Particle Optics – part I

N.Biancacci

With many thanks to J.De Conto, A.Latina, P.Lebrun and E.Métral for all the input in the preparation!

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Objectives of this course

Part I: Light optics

- Understand the basic laws of light propagation
- Being able to apply the principle of geometrical optics to trace rays of light across common optical elements (boundaries, thin and thick lenses, complex optical systems, …)

Strong analogy

Part II: Particle optics

- Understand the basic laws of particle transport
- Being able to apply the principles of light optics to trace particles across common accelerators elements (dipoles, quadrupoles, …)
- Understand the effect of dispersion on spectrometry

Outline of part I

- Recall of Maxwell equations
  - Integral form
  - Local form
  - Boundary conditions
  - The plane wave
  - Law of reflection & refraction

- Ray optics
  - Eikonal equation
  - ABCD matrix for simple elements (mirror, drifts, planar and curved interfaces)
  - Thick and thin lenses
  - Image formation
  - Complex systems: principal planes
  - Light transport in periodic channels: stability condition
  - Limit of the treatment: aberrations

Recall of Maxwell's equations

Short recap, more details in H.Henke's lectures!

juas...

JUAS - TIMETABLE 2020 - WEEK 1

Schedule 2020	Monday Jan 13	Tuesday Jan 14	Wednesday Jan 15	Thursday Jan 16	Friday Jan 17
9:00		Relativity *H. Henke*	Electro-magnetism *H. Henke*	Intro. to Accelerator Design *Ph. Bryant*	Intro. to the Mini-Workshop *Ph. Bryant*
10:00		Coffee Break	Coffee Break	Coffee Break	Coffee Break
10:15		Relativity *H. Henke*	Electro-magnetism *H. Henke*	Intro. to Accelerator Design *Ph. Bryant*	Intro. to the Mini-Workshop *Ph. Bryant*
11:15	12:00 OFFICIAL OPENING (welcome & building visit)	Relativity *H. Henke*	Electro-magnetism *H. Henke*	Intro. to Accelerator Design *Ph. Bryant*	*Bus leaves at 11:15 from JUAS (Lunch at CERN, R3, offered by ESI)*
12:15	13:00 WELCOME LUNCH	BREAK	BREAK	BREAK	13:30 Visit of LHC Magnets Test Hall *M. Bajko*
14:00	14:00 Presentation of JUAS & Introduction of students *P. Lebrun*	Relativity *H. Henke*	Electro-magnetism *H. Henke*	Intro. to Accelerator Design *Ph. Bryant*	15:00 Introduction to CERN & its Accelerator Network Seminar - *R. Alemany*
15:00	History of particle accelerators Seminar *V. Vaccaro*	Particle optics N. Biancacci	Particle optics N. Biancacci	Intro. to Accelerator Design *Ph. Bryant*	
16:00	CHECK-IN AT THE RESIDENCE & SHOPPING FOR GROCERIES	Coffee Break	Coffee Break	Coffee Break	16:30 Visit of CERN Control Center *R. Alemany* Bus leaves at 17:30 from CERN
16:15		Particle optics N. Biancacci	Particle optics N. Biancacci	Intro. to Accelerator Design *Ph. Bryant*	
17:15		Particle optics N. Biancacci			
18:15		AFTER WORK AT ESI			

CERN

Maxwell equations: integral form

In a homogeneous, linear, isotropic medium:

Gauss' law
$$\int_{\partial S} \bar{\mathcal{E}} \cdot dS = \int_V \frac{\rho}{\varepsilon} dV$$

Gauss' law for magnetism
$$\int_{\partial S} \bar{\mathcal{H}} \cdot dS = 0$$

Faraday's law
$$\oint_{\partial S} \bar{\mathcal{E}} \cdot dl = -\frac{d}{dt} \int_{\partial S} \mu \bar{\mathcal{H}} \cdot dS$$

Ampère-Maxwell law
$$\oint_{\partial S} \bar{\mathcal{H}} \cdot dl = \int_S \bar{\mathcal{J}} \cdot dS + \frac{d}{dt} \int_S \varepsilon \bar{\mathcal{E}} \cdot dS$$

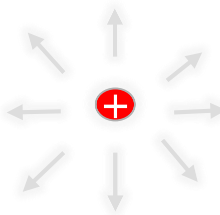
ε: medium permittivity
μ: medium permeability

