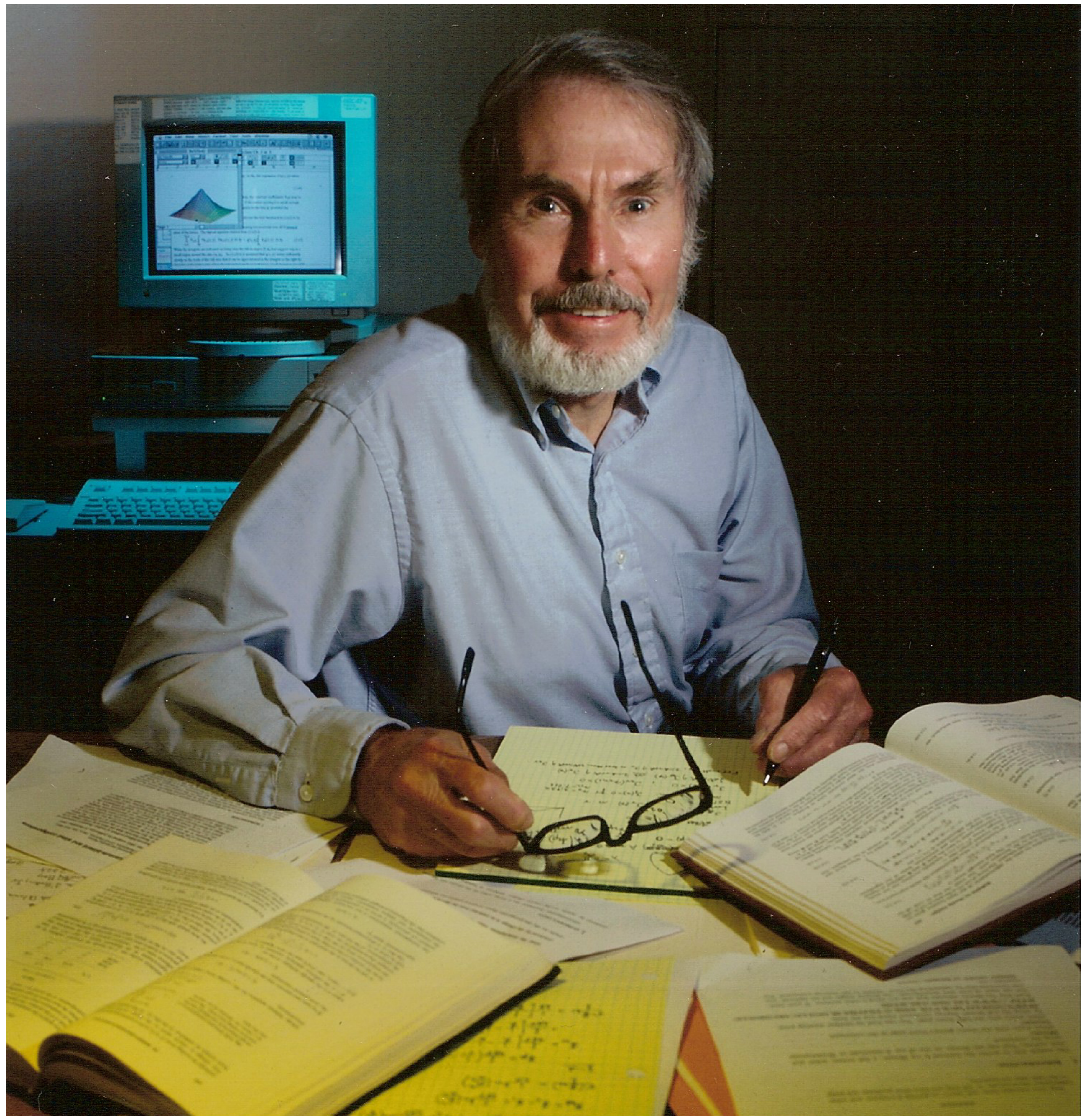
Rasmus Ischebeck

Machine Physics 1

Joint Universities Accelerator School

Radiation Emitted by Electrons in a Synchrotron

- Radiation emitted by electrons in a magnetic fields can be calculated from Maxwell's equations



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Lienard-Wiechert potentials (I)

We want to compute the em field generated by a charged particle in motion on a given trajectory $\bar{x} = \bar{r}(t)$

The charge density and current distribution of a single particle read

$$\rho(\bar{x},t) = q\delta^{(3)}(\bar{x} - \bar{r}(t)) \qquad \bar{J}(\bar{x},t) = q\bar{v}(t)\delta^{(3)}(\bar{x} - \bar{r}(t))$$

We have to solve Maxwell equations driven by such time varying charge density and current distribution.

The general expression for the wave equation for the em potentials (in the Lorentz gauge) reads

$$\bar{\nabla}^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \qquad \bar{\nabla}^2\bar{A} - \frac{1}{c^2}\frac{\partial^2\bar{A}}{\partial t^2} = -\mu_0\bar{J}$$

Lienard-Wiechert potentials (II)

The general solutions for the wave equation driven by a time varying charge and current density read (in the Lorentz gauge) [Jackson Chap. 6]

$$\Phi(\bar{x},t) = \frac{1}{4\pi\varepsilon_0}\int d^3\bar{x}'\int dt' \frac{\rho(\bar{x}',t')}{|\bar{x}-\bar{x}'|}\delta\left(t'+\frac{|\bar{x}-\bar{x}'|}{c}-t\right) \qquad A(\bar{x},t) = \frac{1}{4\pi\varepsilon_0 c^2}\int d^3\bar{x}'\int dt' \frac{\bar{J}(\bar{x}',t')}{|\bar{x}-\bar{x}'|}\delta\left(t'+\frac{|\bar{x}-\bar{x}'|}{c}-t\right)$$

Integrating the Dirac delta in time we are left with

$$\Phi(\bar{x},t) = \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\rho(\bar{x}',t_{ret})}{|\bar{x}-\bar{x}'|}d^3\bar{x}' \qquad \bar{A}(\bar{x},t) = \frac{\mu_0}{4\pi}\iiint_V \frac{\bar{J}(\bar{x}',t_{ret})}{|\bar{x}-\bar{x}'|}d^3\bar{x}'$$

where ret means retarted $t_{ret} = t - \frac{|\bar{x}(t)-\bar{x}(t_{ret})|}{c}$ **(see next slide)**

