Quantum field corrections to the equation of state of freely-streaming cosmological matter

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Classical free-streaming

Flat RFLW space-time:

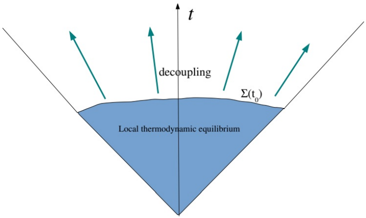
$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right).$$

Fixes the form of the stress-energy tensor SEMT

$$T^{\mu\nu} = (\varepsilon + p)\, u^\mu u^\nu - p g^{\mu\nu}, \quad u^\mu = (1, \mathbf{0})$$

Matter at local thermodynamic equilibrium (LTE) until freeze-out or decoupling.

Instantaneous decoupling: at $t = t_0$ transition from LTE to free evolution.



Boltzmann solution:

$$\varepsilon(t) = \frac{1}{(2\pi)^3 a^4(t)} \int \mathrm{dk}^3 \sqrt{\mathrm{k}^2 + m^2 a^2(t)} \left[\exp\left(\frac{\sqrt{\mathrm{k}^2 + m^2}}{T(t_0)} \right) - 1 \right]^{-1}$$

$$p(t) = \frac{1}{(2\pi)^3 a^4(t)} \int \mathrm{dk}^3 \frac{\mathrm{k}^2}{3\sqrt{\mathrm{k}^2 + m^2 a^2(t)}} \left[\exp\left(\frac{\sqrt{\mathrm{k}^2 + m^2}}{T(t_0)} \right) - 1 \right]^{-1}.$$

- $T(t_0)$: temperature at the decoupling. $\mathrm{k} = const.$ comoving momenta,
- $p > 0$ always. For $m \neq 0$, $p \propto a^{-5}(t)$; for $m = 0$, $p \propto a^{-4}(t)$.
- For $m = 0$, $\varepsilon = 3p$ exactly.

Goal of the work

Calculate ε and p using a full quantum approach.

Quantum scalar field $\widehat{\psi}$. It's free after decoupling:

$$\left(\nabla_\mu \nabla^\mu - m^2 + \xi R\right) \widehat{\psi} = 0,$$

$\xi R \Rightarrow$ Renormalizable, non-minimal coupling with the scalar curvature R.

- $\xi = 0$, minimal coupling.
- $\xi = 1/6$, conformal coupling, $m = 0 \implies T_\mu^\mu = 0$.

The SEMT is the RHS of the Einstein equations:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_\psi}{\delta g_{\mu\nu}}, \quad R_{\mu\nu} - \frac{g_{\mu\nu} R}{2} = 8\pi G T_{\mu\nu}, \quad \nabla_\mu T^{\mu\nu} = 0.$$

In a quantum theory it becomes a quantum operator:

$$\widehat{T}^{\mu\nu} = \nabla^\mu \widehat{\psi} \nabla^\nu \widehat{\psi} - \frac{g^{\mu\nu}}{2} \left(\nabla_\sigma \widehat{\psi} \nabla^\sigma \widehat{\psi} - m^2 \widehat{\psi}^2\right) + \xi \left(G^{\mu\nu} + g^{\mu\nu} \nabla_\sigma \nabla^\sigma - \nabla^\mu \nabla^\nu\right) \widehat{\psi}^2$$

Semiclassical Einstein equations:

The quantum expectation value of the SEMT is the RHS of semiclassical Einstein equations:

$$R_{\mu\nu} - Rg_{\mu\nu}/2 = 8\pi \left\langle \widehat{T}_{\mu\nu} \right\rangle_{ren} .$$

Semiclassical:

- Gravity, LHS, is classical,
- Matter,RHS, is quantum.

The expectation value is usually calculated in the true vacuum:

- The vacuum is not unique in curved space-time,
- The expectation value is divergent: Renormalization.

