Synchrotron Radiation

RASMUS ISCHEBECK

Undulator Radiation

Let's make a few simplifications:

- relativistic electrons : $\beta = \frac{v}{c} \approx 1$, $\gamma = \frac{1}{\sqrt{1-\beta^2}} \gg 1$
- we will look at undulators (more about this later)
- we will look at far-field radiation

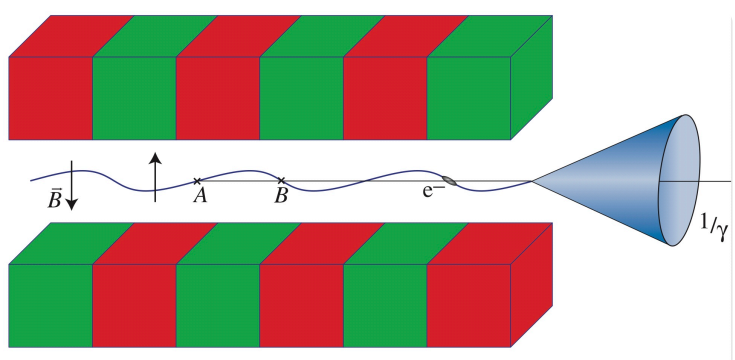
SIDE NOTE:

Fields can be computed without simplifications from Maxwell's equations.

$\rightarrow$ A. Hofmann CAS, Jackson Ch. 12

Here, we will just look at one specific case (which is highly relevant for synchrotron sources).

Assume a periodic magnetic field:



$$\vec{B} = \begin{bmatrix} 0 \\ B_0 \sin(k_u z) \\ 0 \end{bmatrix}$$

SIDE NOTE

In general, this violates Maxwell's Equations, since $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{B} = 0$ in free space.

The correct term would be:

$$\vec{B} = \begin{bmatrix} 0 \\ B_0 \cosh(k_u y) \sin(k_u z) \\ B_0 \sinh(k_u y) \cos(k_u z) \end{bmatrix}$$

we assume that the particles travel close to the axis ($y \approx 0$).

Motion of the electrons:

$$m_e \gamma \frac{d\vec{v}}{dt} = \vec{F} = -e \vec{v} \times \vec{B}$$

$$m_e \gamma \frac{dv_x}{dt} = e v_z B_y = e v_z B_0 \sin(k_u z)$$

Replace t by z: $\left(\frac{dz}{dt} = v_z \right)$

$$\frac{dv_x}{dz} = \frac{e}{m_e \gamma} B_0 \sin(k_u z)$$

assuming that γ is constant

$$\Rightarrow \quad v_x(z) = -\frac{Kc}{\gamma} \cos(k_u z)$$

with $\quad K = \dfrac{e B_0}{m_e c k_u} \simeq 0.934 \cdot B\,[T] \cdot \lambda_u\,[cm]$

$$\Rightarrow \quad x(z) = -\frac{K}{k_u \gamma \beta_z} \sin(k_u z)$$

we find a sinusoidal motion of the particles !

- Radiation is emitted by relativistic particles in a cone $\frac{1}{\gamma}$
- The maximum angular deviation of the particles from the straight orbit is smaller than $\frac{1}{\gamma}$
  (definition of the undulator)
  (i.e. $K \lesssim 1$)

- The horizontal movement causes a reduction in the velocity in z direction:

$$\beta_x^2 + \beta_z^2 = \beta^2 = \text{const.}$$

$$\Rightarrow \beta_z = \beta\left(1 - \frac{K^2}{4\gamma^2} - \frac{K^2}{4\gamma^2}\cos(2k_u z)\right)$$

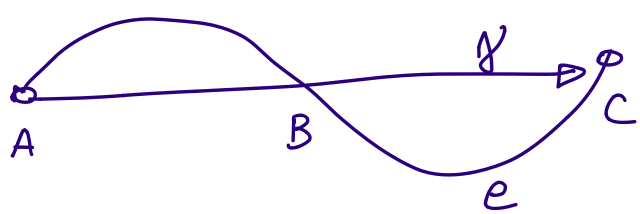
$$\overline{\beta_z} = \beta\left(1 - \frac{K^2}{4\gamma^2}\right)$$

- Calculate the arc length

...