Now we use the charge density and current distribution of a single particle

$$\rho(\bar{x},t) = q\delta^{(3)}(\bar{x}-\bar{r}(t)) \qquad \bar{J}(\bar{x},t) = q\bar{v}(t)\delta^{(3)}(\bar{x}-\bar{r}(t))$$

Lienard-Wiechert potentials (III)

Substituting we get

$$\Phi(\overline{x},t)=\frac{q}{4\pi\varepsilon_0}\iiint_V\frac{\delta^{(3)}[\overline{x}'-\overline{r}(t_{ret})]}{|\overline{x}-\overline{x}'|}d^3\overline{x}' \qquad \overline{A}(\overline{x},t)=\frac{q\mu_0}{4\pi}\iiint_V\frac{\overline{v}(t_{ret})[\overline{x}'-\overline{r}(t_{ret})]}{|\overline{x}-\overline{x}'|}d^3\overline{x}'$$

Using again the properties of the Dirac deltas we can integrate and obtain the Lienard-Wiechert potentials

$$\Phi(\overline{x},t)=\frac{1}{4\pi\varepsilon_0}\left[\frac{e}{(1-\overline{\beta}\cdot\overline{n})R}\right]_{ret} \qquad \overline{A}(\overline{x},t)=\frac{1}{4\pi\varepsilon_0 c}\left[\frac{e\overline{\beta}}{(1-\overline{\beta}\cdot\overline{n})R}\right]_{ret}$$

These are the potentials of the em fields generated by the charged particle in motion.

The trajectory itself is determined by external electric and magnetic fields

Critical Frequency and Critical Angle

$$\frac{d^3 I}{d\Omega\, d\omega} = \frac{e^2}{16\pi^3 \varepsilon_0 c}\left(\frac{2\omega\rho}{3c\gamma^2}\right)^2 \left(1+\gamma^2\theta^2\right)^2 \left[K_{2/3}^2(\xi) + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2} K_{1/3}^2(\xi)\right]$$

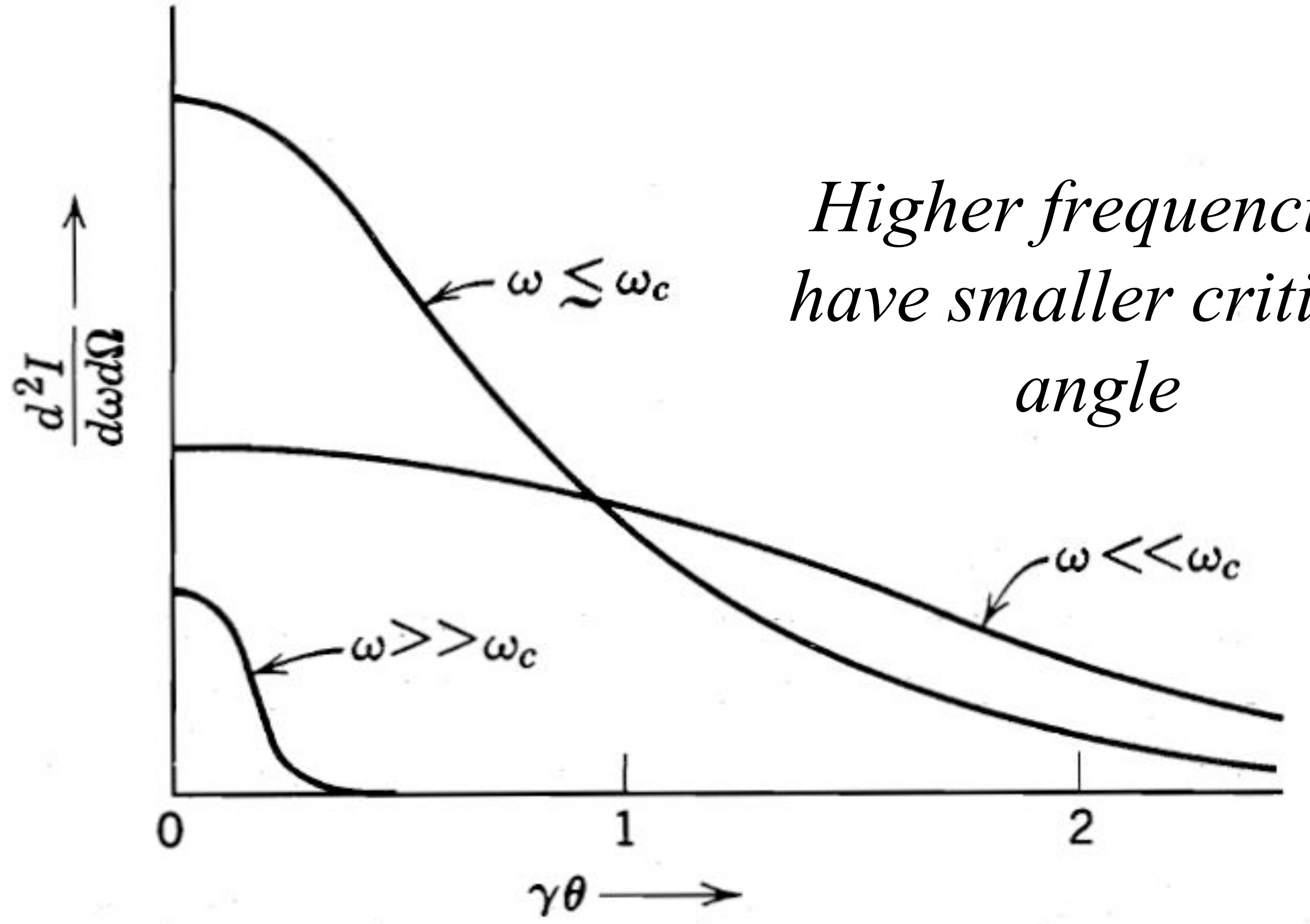
Properties of the modified Bessel function ==> radiation intensity is negligible for x >> 1

$$\xi = \frac{\omega\rho}{3c\gamma^3}\left(1+\gamma^2\theta^2\right)^{3/2} >> 1$$

