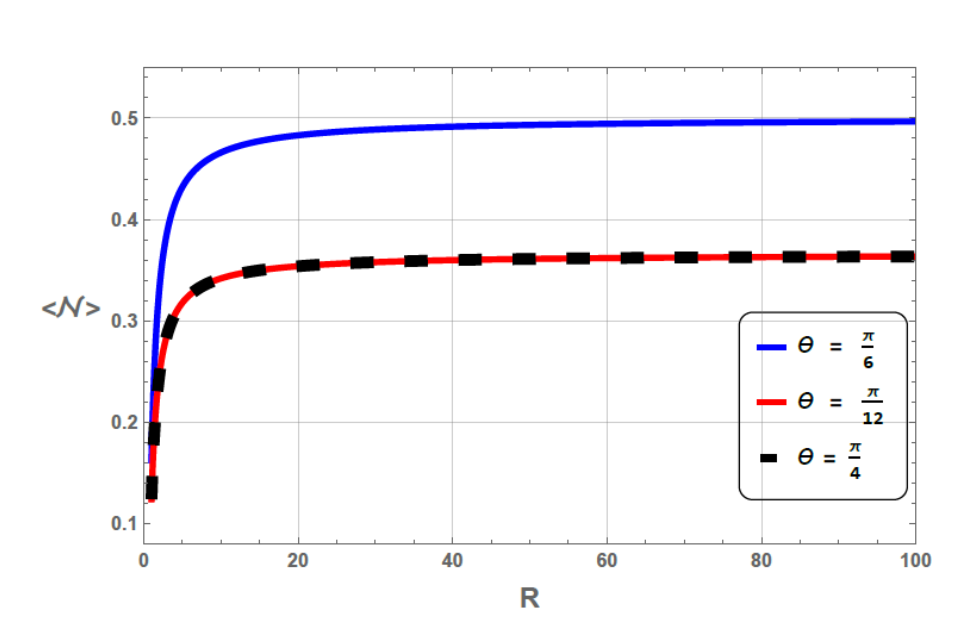
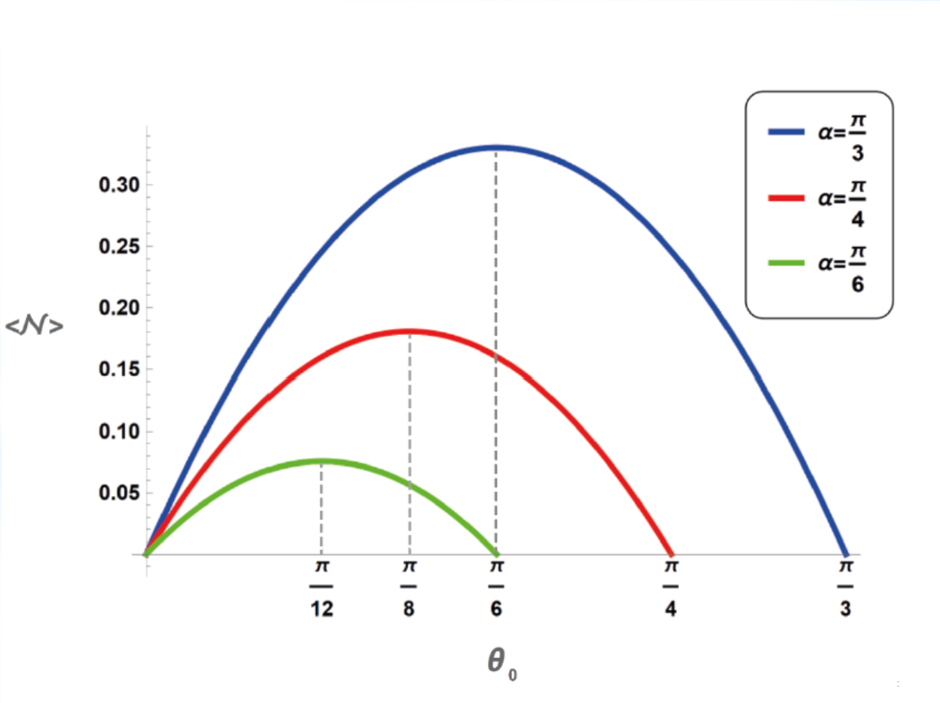
Sector





