Introduction to Transverse Beam Dynamics

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Purpose of this course

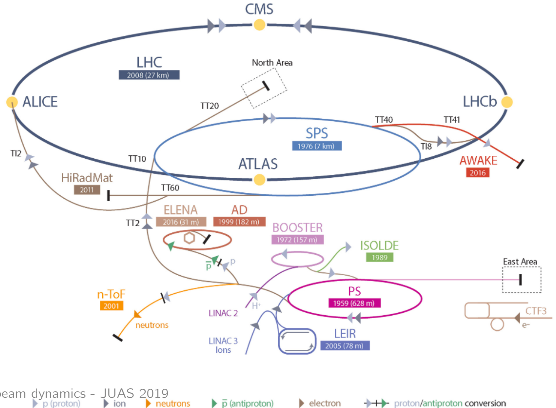
Discuss the oscillations of the particles in the

Transverse planes x and y

of synchrotrons, called

BETATRON OSCILLATIONS

(similarly to the synchrotron oscillations in the longitudinal plane), and derive the basic equations



Some references

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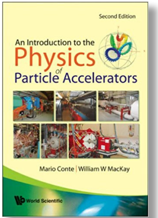
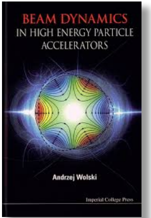
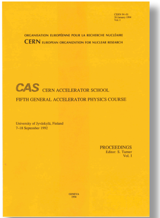
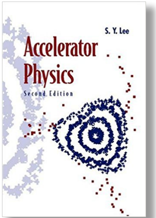
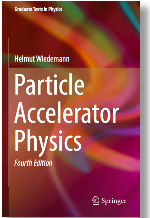










Part 1.

Basics, single-particle dynamics

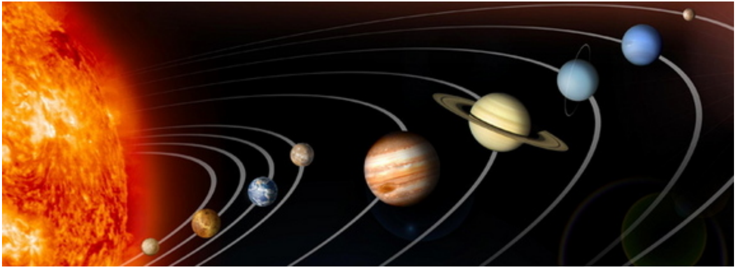
Luminosity run of a typical storage ring

In a storage ring: the protons are accelerated and stored for $\sim 12 - 15$ hours

The distance traveled by particles running at *nearly* the speed of light, $v \approx c$, for 12 hours is

$$\textbf{distance} \approx \mathbf{12 \times 10^9 \ km}$$

$\rightarrow$ this is more than twice the distance from Sun to Pluto!



Forces and fields

It's a circular machine: we need a transverse deflecting force→the Lorentz force

$$\boxed{\vec{F} = q \cdot \left(\vec{E} + \vec{v} \times \vec{B}\right)}$$

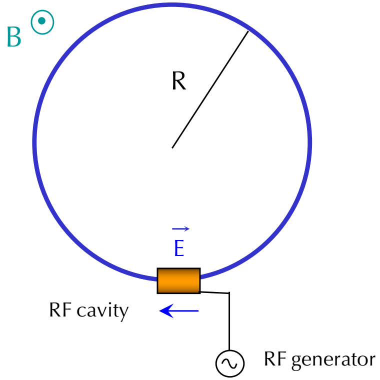
where, in high energy machines, $|\vec{v}| \approx c \approx 3 \cdot 10^8$ m/s. Usually there is no electric field, and the transverse deflection is given by a magnetic field only.

Comparison of electric and magnetic force:

$$\begin{aligned} \left|\vec{E}\right| &= 1 \quad \text{MV/m} \\ \left|\vec{B}\right| &= 1 \quad \text{T} \\ \frac{F_{\text{magnetic}}}{F_{\text{electric}}} &= \frac{evB}{eE} = \frac{\beta c B}{E} \simeq \beta \frac{3 \cdot 10^8}{10^6} = 300\,\beta \end{aligned}$$

⇒ the magnetic force is much stronger then the electric one: in an accelerator, use magnetic fields whenever possible.

Dipole magnets: the magnetic guide



Stable circular motion: centrifugal force + centripetal force = 0

$$\left.\begin{array}{rll} \text{Lorentz force} & F_L & = qvB \\ \text{Centripetal force} & F_{\text{centr}} & = \frac{mv^2}{\rho} \\ & \frac{mv\cancel{^2}}{\rho} & = q\cancel{v}B \end{array}\right\}$$

$$P = mv = m_0\gamma v \text{ "momentum"}$$

$$B\rho = \text{"beam ridigity"}$$

$$\boxed{\frac{P}{q} = B\rho}$$

Rule of thumb, in practical units:

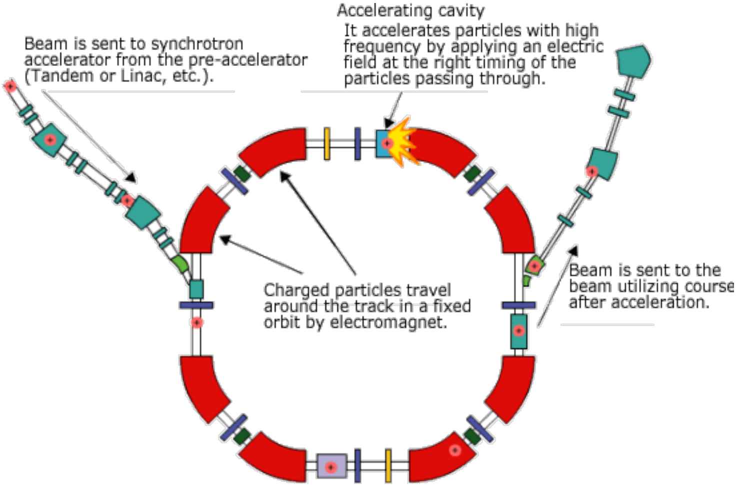
$$\boxed{\frac{1}{\rho\ [m]} \approx 0.3 \frac{B\ [T]}{P\ [GeV/c]/q\ [e]}}$$

Example: In the LHC, $\rho = 2.53$ km. The circumference $2\pi\rho = 15.9$ km $\approx 60\%$ of the entire LHC. ($R = 4.3$ km, and the total circumference is $C = 2\pi R \approx 27$ km)

The field B is $\approx 1 \ldots 8$ T

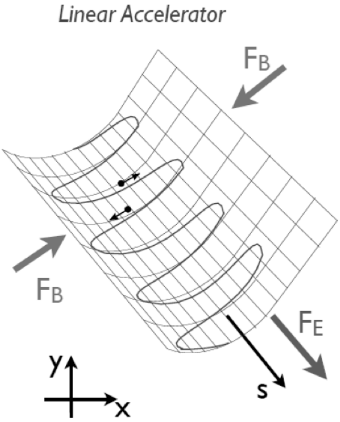
The quantity $\frac{1}{\rho}$ can be seen as a "normalised bending strength", i.e. the bending field normalised to the beam rigidity.

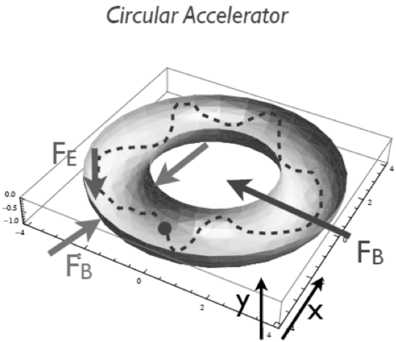
Note: $1/\rho$ is also known as k_0.


Accelerating cavity
It accelerates particles with high frequency by applying an electric field at the right timing of the particles passing through.
Beam is sent to synchrotron accelerator from the pre-accelerator (Tandem or Linac, etc.).
Charged particles travel around the track in a fixed orbit by electromagnet.
Beam is sent to the beam utilizing course after acceleration.

The focusing force

$$\vec{F} = q \cdot \left(\vec{E} + \vec{v} \times \vec{B} \right)$$





Remember the 1D harmonic oscillator: $F = -k\,x$

Reminder: the 1D Harmonic oscillator

Restoring force

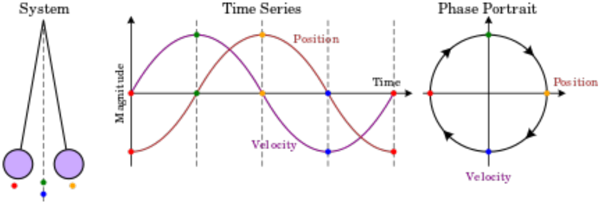
$$F = -k\,x$$

Equation of motion:

$$x'' = -\frac{k}{m}x$$

which has solution:

$$x(t) = A\cos(\omega t + \phi) \underset{\text{or}}{=} a_1 \cos(\omega t) + a_2 \sin(\omega t)$$



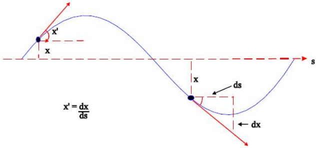
- F , restoring force, N or MeV/m
- k, spring constant or *focusing strength*, N/m or MeV/m^2
- $\omega = \sqrt{\frac{k}{m}} = 2\pi f$, angular velocity, rad/s
- ϕ, initial phase, rad
- f, rotation frequency, Hz
- A, oscillation amplitude, m
- m_0, particle's rest mass, MeV/c^2
- $m = m_0\gamma$, particle's mass, MeV/c^2

Phase-space coordinates

The state of a particle is represented with a 6-dimensional phase-space vector:

$$(x,\ x',\ y,\ y',\ z,\ \delta)$$

where x' and y' are the transverse angles:



with

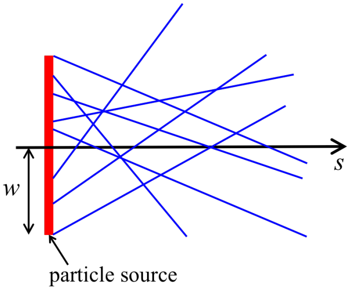
$$\begin{aligned}
x & & & [\mathrm{m}] \\
x' &= \frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{V_x}{V_z} = \frac{P_x}{P_z} \approx \frac{P_x}{P_0} & & [\mathrm{rad}] \\
y & & & [\mathrm{m}] \\
y' &= \frac{\mathrm{d}y}{\mathrm{d}s} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{V_y}{V_z} = \frac{P_y}{P_z} \approx \frac{P_y}{P_0} & & [\mathrm{rad}] \\
z & & & [\mathrm{m}] \\
\delta &= \frac{\Delta P}{P_0} = \frac{P - P_0}{P_0} & & [\#]
\end{aligned}$$

where P_0 is the momentum of the reference particle (reference momentum), and $P = P_0\,(1+\delta)$

Exercise: Phase space representations

1. Consider a cathode, located at position s_0 with radius w, emitting particles. What does the phase space look like for the particles just created? Which portion of the phase space is occupied by the emitted particles?

   Hint: the picture below shows the particle source in the configuration space



Quadrupole magnets: the focusing force

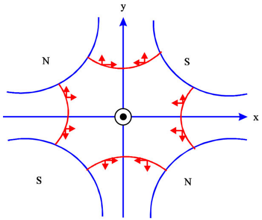
Quadrupole magnets are required to keep the trajectories in vicinity of the ideal orbit

They exert a linearly-increasing Lorentz force, thru a linearly-increasing magnetic field:

$$\begin{aligned} B_x &= G\,y \\ B_y &= G\,x \end{aligned} \quad \Rightarrow \quad \begin{aligned} F_x &= -qv_z B_y = -qv_z G\,x \\ F_y &= qv_z B_x = qv_z G\,y \end{aligned}$$

G is the <u>gradient</u> of the quadrupole magnet:

$$G = \frac{2\mu_0 nI}{r_{\text{aperture}}^2} \left[\frac{T}{m}\right] = \frac{B_{\text{poles}}}{r_{\text{aperture}}} \left[\frac{T}{m}\right]$$



the arrows show the force exerted on a particle

- LHC main quadrupole magnets: $G \approx 25 \ldots 235$ T/m

Normalised focusing strength

Dividing the gradient G by the magnet rigidity P/q one finds k, the "normalised focusing strength"

$$\boxed{k = \frac{G}{P/q}\left[m^{-2}\right]}$$

with

$$G = \left[\frac{T}{m}\right]; \quad q = [e]\,; \quad \frac{P}{q} = \left[\frac{\text{GeV}}{\text{c} \cdot e}\right] = \left[\frac{GV}{c}\right] = [T\ \ m]$$

Another useful rule of thumb: $\boxed{k\left[m^{-2}\right] \approx 0.3\frac{G\ [T/m]}{P\ [GeV/c]/q\ [e]}}$.

Note: k is also known as k_1.

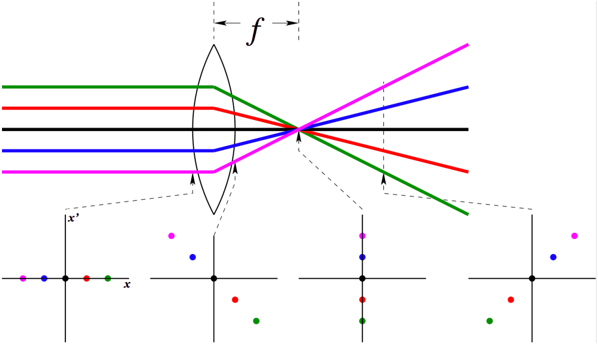
Focal length of a quadrupole

The focal length of a quadrupole is

$$f = \frac{1}{k \cdot L} \text{ [m]}$$

where L is the quadrupole length.

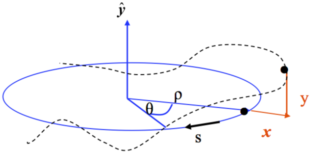
Phase space view:



Towards the equation of motion

Linear approximation:

- the **ideal particle** coincides with the **reference orbit**
- any other particle $\Rightarrow$ has coordinates $x, y, P_x, P_y \neq 0$; $P \neq P_0$ with
  - $x, y \ll \rho$
  - $P_x,\ P_y \ll P_0$
- only linear terms in x and y of B are taken into account



Let's recall some useful relativistic formulæ and definitions:

P_0	$= m_0\,\gamma_0\,v_0 = m_0\,\gamma_0\,\beta_0\,c$	reference momentum
P	$= P_0\,(1+\delta)$	total momentum
δ	$= (P - P_0)\,/\,P_0$	relative momentum offset
E	$= \sqrt{P^2c^2 + m_0^2c^4} = m_0\,\gamma\,c^2 = m_0\,c^2 + K$	total energy
K	$= E - m_0\,c^2$	kinetic energy
β	$= \frac{v}{c} = \frac{Pc}{E}; \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{E}{m_0\,c^2}$	relativistic beta and gamma

Towards the equation of motion

Taylor expansion of the B_y field:

$$B_y(x) = B_{y0} + \frac{\partial B_y}{\partial x} x + \frac{1}{2}\frac{\partial^2 B_y}{\partial x^2} x^2 + \frac{1}{3!}\frac{\partial^3 B_y}{\partial x^3} x^3 + \ldots$$

Now we drop the suffix 'y' and normalise to the magnetic rigidity $P/q = B\rho$

$$\frac{B(x)}{P/q} = \frac{B_0}{B_0\rho} + \frac{G_{\text{quad}}}{P/q} x + \frac{1}{2}\frac{G_{\text{sext}}}{P/q} x^2 + \frac{1}{3!}\frac{G_{\text{oct}}}{P/q} x^3 + \ldots$$

$$= \underbrace{\frac{1}{\rho}}_{\equiv k_0} + k_1 x + \frac{1}{2} k_2 x^2 + \frac{1}{3!} k_3 x^3 + \ldots$$

In the linear approximation, only the terms linear in x and y are taken into account:

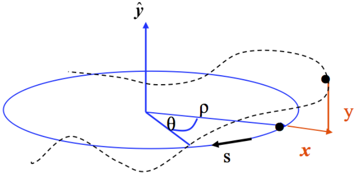
- dipole fields, $1/\rho \equiv k_0$
- quadrupole fields, k_1

It is more practical to use "separate function" magnets, rather than combined ones:

- split the magnets and optimise them regarding their function
  - bending
  - focusing, etc.