Critical frequency $\omega_c = \frac{3}{2}\frac{c}{\rho}\gamma^3$

$$\approx \omega_{rev}\gamma^3$$

Critical angle $\theta_c = \frac{1}{\gamma}\left(\frac{\omega_c}{\omega}\right)^{1/3}$



Higher frequencies have smaller critical angle

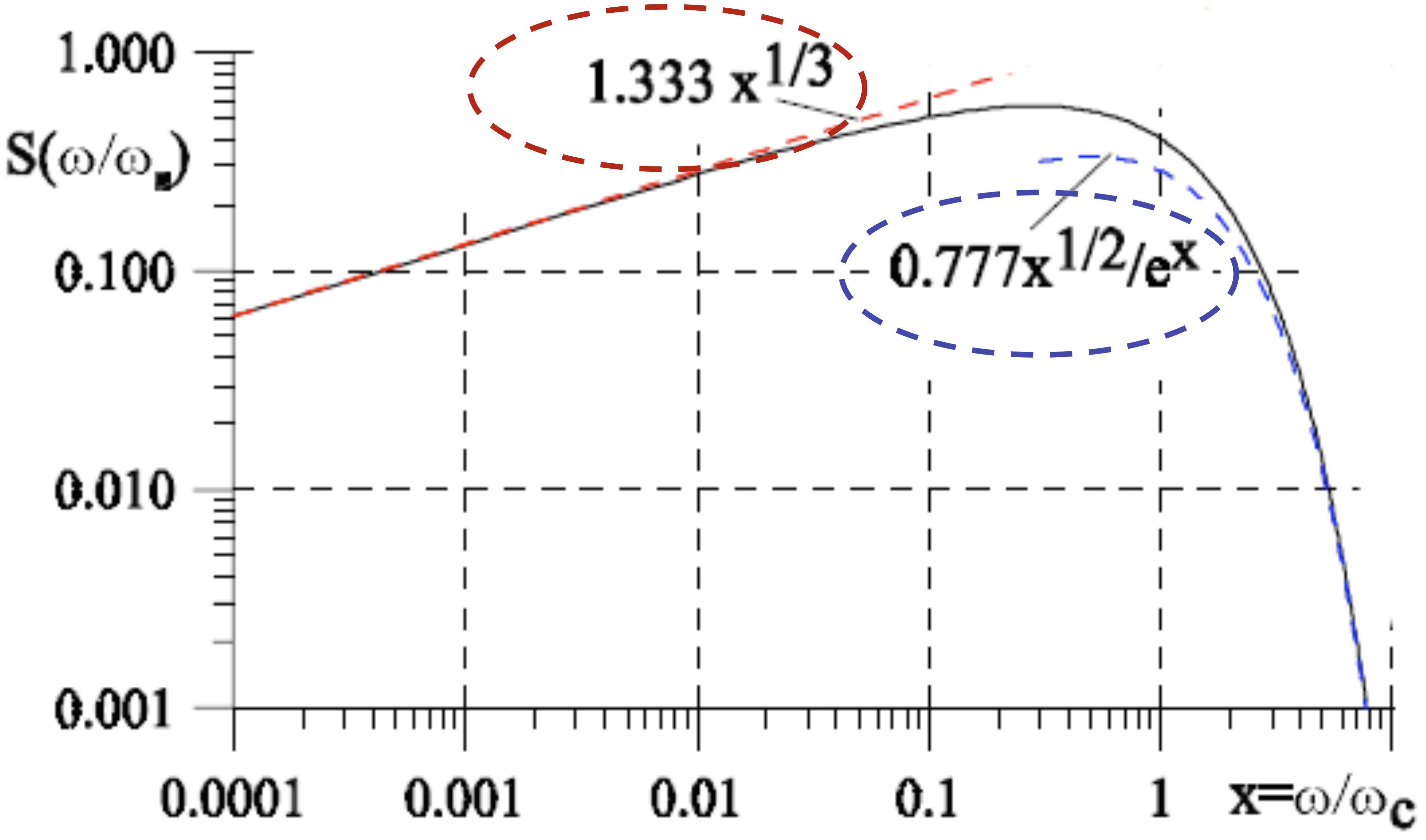
For frequencies much larger than the critical frequency and angles much larger than the critical angle the synchrotron radiation emission is negligible

Spectrum of Synchrotron Radiation

$$\frac{dI}{d\omega} = \iint_{4\pi} \frac{d^3 I}{d\omega\, d\Omega} d\Omega = \frac{\sqrt{3}e^2}{4\pi\varepsilon_0 c}\gamma \frac{\omega}{\omega_C} \int_{\omega/\omega_C}^{\infty} K_{5/3}(x)dx$$

$$\frac{dI}{d\omega} \approx \frac{e^2}{4\pi\varepsilon_0 c}\left(\frac{\omega\rho}{c}\right)^{1/3} \quad \omega \ll \omega_c$$

$$\frac{dI}{d\omega} \approx \sqrt{\frac{3\pi}{2}} \frac{e^2}{4\pi\varepsilon_0 c}\gamma\left(\frac{\omega}{\omega_c}\right)^{1/2} e^{-\omega/\omega_c} \quad \omega \gg \omega_c$$



