Particle Source

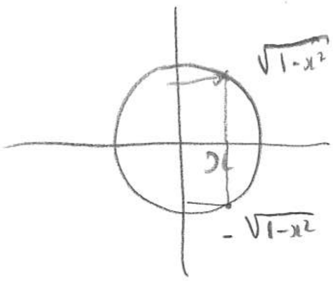
TUTORIAL SOLUTION

1. Probability Density Function for a uniform disk of radius R

1.1 we set $x = \frac{X}{R}$ and $y = \frac{Y}{R}$ to simplify the study

P.D.F. $f(x,y) = \frac{1}{\pi}\begin{cases} 1 & \text{if } x^2+y^2 \leq 1 \\ 0 & \text{elsewhere} \end{cases}$ π = disk surface for a radius = 1

along the x axis, we need to use $g(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy$



$$g(x) = \frac{2}{\pi}\sqrt{1-x^2}$$

$$\langle x \rangle = \int_{-1}^{1} x\, g(x)\, dx = \int_{-1}^{1} \underbrace{x \frac{2}{\pi}\sqrt{1-x^2}}_{\text{odd function}}\, dx = 0$$

$$\langle x^2 \rangle = \frac{2}{\pi}\int_{-1}^{1} x^2 \sqrt{1-x^2}\, dx$$

change of variable $x = \sin\theta$

$dx = \cos\theta\, d\theta$, $x = -1 \Rightarrow \theta = -\frac{\pi}{2}$

$x = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\langle x^2 \rangle = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \sin^2\theta \cos^2\theta\, d\theta$$

$2\sin\theta\cos\theta = \sin 2\theta$, so $\langle x^2 \rangle = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{4}\sin^2 2\theta\, d\theta$

$$\sin^2 2\theta = \left(\frac{e^{2i\theta} - e^{-2i\theta}}{2i}\right)^2 = -\frac{1}{2}\,\frac{e^{4i\theta} + e^{-4i\theta}}{2} + \frac{1}{2} = \frac{1}{2}(1-\cos 4\theta)$$

$$\langle x^2\rangle = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{4}\cdot\frac{1}{2}\,d\theta - \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{4}\cdot\frac{1}{2}\cdot\cos 4\theta\, d\theta$$

$$\langle x^2\rangle = \frac{2}{\pi}\cdot\frac{1}{8}\cdot\Big[\theta\Big]_{-\pi/2}^{\pi/2} - \frac{2}{\pi}\cdot\frac{1}{8}\underbrace{\left[\frac{\sin 4\theta}{4}\right]_{-\pi/2}^{\pi/2}}_{=0}$$

$$\langle x^2\rangle = \frac{1}{4}$$

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{1}{2}$$

$$X = xR \quad \Rightarrow \quad \boxed{\sigma_X = \frac{R}{2}}$$

1.2 $\varepsilon_N = \frac{R}{2}\sqrt{\frac{KT}{m_p c^2}}$

Calculation using eV: $\frac{KT}{m_p c^2} = \frac{8{,}617\cdot 10^{-5}\cdot 300}{938{,}272\cdot 10^{6}} = 2{,}75\cdot 10^{-11}$

calculation using standard units: $\frac{KT}{m_p c^2} = \frac{1{,}38\cdot 10^{-23}\cdot 300}{1{,}672\cdot 10^{-27}\times(3\cdot 10^{8})^2} = 2{,}75\cdot 10^{-11}$

2. $\vec{OM}(0,-L,0)$ at $t=0$

$\vec{v}(v,0,0)$ at $t=0$

$\vec{B}(0,0,B)$

2.1 Newton's law

$$m\frac{d\vec{v}}{dt} = q\vec{v}\times\vec{B} = q\begin{vmatrix} v_x \\ v_y \\ v_z \end{vmatrix} \wedge \begin{vmatrix} 0 \\ 0 \\ B \end{vmatrix}$$

$$\Rightarrow \begin{cases} m\dfrac{dv_x}{dt} = q v_y B & (1) \\ m\dfrac{dv_y}{dt} = -q v_x B & (2) \\ m\dfrac{dv_z}{dt} = 0 & (3) \end{cases} \iff \begin{cases} \dfrac{dv_x}{dt} = \omega v_y & (1) \\ \dfrac{dv_y}{dt} = -\omega v_x & (2) \end{cases}$$