Local form

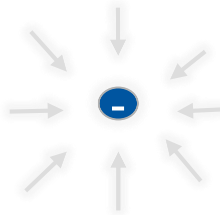
Gauss' law

$$\nabla \cdot \bar{\mathcal{E}} = \frac{\rho}{\varepsilon}$$

$\nabla \cdot \bar{\mathcal{E}} > 0$



$\nabla \cdot \bar{\mathcal{E}} < 0$



Gauss' law for magnetism

$$\nabla \cdot \overline{\mathcal{H}} = 0$$

$\nabla \cdot \overline{\mathcal{H}} = 0$

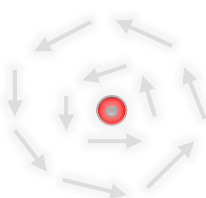
Faraday's law

$$\nabla \times \bar{\mathcal{E}} = -\frac{d}{dt}\mu\overline{\mathcal{H}}$$

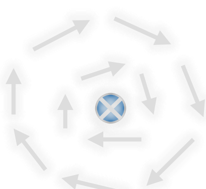
Ampère-Maxwell law

$$\nabla \times \overline{\mathcal{H}} = \bar{\mathcal{J}} + \frac{d}{dt}\varepsilon\bar{\mathcal{E}}$$

$\nabla \times \overline{\mathcal{H}} > 0$



$\nabla \times \overline{\mathcal{H}} < 0$



Local form in frequency domain

Suppose the field is stationary, we can apply the Fourier transform:

$$\bar{E}(\omega) = \int_{-\infty}^{+\infty} \bar{\mathcal{E}}(t) e^{j\omega t} dt$$

With this, $\frac{d}{dt} \rightarrow j\omega$ and dropping the time dependence we have:

Gauss' law $\nabla \cdot \bar{E} = \rho/\varepsilon$

Gauss' law for magnetism $\nabla \cdot \bar{H} = 0$

Faraday's law $\nabla \times \bar{E} = -j\omega\mu\bar{H}$

Ampère-Maxwell law $\nabla \times \bar{H} = j\omega\varepsilon\bar{E} + \bar{J}$

Boundary conditions

Let's now recap the behavior of the fields at the boundaries:

Electric field

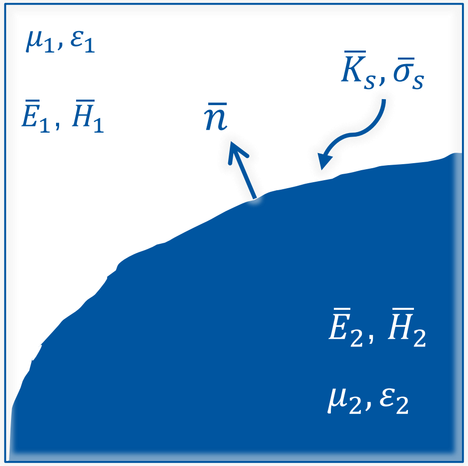
$$\bar{n} \times (\bar{E}_1 - \bar{E}_2) = 0$$

$$\bar{n} \cdot (\varepsilon_1 \bar{E}_1 - \varepsilon_2 \bar{E}_2) = \bar{\sigma}_s$$

Magnetic field

$$\bar{n} \times (\bar{H}_1 - \bar{H}_2) = \bar{K}_s$$

$$\bar{n} \cdot (\mu_1 \bar{H}_1 - \mu_2 \bar{H}_2) = 0$$

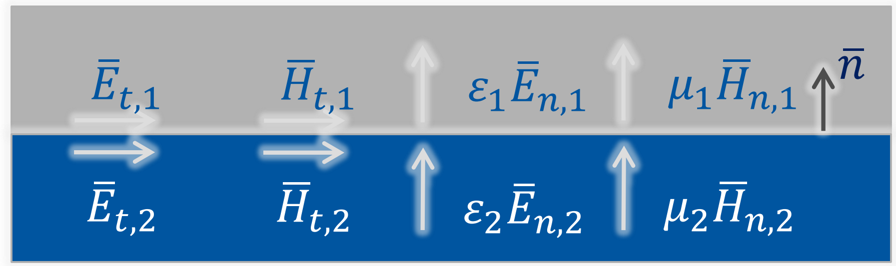


where $\bar{K}_s$ and $\bar{\sigma}_s$ are free surface electric current and charges.