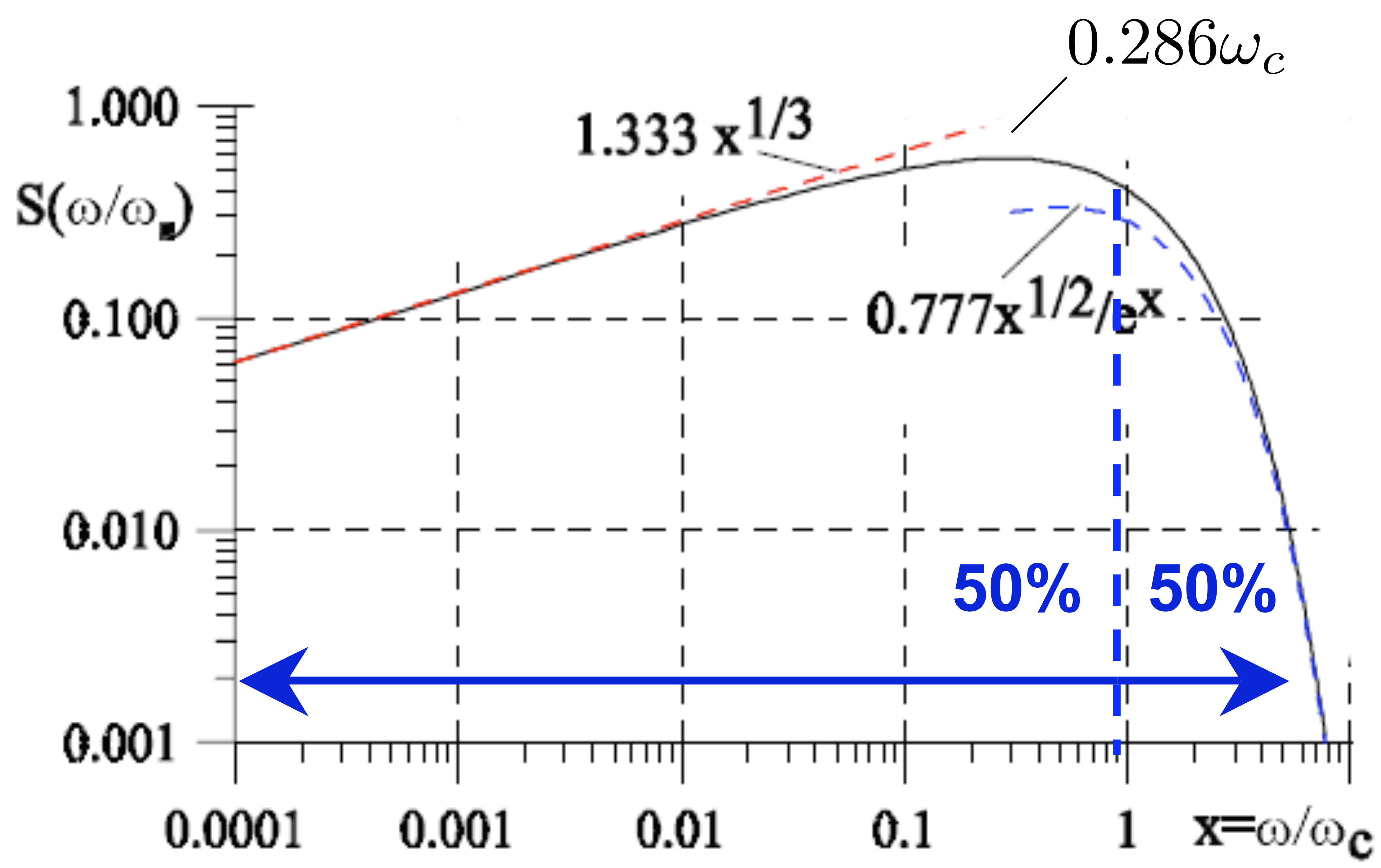
Bill Barletta

Frequency Distribution of Radiation

The integrated spectral density up to the critical frequency contains half of the total energy radiated, the peak occurs approximately at $0.286\omega_c$

where the critical photon energy is

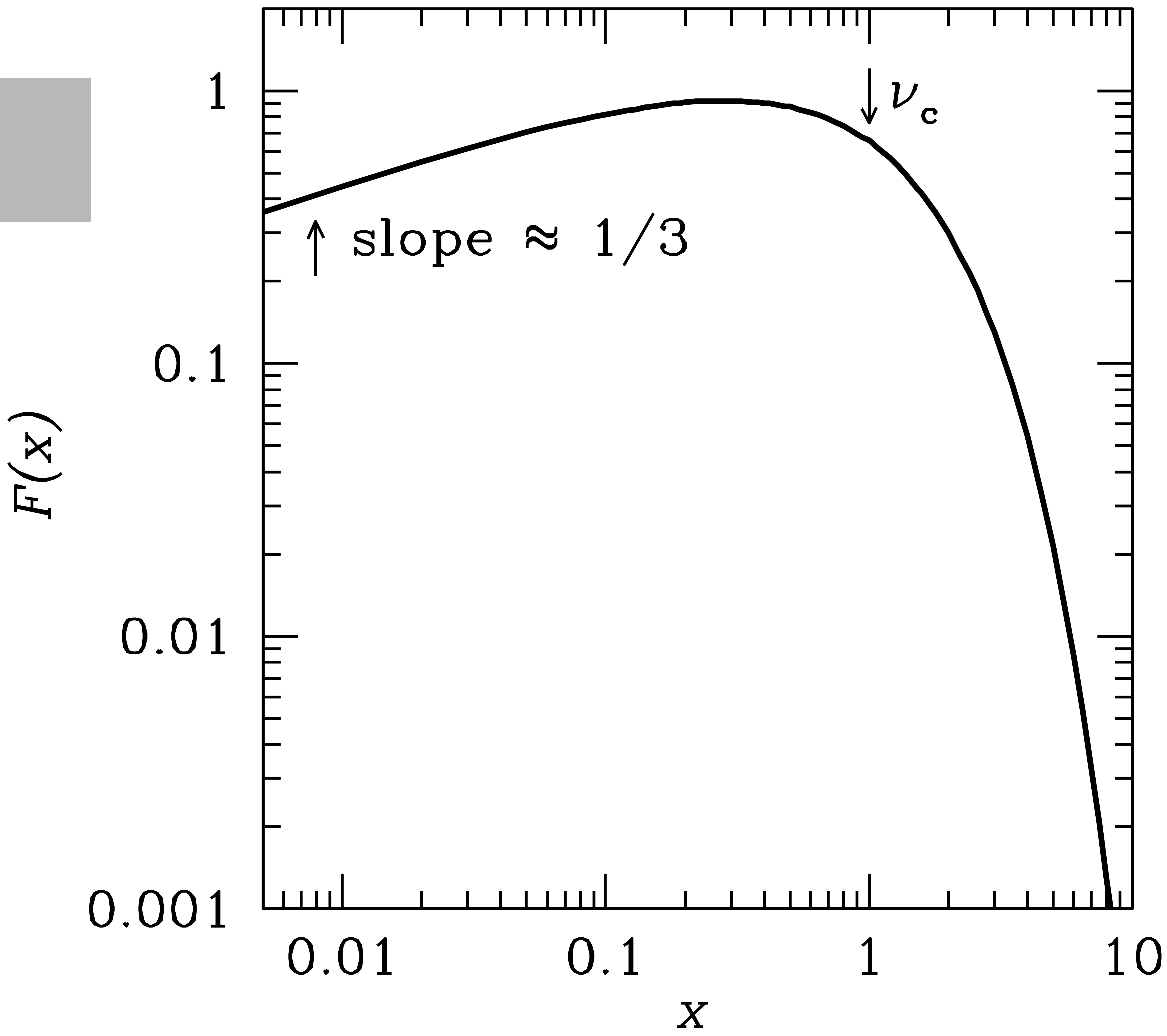
$$\varepsilon_c = \hbar\omega_c = \frac{3}{2}\frac{\hbar c}{\rho}\gamma^3$$