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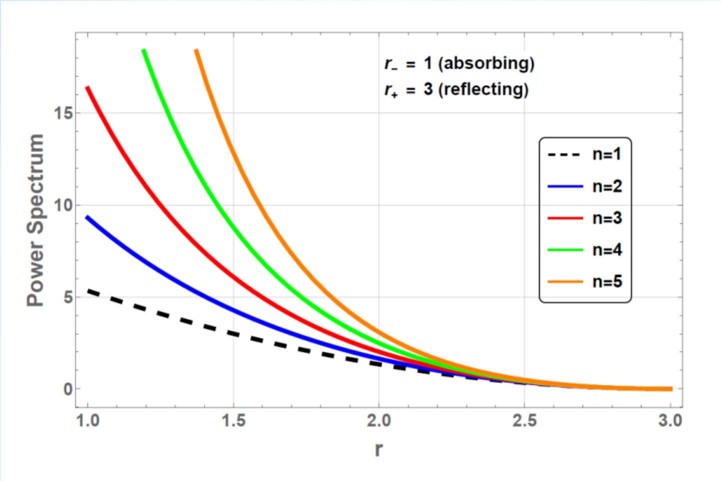
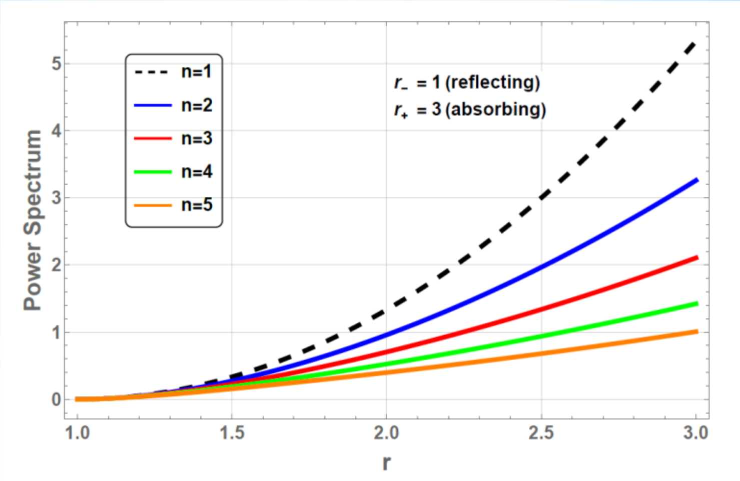




The behaviour of power spectrum versus r for various dimensions and for two different boundary conditions. In the left panel the inner boundary is absorbing while the outer one is reflective. As the dimension of field space increases the power spectrum increases as well. In the right panel the position of the reflective and absorbing boundaries are switched in which we see the opposite trend compared to the left panel.

Primordial Black Holes Formation

The mass fraction of PBHs against the total dark matter density is given by:

$$f_{PBH}(M) = \frac{\Omega_{PBH}}{\Omega_{DM}} = 2.7 \times 10^8 \left(\frac{M_{PBH}}{M_\odot}\right)^{\frac{-1}{2}} \beta(M),$$

$$\beta(M) \sim \int_{\mathcal{R}_c}^{\infty} P(\mathcal{R}) d\mathcal{R}$$

The observational constraints for the heavy PBHs:

$10^{16}\text{gr} < M < 10^{50}\text{gr}$ → $\beta < 10^{-11}$ to $\beta < 10^{-5}$

$10^{9}\text{gr} < M < 10^{16}\text{gr}$ → $\beta < 10^{-24}$ to $\beta < 10^{-17}$