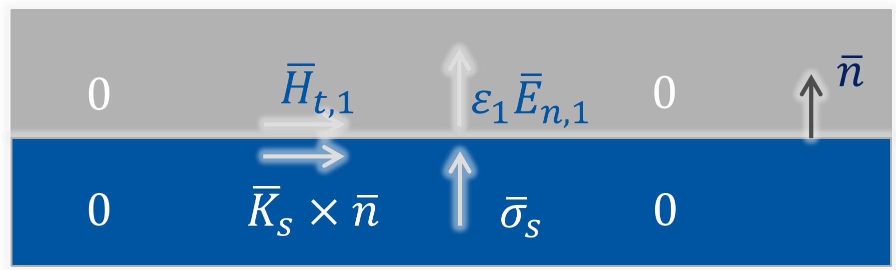
Boundary conditions

Example for dielectrics and perfect electric conductor (PEC)

For a dielectric → $\bar{\sigma}_s, \bar{K}_s$ are null



For a PEC→ $\bar{E}_2, \bar{H}_2$ are null but free current and charges can be present on the surface



Plane wave

For an homogeneous, isotropic medium (e.g. vacuum, dielectric, magnetic material):

$\nabla \times \bar{H} = j\omega\varepsilon\bar{E} + \bar{J}$

$\nabla \times \bar{E} = -j\omega\mu\bar{H}$

$$\nabla \times \left(\frac{1}{-j\omega\mu}\nabla \times \bar{E}\right) = j\omega\varepsilon\bar{E} + \bar{J}$$

$\nabla \times (\nabla \times \bar{E}) = \omega^2\mu\varepsilon\bar{E} - j\omega\mu\bar{J}$

$\nabla \times (\nabla \times \bar{E}) = \nabla(\nabla \cdot \bar{E}) - \nabla^2\bar{E}$

$$\nabla(\nabla \cdot \bar{E}) - \nabla^2\bar{E} = \omega^2\mu\varepsilon\bar{E} - j\omega\mu\bar{J}$$

$\nabla\left(\frac{\rho}{\varepsilon}\right) - \nabla^2\bar{E} = \omega^2\mu\varepsilon\bar{E} - j\omega\mu\bar{J}$

$\nabla \cdot \varepsilon\bar{E} = \rho$

In vacuum ($\rho, \bar{J}$ null):

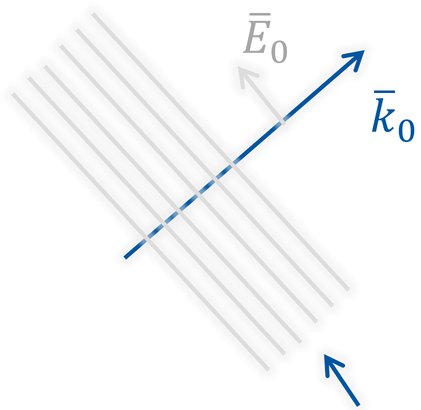
$\nabla^2\bar{E} + \omega^2\mu\varepsilon\bar{E} = 0$

$k^2 = \omega^2\mu\varepsilon \rightarrow k = \frac{\omega}{c}\boldsymbol{n}$ with $\boldsymbol{n} = \sqrt{\varepsilon_r}$ refraction index (real in dielectrics with negligible absorption)

Plane wave

Maxwell equations in free space condense into Helmoltz equation:

$$\nabla^2 \bar{E} + k^2 \bar{E} = 0 \qquad \text{with } k = \frac{\omega}{c} n$$



The solution is a plane wave in free space with propagation vector $k \cdot \bar{k}_0 = k_x \bar{x}_0 + k_y \bar{y}_0 + k_z \bar{z}_0$.

$$\bar{E} = \bar{E}_0 e^{-j(k_x x + k_y y + k_z z)} = \bar{E}_0 e^{-j(\bar{k} \cdot \bar{r})}$$