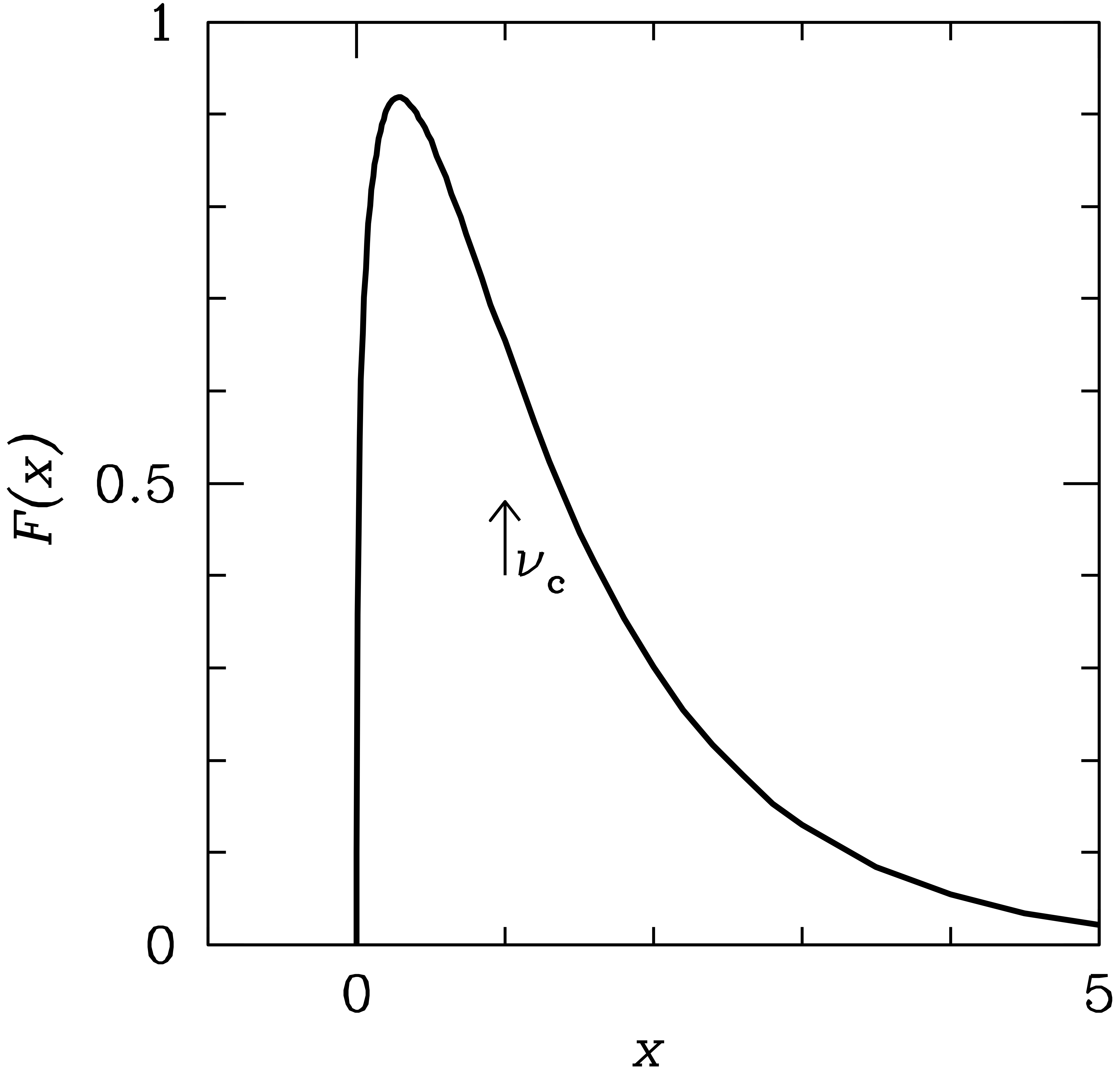


For *electrons*, the critical energy in practical units is

$$\varepsilon_c[keV] = 2.218\frac{E[GeV]^3}{\rho[m]} = 0.665 \cdot E[GeV]^2 \cdot B[T]$$