(3) $\Rightarrow v_z(t) = \text{const.}$ $\quad v_z(0)=0 \Rightarrow v_z(t)=0 \ \forall t$

ω = cyclotronic frequency

$$\frac{d}{dt}(1) \Rightarrow \frac{d^2 v_x}{dt} = \omega\frac{dv_y}{dt} \quad (1)'$$

$$(2) \text{ in } (1)' \Rightarrow \frac{d^2 v_x}{dt} = -\omega^2 v_x$$

general solution $\quad v_x(t) = a\cos\omega t + b\sin\omega t$

$v_x(0) = v \Rightarrow a = v$

$$(1) \Rightarrow v_y(t) = \frac{1}{\omega}\frac{dv_x}{dt} = \frac{1}{\omega}\left(-v.\omega\sin\omega t + \omega b\cos\omega t\right)$$

$v_y(0) = 0 \Rightarrow b = 0$

so

$$\begin{cases} v_x(t) = v\cos(\omega t) \\ v_y(t) = -v\sin(\omega t) \\ v_z(t) = 0 \longrightarrow z(t) = 0 \quad \forall t \end{cases}$$

2.2 $$\underset{-x(0) \atop {}^{"}0}{x(t)} = \int_0^t v_x(t)\,dt = v\int_0^t \cos(\omega t)\,dt = \frac{v}{\omega}\cdot\left[\sin\omega t\right]_0^t$$

$$x(t) = \frac{v}{\omega}\sin(\omega t)$$

$$y(t) - y(0) = \int_0^t v_y(t)\,dt = +\frac{v}{\omega}\left[\cos\omega t\right]_0^t$$

$$y(t) + L = \frac{v}{\omega}\left(\cos\omega t - 1\right)$$

$$\rho = \frac{v}{\omega} \Rightarrow \begin{cases} x(t) = \rho\sin\omega t \\ y(t) = \rho\cos\omega t - \rho - L \end{cases}$$

ρ is the Larmor radius

2.3 $$v = \rho\cdot\omega = \rho\frac{qB}{m}$$

$$\Rightarrow \quad B\rho = \frac{mv}{q} \quad \left(\text{non relativistic calculation}\right)$$

2.4 $$\omega = 2\pi f$$

$$f = \frac{\omega}{2\pi} = \frac{qB}{m2\pi} = \frac{\cancel{e}\cdot 1}{2\pi\cdot 511.10^3\cdot\frac{\cancel{e}}{c^2}} = \frac{(3\,10^8)^2}{511\,10^3\cdot 2\pi} = 28\text{ GHz}$$

3. charge breeding

3.1 ^{39}K ← Atomic number $A = 39 = Z + N$ (proton, neutron)

Atom Mass is $M = A \cdot amu = 39 \times 931{,}49\ MeV/c^2$

Kinetic energy + potential energy = constant of motion

$$T + qV = const. = E$$

in the 1+ source $^{39}K^+$ has a potential of 10 kV and charge 1+

so $E = qV = 10\,KeV$

in the beam pipe after acceleration, the $^{39}K^+$ has a potential $V = 0$

and $T + 0 = E = 10\,KeV$

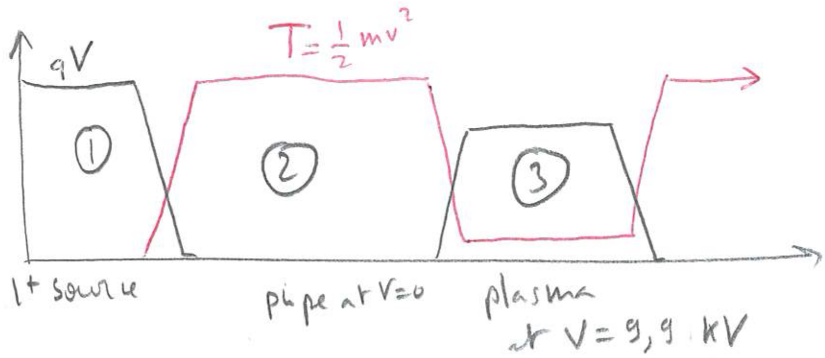
$$T = (\gamma - 1) Mc^2 \Rightarrow \gamma = 1 + \frac{T}{Mc^2}$$

$$\gamma = 1 + \frac{10^4\,\cancel{e}}{39 \times 931{,}49 . 10^6\ \frac{\cancel{e}}{\cancel{c^2}} . \cancel{c^2}} = 1 + 2{,}75 . 10^{-7}$$