Back in time domain this is:

$$\bar{\mathcal{E}} = \bar{E}_0 \cos(\bar{k} \cdot \bar{r} - \omega t + \phi) = \bar{E}_0 \cos\left(\frac{\omega}{c} n \cdot \bar{k}_0 \cdot \bar{r} - \omega t + \phi\right)$$

where the equiphase planes are given by: $\frac{\omega}{c} n \cdot \bar{k}_0 \cdot \bar{r} - \omega t = cost$

and travel in the direction of $\bar{k}_0$ with phase velocity $\frac{dr}{dt} \bar{k}_0 \cdot \bar{r}_0 = \frac{\boldsymbol{c}}{\boldsymbol{n}} = \boldsymbol{v_p}$

Laws of reflection & refraction

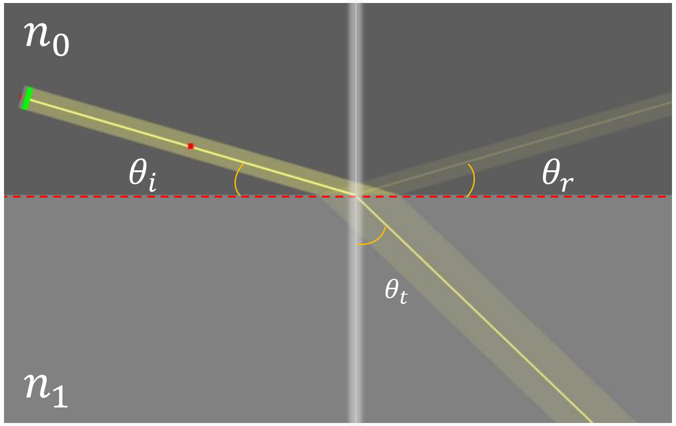
Consider two media with different index n_0 and n_1. In order to satisfy the boundary conditions a reflected and a refracted wave are produced.

- The angle of reflection is equal to the angle of incidence (*law of reflection*):

$$\theta_i = \theta_r$$

- The angle of refraction is given by (*Snell law*):

$$n_0 \sin(\theta_i) = n_1 \sin(\theta_t)$$



E.g. going from a medium with lower n to a larger one, the angle of propagation of the refracted wave gets closer to the normal to the interfaces.

P.S.: Have fun with this nice optics simulator at https://ricktu288.github.io/ray-optics/ ☺

Ray optics

Now let's consider $n(\bar{r})$, i.e. a refraction index varying in space.

$$\nabla^2 \bar{E} + \frac{\omega^2}{c^2} n(\bar{r})^2 \bar{E} = 0 \quad \longrightarrow \quad \nabla^2 \bar{E} + k_0^2 n(\bar{r})^2 \bar{E} = 0$$

We look for a solution as for a plane wave:

$$\bar{E} = \bar{E}_0 e^{-j(k_0 R(\bar{r}))}$$

$$\frac{1}{k_0^2} \nabla^2 \left(\bar{E}_0 e^{-j\left(k_0 R(\bar{r})\right)} \right) + n(\bar{r})^2 \bar{E}_0 e^{-j\left(k_0 R(\bar{r})\right)} = 0$$

The computation of $\nabla^2 \left(\bar{E}_0 e^{-j\left(k_0 R(\bar{r})\right)} \right)$ is rather lengthy and it is left in appendix for curiosity.

Ray optics

The result is this:

$\Big[-\frac{j}{k_0}\, \nabla^2 R(\bar{r}) \bar{\bar{E}}_0 -$

$\hat{\mathbf{z}}\, \nabla E_{0,z} \cdot \nabla R(\bar{r}) \Big)$

Light optics

$$\left[-\frac{j}{k_0}\nabla^2 R(\bar{r})\bar{E}_0 - \frac{2j}{k_0}\left(\hat{\boldsymbol{x}}\,\nabla E_{0,x}\cdot\nabla R(\bar{r}) + \hat{\boldsymbol{y}}\,\nabla E_{0,y}\cdot\nabla R(\bar{r}) + \hat{\boldsymbol{z}}\,\nabla E_{0,z}\cdot\nabla R(\bar{r})\right)\right.$$

For very short wavelengths (light → $400-700$ nm scale), $1/k_0 \to 0$ and the solution simplifies to:

$$|\nabla R(\bar{r})|^2 = n(\bar{r})^2$$

Eikonal equation (εἰκών, image)

Or equivalently:

$$\nabla R(\bar{r}) = n(\bar{r})\hat{s}(r)$$

- The energy flows in the direction of $\nabla R(\bar{r})$, i.e. the gradient of the wavefronts.
- For a homogeneous medium $n = const$ and $R(\bar{r}) = n/k_0(k_x x + k_y y + k_z z)$ represents the plane wavefronts (verify that $|\nabla R(\bar{r})|^2 = n^2$).
- By this simplification we pay the price of not being able to describe phenomena as diffraction, reflection.