To be a proper RHS $\left\langle \widehat{T}^{\mu\nu} \right\rangle_{ren}$ must be conserved:

$$\nabla_{\mu} \left\langle \widehat{T}^{\mu\nu} \right\rangle_{ren} = 0$$

Quantum statistical approach

- Matter originated from a state of LTE at $t = t_0$,
- State of local equilbrium is obtained by maximizing entropy with the constraint of given energy density on some 3D hypersurface Σ:

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_\Sigma \mathrm{d}\Sigma_\mu \widehat{T}^{\mu\nu} \beta_\nu \right], \quad \beta_\mu = \frac{u_\mu}{T}$$

For the flat RFLW metric:

LTE Density operator

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \frac{\widehat{H}(t_0)}{T(t_0)} \right], \quad \widehat{H}(t) \doteq a^3(t) \int \mathrm{d}^3x \, \widehat{T}^{00}(t, \mathbf{x}) \, .$$

Thermal expectation value on $\widehat{\rho}$:

$$\left\langle \widehat{O} \right\rangle \doteq \mathrm{Tr}\left[\widehat{\rho} \widehat{O} \right] .$$

We calculate contribution of the excited (with respect to the vacuum) states to SEMT:

$$\left\langle \widehat{T}^{\mu\nu} \right\rangle \doteq \mathrm{Tr}\left[\widehat{\rho}\widehat{T}^{\mu\nu}\right] - \langle 0_{t_0} | \, \widehat{T}^{\mu\nu} \, | 0_{t_0} \rangle \, .$$

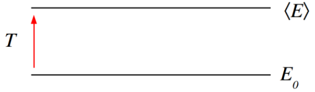
$|0_{t_0}\rangle$: vacuum of the theory at t_0. Is fixed in time $\Longrightarrow \nabla_\mu \left\langle \widehat{T}^{\mu\nu} \right\rangle = 0$.
The vacuum is the state of minimum "energy" E_0 at the decoupling:

$$\widehat{H}(t_0) \, |0_{t_0}\rangle = E_0 \, |0_{t_0}\rangle \, .$$

E_0 is the minimum eigenvalue of $\widehat{H}$:

$$E_0 = \int \mathrm{d}^3\mathrm{k}\,\omega_\xi\left(\mathrm{k}\right), \quad \omega_\xi(\mathrm{k}) = \sqrt{\mathrm{k}^2 + m^2 a^2 + 6\xi\left(1 - 6\xi\right)\frac{a'^2}{a^2}}$$

The subtraction of the minimum vacuum terms amounts to the subtraction of the $T\left(t_0\right) \to 0$ contribution to the SEMT.



Field expansion in normal modes:

$$\widehat{\psi}(\eta) = \frac{1}{a(\eta)\,(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathrm{k} \left(\widehat{a}_k v_k(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} + \mathrm{c.c} \right), \; \eta \doteq \int_{t_0}^{t} \frac{\mathrm{d}t'}{a(t')}$$

$\widehat{a}_k^\dagger, \widehat{a}_k$ creation/annihilation operators: $\left[\widehat{a}_k, \widehat{a}_{k'}^\dagger\right] = \delta^3(\mathrm{k}-\mathrm{k}')$.
$v_k \Rightarrow$ *mode functions*. They solve:

$$\frac{\mathrm{d}^2 v_k}{\mathrm{d}\eta^2} + \left(\mathrm{k}^2 + m^2 a^2(\eta) - (1-6\xi)\frac{a''}{a} \right) v_k(\eta) = 0, \quad \eta(t_0) = 0.$$

with initial conditions:

$$v_k(0) = \frac{1}{\sqrt{2\omega_\xi(0,\mathrm{k})}}, \quad \left.\frac{\mathrm{d}v_k}{\mathrm{d}\eta}\right|_{\eta=0} = -\frac{i}{2v_k(0)} + (1-6\xi)\,a'(0)v_k(0).$$

In flat space-time ($a=1$) we would have:

$$\ddot{v}_k + \omega_k^2 v_k = 0 \implies v_k = \frac{e^{-i\omega_k t}}{\sqrt{2\omega_k}}, \quad \omega_k = \sqrt{\mathrm{k}^2 + m^2}$$

With our choice for the vacuum $\widehat{\rho}$ is diagonal:

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-\frac{1}{T(0)} \int \mathrm{dk}^3 \omega_\xi\,(0,k) \left(\widehat{a}_k^\dagger \widehat{a}_k + \frac{1}{2} \right) \right].$$

Fundamental expectation values are known, e.g:

$$\mathrm{Tr}\left[\widehat{a}_k^\dagger \widehat{a}_{k'} \widehat{\rho} \right] = \left[n_B \left(\frac{\omega_\xi\,(0,k)}{T(0)} \right) + 1 \right] \delta^3 \left(\mathbf{k} - \mathbf{k}' \right)$$

$n_B(x) \Rightarrow$ Bose-Einstein distribution calculated at the decouling $\eta = 0$

$$n_B(x) = \frac{1}{e^x - 1}$$

To calculate ε and p:

- Plug the field expansion in $\widehat{T}^{\mu\nu}$,
- Calculate all the expectation values of $\widehat{a}_k \widehat{a}_k^\dagger$,
- Subtract the vacuum term.