The equation of motion in radial coordinates

Let's consider a local segment of one particle's trajectory:



now recall the radial centrifugal acceleration: $a_r = \dfrac{\mathrm{d}^2\rho}{\mathrm{d}t^2} - \rho\left(\dfrac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = \dfrac{\mathrm{d}^2\rho}{\mathrm{d}t^2} - \rho\omega^2.$

- For an ideal orbit: $\rho = \text{const} \Rightarrow \frac{\mathrm{d}\rho}{\mathrm{d}t} = 0$

  $\Rightarrow$the force is
  $$\begin{aligned} F_{\text{centrifugal}} &= -m\rho\omega^2 = -mv^2/\rho \\ F_{\text{Lorentz}} &= qB_y v = -F_{\text{centrifugal}} \end{aligned} \Rightarrow \qquad \frac{P}{q} = B_y\rho$$

- For a general trajectory: $\rho \to \rho + x$:
  $$F_{\text{centrifugal}} = m\,a_r = -F_{\text{Lorentz}} \quad \Rightarrow \quad m\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\rho + x) - \frac{v^2}{\rho + x}\right] = -qB_y v$$

$$F = \underbrace{m \frac{\mathrm{d}^2}{\mathrm{d}t^2}(\rho + x)}_{\text{term 1}} - \underbrace{\frac{mv^2}{\rho + x}}_{\text{term 2}} = -qB_y v$$

- Term 1: As $\rho =$const...

$$m \frac{\mathrm{d}^2}{\mathrm{d}t^2}(\rho + x) = m \frac{\mathrm{d}^2}{\mathrm{d}t^2} x$$

- Term 2: Remember: $x \approx$ mm whereas $\rho \approx$ m $\rightarrow$ we develop for small x

remember

$$\frac{1}{\rho + x} \approx \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right)$$

Taylor expansion:

$$f(x) = f(x_0) +$$
$$+ (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \cdots$$

$$m \frac{d^2 x}{\mathrm{d}t^2} - \frac{mv^2}{\rho}\left(1 - \frac{x}{\rho}\right) = -qB_y v$$

The guide field in linear approximation $B_y = B_0 + x\frac{\partial B_y}{\partial x} = B_0 + G_{\text{quad}}x$

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \frac{mv^2}{\rho}\left(1-\frac{x}{\rho}\right) = -qv\left\{B_0 + x\frac{\partial B_y}{\partial x}\right\} \qquad \text{let's divide by } m$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \frac{v^2}{\rho}\left(1-\frac{x}{\rho}\right) = -\frac{qvB_0}{m} - x\frac{qv\,G_{\text{quad}}}{m}$$

Let's change the independent variable: $t \to s$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = x'v$$

$$\frac{d^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\underbrace{\frac{\mathrm{d}x}{\mathrm{d}s}}_{x'}\underbrace{\frac{\mathrm{d}s}{\mathrm{d}t}}_{v}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(x'v\right) =$$

$$= \frac{\mathrm{d}}{\mathrm{d}s}\underbrace{\frac{\mathrm{d}s}{\mathrm{d}t}}_{v}\left(x'v\right) = \frac{\mathrm{d}}{\mathrm{d}s}\left(x'v^2\right) = x''v^2 + \cancel{x'2v\frac{\mathrm{d}v}{\mathrm{d}s}}$$

$$x''v^2 - \frac{v^2}{\rho}\left(1-\frac{x}{\rho}\right) = -\frac{qvB_0}{m} - x\frac{vG_{\text{quad}}}{m} \qquad \text{let's divide by } v^2$$

$$x'' - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = -\frac{qB_0}{mv} - x\frac{qG_{\text{quad}}}{mv}$$
$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = -\frac{B_0}{P/q} - \frac{xG_{\text{quad}}}{P/q}$$
$$x'' \cancel{-\frac{1}{\rho}} + \frac{x}{\rho^2} = \cancel{-\frac{1}{\rho}} - kx$$

Remember:

$$mv = p$$

Normalise to the momentum of the particle:

$$\frac{1}{\rho} = \frac{B_0}{P/q}\ [\mathrm{m}^{-1}]; \quad k = \frac{G_{\text{quad}}}{P/q}\ [\mathrm{m}^{-2}]$$

$$\boxed{x'' + x\left(\frac{1}{\rho^2} + k\right) = 0}$$

Equation for the vertical motion

- $\frac{1}{\rho^2} = 0$ usually there are not vertical bends
- $k \longleftrightarrow -k$ quadrupole field changes sign

$$\boxed{y'' - ky = 0}$$

Weak focusing

- "Weak" focusing:

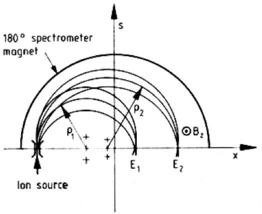
$$x''(s) + \underbrace{\left(\frac{1}{\rho^2} + k\right)}_{\text{focusing effect}} x(s) = 0$$

  there is a focusing force, $\frac{1}{\rho^2}$, even without a quadrupole gradient,

$$k = 0 \quad \Rightarrow \quad x'' = -\frac{1}{\rho^2} x$$

  even without quadrupoles there is retrieving force (focusing) in the bending plane of the dipole magnets

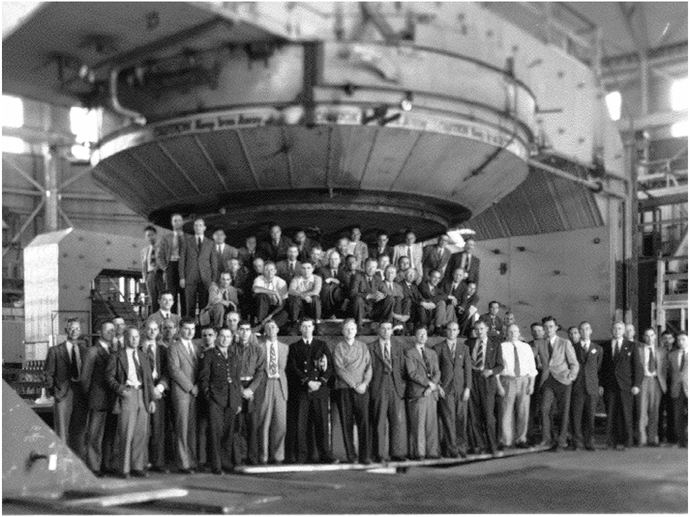
- In large machines, this effect is very weak.



Mass spectrometers entirely rely on weak focusing: they have no quadrupoles; particles are separated according to their energy and focused due to the $1/\rho$ *effect of the dipole*

When weak focusing was the state of the art...

184-inch cyclotron: a single dipole magnet with diameter 467 cm, Berkeley campus, 1942:



Solution of the trajectory equations

Definition:

$$\left.\begin{array}{rll} \text{horizontal plane} & K & = 1/\rho^2 + k \\ \text{vertical plane} & K & = -k \end{array}\right\} \quad x'' + Kx = 0$$

This is the differential equation of a 1D harmonic oscillator with spring constant K. We know that, for $K > 0$, the solution is in the form:

$$x(s) = a_1 \cos(\omega s) + a_2 \sin(\omega s)$$

In fact,

$$\begin{aligned} x'(s) &= -a_1 \omega \sin(\omega s) + a_2 \omega \cos(\omega s) \\ x''(s) &= -a_1 \omega^2 \cos(\omega s) + a_2 \omega^2 \sin(\omega s) = -\omega^2 x(s) \quad \rightarrow \quad \omega = \sqrt{K} \end{aligned}$$

Thus, the general solution is

$$x(s) = a_1 \cos\left(\sqrt{K} s\right) + a_2 \sin\left(\sqrt{K} s\right)$$

for $K > 0$.

We determine a_1, a_2 by imposing the initial conditions:

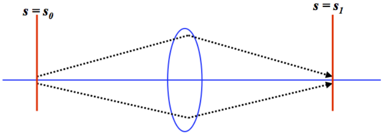
$$s = 0 \quad \rightarrow \quad \begin{cases} x(0) = x_0, & a_1 = x_0 \\ x'(0) = x'_0, & a_2 = \frac{x'_0}{\sqrt{K}} \end{cases}$$

Horizontal focusing quadrupole, $K > 0$:

$$x(s) = x_0 \cos\left(\sqrt{K} s\right) + x'_0 \frac{1}{\sqrt{K}} \sin\left(\sqrt{K} s\right)$$
$$x'(s) = -x_0 \sqrt{K} \sin\left(\sqrt{K} s\right) + x'_0 \cos\left(\sqrt{K} s\right)$$

We can use the matrix formalism:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} = M_{foc} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}_{s_0}$$



For a quadrupole of length L:

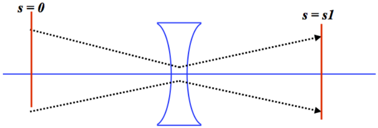
$$M_{\text{foc}} = \begin{pmatrix} \cos\left(\sqrt{K} L\right) & \frac{1}{\sqrt{K}} \sin\left(\sqrt{K} L\right) \\ -\sqrt{K} \sin\left(\sqrt{K} L\right) & \cos\left(\sqrt{K} L\right) \end{pmatrix}$$

Defocusing quadrupole

The equation of motion is

$$x'' + Kx = 0$$

with $K < 0$



Remember: $$f(s) = \cosh(s)$$ $$f'(s) = \sinh(s)$$

The solution is in the form:

$$x(s) = a_1 \cosh(\omega s) + a_2 \sinh(\omega s)$$

with $\omega = \sqrt{|K|}$. For a quadrupole of length L the transfer matrix reads:

$$M_{\text{defoc}} = \begin{pmatrix} \cosh\left(\sqrt{|K|}L\right) & \frac{1}{\sqrt{|K|}} \sinh\left(\sqrt{|K|}L\right) \\ \sqrt{|K|} \sinh\left(\sqrt{|K|}L\right) & \cosh\left(\sqrt{|K|}L\right) \end{pmatrix}$$

Notice that for a drift space, i.e. when $K = 0 \quad \rightarrow \quad M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$

Summary of the transfer matrices

- Focusing quad, $K > 0$

$$M_{\text{foc}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix}$$

- Defocusing quad, $K < 0$

$$M_{\text{defoc}} = \begin{pmatrix} \cosh\left(\sqrt{|K|}L\right) & \frac{1}{\sqrt{|K|}}\sinh\left(\sqrt{|K|}L\right) \\ \sqrt{|K|}\sinh\left(\sqrt{|K|}L\right) & \cosh\left(\sqrt{|K|}L\right) \end{pmatrix}$$

- Drift space, $K = 0$

$$M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

With the assumptions we have made, the motion in the horizontal and vertical planes is independent: the particle motion in x and y is "uncoupled"

Thin-lens approximation of a quadrupole magnet

When the focal length f of the quadrupolar lens is much bigger than the length of the magnet itself, L_Q

$$f = \frac{1}{k \cdot L_Q} \qquad \gg L_Q$$

we can derive the limit for $L \to 0$ while keeping constant f, i.e. $k \cdot L_Q = \text{const.}$

The transfer matrices are

$$M_x = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \qquad M_y = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

focusing, and defocusing respectively.

This approximation is useful for fast calculations.

Alternating-gradient focusing

- One single quadrupole cannot simultaneously focus in both the horizontal and the vertical planes
- Two quadrupoles, separated by a drift of length L, can focus in both directions
- Demonstration in thin-lens approximation:
  If
  $$M_1 = \begin{pmatrix} 1 & 0 \\ \frac{1}{f_1} & 1 \end{pmatrix}; \quad M_2 = \begin{pmatrix} 1 & 0 \\ \frac{1}{f_2} & 1 \end{pmatrix}; \quad D = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$
  The composite system is:
  $$M = M_1 \cdot D \cdot M_2 \cdot D = \begin{pmatrix} \frac{L}{f_2} + 1 & \frac{L^2}{f_2} + 2L \\ \frac{L}{f_1 f_2} + \frac{1}{f_1} + \frac{1}{f_2} & \frac{L^2}{f_1 f_2} + \frac{L}{f_2} + \frac{2L}{f_1} + 1 \end{pmatrix}$$
- This system focuses in both axes if the matrix element $M_{21} < 0$ *always*. This can be achieved imposing $f_2 = -f_1$.

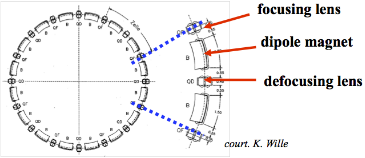
$\Rightarrow$ A system with alternating gradients, always focuses in both axes: $M_{21} = -\frac{L}{f_1^2}$

Transformation through a system of lattice elements

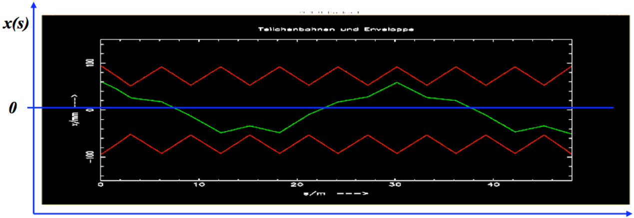
One can compute the solution of a system of elements, by multiplying the matrices of each single element:

$$M_{\text{total}} = M_{\text{QF}} \cdot M_{\text{D}} \cdot M_{\text{Bend}} \cdot M_{\text{D}} \cdot M_{\text{QD}} \cdot \quad \cdots$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = M_{s_1 \rightarrow s_2} \cdot M_{s_0 \rightarrow s_1} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$



In each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator.



...typical values are:

$$x \approx \text{mm}$$

$$x' \leq \text{mrad}$$

Properties of the transfer matrix M

The transfer matrix M has two important properties:

- Its determinant is 1

$$\det(M) = 1$$

  (Liouville's theorem, but only in case of no acceleration)

- Provides a stable motion over N turns, with $N \to \infty$, if and only if:

$$\mathrm{trace}(M) \leq 2$$

  (Stability condition)

Stability condition

Question: Given a periodic lattice with generic transport map M,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

under which condition the matrix M provides stable motion after N turns (with $N \to \infty$)?

$$x_N = \underbrace{M \cdot \ldots \cdot M \cdot M \cdot M}_{N \text{ turns, with } N \to \infty} x_0 = M^N x_0$$

The answer is simple: the motion is stable when all elements of M^N are finite, with $N \to \infty$.

The difficult question is... how do we compute M^N with $N \to \infty$?

Remember:

- $\det(M) = ad - bc = 1$
- $\mathrm{trace}(M) = a + d$

If we diagonalise M, we can rewrite it as:

$$M = U \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot U^T$$

where U is some unitary matrix, λ_1 and λ_2 are the eigenvalues.

Stability condition (cont.)

What happens if we consider N turns?

$$M^N = U \cdot \begin{pmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{pmatrix} \cdot U^T$$

Notice that λ_1 and λ_2 can be complex numbers. Given that $\mathbf{det}\,(M) = 1$, then

$$\lambda_1 \cdot \lambda_2 = 1 \quad \rightarrow \lambda_1 = \frac{1}{\lambda_2} \quad \rightarrow \lambda_{1,2} = e^{\pm i\,x}$$

$\Rightarrow$ to have a stable motion, x must be <u>real</u>: $x \in \mathrm{R}$.

Now we can find the eigenvalues through the characteristic equation:

$$\mathbf{det}\,(M - \lambda I) = \mathbf{det} \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = 0$$

$$\begin{gathered} \lambda^2 - (a + d)\,\lambda + (ad - bc) = 0 \\ \lambda^2 - \mathrm{trace}\,(M)\,\lambda + 1 = 0 \end{gathered}$$

$$\begin{aligned} \mathrm{trace}\,(M) &= \lambda + 1/\lambda = \\ &= e^{ix} + e^{-ix} = 2\cos x \end{aligned}$$

From which derives the stability condition:

$$\text{since } x \in \mathrm{R} \quad \rightarrow \quad |\mathrm{trace}\,(M)| \ \leq \ 2$$

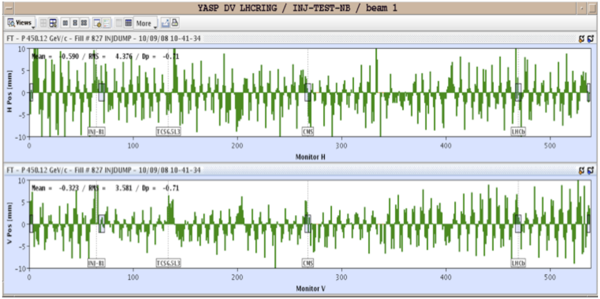
Orbit and tune

Tune: the number of oscillations per turn.

Example:

64.31

59.32



Relevant for beam stability studies is : the non-integer part

Summary

beam rigidity: $B\rho = \frac{P}{q}$

bending strength of a dipole: $\frac{1}{\rho}\ [m^{-1}] = \frac{0.2998 \cdot B_0\ [\mathrm{T}]}{P\ [\mathrm{GeV/c}]}$

focusing strength of a quadruple: $k\ [m^{-2}] = \frac{0.2998 \cdot G}{P\ [\mathrm{GeV/c}]}$

focal length of a quadrupole: $f = \frac{1}{k \cdot L_\mathrm{Q}}$

equation of motion: $x'' + \left(\frac{1}{\rho^2} + k\right) x = 0$

solution of the eq. of motion: $x_{s_2} = M \cdot x_{s_1}$ $\quad \ldots$ with $M \equiv \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$

$$\text{e.g.:} \quad M_\mathrm{QF} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}} \sin\left(\sqrt{K}L\right) \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix},$$

$$M_\mathrm{QD} = \begin{pmatrix} \cosh\left(\sqrt{|K|}L\right) & \frac{1}{\sqrt{|K|}} \sinh\left(\sqrt{|K|}L\right) \\ \sqrt{|K|}\sinh\left(\sqrt{|K|}L\right) & \cosh\left(\sqrt{|K|}L\right) \end{pmatrix}, \quad M_\mathrm{D} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

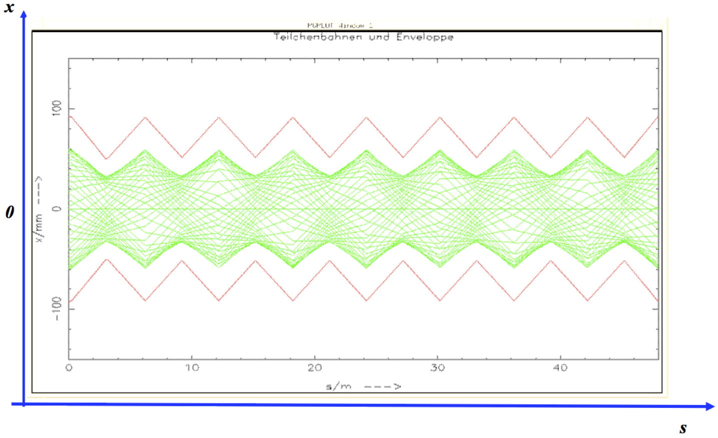
Part 2.

Optics functions and Twiss parameters

Envelope

So far we have studied the motion of a particle.
Question: what will happen, if the particle performs a second turn ?

- … or a third one or … 10^{10} turns …



The Hill's equation

In 19th century George William Hill (1838 – 1914), one of the greatest master of celestial mechanics of his time, studied the differential equation for "motions with periodic focusing properties": the "Hill's equation"

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} + K(s)\, x = 0$$

where:

- $K(s)$ is a non-constant restoring force
- $K(s)$ depends on the position s
- $K(s+L) = K(s)$ periodic function, where L is the period (the "lattice" period, in accelerator physics)

We expect a solution in the form of a quasi harmonic oscillation: amplitude and phase will depend on the position s in the ring.

The "beta" function

General solution of Hill's equation:

$$x(s) = \sqrt{\beta_x(s)\,J_x}\cos\left(\mu_x(s) + \mu_{x,0}\right) \tag{1}$$

J_x, μ_0 =integration constants determined by initial conditions

$\beta_x(s)$ is a periodic function given by the focusing properties of the lattice $\leftrightarrow$ quadrupoles

$$\beta_x(s+L) = \beta_x(s)$$

Inserting Eq. (1) in the equation of motion, we get (Floquet's theorem) the following result

$$\mu_x(s) = \int_0^s \frac{\mathrm{d}s}{\beta_x(s)}$$

where $\mu_x(s)$ is the "phase advance" between the points 0 and s, in the phase space.

For one complete revolution, $\mu_x(s)$ is the number of oscillations per turn, or "tune" when normalised to 2π

$$Q_x = \frac{1}{2\pi}\oint \frac{\mathrm{d}s}{\beta_x(s)}$$

J_x is a constant of motion, called the Courant-Snyder invariant or "action".

Note: β and J are measured in units of length, μ in units of angle.