Applying a Linear Scale





https://www.cv.nrao.edu/course/astr534/SynchrotronSpectrum.html

Damping in Synchrotrons

- Damping
  - Vertical
  - Horizontal
  - Longitudinal
- Quantum excitation

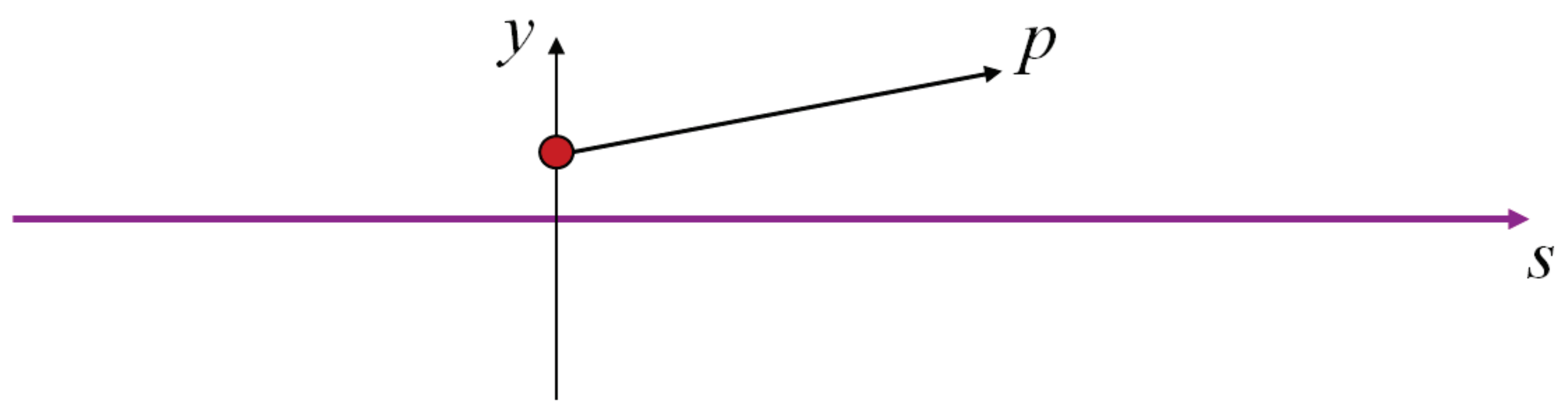
Damping in Synchrotrons

- Remember the canonical variables

$$\begin{bmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{bmatrix}_{s_1} = M_{s_0 \to s_1} \begin{bmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{bmatrix}_{s_0}$$

- *M* is a *symplectic* transformation, if $M^T \cdot S \cdot M = S$, with $S = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$

Radiation Damping of the Vertical Emittance



The radiation emitted by a relativistic particle has an opening angle of $1/\gamma$, where γ is the relativistic factor for the particle.

For an ultra-relativistic particle, $\gamma \gg 1$, and we can assume that the radiation is emitted directly along the instantaneous direction of motion of the particle.

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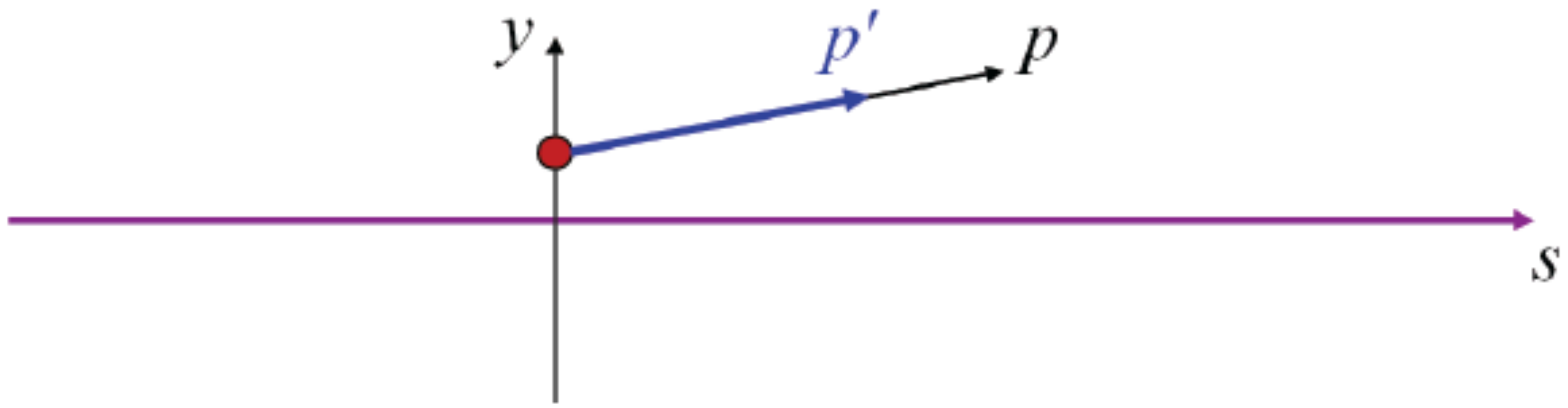
Radiation Damping of the Vertical Emittance



The momentum of the particle after emitting radiation is:

$$p' = p - dp \approx p\left(1 - \frac{dp}{P_0}\right), \tag{14}$$

where dp is the momentum carried by the radiation, and we assume that:

$$p \approx P_0. \tag{15}$$

Since there is no change in direction of the particle, we must have:

$$p'_y \approx p_y\left(1 - \frac{dp}{P_0}\right). \tag{16}$$

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Radiation Damping of the Vertical Emittance

After emission of radiation, the vertical momentum of the particle is:

$$p_y' \approx p_y \left(1 - \frac{dp}{P_0}\right). \quad (17)$$

Now we substitute this into the expression for the vertical betatron action (valid for *uncoupled* motion):

$$2J_y = \gamma_y y^2 + 2\alpha_y y p_y + \beta_y p_y^2, \quad (18)$$

to find the change in the action resulting from the emission of radiation:

$$dJ_y = -\left(\alpha_y y p_y + \beta_y p_y^2\right) \frac{dp}{P_0}. \quad (19)$$

Then, we average over all particles in the beam, to find:

$$\langle dJ_y \rangle = d\varepsilon_y = -\varepsilon_y \frac{dp}{P_0}, \quad (20)$$

where we have used:

$$\langle y p_y \rangle = -\alpha_y \varepsilon_y, \qquad \langle p_y^2 \rangle = \gamma_y \varepsilon_y, \quad \text{and} \quad \beta_y \gamma_y - \alpha_y^2 = 1. \quad (21)$$

Radiation Damping of the Vertical Emittance

For a particle moving round a storage ring, we can integrate the loss in momentum around the ring, to find the total change in momentum in one turn. The emittance is conserved under symplectic transport, so if the non-symplectic (radiation) effects are slow, we can write:

$$d\varepsilon_y = -\varepsilon_y \frac{dp}{P_0} \qquad \therefore \qquad \frac{d\varepsilon_y}{dt} = -\frac{\varepsilon_y}{T_0} \oint \frac{dp}{P_0} \approx -\frac{U_0}{E_0 T_0} \varepsilon_y, \tag{22}$$

where T_0 is the revolution period, and U_0 is the energy loss in one turn. The approximation is valid for an ultra-relativistic particle, which has $E \approx pc$.