Quantum energy density and pressure

For the energy density we have:

$$\varepsilon(\eta) = \frac{1}{(2\pi)^3 a^4(\eta)} \int \mathrm{d}^3\mathrm{k}\,\omega_k(\eta) K_k(\eta) n_B\left(\omega_\xi\left(0,k\right)/T(0)\right),$$

$$\omega_k = \sqrt{\mathrm{k}^2 + m^2 a^2 + (1-6\xi)\frac{a'^2}{a^2}},$$

$$K_k = \frac{1}{\omega_k}\left[\left|v_k'\right|^2 + \omega_k^2\left|v_k\right|^2 - 2\left(1-6\xi\right)\frac{a'}{a}\mathrm{Re}\left(v_k' v_k^*\right)\right]$$

while for the pressure

$$p(\eta) = \frac{1}{(2\pi)^3 a^4(\eta)} \int \mathrm{d}^3\mathrm{k}\,\omega_k(\eta) \Gamma_k(\eta) n_B\left(\omega_\xi\left(0,k\right)/T(0)\right),$$

$$\Gamma_k = \frac{1}{\omega_k}\left[\left|v_k'\right|^2 + \frac{1}{3}\gamma_k\left|v_k\right|^2 - 2\left(1-6\xi\right)\frac{a'}{a}\mathrm{Re}\left(v_k' v_k^*\right)\right]$$

$$\gamma_k = \left(12\xi - 1\right)\mathrm{k}^2 + \left(12\xi - 3\right) m^2 a^2 + \left(3 - 18\xi\right)\frac{a'^2}{a^2} - 12\xi\left(1-6\xi\right)\frac{a''}{a},$$

Discussion

Quantum corrections to ε and p, $(\xi = 0, 1/6)$:

$$\Delta\varepsilon(\eta) = \frac{1}{(2\pi)^3 a^4(\eta)} \int \mathrm{d}^3\mathrm{k} \left[\omega_k(\eta) K_k(\eta) - \sqrt{\mathrm{k}^2 + m^2 a^2}\right] n_B\left[\frac{\omega_k(0)}{T(0)}\right],$$

$$\Delta p(\eta) = \frac{1}{(2\pi)^3 a^4(\eta)} \int \mathrm{d}^3\mathrm{k} \left[\omega_k(\eta)\Gamma_k(\eta) - \frac{\mathrm{k}^2}{3\sqrt{\mathrm{k}^2 + m^2 a^2}}\right] n_B\left[\frac{\omega_k(0)}{T(0)}\right],$$

At the decoupling $(\eta = 0)$ they are precisely 0 but they are non vanishing thereafter. Can they become important at later times?

The behaviour of the corrections depends on the modes v_k which obey:

$$v_k'' + \Omega_k^2 v_k = 0, \quad \Omega_k = \sqrt{\mathrm{k}^2 + m^2 a^2 - (1 - 6\xi)\frac{a''}{a}}$$

Thus they depend on:

- The form of the scale factor $a(t)$,
- The value of the parameters H, m and $T(0)$.
- The value of the coupling ξ.

Negative pressures from field corrections

Corrections to pressure can lead to negative values.

E.g: $\xi = 0$, m>0. Corresponding pressures are asymptotically negative. This is better seen in cosmological time $t(\eta)$:

$$v_k'' + \Omega_k^2 v_k = 0 \implies \ddot{u}_k + 3H\dot{u}_k + \left(\frac{\mathrm{k}^2}{a^2} + m^2\right) u_k = 0, \quad u_k = a^{-1} v_k.$$

If $H \doteq \frac{\dot{a}}{a} \gg m \Rightarrow$ Overdamped oscillator: $u_k \sim \exp\left[-\frac{m^2 t}{3H}\right]$:

$$p \propto \int \mathrm{d}^3\mathrm{k} n_B \left[\frac{\omega_k(0)}{T(0)}\right] \left[|\dot{u}_k|^2 - \left(\frac{\mathrm{k}^2}{3a^2} + m^2\right)|u_k|^2\right], \quad p < 0 \text{ for } |\dot{u}_k| \ll m u_k.$$