The Twiss parameters

General solution of the Hill's equation

$$\begin{cases} x(s) &= \sqrt{\beta_x(s)\, J_x} \cos\left(\mu_x(s) + \mu_{x,0}\right) & (1) \\ x'(s) &= -\dfrac{\sqrt{J_x}}{\sqrt{\beta_x(s)}} \left\{\alpha_x(s) \cos\left(\mu_x(s) + \mu_{x,0}\right) + \sin\left(\mu_x(s) + \mu_{x,0}\right)\right\} & (2) \end{cases}$$

From Eq. (1) we get

$$\cos\left(\mu(s) + \mu_0\right) = \frac{x(s)}{\sqrt{J_x}\sqrt{\beta_x(s)}}$$

$$\alpha_x(s) = -\frac{1}{2}\beta_x'(s)$$

$$\gamma_x(s) = \frac{1 + \alpha_x(s)^2}{\beta_x(s)}$$

Insert into Eq. (2) and solve for J

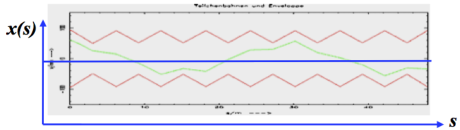
$$\boxed{J_x = \gamma_x(s)\, x(s)^2 + 2\alpha_x(s)\, x(s)\, x'(s) + \beta_x(s)\, x'(s)^2}$$

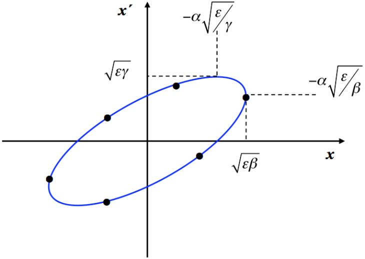
- J_x is a constant of the motion, i.e. the Courant-Snyder invariant or Action
- it is a parametric representation of an ellipse in the xx' space
- the shape and the orientation of the ellipse are given by α_x, β_x, and γ_x $\Rightarrow$ these are the Twiss parameters

The Twiss ellipse

$$J_x = \gamma_x(s)\,x(s)^2 + 2\alpha_x(s)\,x(s)\,x'(s) + \beta_x(s)\,x'(s)^2$$

Liouville's theorem: in an ideal storage ring, if there is no beam energy change, the area of the ellipse in the phase space $x - x'$ is constant

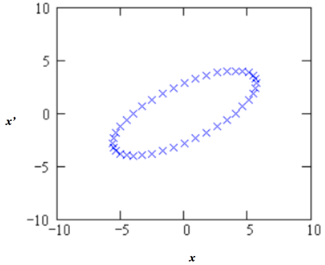




The area of ellipse, $\pi \cdot J_x$, is an intrinsic beam parameter and cannot be changed by the focal properties.

Particle motion and Twiss ellipse

For each turn x, x' at a given position s_1 in the phase-space diagram is



Note: The equation of the beam ellipse can be written also in matrix form:

$$X^T \Omega^{-1} X = J_x$$

with $X = \begin{pmatrix} x \\ x' \end{pmatrix}$ and

$$\Omega = \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix}$$

Ω is the "Twiss matrix".

β function and beam properties

Given the particle trajectory:

$$x(s) = \sqrt{\beta_x(s)\,J_x}\cos\left(\mu(s) + \mu_0\right)$$

- the max. amplitude is:

$$\hat{x}(s) = \sqrt{\beta_x(s)\,J_x}$$

- the corresponding angle, in $\hat{x}(s)$, can be found putting $\hat{x}(s) = \sqrt{\beta_x(s)\,J_x}$ in Eq.

$$J_x = \gamma_x(s)\,x(s)^2 + 2\alpha_x(s)\,x(s)\,x'(s) + \beta_x(s)\,x'(s)^2$$

  and solving for x':

$$J_x = \gamma_x(s)\cdot\beta_x(s)\,J_x + 2\alpha_x(s)\sqrt{\beta_x J_x}\cdot x' + \beta_x(s)\,x'^2$$

$$\rightarrow \quad \hat{x}' = -\alpha_x(s)\sqrt{\frac{J_x}{\beta_x(s)}} \quad \leftarrow$$

Important remarks:

- A large β-function corresponds to a large beam size and a small beam divergence
- Wherever β reaches a maximum or a minimum, $\alpha = 0$ (and $x' = 0$)

Covariance matrix of a particle distribution and geometric emittance

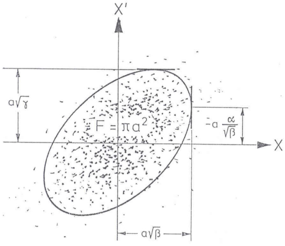
Normally in the phase space realistic particle distributions are Gaussian and this can be described using a covariance matrix, Σ. This is called also "beam matrix".

Σ is the covariance matrix of the particles distribution:

$$\Sigma = \begin{pmatrix} \sigma_x^2 & \sigma_{xx'} \\ \sigma_{xx'} & \sigma_{x'}^2 \end{pmatrix} = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'^2 \rangle \end{pmatrix}$$

The square root of the determinant of the covariance matrix is proportional to the area of the distribution in the phase space. Where

$$\det \Sigma = \sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2$$



The *geometric emittance* ϵ is defined as the square root of the determinant of Σ:

$$\boxed{\epsilon = \sqrt{\det \Sigma}}$$

$\Rightarrow$ ϵ is the area of the distribution in the phase space.

Geometric emittance and Twiss matrix

The *geometric emittance* ϵ is the square root of the determinant of Σ:

$$\boxed{\text{geometric emittance} \quad \epsilon = \sqrt{\det \Sigma}}$$

Notice that one can write:

$$\Sigma = \epsilon\,\Omega$$

where Ω is the Twiss matrix, previously defined.

Demonstration:

$$\Sigma = \begin{pmatrix} \sigma_x^2 & \sigma_{xx'} \\ \sigma_{xx'} & \sigma_{x'}^2 \end{pmatrix} = \underbrace{\epsilon \underbrace{\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}}_{\det\ \Omega = 1}}_{\det\ \epsilon\Omega = \epsilon^2}$$

From which: $\epsilon = \sqrt{\det \Sigma}$.

Liouville's theorem

Named after the French mathematician Joseph Liouville (1809 – 1882), it's a key theorem in classical statistical and Hamiltonian mechanics.

The **Liouville equation** describes the time evolution of the phase space distribution function, ρ, and asserts that such phase-space distribution function is constant along the trajectories of the system — that is, the density of system points in the vicinity of a given system point traveling through phase-space is constant with time.

In equations, the Liouville's theorem states that:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial \rho}{\partial t} + \sum_{i=1}^{N} \left(\frac{\partial \rho}{\partial q_i} \dot{q}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i \right) = 0$$

when

q_i are the canonical coordinates

p_i are the conjugate momenta

$i = 1, \ldots, N$ (where N is the number of particles)

and the system is Hamiltonian (that is, it's governed by the Hamilton's equations).

$\Rightarrow$ This is the case for planetary systems and charged particles in electromagnetic fields.

Symplectic condition

In terms of phase space, the Liouville's theorem corresponds to saying that the system's volume in the phase space is invariant under "Hamiltonian" flows.

Without entering the details, it can be demonstrated that the Liouville's theorem is verified if the so called "Symplectic condition" is true. **An arbitrary 6×6 transfer matrix**, **M**, **is said to be symplectic** if the following condition is true:

$$M^T J M = J$$

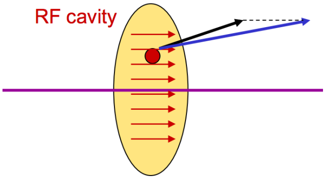
where J is the **symplectic matrix**:

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Geometric and normalised emittance

The *geometric emittance* is a constant of motion only if the beam energy is preserved:

- e.g. in case of no acceleration ($P =$ constant)
- in absence of dissipative forces (e.g. synchrotron radiation, intra-beam scattering, etc.)



In presence of acceleration $P_z \to P_z + \Delta P_z$, so that $x' = \frac{P_x}{P_z}$ goes to $x' = \frac{P_x}{P_z + \Delta P_z}$, and the area of the phase space shrinks. We therefore define the *normalised emittance*:

$$\boxed{\epsilon_{normalized} \stackrel{\text{def}}{=} \beta_{\text{rel}} \cdot \gamma_{\text{rel}} \cdot \epsilon_{\text{geometric}}}$$

The normalised emittance is a constant of motion *also* in case of acceleration.

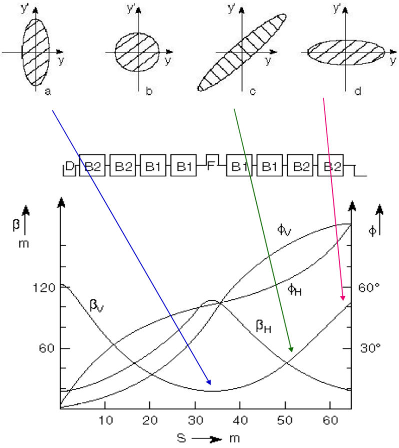
- The beam size and the beam divergence are:

$$\boxed{\sigma_x = \sqrt{\epsilon_{\text{geometric}} \cdot \beta_{x\ \text{Twiss}}}} \qquad \boxed{\sigma_{x'} = \sqrt{\epsilon_{\text{geometric}} / \beta_{x\ \text{Twiss}}}}$$

Twiss parameters in a FODO cell

Let's repeat the remarks:

- A large β-function corresponds to a large beam size and a small beam divergence
- In the middle of a quadrupole, β is maximum, and $\alpha = 0 \Rightarrow x' = 0$



[VIDEOS!]

The transfer matrix in terms of Twiss parameters

As we have already seen, a general solution of the Hill's equation is:

$$x(s) = \sqrt{\beta_x(s)\, J_x} \cos\left(\mu_x(s) + \mu_{x,0}\right)$$

$$x'(s) = -\sqrt{\frac{J_x}{\beta_x(s)}} \left[\alpha_x(s) \cos\left(\mu_x(s) + \mu_{x,0}\right) + \sin\left(\mu_x(s) + \mu_{x,0}\right)\right]$$

Let's remember some trigonometric formulæ:

$$\begin{aligned}
&\sin(a \pm b) = \sin a \, \cos b \pm \cos a \, \sin b, \\
&\cos(a \pm b) = \cos a \, \cos b \mp \sin a \, \sin b, \ \ldots
\end{aligned}$$

then,

$$\begin{aligned}
x(s) &= \sqrt{\beta_x(s)\, J_x} \left(\cos\mu_x(s) \, \cos\mu_{x,0} - \sin\mu_x(s) \, \sin\mu_{x,0}\right) \\
x'(s) &= -\sqrt{\frac{J_x}{\beta_x(s)}} \left[\alpha_x(s) \left(\cos\mu_x(s) \, \cos\mu_{x,0} - \sin\mu_x(s) \, \sin\mu_{x,0}\right) + \right. \\
&\qquad \left. + \sin\mu_x(s) \, \cos\mu_{x,0} + \cos\mu_x(s) \, \sin\mu_{x,0}\right]
\end{aligned}$$

At the starting point, $s(0) = s_0$, we put $\mu(0) = 0$. Therefore we have

$$\cos\mu_0 = \frac{x_0}{\sqrt{\beta_0 J_x}}$$
$$\sin\mu_0 = -\frac{1}{\sqrt{J_x}}\left(x_0'\sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right)$$

If we replace this in the formulæ, we obtain:

$$\underline{x(s)} = \sqrt{\frac{\beta_s}{\beta_0}}\left\{\cos\mu_s + \alpha_0\sin\mu_s\right\}\underline{x_0} + \left\{\sqrt{\beta_s\beta_0}\sin\mu_s\right\}\underline{x_0'}$$

$$\underline{x'(s)} = \frac{1}{\sqrt{\beta_s\beta_0}}\left\{(\alpha_0 - \alpha_s)\cos\mu_s - (1+\alpha_0\alpha_s)\sin\mu_s\right\}\underline{x_0} + \sqrt{\frac{\beta_0}{\beta_s}}\left\{\cos\mu_s - \alpha_s\sin\mu_s\right\}\underline{x_0'}$$

The linear map follows easily,

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M\begin{pmatrix} x \\ x' \end{pmatrix}_0 \rightarrow M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\mu_s + \alpha_0\sin\mu_s\right) & \sqrt{\beta_s\beta_0}\sin\mu_s \\ \frac{(\alpha_0-\alpha_s)\cos\mu_s - (1+\alpha_0\alpha_s)\sin\mu_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}\left(\cos\mu_s - \alpha_s\sin\mu_s\right) \end{pmatrix}$$

- We can compute the single particle trajectories between two locations in the ring, if we know the α, β, and γ at these positions!
- Exercise: prove that $\det(M) = 1$

Periodic lattices, 1-turn map

The transfer matrix for a particle trajectory

$$M_{0\to s} = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\mu_s + \alpha_0 \sin\mu_s\right) & \sqrt{\beta_s\beta_0}\sin\mu_s \\ \frac{(\alpha_0-\alpha_s)\cos\mu_s-(1+\alpha_0\alpha_s)\sin\mu_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}\left(\cos\mu_s - \alpha_s\sin\mu_s\right) \end{pmatrix}$$

simplifies considerably if we consider one complete turn:

$$\boxed{M = \begin{pmatrix} \cos\mu_L + \alpha_s\sin\mu_L & \beta_s\sin\mu_L \\ -\gamma_s\sin\mu_L & \cos\mu_L - \alpha_s\sin\mu_L \end{pmatrix}}$$

where μ_L is the phase advance per period

$$\mu_L = \int_s^{s+L} \frac{ds}{\beta(s)}$$

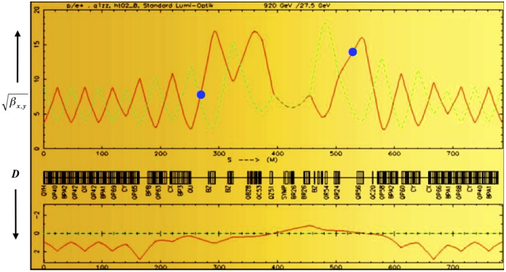
Remember: the tune is the phase advance in units of 2π:

$$Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)} = \frac{\mu_L}{2\pi}$$

Evolution of α, β, and γ

Consider two positions in the storage ring: s_0, s

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} \quad \text{with} \quad \begin{array}{l} M = M_{\text{QF}} \cdot M_{\text{D}} \cdot M_{\text{Bend}} \cdot M_{\text{D}} \cdot M_{\text{QD}} \cdot \quad \cdots \\ M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \qquad M^{-1} = \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix} \end{array}$$



Since the Liouville's theorem holds, $J = \text{const}$:

$$J_x = \beta x'^2 + 2\alpha x x' + \gamma x^2$$

$$J_x = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0^2$$

We express x_0 and x_0' as a function of x and x':

$$\left(\begin{array}{c} x \\ x' \end{array}\right)_{s_0} = M^{-1} \left(\begin{array}{c} x \\ x' \end{array}\right)_{s} \quad \Rightarrow \quad \begin{array}{l} x_0 = S'x - Sx' \\ x_0' = -C'x + Cx' \end{array}$$

Substituting x_0 and x_0' into the expression of J, we obtain:

$$J_x = \beta x'^2 + 2\alpha x x' + \gamma x^2$$
$$J_x = \beta_0 \left(-C'x + Cx'\right)^2 + 2\alpha_0 \left(S'x - Sx'\right)\left(-C'x + Cx'\right) + \gamma_0 \left(S'x - Sx'\right)^2$$

We need to sort by x and x':

$$\beta(s) = C^2 \beta_0 - 2SC\alpha_0 + S^2 \gamma_0$$
$$\alpha(s) = -CC'\beta_0 + \left(SC' + S'C\right)\alpha_0 - SS'\gamma_0$$
$$\gamma(s) = C'^2 \beta_0 - 2S'C'\alpha_0 + S'^2 \gamma_0$$

Evolution of α, β, and γ in matrix form

The beam ellipse transformation in matrix notation:

$$T_{0\to s} = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix}$$

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = T_{0\to s} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

This expression is important, and useful:

1. given the twiss parameters α, β, γ at any point in the lattice we can transform them and compute their values at any other point in the ring
2. the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to compute single particle trajectories

Exercise: Twiss transport matrix, T

Compute the Twiss transport matrix, T,

$$T = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix}$$

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = T \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

for:

1. the identity matrix: $M = \pm \mathbf{I}$
2. a drift of length L
3. a thin quadrupole with focal length $\pm f$

Beam ellipse evolution (another approach)

Let's start from the equation of the Twiss matrix Ω seen before, now for x_0:

$$X_0^T \Omega_0^{-1} X_0 = J_x \qquad \text{with:} \quad \Omega_0 = \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix}$$

$$X_1^T \Omega_1^{-1} X_1 = J_x$$

At a later point if the lattice the coordinates of an individual particle are given using the transfer matrix M from s_0 to s_1:

$$X_1 = M \cdot X_0$$

Solving for X_0 , i.e. $X_0 = M^{-1} \cdot X_1$, and inserting in the first equation above, one obtains:

$$\left(M^{-1} \cdot X_1\right)^T \Omega_0^{-1} \left(M^{-1} \cdot X_1\right) = J_x$$

$$\left(X_1^T \cdot \left(M^T\right)^{-1}\right) \Omega_0^{-1} \left(M^{-1} \cdot X_1\right) = J_x$$

$$X_1^T \cdot \underbrace{\left(M^T\right)^{-1} \Omega_0^{-1} M^{-1}}_{\Omega_1^{-1}} \cdot X_1 = J_x$$

Which gives

$$\Omega_1 = M \cdot \Omega_0 \cdot M^T$$

Normalised coordinates

The transfer matrix between two locations 1 and 2, with phase advance ϕ between them, in Twiss form is:

$$M_{1\to 2} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\phi + \alpha_1 \sin\phi) & \sqrt{\beta_1\beta_2}\sin\phi \\ \frac{(\alpha_1-\alpha_2)\cos\phi-(1+\alpha_1\alpha_2)\sin\phi}{\sqrt{\beta_2\beta_1}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\phi - \alpha_2\sin\phi) \end{pmatrix}$$

This matrix can be decomposed into three matrices

$$M_{1\to 2} = U^{-1}(\beta_2,\alpha_2)\cdot R(\phi)\cdot U(\beta_1,\alpha_1) =$$

$$= \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ \frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix}$$

The coordinates obtained by the tranformation $U(\beta,\alpha) = \begin{pmatrix} 1/\sqrt{\beta} & 0 \\ \alpha/\sqrt{\beta} & \sqrt{\beta} \end{pmatrix}$

$$\begin{pmatrix} x_n \\ x'_n \end{pmatrix} = U(\beta_1,\alpha_1)\begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}$$

are called *normalised coordinates*. [Question: what are the units of x_n and x'_n ?]