We define the damping time τ_y:

$$\tau_y = 2 \frac{E_0}{U_0} T_0, \tag{23}$$

so the evolution of the emittance is:

$$\varepsilon_y(t) = \varepsilon_y(t) \exp\left(-2 \frac{t}{\tau_y} \right). \tag{24}$$

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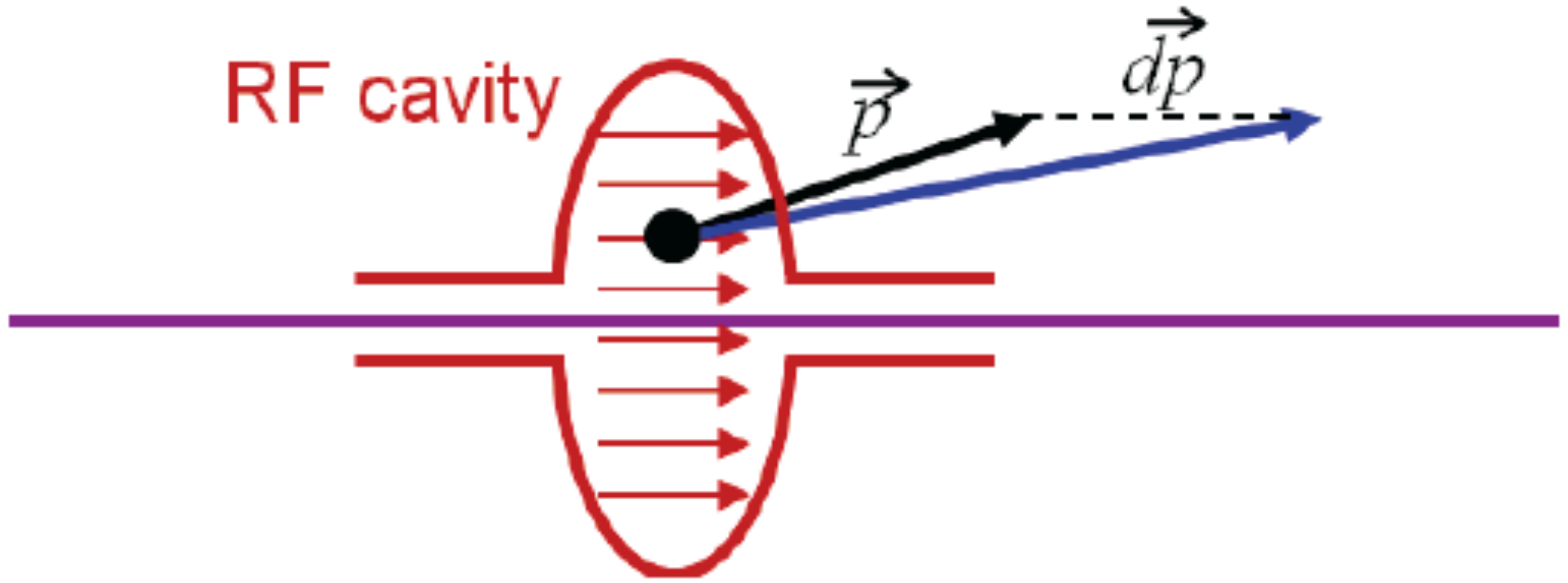
Radiation Damping of the Vertical Emittance

Typically, in an electron storage ring, the damping time is of order several tens of milliseconds, while the revolution period is of order of a microsecond. Therefore, radiation effects are indeed "slow" compared to the revolution frequency.

But note that we made the assumption that the momentum of the particle was close to the reference momentum, i.e. $p \approx P_0$.

If the particle continues to radiate without any restoration of energy, we will reach a point where this assumption is no longer valid. However, electron storage rings contain RF cavities to restore the energy lost through synchrotron radiation. But then, we should consider the change in momentum of a particle as it moves through an RF cavity.

Radiation Damping of the Vertical Emittance



Fortunately, RF cavities are usually designed to provide a longitudinal electric field, so that particles experience a change in longitudinal momentum as they pass through, without any change in transverse momentum.

This means that we do not have to consider explicitly the effects of RF cavities on the emittance of the beam.

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Synchrotron Radiation Energy Loss

To complete our calculation of the the vertical damping time, we need to find the energy lost by a particle through synchrotron radiation on each turn through the storage ring. We quote the (classical) result that the power radiated by a particle of charge e and energy E in a magnetic field B is given by:

$$P_\gamma = \frac{C_\gamma}{2\pi} c^3 e^2 B^2 E^2. \qquad (25)$$

C_γ is a physical constant given by:

$$C_\gamma = \frac{e^2}{3\varepsilon_0 (mc^2)^4} \approx 8.846 \times 10^{-5}\,\mathrm{m/GeV^3}. \qquad (26)$$

Synchrotron Radiation Energy Loss

A charged particle with energy E in a magnetic field B follows a circular trajectory with radius ρ, given by:

$$B\rho = \frac{p}{q}. \tag{27}$$

For an ultra-relativistic electron, $E \approx pc$:

$$B\rho \approx \frac{E}{ec}. \tag{28}$$

Hence, the synchrotron radiation power can be written:

$$P_\gamma \approx \frac{C_\gamma}{2\pi} c \frac{E^4}{\rho^2}. \tag{29}$$

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Synchrotron Radiation Energy Loss

For a particle with the reference energy, travelling at (close to) the speed of light along the reference trajectory, we can find the energy loss by integrating the radiation power around the ring:

$$U_0 = \oint P_\gamma dt = \oint P_\gamma \frac{ds}{c}. \qquad (30)$$

Using the previous expression for P_γ, we find:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 \oint \frac{1}{\rho^2} ds, \qquad (31)$$

where ρ is the radius of curvature of the reference trajectory.

Note that for these expressions to be valid, we require that the reference trajectory be a real physical trajectory of a particle.

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The Second Synchrotron Radiation Integral

Following convention, we define the *second synchrotron radiation integral*, I_2:

$$I_2 = \oint \frac{1}{\rho^2} ds. \qquad (32)$$

In terms of I_2, the energy loss per turn U_0 is written:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2. \qquad (33)$$

Note that I_2 is a property of the lattice (actually, a property of the reference trajectory), and does not depend on the properties of the beam.