In general quantum effects are dominant in non-adiabatic regimes:

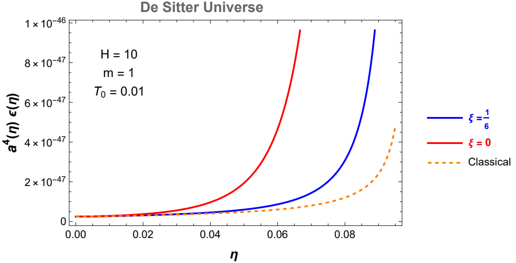
$$\frac{\Omega_k'}{\Omega_k^2} \gtrsim 1 \implies \frac{H}{m} \gtrsim 1.$$

In an adiabatic regime, slowly changing universe$\Rightarrow$ suppressed corrections

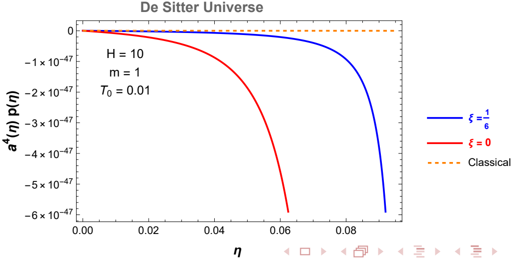
An example: Non adiabatic de Sitter Universe

$$a(t) \propto \exp(Ht) \implies a(\eta) = (1 - H\eta)^{-1}, \text{ constant rate: } H = const.$$

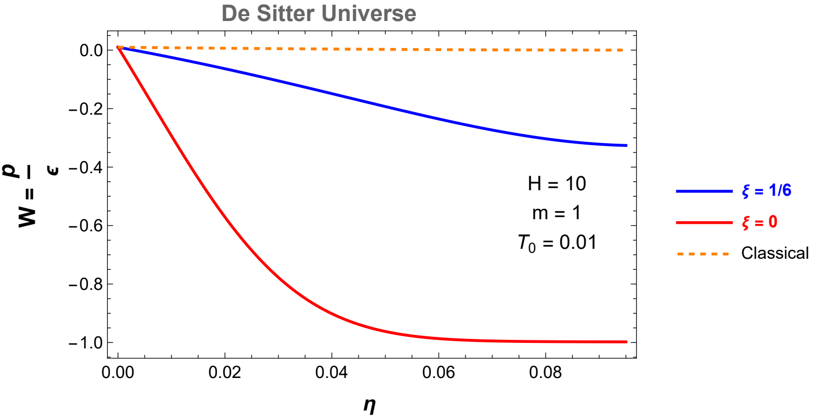
The energy densities is enhanced due to the cosmological particle production



The pressure is decreased by quantum effects and is negative at late times

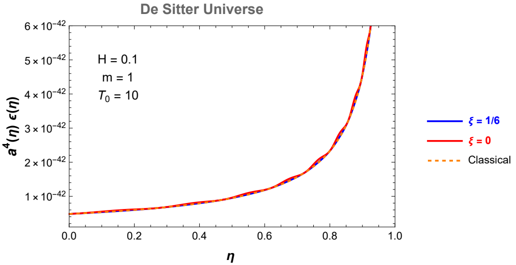


The resulting EoS for massive particles is significantly modified by quantum effects respect to the classical case:

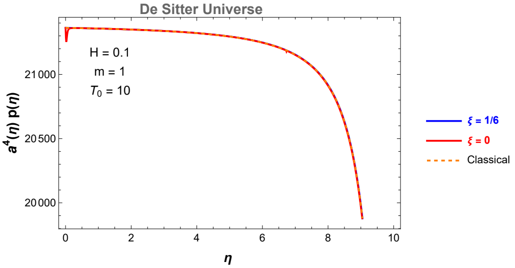


Adiabatic de Sitter Universe

The energy density coincides with the classical one for both the couplings:



The same for the pressure



In conclusion:

- Quantum corrections to ε and p arise due to the combined use of quantum field theory and quantum statistical mechanic.
- Such corrections can dominate at large times and strongly modify the equation of state.

Outlook and future perspectives:

- Find analytic and numeric solutions for more scale factors,
- Find the backreaction to the metric studying the self-consistent solution to the equation of motion.