Summary

Hill's equation: $x''(s) + K(s)\,x(s) = 0, \quad K(s) = K(s+L)$

general solution of the Hill's equation: $x(s) = \sqrt{J\beta(s)}\cos(\mu(s) + \mu_0)$

phase advance & tune: $\mu_{12} = \int_{s_1}^{s_2} \frac{ds}{\beta(s)}, \quad Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$

beam ellipse: $J = \gamma(s)\,x(s)^2 + 2\alpha(s)\,x(s)\,x'(s) + \beta(s)\,x'(s)^2$

geometric emittance: $\epsilon = \text{Area of the beam ellipse} = \sqrt{\det\left(\mathrm{cov}(\mathbf{x}, \mathbf{x}')\right)}$

transfer matrix $s_1 \to s_2$:
$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\mu_s + \alpha_0\sin\mu_s\right) & \sqrt{\beta_s\beta_0}\sin\mu_s \\ \frac{(\alpha_0-\alpha_s)\cos\mu_s-(1+\alpha_0\alpha_s)\sin\mu_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}\left(\cos\mu_s - \alpha_s\sin\mu_s\right) \end{pmatrix}$$

stability criterion: $|\mathrm{trace}(M)| \leq 2$

Summary: beam matrix, emittance, and Twiss parameters

- The beam matrix is the covariance matrix of the particle distribution

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle \\ \langle x' x \rangle & \langle x'^2 \rangle \end{pmatrix}$$

  this matrix can be also expressed in terms of Twiss parameters α, β, γ and of the emittance ϵ:

$$\mathbf{\Sigma} = \begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle \\ \langle x' x \rangle & \langle x'^2 \rangle \end{pmatrix} = \epsilon \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

- Given $M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}_{0 \to s}$, we can transport the beam matrix, or the twiss parameters, from 0 to s in two equivalent ways:
  1. Twiss 3×3 transport matrix:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

  2. Recalling that $\mathbf{\Sigma}_s = M \, \mathbf{\Sigma}_0 \, M^T$:

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_s = M \cdot \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_0 \cdot M^T$$

Part 3.

Lattice design

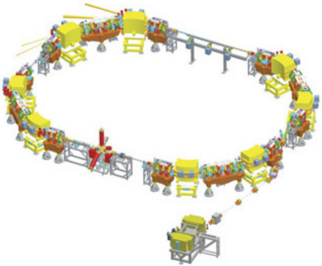
Lattice design in particle accelerators

Or..."how to build a storage ring"

High energy accelerators — are mostly circular machines
we need to juxtapose a number of **dipole** magnets,
to bend the design orbit to a closed ring, then add
quadrupole magnets (FODO cells) to focus the beam
transversely

The geometry of the system is determined by the following equality

centrifugal force = Lorentz force



$$
\begin{aligned}
\text{Lorentz force} \quad & F_L && = evB \\
\text{Centrifugal force} \quad & F_{\text{centr}} && = \frac{\gamma m v^2}{\rho} \\
& \frac{\gamma m v^{\cancel{2}}}{\rho} && = e\cancel{v}B
\end{aligned}
$$

$$\frac{P}{q} = B\rho$$

$B\rho$ is the well known beam ridigity

Note that ρ is different from R the physical radius of the machine (typically $\rho < R$).

7000 GeV proton storage ring

$N = 1232$ **dipole magnets**

$L_{\text{Bend}} = 15$ **m**

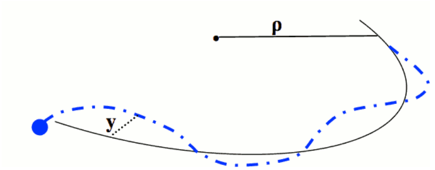
$q = +e$

$$\int B\,\mathrm{d}l \approx N L_{\text{Bend}} B = 2\pi p/e$$

$$B \approx \frac{2\pi \cdot 7000 \cdot 10^9\ \mathbf{eV}}{1232 \cdot 15\ \mathbf{m} \cdot 3 \cdot 10^8\ \frac{m}{s}\, e} = 8.3\ \mathbf{T}$$

Focusing force

$$x'' + Kx = 0$$



$$K = 1/\rho^2 + k \qquad \text{hor. plane}$$
$$K = -k \qquad \text{vert. plane}$$

$$\left.\begin{array}{lll} \text{dipole magnet} & \frac{1}{\rho} & = \frac{B}{P/q} \\ \text{quadrupole magnet} & k & = \frac{g}{P/q} \end{array}\right\}$$

Example: the LHC ring

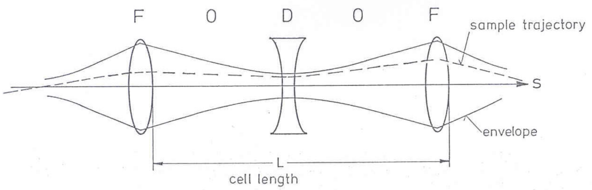
Bending radius:	$\rho = 2.53$ km
Quad gradient:	$g = 220$ T/m

$$k = 9.4 \cdot 10^{-3}\ \mathrm{m}^{-2}$$
$$1/\rho^2 = 1.3 \cdot 10^{-7}\ \mathrm{m}^{-2}$$

For estimates, in large accelerators, the weak focusing term $1/\rho^2$ can in general be neglected

The FODO lattice

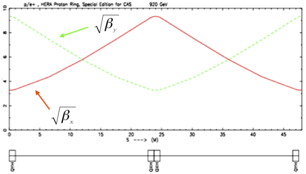
- Most high-energy accelerators, or storage rings, have a periodic sequence of quadrupole magnets of alternating polarity in the arcs

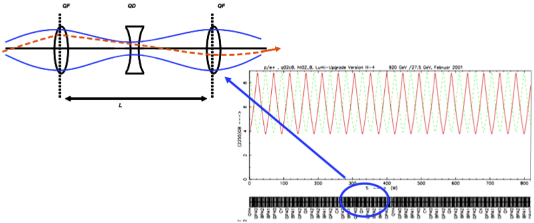


- A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with "nothing" in between

- Nota bene: "nothing" here means the elements that can be neglected on first sight: drift, bending magnet, RF structures ... and experiments...

Periodic solution in a FODO Cell





Output of MAD-X

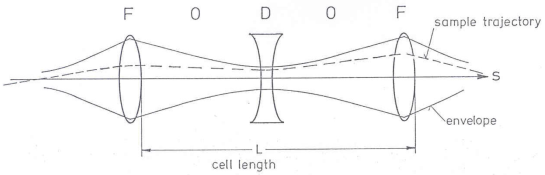
Nr	Type	Length m	Strength 1/m2	β_x m	α_x	φ_x 1/2π	β_z m	α_z	φ_z 1/2π
0	IP	0,000	0,000	11,611	0,000	0,000	5,295	0,000	0,000
1	QFH	0,250	-0,541	11,228	1,514	0,004	5,488	-0,781	0,007
2	QD	3,251	0,541	5,488	-0,781	0,070	11,228	1,514	0,066
3	QFH	6,002	-0,541	11,611	0,000	0,125	5,295	0,000	0,125
4	IP	6,002	0,000	11,611	0,000	0,125	5,295	0,000	0,125

QX= 0,125 QZ= 0,125

$$0.125 * 2\pi = 45^0$$

The FODO cell

The transfer matrix gives all the information we need.



In thin-lens approximation, we have:

$$M_{\mathrm{F}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}; \qquad M_{\mathrm{O}} = \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix}; \qquad M_{\mathrm{D}} = \begin{pmatrix} 1 & 0 \\ +\frac{1}{f} & 1 \end{pmatrix}$$

the transformation matrix of the cell is:

$$M_{\mathrm{FODO}} = M_{\mathrm{F}} \cdot M_{\mathrm{O}} \cdot M_{\mathrm{D}} \cdot M_{O}$$

(notice that you can also write $M = M_{\mathrm{F}/2} \cdot M_{\mathrm{O}} \cdot M_{\mathrm{D}} \cdot M_{O} \cdot M_{\mathrm{F}/2}$, or other cyclic permutation), which corresponds to

$$M_{\mathrm{FODO}} = \begin{pmatrix} 1 + \frac{L}{2f} & L + \frac{L^2}{4f} \\ -\frac{L}{2f^2} & 1 - \frac{L}{2f} - \frac{L^2}{4f^2} \end{pmatrix}$$

The FODO cell (cont.)

If we compare the previous matrix with the Twiss representation over one period,

$$M_{\text{FODO}} = \begin{pmatrix} 1 + \frac{L}{2f} & L + \frac{L^2}{4f} \\ -\frac{L}{2f^2} & 1 - \frac{L}{2f} - \frac{L^2}{4f^2} \end{pmatrix}$$

$$M_{\text{Twiss}} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} = \cos\mu \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathrm{I}} + \sin\mu \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{\mathrm{J}}$$

we can derive interesting properties.

- Phase advance

$$\cos\mu = \frac{1}{2}\text{trace}\,(M) = 1 - \frac{L^2}{8f^2}$$

remembering that $\cos\mu = 1 - 2\sin^2\frac{\mu}{2}$

$$\left|\sin\frac{\mu}{2}\right| = \frac{L}{4f}$$

This equation allows to compute the phase advance per cell from the cell length and the focal length of the quadrupoles.

The FODO cell (cont.)

- Example: compute the focal length in order to have a phase advance of 90° per cell

$$f = \frac{1}{\sqrt{2}}\frac{L}{2}$$

e.g. an emittance measurement station

- Stability requires that $|\cos\mu| < 1$, that is

$$\frac{L}{4f} < 1 \qquad \rightarrow \qquad \text{stability is for:} \qquad f > L/4 \quad (\text{or } L < 4f)$$

- Compute the phase advance per cell from the transfer matrix: From $\cos\mu = \frac{1}{2}\text{trace}\,(M)$

$$\mu = \arccos\left(\frac{1}{2}\text{trace}\,(M)\right)$$

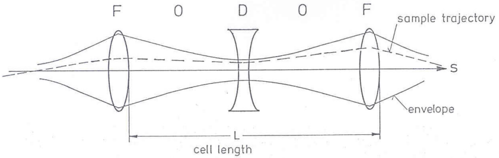
- Compute β-function and α parameter

$$\beta = \frac{M_{12}}{\sin\mu}$$

$$\alpha = \frac{M_{11} - \cos\mu}{\sin\mu}$$

The FODO cell: useful formulæ

For a FODO cell like in figure, with two quads separated by length $L/2$



one has:

$$f = \frac{1}{k_1 L_{\text{quad}}} = \frac{L_{\text{cell}}}{4 \sin \frac{\mu}{2}}$$

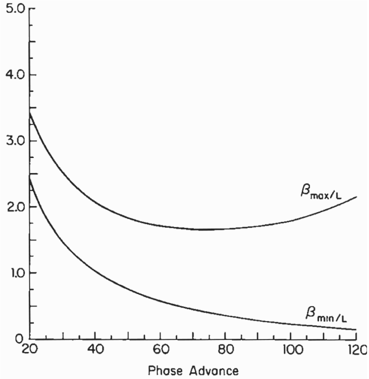
$$\beta^{\pm} = \frac{L_{\text{cell}} \left(1 \pm \sin \frac{\mu}{2}\right)}{\sin \mu}$$

$$\alpha^{\pm} = \frac{\mp 1 - \sin \frac{\mu}{2}}{\cos \frac{\mu}{2}}$$

$$D^{\pm} = \frac{L_{cell} \theta \left(1 \pm \frac{1}{2} \sin \frac{\mu}{2}\right)}{4 \sin^2 \frac{\mu}{2}}$$

θ is the total bending angle of the whole cell.

β_{max} and β_{min} as a function of μ

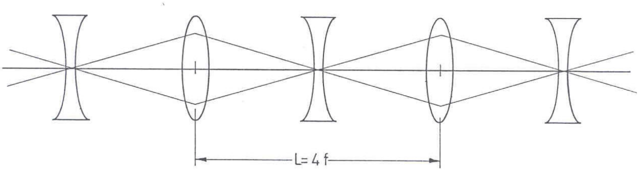


- The minimum of β_{max} can be found at $\mu_{\text{min}} = 76.345°$. (Remember: μ_{min} is such that $\frac{d\beta(\mu_{\text{min}})}{d\mu} = 0$) $\Leftarrow$ this applies only for the cases where $\epsilon_y \gg \epsilon_x$, or $\epsilon_x \gg \epsilon_y$.
- In cases where $\epsilon_x \approx \epsilon_y$ one needs to minimise $\beta_x + \beta_y$ (i.e. find the zero of $\frac{d(\beta_x+\beta_y)}{d\mu}$), which has solution $\mu_{\text{min}} = 90°$.

Stability of the FODO cell

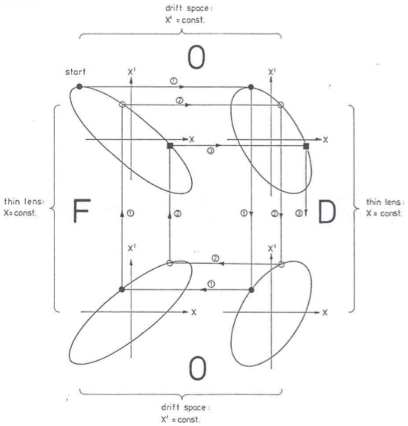
A FODO cell is stable if $4f \geq L$. This has a simple interpretation:

- It is well known from optics that an object at a distance $a = 2f$ from a focusing lens has its image at $b = 2f$



- The defocusing lenses have no effect if a point-like object is located exactly on the axis at distance $2f$ from a focusing lens, because they are traversed on the axis
- If however the lens system is moved further apart $(L > 4f)$, this is no more true and the divergence of the light or particle beam is increased by every defocusing lens

Phase space evolution



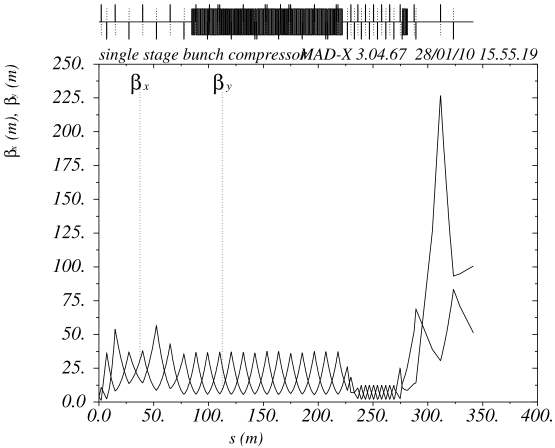
- Phase space dynamics in a simple circular accelerator consisting of one FODO cell with two 180° bending magnets located in the drift spaces (the O's)
- The periodicity of α, β, and γ is reflected by the fact that the phase-space ellipse is transformed into itself after each turn
- An individual particle trajectory, however, which starts, for instance, somewhere on the ellipse at the exit of the focusing quadrupole (small circle), is seen to move on the ellipse from turn to turn as determined by the phase angle μ
- Thus, an individual particle trajectory is not periodic, while the envelope of a whole beam is

Non-periodic lattices

- In the previous sections the Twiss parameters α, β, γ, and μ have been derived for a periodic, circular accelerator. The condition of periodicity was essential for the definition of the beta function (Hill's equation)
- Often, however, a particle beam moves only **once** along a **beam transfer line**, but one is nonetheless interested in quantities like beam envelopes and beam divergence
- In a circular accelerator α, β, and γ are completely determined by the magnet optics and the condition of periodicity (beam properties are not involved - only the beam emittance is chosen to match the beam size)
- In a transfer line, α, β, and γ are no longer uniquely determined by the transfer matrix, but they also depend on initial conditions which have to be specified in an adequate way

Example: ILC bunch compressor

Optics of a non-periodic system including non-periodic optics. "Matching" sections connect parts with different periodic conditions.



The matrix

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = M_{3\times 3} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

with

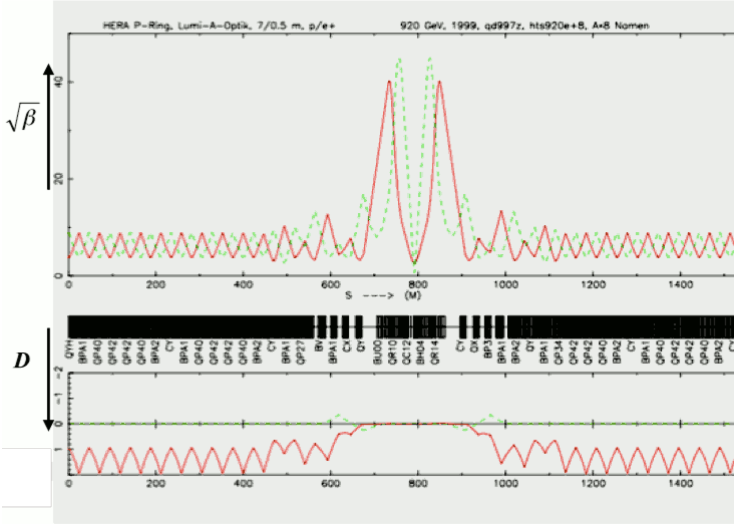
$$M_{3\times 3} = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix}$$

allows to compute the magnets parameters for the matching sections

Note: even if the β functions are very large, the beam size keeps small: $\sigma = \sqrt{\beta \epsilon}$, with

$$\epsilon_y = \frac{\epsilon_{y,N}}{\gamma_{\text{rel}}} = \frac{5 \times 10^{-9}\ \text{m}}{5\ \text{GeV} / 0.5\ \text{MeV}} = 10^{-13}\ \text{m}$$

Example: final focus of a HEP experiment



Summary

integrated dipole field over a turn	$\int B dl \approx N L_{\text{Bend}} B = 2\pi \frac{P_0}{q}$
transfer matrix of a FODO cell	$M_{\text{FODO}} = \begin{pmatrix} 1 + \frac{L}{2f} & L + \frac{L^2}{4f} \\ -\frac{L}{2f^2} & 1 - \frac{L}{2f} - \frac{L^2}{4f^2} \end{pmatrix}$
stability in a FODO cell	$f > L/4$
phase advance in a FODO cell	$\mu = \textbf{arccos}\left(\frac{1}{2}\text{trace}\,(M)\right)$
matching sections provide	$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = M_{3\times 3} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$

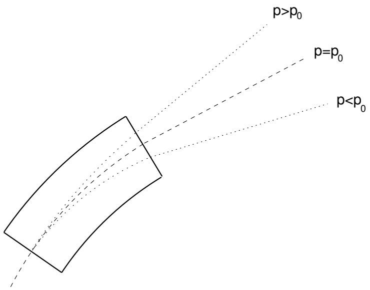
Part 4.