Number of Photons Emitted

- Let's assume a uniform radius of curvature *ρ*
- The energy lost per turn can then be simplified to:

$$U_0 = \frac{e^2\gamma^4}{3\varepsilon_0\rho}$$

- And average energy per photon is the

$$\langle \varepsilon_\gamma \rangle \approx \frac{1}{3}\varepsilon_c = \frac{\hbar\omega_c}{3} = \frac{1}{2}\frac{\hbar c}{\rho}\gamma^3$$

- The average number of photons emitted per revolution is

$$\langle n_\gamma \rangle \approx 2\pi\alpha_{fine}\gamma$$

Radiation Damping of the Horizontal Emittance

Analysis of radiation effects on the vertical emittance was relatively straightforward. When we consider the horizontal emittance, there are three complications that we need to address:

- The horizontal motion of a particle is often strongly coupled to the longitudinal motion.
- Where the reference trajectory is curved (usually, in dipoles), the path length taken by a particle depends on the horizontal coordinate with respect to the reference trajectory.
- Dipole magnets are sometimes built with a gradient, so that the vertical field seen by a particle in a dipole depends on the horizontal coordinate of the particle.

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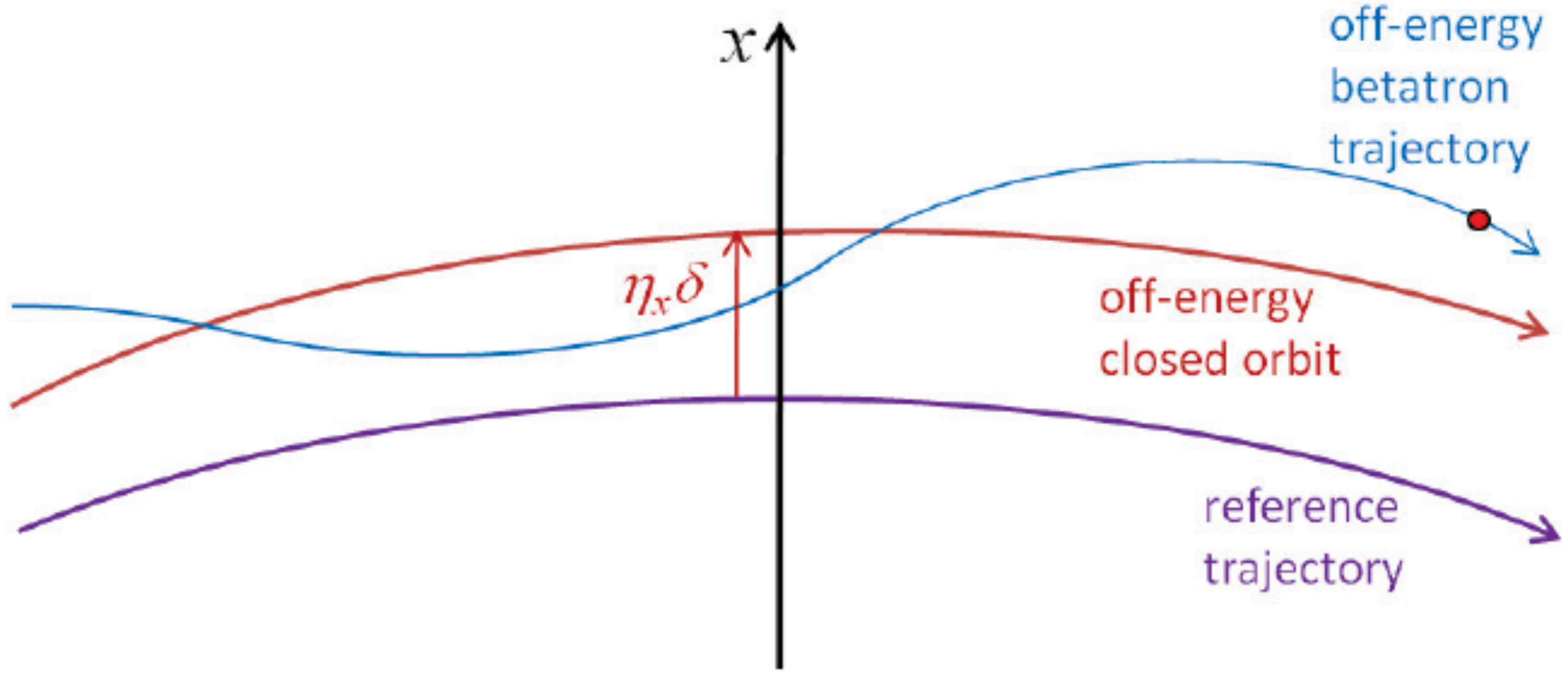
Coupling Between Horizontal and Longitudinal Phase Spaces



Coupling between transverse and longitudinal planes in a beam line is usually represented by the dispersion, η_x. So, in terms of the horizontal dispersion and betatron action, the horizontal coordinate and momentum of a particle are given by:

$$x = \sqrt{2\beta_x J_x}\cos\phi_x + \eta_x\delta \tag{37}$$

$$p_x = -\sqrt{\frac{2J_x}{\beta_x}}(\sin\phi_x + \alpha_x\cos\phi_x) + \eta_{px}\delta. \tag{38}$$

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Coupling Between Horizontal and Longitudinal Phase Spaces

When a particle emits radiation, we have to take into account:

- the change in momentum of the particle;
- the change in coordinate x and momentum p_x, resulting from the change in the energy deviation δ.

When we analysed the vertical motion, we ignored the second effect, because we assumed that the vertical dispersion was zero.

Damping of Horizontal Emittance

Taking all the above effects into account, we can proceed along the same lines as for the analysis of the vertical emittance. That is:

- Write down the changes in coordinate x and momentum p_x resulting from an emission of radiation with momentum dp (taking into account the additional effects of dispersion).
- Substitute expressions for the new coordinate and momentum into the expression for the horizontal betatron action, to find the change in action resulting from the radiation emission.
- Average over all particles in the beam, to find the change in the emittance resulting from radiation emission from each particle.
- Integrate around the ring (taking account of changes in path length and field strength with x in the bends) to find the change in emittance over one turn.

The algebra gets somewhat cumbersome, and is not especially enlightening. See Appendix A* for more details. Here, we just quote the result...

*** https://uspas.fnal.gov/materials/13CSU/Lecture1.pdf**

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Damping of Horizontal Emittance

The horizontal emittance decays exponentially:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x, \qquad (39)$$

where the horizontal damping time is given by:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0. \qquad (40)$$

The horizontal damping partition number j_x is given by:

$$j_x = 1 - \frac{I_4}{I_2}, \qquad (41)$$

where the fourth synchrotron radiation integral is given by:

$$I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds, \qquad k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x}. \qquad (42)$$

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