Examination of Transverse Beam Dynamics (Solutions)

JUAS - 12$^{\text{th}}$ February 2020

Remark: Below you will find 5 exercises. The total number of points of the 5 exercises amounts to 25. The results will be normalised to 20.

1 Exercise: Lattice properties [5pt]

Consider a ring composed by 32 identical FODO cells. Let the basic cell consist of a thin quadrupole, a bending magnet, another thin quadrupole, and another bending magnet.

Assume that the focal length of the quadrupoles is $|f| = 20$ m and that the bending radius is $\rho = 100$ m. Compute:

1. The horizontal and vertical betatron tunes, Q_x and Q_y.
2. Evaluate $\beta_{x,min}$ and $\beta_{x,max}$, D_{min} and D_{max} , and plot them in relation to the cell elements.
3. Compute the momentum compaction of the ring. Compare with the formula $\alpha_C = 1/Q_x^2$.

Solution:

1. In order to compute the tune, we need to know the phase advance per cell.

   The ring is composed by 32 cells. The length of each cell is:

$$L_{\text{cell}} = \frac{C}{32}$$
$$C = 2\pi\rho = 628.3 \text{ m}$$
$$L_{\text{cell}} = \frac{C}{32} = 19.63 \text{ m}$$

   In a FODO cell

$$f = \frac{L_{\text{cell}}}{4\sin\frac{\mu}{2}}$$

   which gives:

$$\mu = 2\arcsin\left(\frac{L_{\text{cell}}}{4f}\right).$$

   Then

$$Q = \frac{1}{2\pi}\sum_{\text{cells}}\mu_{\text{cell}} = \frac{32}{2\pi}\mu_{\text{cell}}$$

   We need to compute μ_{cell}.

$$\mu_{\text{cell}} = 2\arcsin\left(\frac{19.63 \text{ m}}{4\times 20 \text{ m}}\right) = 0.4958 \text{ radian} = 28.40 \text{ degree}$$

   Hence the tune is:

$$Q_x = Q_y = \frac{32}{2\pi}\mu_{\text{cell}} = 2.52$$

2. Evaluate $\beta_{x,min}$ and $\beta_{x,max}$, D_{min} and D_{max}

$$\beta_{\pm} = \frac{L_{\text{cell}}\left(1 \pm \sin\frac{\mu}{2}\right)}{\sin\mu} = \begin{cases}\beta_{max} & = 51.38 \text{ m}\\ \beta_{min} & = 31.13 \text{ m}\end{cases}$$
$$D_{\pm}\frac{L_{cell}\theta\left(1 \pm \frac{1}{2}\sin\frac{\mu}{2}\right)}{4\sin^2\frac{\mu}{2}} = \begin{cases}D_{max} & = 17.96 \text{ m}\\ D_{min} & = 14.04 \text{ m}\end{cases}$$

3. Compute the momentum compaction of the ring:

$$\begin{aligned}
\alpha_C &= \frac{1}{C}\oint \frac{D(s)}{\rho} ds. \\
&= \frac{1}{628.3 \text{ m}} \oint \frac{\langle D(s) \rangle}{\rho} ds \\
&= \frac{1}{628.3 \text{ m}} \frac{\frac{1}{2}\left(D_{max} + D_{min}\right)}{\rho} \oint ds \\
&= \frac{\frac{1}{2}\left(D_{max} + D_{min}\right)}{\rho} = 0.16 \\
\frac{1}{Q_x^2} &= 0.157
\end{aligned}$$

2 Exercise: Action amplification [5pt]

The electromagnetic interaction of a bunch of charged particles with the accelerator environment can give rise to electric fields that persist in the beamline long enough to deflect the trailing bunches transversely.

One such effect is called wakefield, where the kick imparted from one bunch ("1st" bunch) to a trailing one ("2nd" bunch) is proportional to a constant, K, multiplied by the position of the first bunch at the wakefield source, $x_{1\text{st}}$. That is, the kick felt by the 2nd bunch is:

$$\Delta x'_{2\text{nd}} = K \cdot x_{1\text{st}}.$$

In the phase space, the effect of this kick is to displace the 2nd bunch along the angle axis by a quantity $\Delta x'_{2\text{nd}}$. The wakefield kick increases the "action" of the 2nd bunch, $J_{2\text{nd}}$.

One can define an "action amplification factor", A, as

$$A = \frac{J_{2\text{nd},\, final}}{J_{2\text{nd},\, initial}}.$$

Assuming that:

- the phase space coordinates of the 1st bunch are

$$\begin{pmatrix} x_{1\text{st}} \\ 0 \end{pmatrix}$$

- the initial action of both bunches is the same:

$$J_{2\text{nd},\, initial} = J_{1\text{st},\, initial}$$

- the wakefield source is located at a beam waist (that is, the Twiss parameters are $\beta \neq 0$ and $\alpha = 0$)

estimate what is the maximum amplification factor, A, one can expect from the wakefield kick in normalised coordinates.