Dispersion

Dispersion

So far we have studied monochromatic beams of particles, but this is slightly unrealistic: We always have some (small?) momentum spread among all particles: $\Delta P = P - P_0 \neq 0$.

Example: Consider three particles with P respectively: less than, greater than, and equal to P_0 , traveling through a dipole. Remembering $B\rho = \frac{P}{q}$:



The dipole introduces a linear correlation between transverse position and momentum, called $D(s)$:

$$x(s) = D(s)\,\frac{\Delta P}{P_0}$$

This correlation is known as **dispersion function**, which can be seen as an intrinsic property of the dipole magnets.

The Inhomogeneous Hill's equation

Let's go back to the magnetic rigidity. If $P \neq P_0$ (define $\delta = \frac{P-P_0}{P_0} = \frac{\Delta P}{P_0}$) we can work out how the bending radius ρ depends on the particle momentum, w.r.t. ρ_0:

$$\Rightarrow B\rho = \frac{P}{q} = \frac{P_0\,(1+\delta)}{q} = B\rho_0\,(1+\delta) \qquad \Rightarrow \qquad \rho = \rho_0\,(1+\delta)\,.$$

When we derived the equation of motion at some point we had (slide 21):

$$\underbrace{x''}_{\text{term 1}} - \underbrace{\frac{1}{\rho+x}}_{\text{term 2}} = -\frac{B_y}{P/q} \quad \text{that later became: } x'' + \left(\frac{1}{\rho^2}+k\right)x = 0$$

On the way we had "Taylor expanded" term 2: $\displaystyle \frac{1}{\rho+x} \approx \frac{1}{\rho}\left(1-\frac{x}{\rho}\right)$.

Now we need to redo it for ρ as $\rho_0\,(1+\delta)$: $\displaystyle \frac{1}{\rho+x} = \frac{1}{\rho_0\,(1+\delta)+x} \approx \frac{1}{\rho_0}\left(1-\frac{x}{\rho_0}-\delta\right)$

and the equation of motion becomes:

$$x'' + \left(\frac{1}{\rho_0^2}+k\right)x - \frac{\delta}{\rho_0} = 0.$$

If we drop the suffix 0 and explicit δ, this is "the inhomogeneous Hill's equation":

$$\boxed{x''+\left(\frac{1}{\rho^2}+k\right)x=\frac{1}{\rho}\frac{\Delta P}{P_0}}$$

Solution of the inhomogeneous Hill's equation

A particle with $\Delta P = P - P_0 \neq 0$ satisfies the inhomogeneous Hill equation for the horizontal motion:

$$x''(s) + K(s)\,x(s) = \frac{1}{\rho}\frac{\Delta P}{P_0}$$

the total deviation of the particle from the reference orbit can be written as

$$x(s) = x_\beta(s) + x_D(s)$$

where:

- $x_\beta(s)$ describes the betatron oscillation around the new closed orbit, and it's the solution of the homogeneous equation $x''_\beta(s) + K(s)\,x_\beta(s) = 0$
- $x_D(s)$ describes the deviation of the closed orbit for an off-momentum particle. It is rewritten as $x_D(s) = D(s)\,\frac{\Delta P}{P_0}$, where $D(s)$ is the solution of the equation

  $$D''(s) + K(s)\,D(s) = \frac{1}{\rho}$$

  is that special orbit that an ideal particle would have for $\Delta P/P_0 = 1$

$D(s)$ is the *dispersion function*.

Dispersion function and orbit

The dispersion function $D(s)$ is the solution of the inhomogeneous Hill's equation:

$$D''(s) + K(s)\,D(s) = \frac{1}{\rho}$$

It can be shown that the solution is:

$$D(s) = S(s)\int_0^s \frac{1}{\rho(s')}C(s')\,\mathrm{d}s' - C(s)\int_0^s \frac{1}{\rho(s')}S(s')\,\mathrm{d}s'$$

Once we know $D(s)$, the orbit $x(s) = x_\beta(s) + x_D(s)$, with $x_D(s) = D(s)\,\frac{\Delta P}{P_0}$, can be rewritten as

$$\begin{aligned} x(s) &= x_\beta(s) + x_D(s) \\ &= C(s)\,x_0 + S(s)\,x_0' + D(s)\,\frac{\Delta P}{P_0} \end{aligned}$$

Dispersion function and orbit

The equation of motion:

$$
\begin{aligned}
x(s) &= x_\beta(s) + x_D(s) \\
&= C(s)\,x_0 + S(s)\,x'_0 + D(s)\,\frac{\Delta P}{P_0}
\end{aligned}
$$

can be written in matrix form:

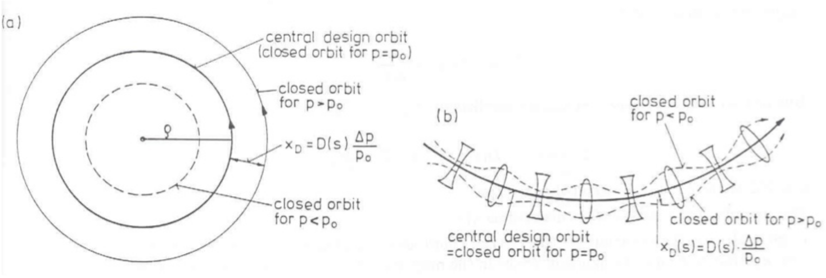
$$
\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta P}{P_0} \begin{pmatrix} D \\ D' \end{pmatrix}_0
$$

Or, in a more compact way:

$$
\begin{pmatrix} x \\ x' \\ \Delta P/P_0 \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \Delta P/P_0 \end{pmatrix}_0
$$

Closed orbit of off-momentum particles

Orbit $x(s) = x_\beta(s) + D(s)\,\frac{\Delta P}{P_0}$.



Closed orbit for particles with momentum $P \neq P_0$ in
a weakly (a) and strongly (b) focusing circular accelerator.

- $x_D(s)$ describes the deviation from the reference orbit of an off-momentum particle with $P = P_0 + \Delta P$
- $x_\beta(s)$ describes the betatron oscillation around the orbit $x_D(s)$

Dispersion and orbit propagation

The dispersion orbit is solution of $D''(s) + K(s)\,D(s) = \frac{1}{\rho}$:

$$D(s) = S(s)\int_0^s \frac{1}{\rho(s')}C(s')\,\mathrm{d}s' - C(s)\int_0^s \frac{1}{\rho(s')}S(s')\,\mathrm{d}s'$$

Now the orbit:

$$x(s) = x_\beta(s) + x_D(s)$$
$$x(s) = C(s)\,x_0 + S(s)\,x_0' + D(s)\,\frac{\Delta P}{P_0}$$

In matrix form

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta P}{P_0}\begin{pmatrix} D \\ D' \end{pmatrix}_0$$

We can rewrite the solution in matrix form:

$$\begin{pmatrix} x \\ x' \\ \Delta P/P_0 \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ x' \\ \Delta P/P_0 \end{pmatrix}_0$$

Exercise: show that $D(s)$ is a solution for the equation of motion, with the initial conditions $D_0 = D_0' = 0$.

Examples of dispersion function

Let's study, for different magnetic elements, the solution of:

$$D(s) = S(s)\int_0^s \frac{1}{\rho(s')} C(s')\,\mathrm{d}s' - C(s)\int_0^s \frac{1}{\rho(s')} S(s')\,\mathrm{d}s'$$

at the exit of the element: that is, $D(s)$ with $s = L_{\text{magnet}}$

- Drift space:

$$M_{\text{Drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$C(t) = 1,\ S(t) = L,\ \rho(t) = \infty \quad \Rightarrow$ the integrals cancel

$$M_{\text{Drift}} = \begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Dispersion function in a quadrupole

- Focusing quadrupole, $K > 0$:

$$M_{\mathrm{QF}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) & 0 \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

- Defocusing quadrupole, $K < 0$:

$$M_{\mathrm{QD}} = \begin{pmatrix} \cosh\left(\sqrt{|K|}L\right) & \frac{1}{\sqrt{|K|}}\sinh\left(\sqrt{|K|}L\right) & 0 \\ \sqrt{|K|}\sinh\left(\sqrt{|K|}L\right) & \cosh\left(\sqrt{|K|}L\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Dispersion function in a sector dipole

- Sector dipole:
  $K = \frac{1}{\rho^2}$:

$$M_{\text{Dipole}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix} = \begin{pmatrix} \cos\frac{L}{\rho} & \rho\sin\frac{L}{\rho} \\ -\frac{1}{\rho}\sin\frac{L}{\rho} & \cos\frac{L}{\rho} \end{pmatrix}$$

  which gives

$$D\left(L\right) = \rho\left(1 - \cos\frac{L}{\rho}\right)$$

$$D'\left(L\right) = \sin\frac{L}{\rho}$$

  therefore

$$M_{\text{dipole}} = \begin{pmatrix} \cos\frac{L}{\rho} & \rho\sin\frac{L}{\rho} & \rho\left(1 - \cos\frac{L}{\rho}\right) \\ -\frac{1}{\rho}\sin\frac{L}{\rho} & \cos\frac{L}{\rho} & \sin\frac{L}{\rho} \\ 0 & 0 & 1 \end{pmatrix}$$

$\phi = \frac{L}{\rho}$ is the bending angle, L is the length of magnet.

Exercise: Thin-lens approximation

- Starting from the transfer matrix of a thick dipole magnet of *small* bending angle, ϕ

$$M_{\text{dipole}} = \begin{pmatrix} \cos\phi & \rho\sin\phi & \rho\left(1-\cos\phi\right) \\ -\frac{1}{\rho}\sin\phi & \cos\phi & \sin\phi \\ 0 & 0 & 1 \end{pmatrix}$$

derive its thin-lens approximation. L is the length of the dipole

[Hint: compute the limit for $L \to 0$, while keeping the bending angle, $\phi = \frac{L}{\rho}$, constant]

Dispersion propagation through the lattice

- The equation:

$$D(s) = S(s) \int_0^s \frac{1}{\rho(s')} C(s')\,\mathrm{d}s' - C(s) \int_0^s \frac{1}{\rho(s')} S(s')\,\mathrm{d}s'$$

  allows to compute **the dispersion inside a (dipole) magnet**, which does not depend on the dispersion that might have been generated by the upstreams magnets.

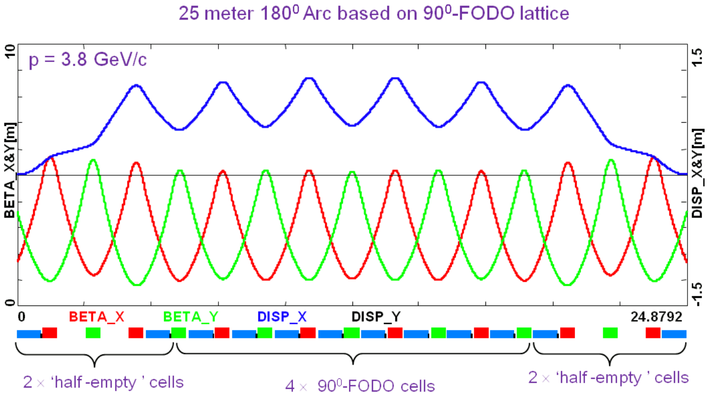
  - At the exit of a magnet of length L_{m} the dispersion reaches the value $D(L_{\mathrm{m}})$

- The dispersion (also indicated as η, with its derivative η') propagates from there, through the rest of the machine, just like a particle with $\Delta P/P = 1$:

$$\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}_0$$

Periodic dispersion

In a periodic lattice, also the dispersion must be periodic:



Aperture radius: r = 15 cm

12 × Dipoles:	field: 3.9 Tesla	length: 85 cm
15 × Quads:	gradient: 25 Tesla/m (3.8 Tesla at the pole)	length: 50 cm

Periodic dispersion

That is, for $\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}$ we need to have:

$$\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}$$

Let's rewrite this in 2×2 form:

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \end{pmatrix} + \begin{pmatrix} D \\ D' \end{pmatrix}$$

$$\begin{pmatrix} 1-C & -S \\ -C' & 1-S' \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} D \\ D' \end{pmatrix}$$

The solution is:

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \frac{1}{(1-C)(1-S') - C'S} \begin{pmatrix} 1-S' & S \\ C' & 1-C \end{pmatrix} \begin{pmatrix} D \\ D' \end{pmatrix}$$

Dispersion function in a FODO lattice

The dispersion function in a FODO cell is a periodic function with maxima at the focusing quadrupoles and minima at the defocusing quadrupoles:

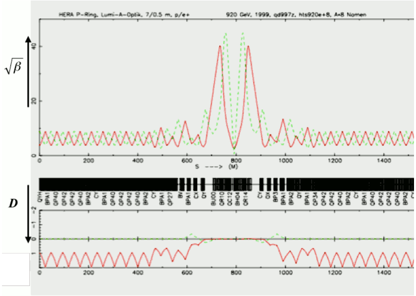
$$D^{\pm} = \frac{L\phi\left(1 \pm \frac{1}{2}\sin\frac{\mu}{2}\right)}{4\sin^2\frac{\mu}{2}}$$

where:

- L is the total length of the cell
- ϕ is the total bending angle of the cell
- μ is the phase advance of the cell

Impact of dispersion on the beam size

In this example from the HERA storage ring (DESY) we see the Twiss parameters and the dispersion near the interaction point. In the periodic region,



$$x_\beta(s) = 1 \ldots 2 \text{ mm}$$

$$D(s) = 1 \ldots 2 \text{ m}$$

$$\Delta P / P_0 \approx 1 \cdot 10^{-3}$$

Remember:

$$x(s) = x_\beta(s) + D(s)\,\frac{\Delta P}{P_0}$$

Beware: the dispersion contributes to the beam size:

$$\sigma_x = \sqrt{\sigma_{x_\beta}^2 + \text{std}\left(D \cdot \frac{\Delta P}{P_0}\right)^2} = \sqrt{\epsilon_{\text{geometric}} \cdot \beta + D^2 \cdot \frac{\sigma_P^2}{P_0^2}}$$

- We need to suppress the dispersion at the IP !
- We need a special insertion section: a *dispersion suppressor*
- Remember: $\epsilon_{\text{geometric}} = \dfrac{\epsilon_{\text{normalised}}}{\beta_{\text{rel}}\gamma_{\text{rel}}}$

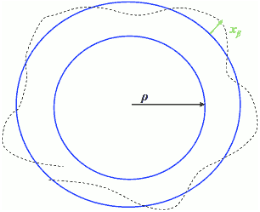
The momentum compaction factor

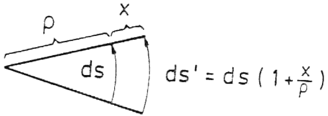
The dispersion function relates the momentum error of a particle to the horizontal orbit coordinate

The general solution of the equation of motion is

$$x(s) = x_\beta(s) + D(s)\frac{\Delta P}{P_0}$$

The dispersion changes also the length of the off-energy orbit.





particle with offset x w.r.t. the design orbit:

$$\frac{ds'}{ds} = \frac{\rho + x}{\rho} \qquad \rightarrow \qquad ds' = \left(1 + \frac{x}{\rho}\right) ds$$

The circumference change is ΔC, that is $C' = \oint \left(1 + \frac{x}{\rho}\right) ds = C + \Delta C$

We define α_p as "momentum compaction factor", such that:

$$\frac{\Delta C}{C} = \alpha_p \frac{\Delta P}{P_0} \qquad \rightarrow \text{to the lowest order in } \Delta P/P_0: \quad \alpha_p = \frac{1}{C}\oint \frac{D(s)}{\rho} ds \approx \frac{1}{Q_x^2}$$

Summary

inhomogeneous Hill's equation	$x'' + K(s)\,x = \frac{1}{\rho}\frac{\Delta P}{P_0}$
...and its solution	$x(s) = x_\beta(s) + D(s)\,\frac{\Delta P}{P_0}$
new closed orbit of off-momentum particle	$x_D(s) = D(s)\,\frac{\Delta P}{P_0}$
dispersion function	$D(s)$ [m] (closed orbit for a particle with $\frac{\Delta P}{P_0} = 1$)
how to compute dispersion in an element	$D(s) = S(s)\int_0^s \frac{1}{\rho(t)} C(t)\,\mathrm{d}t - C(s)\int_0^s \frac{1}{\rho(t)} S(t)\,\mathrm{d}t$
definition of momentum compaction, α_P	$\frac{\Delta C}{C} = \alpha_p \frac{\Delta P}{P_0}$

Part 5.

Imperfections, chromaticity

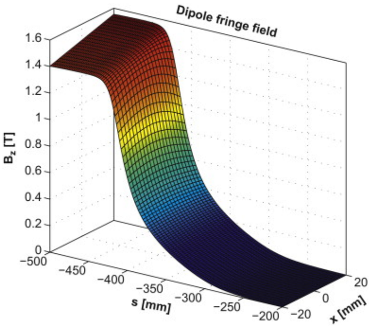
Fringe fields

- We use a "hard-edge" model:

$$x''(s) + \left(\frac{1}{\rho^2} + k\right) x(s) = 0$$

(e.g. $\rho \neq 0$ inside bending dipoles, $\rho = 0$ outside of them) but this cannot be *really* correct, because it would violate the Maxwell equations at the magnet edges

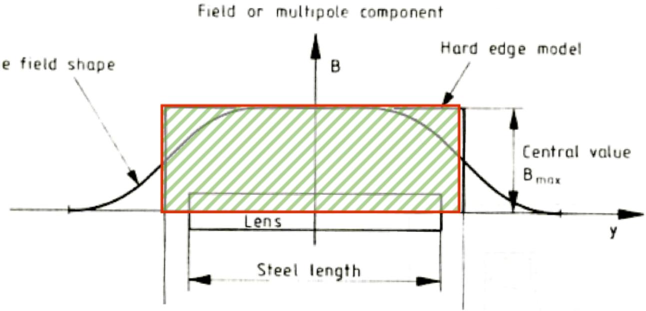
- At the edges, bending and focusing fields depend on the position s smoothly



Fringe field of a dipole magnet (in this case: a combined dipole + quadrupole magnet, notice the slope of the field along the x axis)

Effective length

$$B_0 \cdot L_{eff} = \int_0^{l_{mag}} B\,(s)\,ds$$

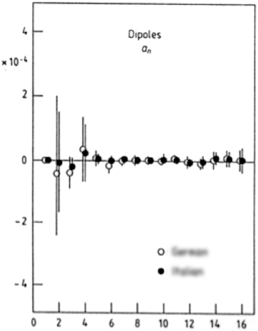


Magnetic imperfections

High-order multipolar components and misalignments

Taylor expansion of the B field:

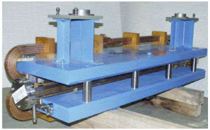
$$B_y(x) = \underbrace{B_{y0}}_{\text{dipole}} + \underbrace{\frac{\partial B_y}{\partial x}}_{\text{quad}} x + \frac{1}{2} \underbrace{\frac{\partial^2 B_y}{\partial x^2}}_{\text{sextupole}} x^2 + \frac{1}{3!} \underbrace{\frac{\partial^3 B_y}{\partial x^3}}_{\text{octupole}} x^3 + \dots \qquad \text{divide by } B_{y0}$$



There can be undesired multipolar components, due to small fabrication defects

Or also errors in the windings, in the gap h, ...

remember: $B = \dfrac{\mu_0 n I}{h}$



Moreover: **"feed-down"** effect $\Rightarrow$ a misalign magnet of order n, behaves like a magnet of order n, plus a magnet of order $n-1$ overlapped

Dipole magnet errors

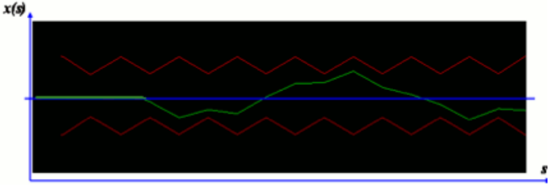
Let's imagine to have a magnet with $B = B_0 + \Delta B$. This will give an additional kick to each particle, and will distort the ideal design orbit

$$F_x = ev\,(B_0 + \Delta B)\,; \qquad \Delta x' = \Delta B \mathrm{d}s / B\rho$$

A dipole error will cause a distortion of the closed orbit, that will „run around" the storage ring, being observable everywhere. If the distortion is small enough, it will still lead to a closed orbit.