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} \simeq 0{,}000742$$ non relativistic

$$V = 2{,}23 . 10^5\ m/s$$

3.2



$$E = T + qV$$

in 1: $T_1 = 0$ $\quad qV_1 = 10\ \text{KeV}$

in 2 $\quad T_2 = 10\ \text{KeV}$ and $V_2 = 0$

in 3 $\quad V_3 = 9{,}9\ \text{kV} \Rightarrow E = T_3 + qV_3 \Rightarrow T_3 = E - qV_3 = q(V_1 - V_3)$

$$\boxed{T_3 = 100\ \text{eV}}$$

non relativistic case $T_3 = \frac{1}{2}mv^2$

$$v = \sqrt{\frac{2T_3}{m}} = \sqrt{\frac{200\ \cancel{e}}{39.931{,}49.10^6\ \frac{\cancel{e}}{c^2}}} = \sqrt{\frac{200}{39.931{,}49.10^6}}\ .3.10^8$$

$$v = 2{,}23.10^4\ \text{m/s}$$

3.3 electron kinetic energy in the plasma is $90\ \text{eV} = T_e$

see Table 1 : $\quad \sigma_{1^+ \to 2^+} = 4{,}5.10^{-14}.g_i.\dfrac{\ln(T_e/P_i)}{T.P_i}\ \text{cm}^2$

$T_e = 90\ \text{eV}$

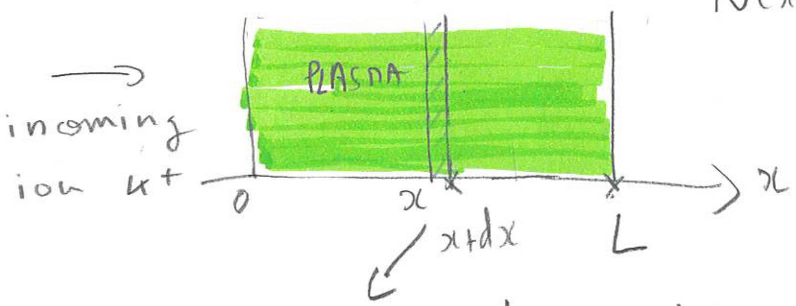
$P_i = 31{,}63\ \text{eV} \quad g_i = 6$

$$\sigma = 4{,}31.10^{-17}\ \text{cm}^2$$

$$\frac{1}{\lambda} = \sigma n = 4{,}3.10^{-17} \times 10^{12}$$

$$\lambda = 231{,}72 \text{ m}$$

$N(x)$ = number of ions $^{39}K^{+}$



$$N(x+dx) - N(x) = -\underbrace{N \cdot n\sigma\, dx}_{\text{number of collisions in the slice } dx}$$

negative because $N(x+dx) < N(x)$ as collisions "destroy" the 1^+ ions and create a $2+$ ion

$$\frac{dN}{N} = -n\sigma\, dx \Rightarrow \int_{N(0)}^{N(x)} \frac{dN}{N} = -\int_{x=0}^{x} n\sigma\, dx$$

$$\Rightarrow \ln\left(\frac{N(x)}{N(0)}\right) = -n\sigma x = -\frac{x}{\lambda}$$

$$\Rightarrow N(x) = N(0)\, e^{-\frac{x}{\lambda}}$$

$$\frac{N(L)}{N(0)} = e^{-\frac{L}{\lambda}} = 0{,}9978$$

$$N(0) = \text{constant} = N_{K^+}(x) + N_{K^{2+}}(x)$$

$$\frac{N_{K^{2+}}(L)}{N(0)} = 1 - \frac{N(L)}{N(0)} = 0{,}0022$$