[**Recall.** The transformation to normalised coordinates is, U:

$$U = \begin{pmatrix} 1/\sqrt{\beta} & 0 \\ \alpha/\sqrt{\beta} & \sqrt{\beta} \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} x_n \\ x'_n \end{pmatrix}_{\text{nornalised}} = U \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{normal}}$$

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Solution:

- Given that $\alpha = 0$, the matrix transformation to go from normal coordinates to normalised coordinates is:

$$U = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ 0 & \sqrt{\beta} \end{pmatrix}$$

- To calculate the action of the first bunch we need to compute the coordinates of the first bunch in normalised phase space:

$$J_{1\text{st}} = \left\| U \cdot \begin{pmatrix} x_{1\text{st}} \\ 0 \end{pmatrix} \right\| = \frac{x_{1\text{st}}}{\sqrt{\beta}}$$

- The action of the second bunch, before the kick is

$$J_{2\text{nd}} = J_{1\text{st}}$$

- The maximum increase of $J_{2\text{nd}}$ occurs when the second particle has coordinates

$$\begin{pmatrix} 0 \\ x'_{2\text{nd}} \end{pmatrix}.$$

  The kick increases its action from:

$$\begin{aligned} J_{2\text{nd}} = J_{1\text{st}} \rightarrow J_{1\text{st}} + \left\| U \cdot \begin{pmatrix} 0 \\ K \cdot x_{1\text{st}} \end{pmatrix} \right\| &= \\ = J_{1\text{st}} + \sqrt{\beta} K \cdot x_{1\text{st}} & \\ = \frac{x_{1\text{st}}}{\sqrt{\beta}} + \sqrt{\beta} K \cdot x_{1\text{st}} & \end{aligned}$$

- Therefore:

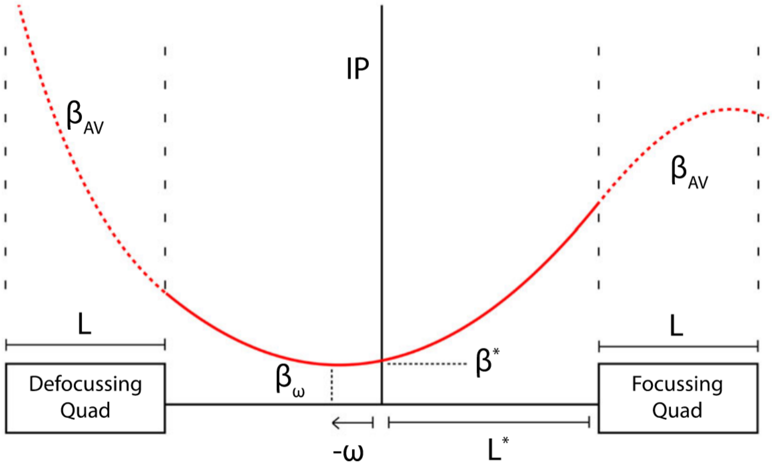
$$A = \frac{\frac{x_{1\text{st}}}{\sqrt{\beta}} + \sqrt{\beta}K \cdot x_{1\text{st}}}{\frac{x_{1\text{st}}}{\sqrt{\beta}}} = K \cdot \beta + 1.$$

3 Exercise: Low β^* insertion: measuring β^* using K-modulation [5pt]

In the LHC, the "K-modulation" is the method used to measure the value of β-function at the IP. By means of the modulation of the strength of a single quadrupole a tune shift is induced. As seen in the lectures, the relationship between the tune shift ΔQ, the quadrupole strength variation ΔK and the β-function at the quadrupole location is given approximately by,

$$\beta_{\rm av} = \frac{4\pi\Delta Q}{\Delta K}$$

From the value of the β-function at the quadrupole we can easily obtain the value at the IP by transporting the above value through the drift space (L^*) to the IP.



Due to unavoidable errors, the position of the waist of the β-function, w, usually doesn't correspond exactly with the IP. For this reason, we distinguish between the β-function at the waist β_w and the β-function at the IP β^*. (If $w = 0$, $\beta_w = \beta^*$).

Assume that the strength modulation is performed in the quadrupole closer to the IP (left side in this case) with a strength variation of $\Delta K = 10^{-5}{\rm m}^{-1}$. For this strength change a tune shift of $\Delta Q = 6.5 \cdot 10^{-3}$ is observed. Calculate:

1. The β-function at the IR face of the quadrupole.

$$\beta_{\rm av} = \frac{4\pi \cdot 6.5 \cdot 10^{-3}}{10^{-5}} = 8168 \text{ m}$$

2. It was observed a waist of $w = 1.2$ cm. Taking $L^* = 23$m, find the β-function at this location (β_w).

   The β-function at the quadrupole location can be obtained by transporting the β-function at the minima:

$$\beta_{\rm av} = \beta_w + \frac{(L^\star - w)^2}{\beta_w}$$

   from here we can calculate the β-function at the minimum.

$$\beta_w^2 - \beta_{\rm av}\beta_w + (L^\star - w)^2 = 0$$

   From where we find two different solutions. We are interested in the smallest β_w so we take the "-" solution.