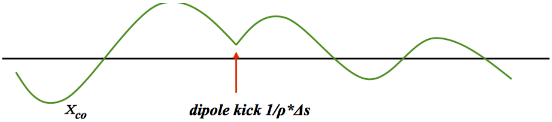
Example: 1 single dipole error

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M_{\text{lattice}} \begin{pmatrix} 0 \\ \Delta x' \end{pmatrix}_0$$



In order to have bounded motion the tune Q must be non-integer, $Q \neq 1$. We see that even for particles with reference momentum P_0 an integer Q value is forbidden, since small field errors are always present.

Orbit distortion for a single dipole field error



We consider a single thin dipole field error at the location $s = s_0$, with a kick angle $\Delta x'$.

$$X_- = \begin{pmatrix} x_0 \\ x_0' + \Delta x' \end{pmatrix}, \quad X_+ = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

are the phase space coordinates before and after the kick located at s_0. The closed-orbit condition becomes

$$M_{\text{Lattice}} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \begin{pmatrix} x_0 \\ x_0' + \Delta x' \end{pmatrix}$$

The resulting closed orbit at s_0 is

$$x_0 = \frac{\beta_0 \Delta x'}{2 \sin \pi Q} \cos \pi Q; \quad x_0' = \frac{\Delta x'}{2 \sin \pi Q} \left(\sin \pi Q - \alpha_0 \cos \pi Q \right)$$

where Q is the tune. The orbit at any other location s is

$$x(s) = \frac{\sqrt{\beta_s \beta_0} \Delta x'}{2 \sin \pi Q} \cos \left(\pi Q - |\mu_s - \mu_0| \right)$$

(see the references for a demonstration)

Orbit distortion for distributed dipole field errors

One single dipole field error

$$x(s) = \frac{\sqrt{\beta_s \beta_0}\,\Delta x'}{2 \sin \pi Q} \cos\left(\pi Q - |\mu_s - \mu_0|\right)$$

Distributed dipole field errors

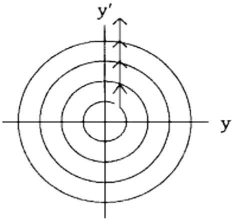
$$x(s) = \frac{\sqrt{\beta_s}}{2 \sin \pi Q} \sum_i \sqrt{\beta_i}\,\Delta x'_i \cos\left(\pi Q - |\mu_s - \mu_i|\right)$$

- orbit distortion is visible at any position s in the ring, even if the dipole error is located at one single point s_0
- the β function describes the sensitivity of the beam to external fields
- the β function acts as amplification factor for the orbit amplitude at the given observation point
- there is a singularity at the denominator when Q integer $\Rightarrow$ it's called resonance

Tune and resonances

The particles – oscillating under the influence of the external magnetic fields – can be excited in case of resonant tunes to infinite high amplitudes.

There is particle loss within a short number of turns.



ν_y = integer

The cure:

1. **avoid** large magnet errors
2. **avoid forbidden tune values** in both planes

$$m \cdot Q_x + n \cdot Q_y \neq p$$

with m, n, p integer numbers

Quadrupole errors: tune shift

Orbit perturbation described by a thin lens quadrupole:

$$M_{\mathrm{Perturbed}} = \underbrace{\begin{pmatrix} 1 & 0 \\ \Delta k ds & 1 \end{pmatrix}}_{\text{perturbation}} \underbrace{\begin{pmatrix} \cos\mu_0 + \alpha\sin\mu_0 & \beta\sin\mu_0 \\ -\gamma\sin\mu_0 & \cos\mu_0 - \alpha\sin\mu_0 \end{pmatrix}}_{\text{ideal ring}}$$

Let's see how the tunes changes: one-turn map

$$M_{\mathrm{Perturbed}} = \begin{pmatrix} \cos\mu_0 + \alpha\sin\mu_0 & \beta\sin\mu_0 \\ \Delta k ds \left(\cos\mu_0 + \alpha\sin\mu_0\right) - \gamma\sin\mu_0 & \Delta k ds \beta\sin\mu_0 + \cos\mu_0 - \alpha\sin\mu_0 \end{pmatrix}$$

with $\mu_0 = 2\pi Q$. Remember the rule for computing the tune:

$$2\cos\mu = \mathrm{trace}\left(M\right) = 2\cos\mu_0 + \Delta k ds \beta\sin\mu_0$$

Quadrupole errors: tune shift (cont.)

We rewrite $\cos\mu = \cos(\mu_0 + \Delta\mu)$

$$\cos(\mu_0 + \Delta\mu) = \cos\mu_0 + \frac{1}{2}\Delta k \mathrm{d}s \beta \sin\mu_0$$

from which we can compute that

$$\Delta\mu = \frac{\Delta k \,\mathrm{d}s\, \beta}{2} \quad \text{shift in the phase advance}$$

$$\Delta Q = \oint_{\text{quads}} \frac{\Delta k(s)\,\beta(s)\,\mathrm{d}s}{4\pi} \quad \text{tune shift}$$

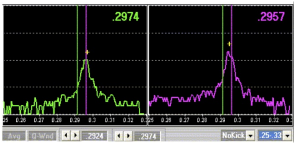
Important remarks:

- the tune shift if proportional to the β-function at the location of the quadrupole
  - field quality, power supply tolerances etc. are much tighter at places where β is large
- β is a measurement of the sensitivity of the beam

Quadrupole errors: tune shift example

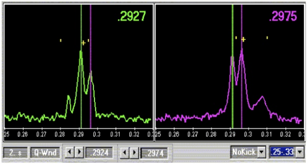
Deliberate change of a quadrupole strength in a synchrotron:

$$\Delta Q = \oint_{\text{quads}} \frac{\Delta K(s)\,\beta(s)\,\mathrm{d}s}{4\pi} \approx \frac{\Delta K(s)\; L_{\text{quad}}\,\overline{\beta}}{4\pi}$$



The tune is measured permanently

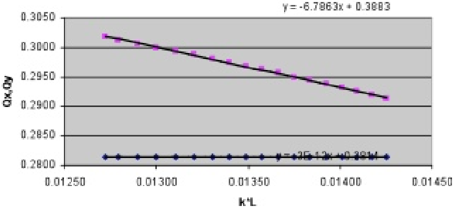
$\Rightarrow$



We change the strength of "trim" quads to fix Q

Horizontal axis is a scan of K_1 (quad integrated focusing strength):

- tune shift is proportional to β through $\Delta Q \propto \Delta K \cdot \beta$
- En passant, we use this to measure β.



Tune shift correction

Errors in the quadrupole fields induce tune shift:

$$\Delta Q = \oint_{\text{quads}} \frac{\Delta k\,(s)\,\beta\,(s)\,\mathrm{d}s}{4\pi}$$

Cure: we compensate the quad errors using other (correcting) quadrupoles

- If you use only one correcting quadrupole, with $1/f = \Delta k_1 L$
  - it changes both Q_x and Q_y:

$$\Delta Q_x = \frac{\beta_{1x}}{4\pi f_1} \quad \text{and} \quad \Delta Q_y = -\frac{\beta_{1y}}{4\pi f_1}$$

- We need to use two independent correcting quadrupoles:

$$\Delta Q_x = \frac{\beta_{1x}}{4\pi f_1} + \frac{\beta_{2x}}{4\pi f_2} \qquad \Delta Q_y = -\frac{\beta_{1y}}{4\pi f_1} - \frac{\beta_{2y}}{4\pi f_2}$$

$$\begin{pmatrix} \Delta Q_x \\ \Delta Q_y \end{pmatrix} = \frac{1}{4\pi} \begin{pmatrix} \beta_{1x} & \beta_{2x} \\ \beta_{1y} & \beta_{2y} \end{pmatrix} \begin{pmatrix} 1/f_1 \\ 1/f_2 \end{pmatrix}$$

- Solve by inversion:

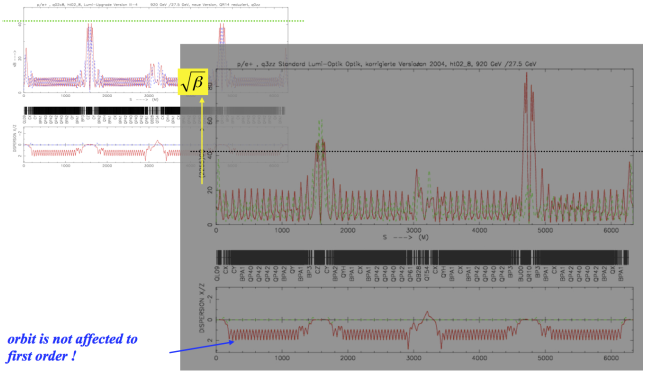
$$\begin{pmatrix} 1/f_1 \\ 1/f_2 \end{pmatrix} = \frac{4\pi}{\beta_{1x}\beta_{2y} - \beta_{2x}\beta_{1y}} \begin{pmatrix} \beta_{2y} & -\beta_{2x} \\ -\beta_{1y} & \beta_{1x} \end{pmatrix} \begin{pmatrix} \Delta Q_x \\ \Delta Q_y \end{pmatrix}$$

Quadrupole errors: beta beat

A quadrupole error at s_0 causes distortion of β-function at s: $\Delta\beta(s)$ due to the errors of all quadrupoles:

$$\frac{\Delta\beta_s}{\beta_s} = \frac{1}{2\sin 2\pi Q}\sum_i \beta_i \Delta k_i \cos\left(2\pi Q - 2\left(\mu_i - \mu_s\right)\right)$$

Note: Unstable betatron motion if tune is half integer!



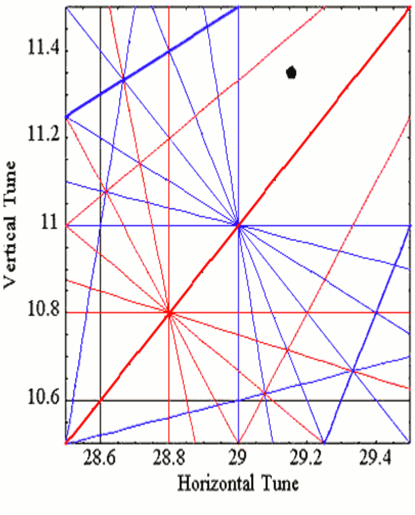
This imperfection can be corrected with an appropriate distribution of tuneable sextupoles.

Resonance diagram

The close orbit is stable when

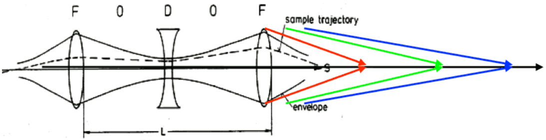
$$m \cdot Q_x + n \cdot Q_y \neq p$$

where m, n, and p are integer numbers.



Quadrupole errors: chromaticity, ξ

[VIDEO!] Chromaticity is an *optical aberration* occurring in quadrupoles when $\Delta P/P_0 \neq 0$:



The chromaticity ξ is the variation of tune ΔQ with the relative momentum deviation:

$$\xi = \frac{\Delta Q}{\Delta P/P_0} \quad \Rightarrow \quad \Delta Q = \xi \frac{\Delta P}{P_0}$$

Remember the quadrupole strength:

$$k = \frac{G}{P/q} \qquad \text{with } P = P_0 + \Delta P = P_0\,(1+\delta)$$

then

$$k = \frac{qG}{P_0 + \Delta P} = \frac{k_0}{1+\delta} \approx \frac{q}{P_0}\left(1 - \frac{\Delta P}{P_0}\right) G = k_0 + \Delta k$$

$$\Delta k = -\frac{\Delta P}{P_0} k_0$$

Quadrupole errors: chromaticity (cont.)

$$\Delta k = -\frac{\Delta P}{P_0} k_0$$

$\Rightarrow$ Chromaticity acts like a quadrupole error and leads to a *tune spread:*

$$\Delta Q_{\text{one quad}} = -\frac{1}{4\pi}\frac{\Delta P}{P_0} k_0 \beta(s)\,\mathrm{d}s \qquad \Rightarrow \Delta Q_{\text{all quads}} = -\frac{1}{4\pi}\frac{\Delta P}{P_0}\oint k(s)\,\beta(s)\,\mathrm{d}s$$

Therefore the definition of chromaticity ξ is

$$\xi = -\frac{1}{4\pi}\oint_{\text{quads}} k(s)\,\beta(s)\,\mathrm{d}s$$

The peculiarity of chromaticity is that it isn't due to external agents, it is generated by the lattice itself!

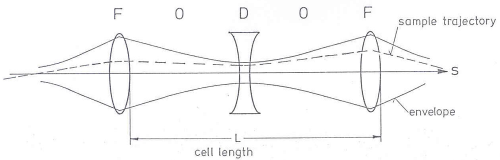
Remarks:

- ξ is a number indicating the size of the tune spot in the working diagram
- ξ is always created by the focusing strength k of **all** quadrupoles
- **natural chromaticity of a focusing quad is always negative**

In other words, because of chromaticity the tune is not a sharp point, but is a **spot**

Example: Chromaticity of the FODO cell

Consider a FODO cells like in figure, with two thin quads, each with focal length f, separated by length $L/2$, and total phase advance μ:



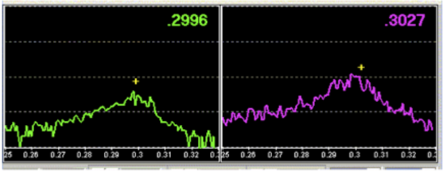
The natural chromaticity ξ_N of the cell is:

$$
\begin{aligned}
\xi_N &= -\frac{1}{4\pi}\oint \beta(s)k(s)ds \\
&= -\frac{1}{4\pi}\int_{\text{cell}} \beta(s) \underbrace{k(s)ds}_{k(s)ds=KL=\frac{1}{f}} \\
&= -\frac{1}{4\pi}\left[\frac{\beta^+}{f} - \frac{\beta^-}{f}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{4\pi\sin\mu}\left[\left(L+\frac{L^2}{4f}\right)\frac{1}{f} - \left(L-\frac{L^2}{4f}\right)\frac{1}{f}\right] \\
&= -\frac{1}{4\pi\sin\mu}\left[\frac{L}{f} - \frac{L}{f} + \frac{L^2}{2f^2}\right] \\
&= -\frac{1}{8\pi\sin\mu}\frac{L^2}{f^2} \simeq -\frac{1}{\pi}\tan\frac{\mu}{2}
\end{aligned}
$$

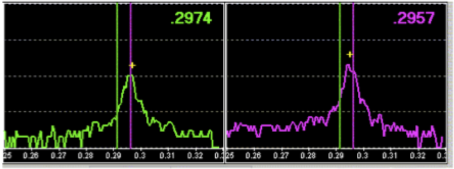
For N_{cell} cells, the total chromaticity is N_{cell} times the chromaticity of each cell $\xi_{N,N_{\text{cell}}} = -\frac{N_{\text{cell}}}{\pi}\tan\frac{\mu}{2}$.

Quadrupole errors: chromaticity



Tune signal for a nearly uncompensated cromaticity (Q' ≈ 20)

Ideal situation: cromaticity well corrected, (Q' ≈ 1)

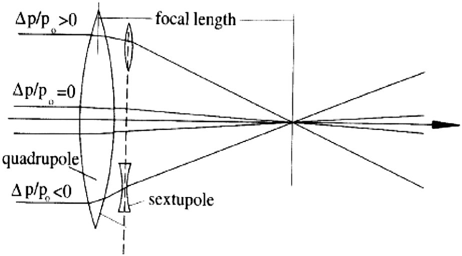


Chromaticity correction

Remember what is chromaticity: the quadrupole focusing experienced by particles changes with energy

- it induces tune shift, which can cause beam lifetime reduction due to resonances

Cure: we need additional, energy-dependent, focusing. This is given by sextupoles



- The sextupole magnetic field rises quadratically:

$$B_x = \tilde{G}xy \qquad \Rightarrow \frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x} = \tilde{G}x \quad \text{a "moving" quadrupole gradient}$$

$$B_y = \frac{1}{2}\tilde{G}\left(x^2 - y^2\right)$$

it provides a linearly increasing quadrupole gradient

Chromaticity correction (cont.)