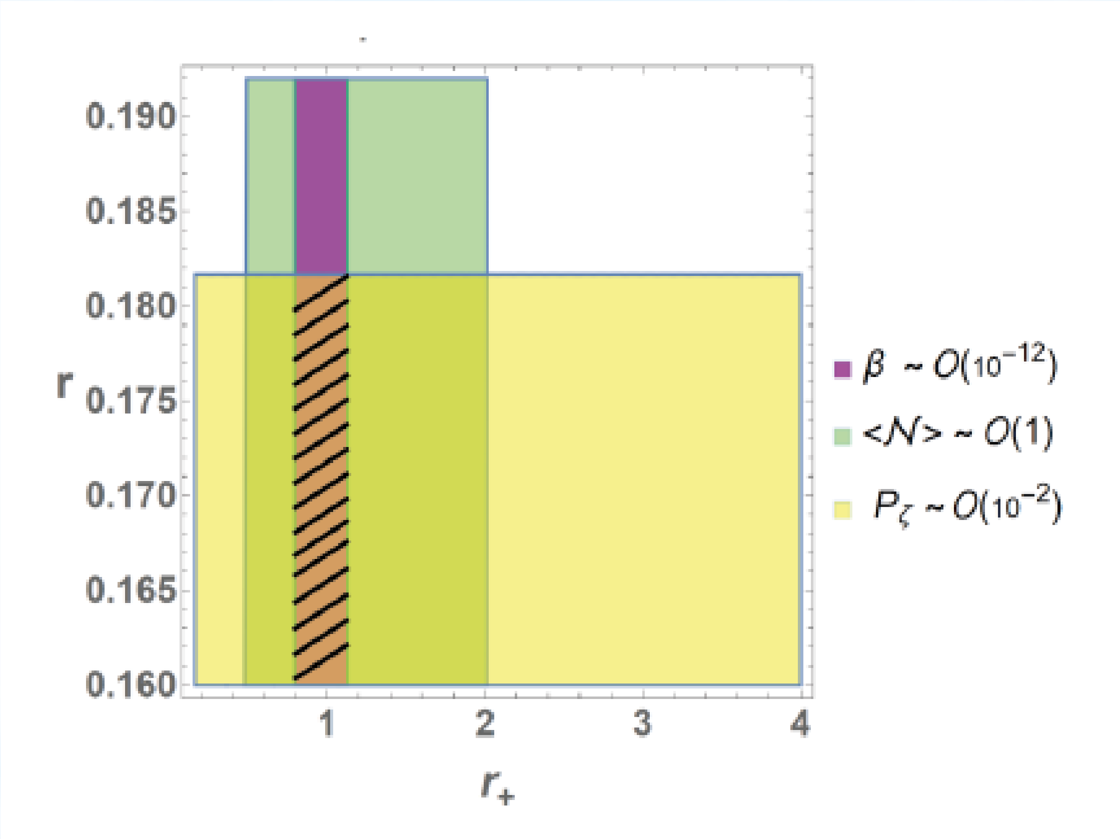
PDF of n-Sphere:

$$P(\mathcal{N}, r) \simeq \frac{\pi \sin\left(\frac{\pi r}{r_+}\right)}{r r_+} e^{-\frac{\pi^2}{2r_+^2}\mathcal{N}}$$

PDF of the Sector:

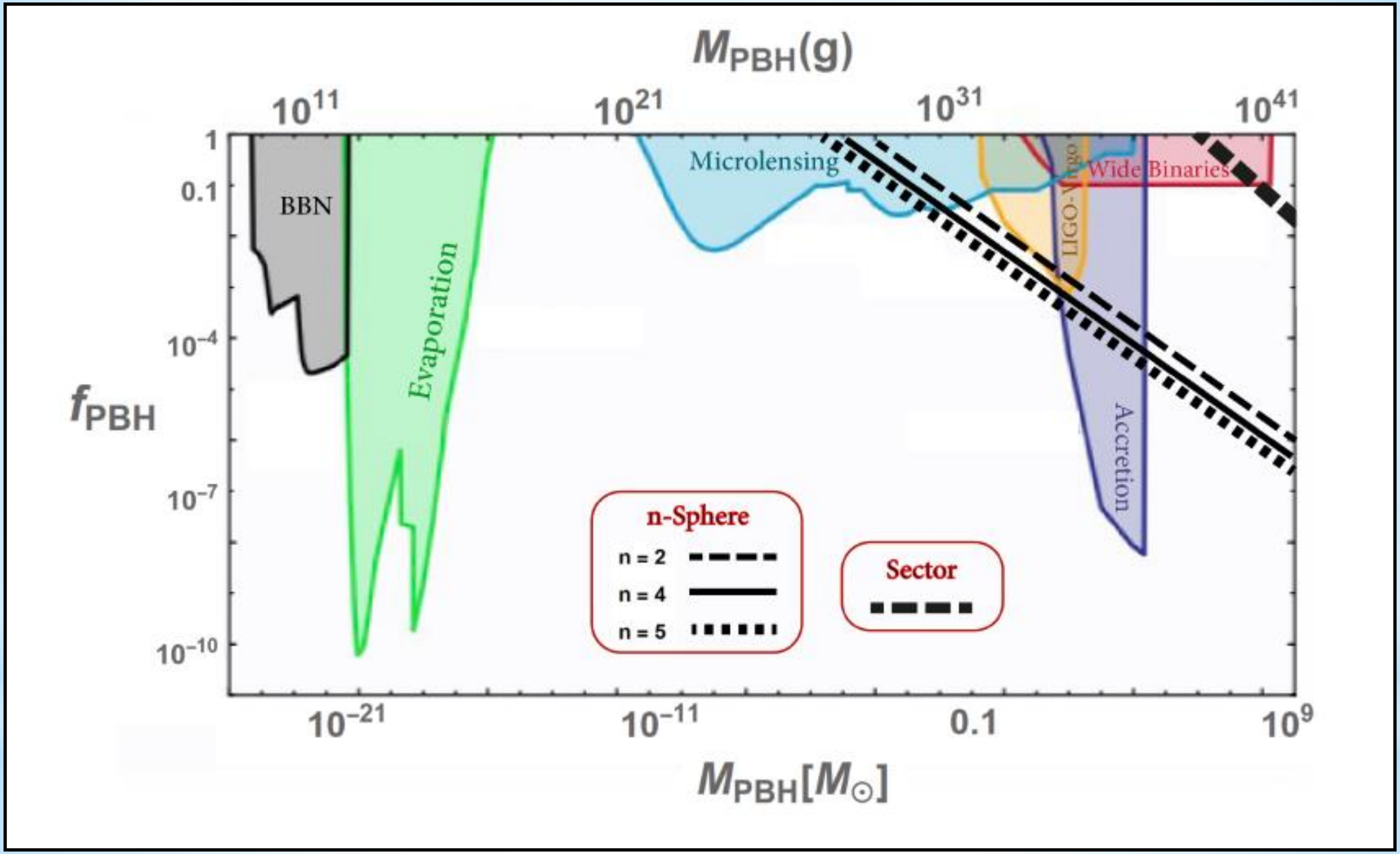
$$P(\mathcal{N}) \simeq -\frac{2^{3-\frac{\pi}{2\alpha}} r_{01}^{\frac{\pi}{2\alpha}} \cos\left(\frac{\pi\theta}{2\alpha}\right) e^{-\frac{\mathcal{N} r_{01}^2}{2R^2}} J_{\frac{\pi}{2\alpha}}\left(\frac{r r_{01}}{R}\right) {}_1F_2\left(\frac{\pi}{4\alpha}; 1+\frac{\pi}{4\alpha}, 1+\frac{\pi}{2\alpha}; -\frac{r_{01}^2}{4}\right)}{\alpha R^2 \Gamma\left(1+\frac{\pi}{2\alpha}\right)\left[-2J_{\frac{\pi}{2\alpha}}(r_{01}) + J_{\frac{\pi}{2\alpha}-2}(r_{01}) + J_{2+\frac{\pi}{2\alpha}}(r_{01})\right] J_{\frac{\pi}{2\alpha}}(r_{01})}$$

K. A. , A. Nassiri-Rad, H. Firouzjahi, arXiv:2304.00577



K. A. , A. Nassiri-Rad, H. Firouzjahi, arXiv:2304.00577

The behavior of f_{PBH} as a function of the mass of the formed PBHs



K. A. , A. Nassiri-Rad, H. Firouzjahi, arXiv:2304.00577

Summary and Conclusion:

we have studied multiple fields inflation in the diffusion dominated regime using the stochastic δN formalism.

Our main motivation for this purpose was to consider the multiple fields USR setup in its final stages. We studied symmetric and asymmetric Boundaries

We finally studied the PBHs formation within our setup for various cases. As in single field USR setup, one expects that PBHs to form in this setup which may comprise all or part of the dark matter energy density.

$$f_{PBH} \sim 10^{-2} - 10^{-5}$$

Thanks for your attention