$$\beta_w = 64.6 \text{ cm}$$

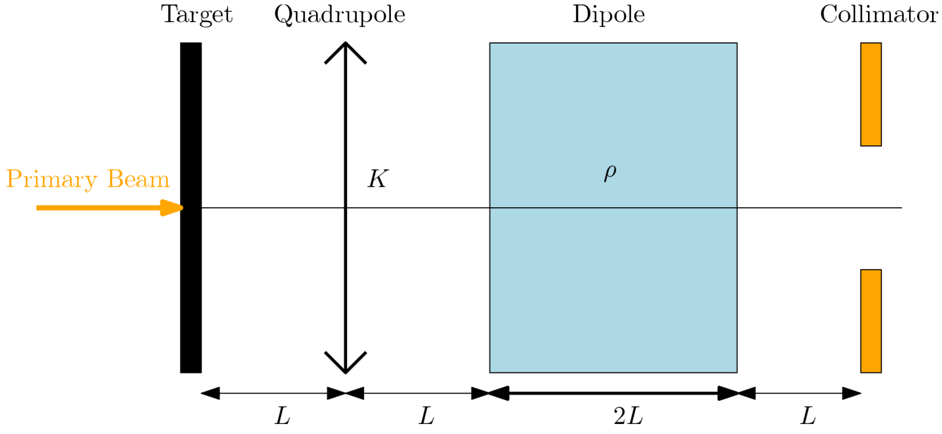
3. Find the actual β-function at the IP location (β^*).

   The solution at the IP can be found just transporting the found $\beta_w$to the IP location:

$$\beta^* = \beta_w + \frac{w^2}{\beta_w} = 70.8 \text{ cm}$$

4 Exercise: Fixed-target transport line [7pt]

Let's consider a simple set up for a fixed target experiment. It consists of a transfer line with a quadrupole of focal length f and a dipole with bending radius ρ and length $2L$ following the scheme shown in the figure:



1. Write the transport matrix of the system from the target to the collimator.

$$\theta = \frac{2L}{\rho}$$

$$M = M_L M_D M_L M_Q M_L$$

$$M = \begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 - 2\frac{L}{f} & 3L - 2\frac{L^2}{f} & L\theta \\ -\frac{1}{f} & 1 - \frac{L}{f} & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

2. If the initial dispersion is $\eta_x = 0$, calculate the dispersion function at the collimator location.

$$\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix} = M \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} L\theta \\ \theta \\ 1 \end{pmatrix}$$

3. Consider an on-momentum particle generated on axis at the target location, with initial transverse momentum x'_0. What is the transverse position at the collimator location?

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix} = M \begin{pmatrix} 0 \\ x'_0 \\ 0 \end{pmatrix} = \begin{pmatrix} (3L - 2\frac{L^2}{f})x'_0 \\ (1 - \frac{L}{f})x'_0 \\ 0 \end{pmatrix}$$

4. What would be the transverse position if the same particle had a momentum deviation $\Delta p/p$?

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix} = M \begin{pmatrix} 0 \\ x'_0 \\ \Delta p/p \end{pmatrix} = \begin{pmatrix} (3L - 2\frac{L^2}{f})x'_0 + L\theta\frac{\Delta p}{p} \\ (1 - \frac{L}{f})x'_0 + \theta\frac{\Delta p}{p} \\ \frac{\Delta p}{p} \end{pmatrix}$$

5. Assuming $L = 1$ m, $f = 10$ m, and a bending radius $\rho = 200$ m, calculate the transverse displacement for points (3) and (4) for a particle with initial transverse momentum $x'_0 = 0.024$ and momentum deviation $\Delta p/p = 0.4$.

$$x = \left(3L - 2\frac{L^2}{f}\right) x'_0 = 6.7 \text{ cm}$$

$$x = \left(3L - 2\frac{L^2}{f}\right) x'_0 + L\theta\frac{\Delta p}{p} = 7.1 \text{ cm}$$

6. Would do these particles pass through the collimator if the collimator half-gap is 7 cm?

   The on-momentum particle will go through while the off-momentum particle will be stopped at the collimator.

7. Same as question 6 in absence of the quadrupole.

   In the absence of the quadrupole both particles will be stopped at the collimator.

$$x = 3Lx'_0 = 7.2 \text{ cm}$$

$$x = 3Lx'_0 + L\theta\frac{\Delta p}{p} = 7.6 \text{ cm}$$

5 Exercise: Quadrupole magnet [3pt]

Consider a quadrupole magnet focusing electrons with $E = 1$ GeV. Its aperture (diameter) is 20 mm, and its length is 40 mm. The magnetic field at the aperture radius is 0.1 T.

(i) What is the field gradient?

$$G = \frac{B}{d/2} = 10 \text{ T/m}$$

(ii) Calculate the normalised quadrupole field strength, k.

$$k = \frac{G}{p/q} = 3 \text{ m}^{-2}$$

(iii) What is the maximum angular deflection that this quadrupole can impart to a particle? (use thin-lens approximation)

$$\Delta x' = L\,k\,x = 0.04 \cdot 3 \cdot 0.01 = 1.2 \text{ mrad}$$