Now remember:

- Normalised quadrupole strength is

$$k_1 = \frac{G}{P_0/q} \ [\mathrm{m}^{-2}]$$

- Sextupoles are characterised by a normalised sextupole strength k_2, which carries a focusing quadrupolar component k_1:

$$k_2 = \frac{\tilde{G}}{P_0/q} \ [\mathrm{m}^{-3}]; \qquad \tilde{k}_1 = \frac{\tilde{G}x}{P_0/q} = k_2 x \ [\mathrm{m}^{-2}]$$

Cure for chromaticity: we need sextupole magnets installed in the storage ring in order to increase the focusing strength for particles with larger energy

- A sextupole at a location with dispersion does the trick: $x = D \cdot \frac{\Delta P}{P_0}$

$$\tilde{k}_1 = \frac{\tilde{G}\left(D\frac{\Delta P}{P_0}\right)}{P/q}\ [\mathrm{m}^{-2}]$$

- for $x = 0$ it corresponds to an energy-dependent focal length

$$\frac{1}{f_{\text{sext}}} = \tilde{k}_1 L_{\text{sext}} = \overbrace{\underbrace{\frac{\tilde{G}}{P/q}}_{k_2} \underbrace{D\frac{\Delta P}{P_0}}_{[\mathrm{m}]}}^{\tilde{k}_1} \cdot L_{\text{sext}} = k_2 D \cdot \frac{\Delta P}{P_0} \cdot L_{\text{sext}}$$

Now the formula for the chromaticity rewrites:

$$\xi = \underbrace{-\frac{1}{4\pi}\oint k(s)\,\beta(s)\,\mathrm{d}s}_{\text{chromaticity due to quadrupoles}} + \underbrace{\frac{1}{4\pi}\oint k_2(s)\,D\beta(s)\,\mathrm{d}s}_{\text{chromaticity due to sextupoles}}$$

Design rules for sextupole scheme

- Chromatic aberrations must be corrected in both planes $\Rightarrow$ you need at least two sextupoles, S_F and S_D (sextupole strengths)

- In each plane the sextupole fields contribute with different signs to the chromaticity ξ_x and ξ_y:

$$\xi_x = -\frac{1}{4\pi} \oint \beta_x(s) \left[\quad k(s) - S_F D_x(s) + S_D D_x(s) \right] \mathrm{d}s$$
$$\xi_y = -\frac{1}{4\pi} \oint \beta_y(s) \left[-k(s) + S_F D_x(s) - S_D D_x(s) \right] \mathrm{d}s$$

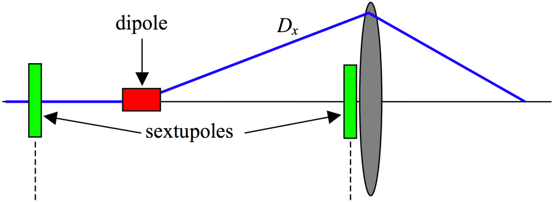
- To minimise chromatic sextupoles strengths, sextupoles should be located near quadrupoles where $\beta_x D_x$ and $\beta_y D_x$ are large

- For optimal independent chromatic correction S_F should be located where the ratio β_x/β_y is large, S_D where β_y/β_x is large.

Example of chromaticity correction scheme

- *Chromatic aberrations* introduced by quadrupoles are locally cancelled by sextupoles placed near the quadrupoles, in dispersive regions (in straight sections dispersion is generated using an upstream bending magnet)

- Notice that the sextupoles affect also the on-momentum particles: i.e. they introduce geometric *aberrations*. These can be cancelled by adding one additional sextupoles (per each direction), in opposite phase with them ($\Delta\mu = \pi$)



The phase advance between the two sextupoles S_1 and S_2 must be π, so that:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} \rightarrow \underbrace{\begin{array}{c} \Delta\mu = \pi \\ \Updownarrow \\ M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}}_{s_1 \rightarrow s_2} \rightarrow \begin{pmatrix} -x \\ -x' \end{pmatrix}_{s_2}$$

Summary of imperfections

Error	Effect	Cure
fabrication imperfections	unwanted multipolar components	better fabrication / multipolar corrector coils
transverse offsets	"feed-down" effect	better alignment / corrector kickers
roll effects	*couplings* $x - y$	*skew quads*
dipole kicks along the ring	orbit distortion $\propto \beta_{\text{kick location}}$, residual dispersion	corrector kickers
quad field errors	tune shift	trim special quadrupoles
chromaticity	tune spread	design / sextupoles
power supplies	closed orbit distortion tune shift / spread	try to correct / improve power supplies

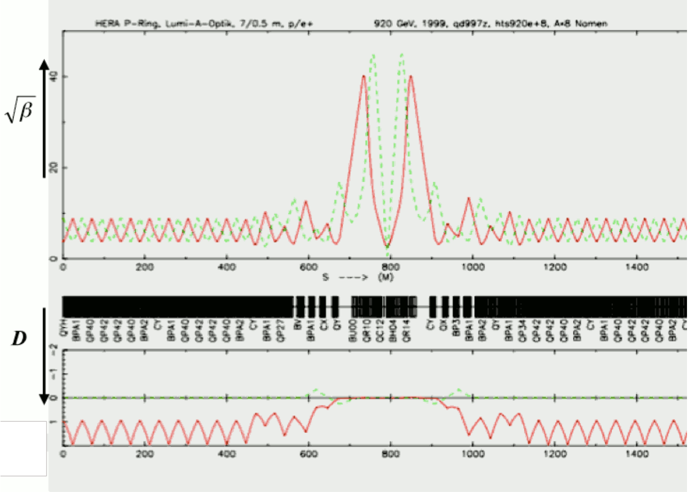
Summary

stability condition & resonances	$m \cdot Q_x + n \cdot Q_y \neq p$ with n, m, p integers
closed orbit distortion due to dipole errors	$x(s) = \frac{\sqrt{\beta_s}}{2 \sin \pi Q} \sum_i \sqrt{\beta_i} \Delta x_i' \cos(\pi Q - \lvert\mu_s - \mu_i\rvert)$
tune shift	$\Delta Q = \frac{1}{4\pi} \oint_{\text{quads}} \Delta k(s)\, \beta(s)\, \mathrm{d}s$
beta beat	$\frac{\Delta\beta(s)}{\beta(s)} = \frac{1}{2 \sin 2\pi Q} \cdot \oint \beta(t)\, \Delta k(t) \cos(2\pi Q - 2(\mu(t) - \mu(s)))\, \mathrm{d}t$
chromaticity	$\xi = \frac{\Delta Q}{\Delta P / P_0} = -\frac{1}{4\pi} \oint_{\text{quads}} k(s)\, \beta(s)\, \mathrm{d}s$

Part 6.

Insertions

Insertions

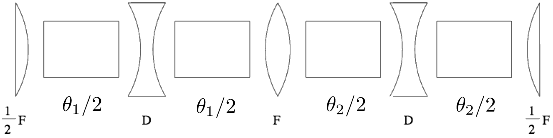


$$L = \frac{N_b N_{e^-} N_{e^+} f_{rev}}{4\pi\sigma_x^* \sigma_y^*} \quad [\mathrm{cm}^{-2}\mathrm{s}^{-1}]$$

Dispersion suppressor

In an arc, the FODO dispersion is non-zero everywhere. However, in straight sections, we often want to have $\eta = \eta' = 0$. $\Rightarrow$ for instance to keep small the beam size at the interaction point.

We can "match" between these two conditions with a "dispersion suppressor": a non-periodic set of magnets that transforms FODO η, η' to zero



Consider two FODO cells with length L and different total bend angles: θ_1, θ_2: we want to have

$$\left(\begin{array}{c}\eta\\ \eta'\end{array}\right)_{\text{entrance}} \equiv \left(\begin{array}{c}\eta_0\\ 0\end{array}\right) \quad \text{and} \quad \left(\begin{array}{c}\eta\\ \eta'\end{array}\right)_{\text{exit}} \equiv \left(\begin{array}{c}0\\ 0\end{array}\right)$$

Note:

- the two cells have the same quadrupole strengths, so that they have also the same β, and μ (phase advance per cell)
- remember that $\alpha = 0$ at both ends, and that, if the incoming beam comes from a FODO cell with the same length L, phase advance μ, and with a total bending angle θ, then the initial dispersion is

$$\eta_0 = \eta^+_{\text{FODO}}$$

[$\eta^+_{\text{FODO}} \approx \frac{4f^2}{L}\left(1 + \frac{L}{8f}\right)\theta$ in thin-lens approximation]

Dispersion suppressor (cont.)

Transport for the dispersion:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix}_{suppressor} \begin{pmatrix} \eta_0 \\ 0 \\ 1 \end{pmatrix}$$

In 2×2 form reads

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} \eta_0 \\ 0 \end{pmatrix} + \begin{pmatrix} D \\ D' \end{pmatrix}$$

which has solution

$$\begin{pmatrix} D \\ D' \end{pmatrix} = - \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} \eta_0 \\ 0 \end{pmatrix}$$

The transfer matrix for the suppressor is

$$M_{\text{suppressor}} = M_{\text{FODO 2}} \cdot M_{\text{FODO 1}}$$

For each FODO cell, $M_{\text{FODO}} = M_{1/2\text{F}} \cdot M_{\text{dipole}} \cdot M_{\text{D}} \cdot M_{\text{dipole}} \cdot M_{1/2\text{F}}$, in thin-lens approximation:

$$M_{FODO\,j} = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L\left(1 + \frac{l}{4f}\right) & \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_j \\ -\frac{L}{4f^2}\left(1 - \frac{L}{4f}\right) & 1 - \frac{L^2}{8f^2} & \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_j \\ 0 & 0 & 1 \end{pmatrix}$$

where $j = 1, 2$ (1=first cell, 2=second cell)

Dispersion suppressor (cont.)

If we do the math, we find the expressions that we have to set to zero:

$$\begin{cases} D\left(s\right) = \dfrac{L}{2}\left(1+\dfrac{L}{8f}\right)\left[\left(3-\dfrac{L^2}{4f^2}\right)\theta_1+\theta_2\right] \\ D'\left(s\right) = \left(1-\dfrac{L}{8f}-\dfrac{L^2}{32f^2}\right)\left[\left(1-\dfrac{L^2}{4f^2}\right)\theta_1+\theta_2\right] \end{cases}$$

From lecture 3, we remember that the phase advance μ for a FODO cell, in terms of the length L and the focal length f, is

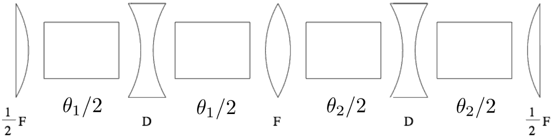
$$\left|\sin\frac{\mu}{2}\right| = \frac{L}{4f}$$

Thus, one can write the solution as a function of the phase advance μ, and of $\theta = \theta_1 + \theta_2$:

$$\begin{cases} \theta_1 = \left(1-\dfrac{1}{4\sin^2\frac{\mu}{2}}\right)\theta \\ \theta_2 = \dfrac{1}{4\sin^2\frac{\mu}{2}}\theta \end{cases}$$

Dispersion suppressor (summary)

Dispersion suppressor, a non-periodic set of magnets that transforms FODO η, η' to zero:



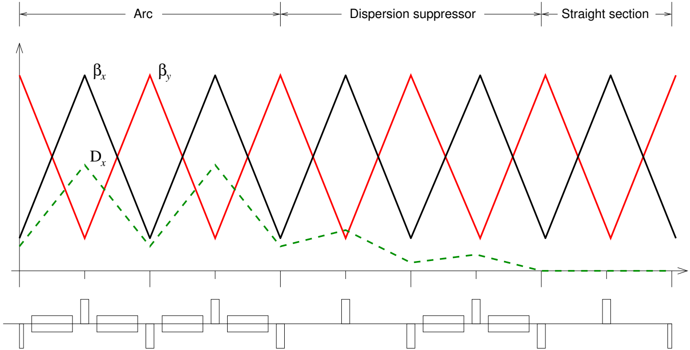
One possibility: two FODO cells with length L, phase advance μ, and different total bend angles: θ_1, θ_2:

$$\begin{cases} \theta_1 = \left(1 - \dfrac{1}{4\sin^2\frac{\mu}{2}}\right)\theta \\ \theta_2 = \dfrac{1}{4\sin^2\frac{\mu}{2}}\theta \end{cases}$$

An interesting solution is for $\mu = 60^\circ$: in this case

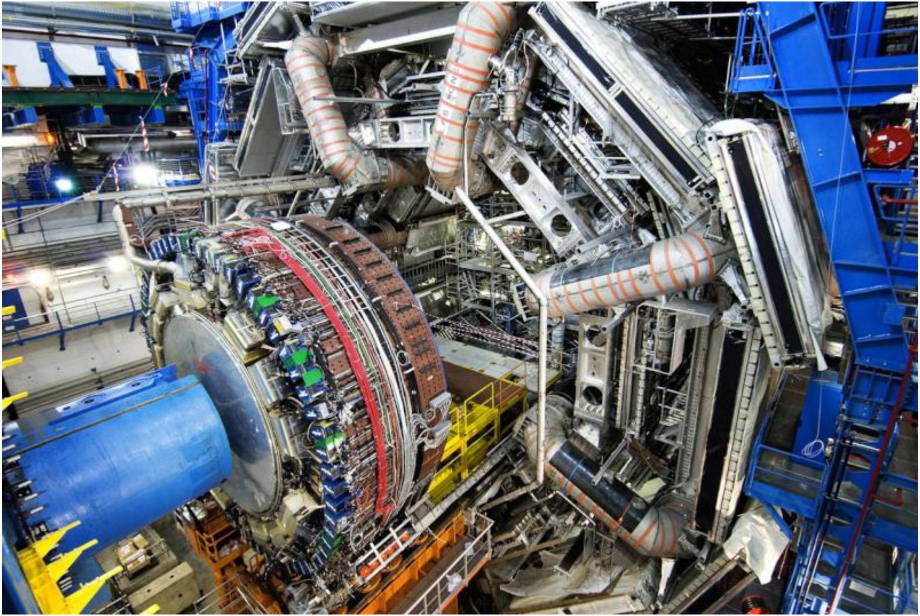
- then $\theta_1 = 0$, and $\theta_2 = \theta \Rightarrow$ we just leave out two dipole magnets in the first FODO cell insertion
- this is called the "missing-magnet" scheme

Optics functions in the dispersion suppressor, with $\mu = 60^\circ$



This is the "missing-magnet" scheme.

Often the insertions are **bigger** than few meters...



The most problematic insertion: the drift space

The most problematic insertion is the drift space !

Let's see what happens to the Twiss parameters α, β, and γ if we stop focusing for a while

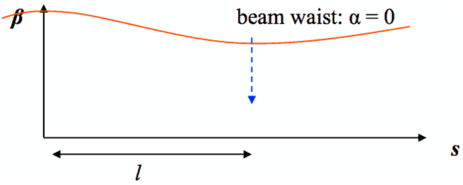
$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

for a drift:

$$M_{\text{drift}} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \begin{cases} \beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2 \\ \alpha(s) = \alpha_0 - \gamma_0 s \\ \gamma(s) = \gamma_0 \end{cases}$$

Let's find the location of the waist: $\alpha = 0$

- the location of the point of smallest beam size, $\beta^\star$



Beam waist:

$$\alpha(s) = \alpha_0 - \gamma_0 s = 0 \qquad \rightarrow \qquad s = \frac{\alpha_0}{\gamma_0} = l_{\text{waist}}$$

Beam size at that point

$$\left.\begin{array}{l} \gamma(l) = \gamma_0 \\ \alpha(l) = 0 \end{array}\right\} \qquad \rightarrow \gamma(l) = \frac{1 + \alpha^2(l)}{\beta(l)} = \frac{1}{\beta(l)} \qquad \rightarrow \beta_{\text{min}} = \frac{1}{\gamma_0}$$

This beta, at $l = l_{\text{waist}}$, is also called "beta star":

$$\Rightarrow \beta^\star = \beta_{\text{min}}$$

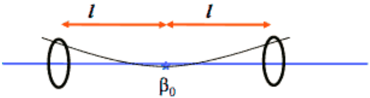
It's at $l = l_{\text{waist}}$ that the interaction point (IP) is located.

A drift space with $L = l_{\text{waist}}$: the Low β-insertion

We can assume we have a symmetry point at a distance l_{waist}:

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2, \text{ at } \alpha(s) = 0 \quad \rightarrow \beta^\star = \frac{1}{\gamma_0}$$

On each side of the symmetry point

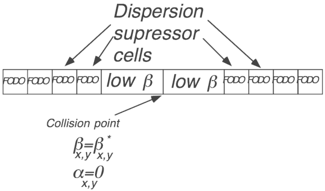


we have

$$\beta(s) = \beta^\star + \frac{s^2}{\beta^\star}$$

$\Rightarrow$ β grows quadratically with s.

A drift space at the interaction point, with length $L = l_{\text{waist}}$, is called "low-β insertion":



typical low-β insertion suitable to accommodate a detector

Phase advance in a low-β insertion

We have:

$$\beta(s) = \beta^\star + \frac{s^2}{\beta^\star}$$

The phase advance across the straight section is:

$$\Delta\mu = \int_{-L_{\text{waist}}}^{L_{\text{waist}}} \frac{\mathrm{d}s}{\beta^\star + \frac{s^2}{\beta^\star}} = 2\arctan\frac{L_{\text{waist}}}{\beta^\star}$$

which is close to $\Delta\mu = \pi$ for $L_{\text{waist}} \gg \beta^\star$.

In other words: in the interaction region the tune increases by half an integer!