- Interference between radiation emitted by <u>the same electron</u> in two subsequent periods

→ Radiation emitted in phase adds coherently



$\Delta\varphi = 2\pi$ from A to C

$\Delta\varphi = \pi$ from A to B

Let's call the arc length $\widetilde{AB}$

Resonance condition:

$$\frac{\lambda}{2c} = \frac{\widetilde{AB}}{v} - \frac{\overline{AB}}{c}$$

From Math textbook, calculate the arc length

$$\widetilde{AB} = \int_0^{\lambda_u/2} \sqrt{1 + \left(\frac{dx}{dz}\right)^2}\, dz$$

$$\approx \int_0^{\lambda_u/2} \left(1 + \frac{1}{2}\left(\frac{dx}{dz}\right)^2\right) dz$$

$$= \int_0^{\lambda_u/2} \left(1 + \frac{K^2}{2\gamma^2}\cos^2(k_u z)\right) dz$$

$$= \frac{\lambda_u}{2}\left(1 + \frac{K^2}{4\gamma^2}\right)$$

from resonance condition:

$$\Rightarrow \frac{\lambda}{2c} = \frac{\lambda_u}{2\beta c}\left(1 + \frac{K^2}{4\gamma^2}\right) - \frac{\lambda_u}{2c}$$

$$\lambda = \frac{\lambda_u}{\beta}\left(1 + \frac{K^2}{4\gamma^2}\right) - \lambda_u\left(1 - \frac{\theta^2}{2}\right)^{*}$$

$$\underbrace{\beta}_{\approx 1}\lambda = \lambda_u\left(1 + \frac{K^2}{4\gamma^2}\right) - \underbrace{\beta}_{\approx 1 - \frac{1}{2}\gamma^{-2}}\lambda_u\left(1 - \frac{\theta^2}{2}\right)$$

$$\lambda = \lambda_u\left(1 - 1 + \frac{K^2}{4\gamma^2} + \frac{1}{2\gamma^2}\right)$$

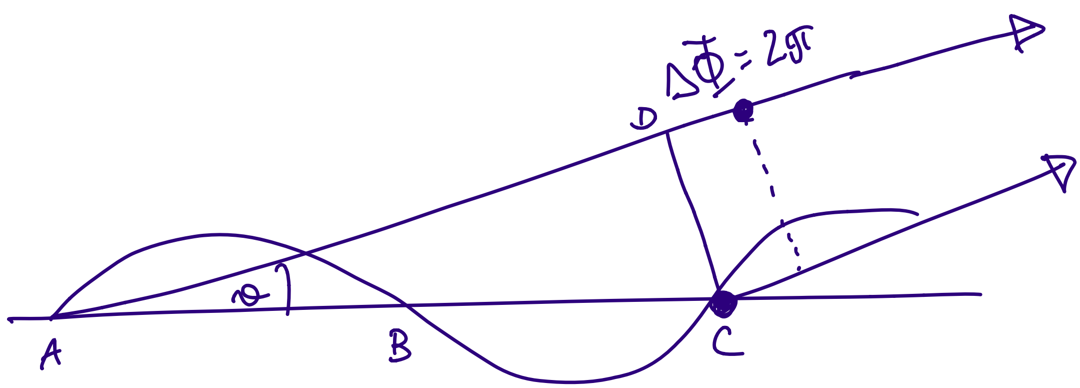
$$\lambda = \lambda_u\left(\frac{K^2}{4\gamma^2} + \frac{2}{4\gamma^2}\right)$$

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(\frac{K^2}{2} + 1\right)$$

insert $\beta = \sqrt{1-\gamma^{-2}} \approx 1 - \frac{1}{2}\gamma^{-2}$

$$\rightarrow \boxed{\lambda = \frac{\lambda_u}{2\gamma^2}\left(1+\frac{K^2}{2}\right)}$$

More generally, consider radiation emitted at an angle θ:



$\overline{AC}$

$\widetilde{ABC}$

$$\overline{AD} = \overline{AC}\cos\theta \approx \overline{AC}\left(1-\frac{\theta^2}{2}\right)$$

$$\frac{\lambda}{c} = \frac{\widetilde{ABC}}{v} - \frac{\overline{AD}}{c}$$

$$= \frac{\lambda_u}{\beta}\left(1+\frac{K^2}{4\gamma^2}\right) - \frac{\lambda_u}{c}\left(1-\frac{\vartheta^2}{2}\right)$$

$$\boxed{\lambda = \frac{\lambda_u}{2\gamma^2}\left(1+\frac{K^2}{2}+\gamma^2\vartheta^2\right)}$$

Conclusions:

- The resonance condition is also fulfilled for integer parts of the wavelength
  → harmonic radiation
- The wavelength of the radiation is much shorter than the undulator period

  $\lambda_u \approx 10^{-2}\,m,\ \gamma \approx 10^4 \Rightarrow \lambda \approx 10^{-10}\,m$

- The wavelength can be varied by
  - changing γ
  - changing K (smaller K → higher photon energy)

(e) The wavelength varies with angle.
For a small observation angle, however,
it is almost monochromatic