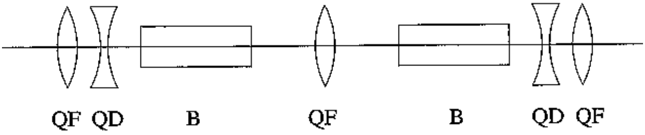
Achromatic insertions

There exist insertions (arcs) that don't introduce dispersion: they are called *achromatic arcs*

- In principle, dispersion can be suppressed by one focusing quadrupole and one bending magnet
- With one focusing quad in between two dipoles, one can get achromat condition: In between two bends, we call it arc section. Outside the arc section, we can match dispersion to zero. This is called "Double Bend Achromat" (DBA) structure
- We need quads outside the arc section to match the betatron functions, tunes, etc.
- Similarly, one can design "Triple Bend Achromat" (TBA), "Quadruple Bend Achromat" (QBA), and "Multi Bend Achromat" (MBA or nBA) structure
- For FODO cells structure, dispersion suppression section at both ends of the standard cells (see previous slides)

The Double Bend Achromat lattice (DBA)

Consider a simple DBA cell with a single quadrupole in the middle (plus external quadrupoles for matching).



$$M_{\text{DBA}} = M_{\text{B}} \cdot M_{\text{drift}} \cdot \underbrace{M_{1/2\text{F}} \cdot M_{1/2\text{F}}}_{M_{\text{F}}} \cdot M_{\text{drift}} \cdot M_{\text{B}}$$

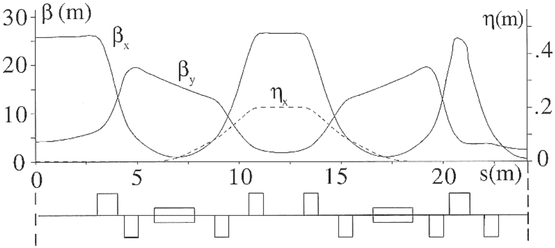
In thin-lens approximation, the dispersion matching condition:

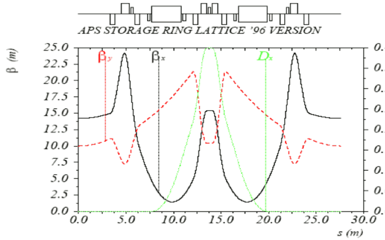
$$\begin{pmatrix} D_{\text{center}} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & L\theta/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

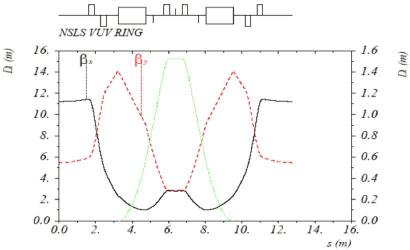
where f is the focal length of the quad, θ and L are the bend angle and the length of the dipole, and L_1 is the distance between the dipole and the centre of the quad.

$$f = \frac{1}{2}\left(L_1 + \frac{1}{2}L\right); \qquad D_{\text{center}} = \left(L_1 + \frac{1}{2}L\right)\theta$$

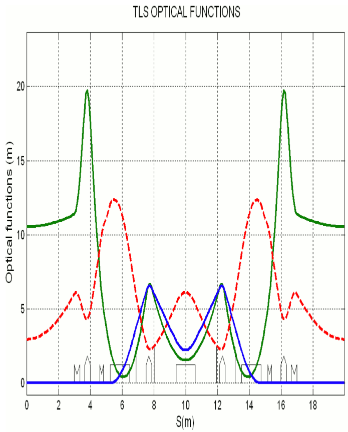
DBA optical functions

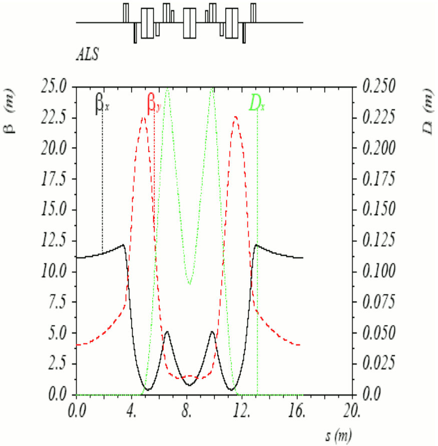






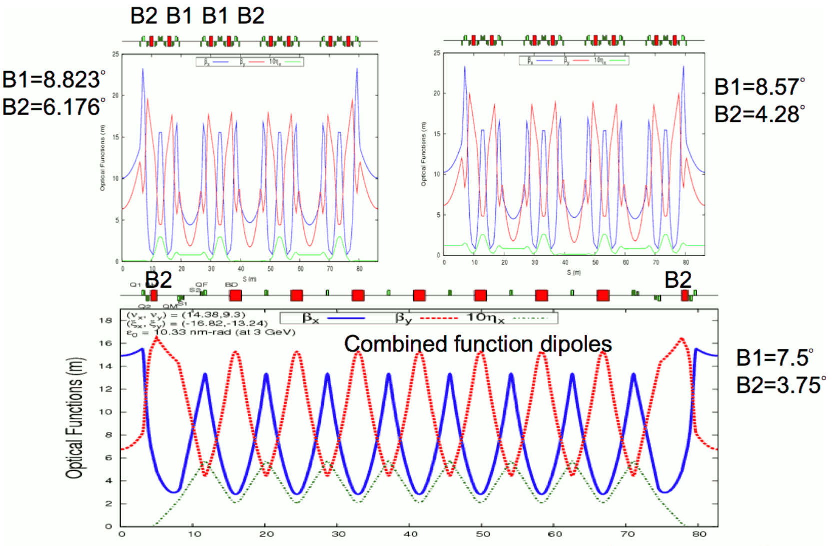
Triple Bend Achromat (TBA)





Combined function dipoles

QBA, OBA, and nBA



Completing the picture: 6D phase space

In real life the state vector is six-dimensional:

$$\begin{pmatrix} x & x' & y & y' & z & \Delta P/P_0 \end{pmatrix}^T$$

and the transfer matrix (typically) reads

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \frac{\Delta P}{P_0} \end{pmatrix}_s = \begin{pmatrix} R_{11} & R_{12} & \mathbf{0} & \mathbf{0} & 0 & R_{16} \\ R_{21} & R_{22} & \mathbf{0} & \mathbf{0} & 0 & R_{26} \\ \mathbf{0} & \mathbf{0} & R_{33} & R_{34} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & R_{43} & R_{44} & 0 & 0 \\ R_{51} & R_{52} & 0 & 0 & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \frac{\Delta P}{P_0} \end{pmatrix}_0$$

· In bold the elements that would couple the $x - y$ motion.

· In a ring: $R_{56} = -C\,\alpha$ (circumference $\times$ momentum compaction).

Nota bene: this matrix can still represent **only** linear elements.

- if we want to consider high-order elements: e.g. sextupoles, octupoles, etc. $\Rightarrow$ we need computer simulations ! "particle tracking" or "maps" (MAD-X, for instance)
- because such elements introduce non-linear motion, which is too difficult to treat analytically

Transfer matrix of a solenoid magnet

Solenoids are magnets just with $B_z = \text{const} \neq 0$. Their transfer matrix reads

$$M_{\text{solenoid}} = \begin{pmatrix} C^2 & \frac{SC}{K} & SC & \frac{S^2}{K} \\ -KSC & C^2 & -KS^2 & SC \\ -SC & -\frac{S^2}{K} & C^2 & \frac{SC}{K} \\ KS^2 & -SC & -KSC & C^2 \end{pmatrix}$$

with: $L =$ effective length of the solenoid, $K = B_z / (2B\rho) = B_z / (2P/q)$, $C = \cos KL$, $S = \sin KL$.

Remark: a rotation of the transverse coordinates x and y about the optical axis at the exit of the solenoid by an angle $-KL$, decouples the x and y first order terms:

$$\begin{pmatrix} C & \frac{S}{K} & 0 & 0 \\ -KS & C & 0 & 0 \\ 0 & 0 & C & \frac{S}{K} \\ 0 & 0 & -KS & C \end{pmatrix} = R_{\text{rot}}(-KL) \times M_{\text{solenoid}}$$

$\Rightarrow$ a solenoid behaves like a rotating quadrupole that focuses in both x and y.

Coupled motion: skew quadrupoles

Certain elements might be used to intentionally couple horizontal and vertical motion, for example: skew quadrupoles

$$M_{\text{skew quad}} = R_{\text{rot}}\left(45^\circ\right) \times M_{\text{quad}} \times R_{\text{rot}}\left(-45^\circ\right) =$$

$$= R_{\text{rot}}\left(45^\circ\right) \times \begin{pmatrix} \cos\sqrt{K}L & \frac{1}{\sqrt{K}}\sin\sqrt{K}L & 0 & 0 \\ -\sqrt{K}\sin\sqrt{K}L & \cos\sqrt{K}L & 0 & 0 \\ 0 & 0 & \cosh\sqrt{|K|}L & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}L \\ 0 & 0 & \sqrt{|K|}\sinh\sqrt{|K|}L & \cosh\sqrt{|K|}L \end{pmatrix} \times R_{\text{rot}}\left(-45^\circ\right)$$

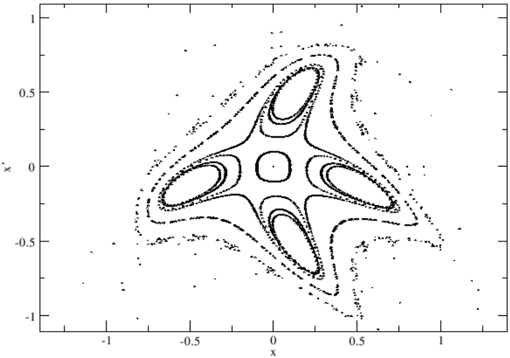
A skew-quadrupole is a quadrupole rotated around the longitudinal axis by 45 degrees.

With:

$$R_{\text{rot}}(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & \cos\phi & 0 & \sin\phi \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & -\sin\phi & 0 & \cos\phi \end{pmatrix}$$

Notice: coupling can be induced even by normal elements, including quadrupoles and dipoles, just because of alignment errors ("roll error", i.e. small angles about the optical axis).

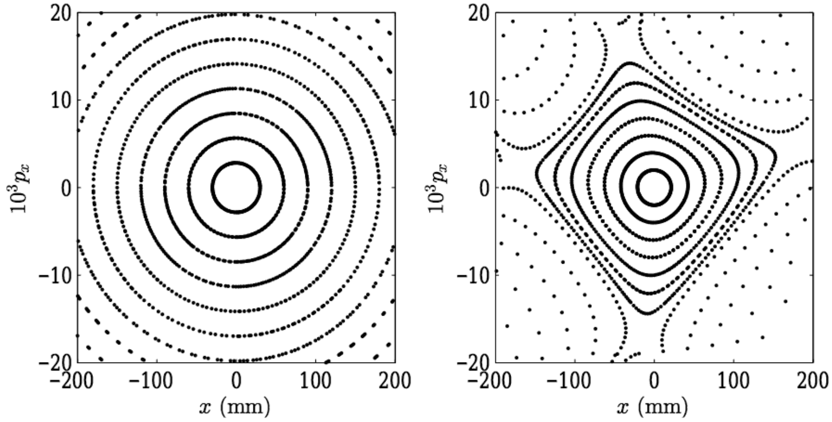
Non-linear dynamics



$$\begin{pmatrix} x_{n+1} \\ x'_{n+1} \end{pmatrix} = \begin{pmatrix} \cos(2\pi Q) & \sin(2\pi Q) \\ -\sin(2\pi Q) & \cos(2\pi Q) \end{pmatrix} \begin{pmatrix} x_n \\ x'_n + x_n^2 \end{pmatrix}$$

- Q=0.2516
- linear motion near center (circles)
- More and more square
- Non-linear tuneshift
- Islands
- Limit of stability
- Dynamic Aperture
- Crucial if strong quads and chromaticity correction in s.r. light sources
- many non-linearities in LHC due to s.c. magnet and finite manufacturing tolerances

Dynamic aperture in a FODO



Phase space portraits of a FODO storage ring without (left) and with (right) sextupoles for correction of chromaticity.

Particle tracking and Dynamic aperture

Dynamic aperture: *is a method used to calculate the amplitude threshold of stable motion of particles. Numerical simulations of particle tracking aim at determining the "dynamic aperture".*

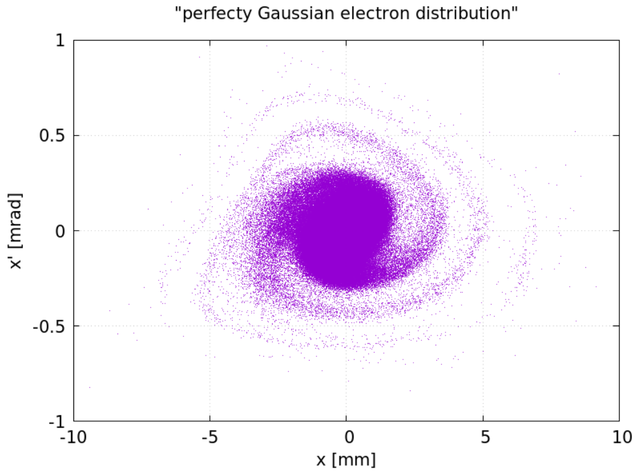
Dynamic aperture for hadrons

- in the case of protons or heavy ion accelerators, (or synchrotrons, or storage rings), there is minimal radiation, and hence the dynamics is symplectic
- for long term stability, a tiny dynamical diffusion can lead an initially stable orbit slowly into an unstable region
- this makes the dynamic aperture problem particularly challenging: One may need to consider the stability over billions of turns

For the case of electrons

- in bending magnetic fields, the electrons radiate which causes a damping effect.
- this means that one typically only cares about stability over few (~thousands) of turns

Emittance growth



An initially Gaussian electron bunch, *filamenting* after traveling through the CLIC Drive Beam Recombination Complex, under the effects of non-linear fields, chromaticity, and synchrotron radiation.

6D transfer matrix for an accelerating structure (linacs)

Let's find the full 6×6 transfer matrix:

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \frac{\Delta P}{P_0} \end{pmatrix}_s = \begin{pmatrix} ? & ? & 0 & 0 & 0 & 0 \\ ? & ? & 0 & 0 & 0 & 0 \\ 0 & 0 & ? & ? & 0 & 0 \\ 0 & 0 & ? & ? & 0 & 0 \\ 0 & 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & 0 & ? & ? \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \frac{\Delta P}{P_0} \end{pmatrix}_0$$

To satisfy the Maxwell's equations, we need to split this matrix in three parts:

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \frac{\Delta P}{P_0} \end{pmatrix}_L = \begin{pmatrix} \text{exit} \\ \text{field} \end{pmatrix} \cdot \begin{pmatrix} \text{body} \\ \text{[acceleration]} \end{pmatrix} \cdot \begin{pmatrix} \text{entrance} \\ \text{field} \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \frac{\Delta P}{P_0} \end{pmatrix}_0$$

where the end fields are thin elements.

Transfer matrix for an accelerating structure: body

- Assume an accelerating RF structure of length L; creating a constant longitudinal electric field $E_z = G$ (accelerating gradient, V/m)
  The energy gain is:

  $$\Delta E = \int \vec{F} \cdot \vec{v}\, dt = \int_0^L \vec{F} \cdot d\vec{s} = \int_0^L qE_z \cdot ds = q \cdot G \cdot L$$

  This energy gain corresponds to a momentum gain the longitudinal direction

  $$\Delta P_z \approx \Delta E / v_z \qquad \Rightarrow \Delta P_z \approx \Delta E / c \quad \text{at relativistic velocity}$$

- We define the normalized momentum gain

  $$\boxed{\Delta = \frac{\Delta P_z}{P_0} \approx \frac{\Delta E}{E_0} = \frac{q\,G\,L}{E_0} \approx \frac{q\,G\,L}{P_0\,c}.}$$

- Since the reference momentum goes from P_0 to $P_0 + \Delta P_z$, a particle's relative-momentum, δ, changes accordingly:

  $$\left(\frac{\Delta P}{P_0}\right)_L = \frac{P_0}{P_0 + \Delta P_z}\left(\frac{\Delta P}{P_0}\right)_0 \qquad \Rightarrow \qquad \delta_L = \frac{1}{1+\Delta}\delta_0$$

  therefore, the longitudinal transfer matrix reads:

  $$\boxed{M_z = \begin{pmatrix} 1 & \frac{L}{\beta^2\gamma^2} \\ 0 & \frac{1}{1+\Delta} \end{pmatrix}}$$

  as $\frac{P_0}{P_0+\Delta P_z} \equiv \frac{1}{1+\Delta}$.

Transfer matrix for an accelerating structure: body

- The transverse angles can be found recalling the conservation of the transverse momentum

$$x'(s) = x'(0)\,\frac{P_0}{P_0 + q\,G\,s/v_z} \quad\approx\quad x'(0)\,\frac{1}{1+\Delta}$$

- The transverse positions can be found integrating the above expression over s, from 0 to L:

$$x(L) = x(0) + \int_0^L x'(s)\,ds = x(0) + L\,\frac{\log\left(1+\frac{q\,G\,L}{P_0\,c}\right)}{\frac{q\,G\,L}{P_0\,c}}\,x'(0) = x(0) + L\,\frac{\log(1+\Delta)}{\Delta}\,x'(0)$$

Then, the transfer matrix for $\begin{pmatrix} x, & x' \end{pmatrix}$ and $\begin{pmatrix} y, & y' \end{pmatrix}$ is:

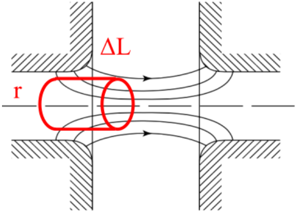
$$\boxed{M_{x,y} = \begin{pmatrix} 1 & L\,\frac{\log(1+\Delta)}{\Delta} \\ 0 & \frac{1}{1+\Delta} \end{pmatrix}}$$

Transfer matrix for an accelerating structure: end fields

- Recall the Gauss' law:

$$\Phi_{\text{total}} = \frac{Q}{\epsilon_0} = 0$$

where Φ_{total} is the total flux of the electric field, and Q is the total charge inside the cylinder, in our case $Q = 0$.



- (At entrance) The total flux is $\Phi_{\text{total}} = \Phi_{\parallel} + \Phi_{\perp}$, with

$$\begin{aligned}\Phi_{\parallel} &= G\,\pi r^2 \\ \Phi_{\perp} &= G_{\perp}\,2\pi r\,\Delta L\end{aligned}$$

Since $\Phi_{\text{total}} = 0$, then $\Phi_{\perp} = -\Phi_{\parallel}$ and $G_{\perp} = -\dfrac{G\,r}{2\Delta L}$

- The transferred transverse momentum is then $\Delta P_{\perp} = q\,G_{\perp}\Delta L/c = -q\,\dfrac{G\,r}{2c}$, which corresponde to an entrance transverse kick:

$$\Delta r' = \frac{\Delta P_{\perp}}{P_0} = -q\,\frac{G\,r}{2P_0\,c} = -\frac{\Delta \cdot r}{2L}$$

(with r either x or y; Note: the exit kick has opposite sign and must be divided by $(1+\Delta)$, since the reference momentum has increased)

- This applies to both x and y, so:

$$\boxed{M_{\text{x,y, entrance}} = \begin{pmatrix} 1 & 0 \\ -\frac{\Delta}{2L} & 1 \end{pmatrix} \qquad M_{\text{x,y, exit}} = \begin{pmatrix} 1 & 0 \\ \frac{\Delta}{2L(1+\Delta)} & 1 \end{pmatrix}}$$

and corresponds to a focusing effect at entrance, and a defocusing at exit.

Transfer matrix for an accelerating structure

The full 6×6 transfer matrix is:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{\Delta}{2L(1+\Delta)} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\Delta}{2L(1+\Delta)} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{\text{exit}} \cdot \quad \cdots$$

$$\cdots \quad \cdot \begin{pmatrix} 1 & L\frac{\log(1+\Delta)}{\Delta} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{1+\Delta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L\frac{\log(1+\Delta)}{\Delta} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{1+\Delta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta^2\gamma^2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{1+\Delta} \end{pmatrix}_{\text{body}} \cdot \quad \cdots$$

$$\cdots \quad \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\Delta}{2L} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\Delta}{2L} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{\text{entrance}}$$

where Δ is the the normalized momentum / energy gain $\Delta = \Delta P_z / P_0 \approx \Delta E / E_0$.

...The End!

Thank you

for your attention!