Daylife example

Consider a very hot day in summer, asphalt gets hot and the air close to it expands lowering the refraction index.

$$n(y) = n_0\sqrt{1+\frac{y^2}{h^2}}$$

Source: https://www.scienceabc.com

$n(y)$

We can compute the ray direction:

$$|\nabla R(\bar{r})|^2 = n(\bar{r})^2$$

$$\left(\frac{\partial R}{\partial x}\right)^2 + \left(\frac{\partial R}{\partial y}\right)^2 = n(\bar{r})^2 = n_0^2\left(1+\frac{y^2}{h^2}\right)$$

$$R(x,y) = (n_0 x)\hat{x}_0 + \left(1+\frac{y^2}{2h^2}\right)\hat{y}_0$$

The direction of light rays is then: $\hat{s}(x,y) = \frac{\nabla R(x,y)}{|\nabla R(x,y)|} = \frac{\left(\frac{y}{h}\hat{y}_0+\hat{x}_0\right)}{\sqrt{1+\frac{y^2}{h^2}}}$

Daylife example

The ray covers a distance dx with direction given by $\hat{s}(x, y)$. The gained altitude dy can be then approximated by:

$$dy = \frac{s_y}{s_x} dx$$

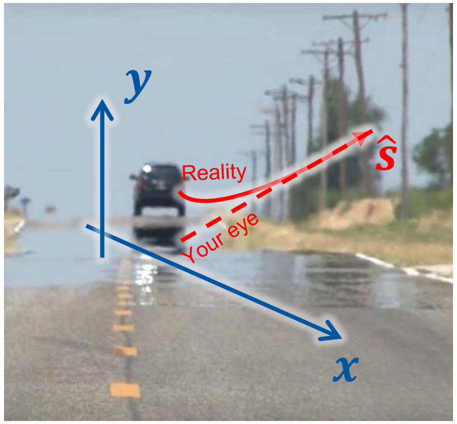
From this we have

$$\frac{dy}{dx} = \frac{s_y}{s_x} = \frac{y}{h}$$

With solution

$$y = e^{\pm(x - x_0)/h}$$

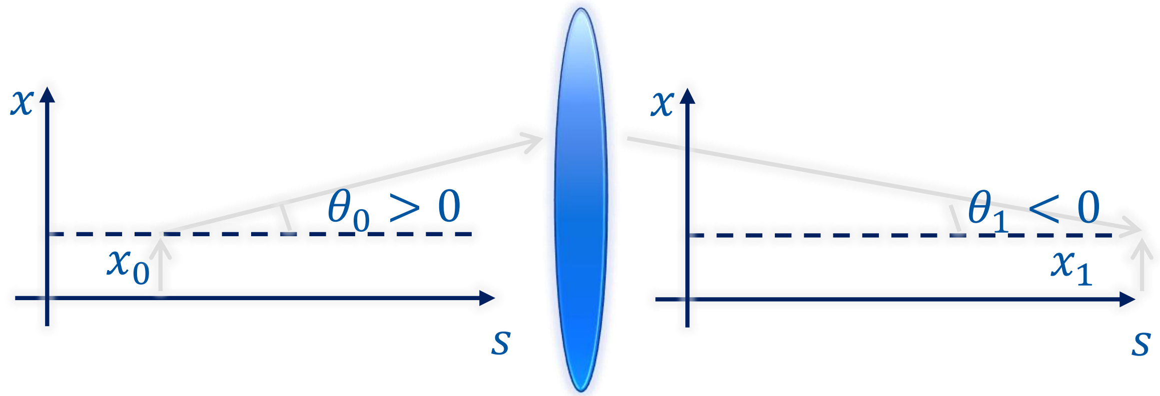
Light bends up (solution with $+$) emulating a reflection from a wet surface.



Source: https://www.scienceabc.com

Matrix treatment

The interaction of a wave can be reduced to the study of the interaction of the ray direction at an object boundary.



Element's matrix:

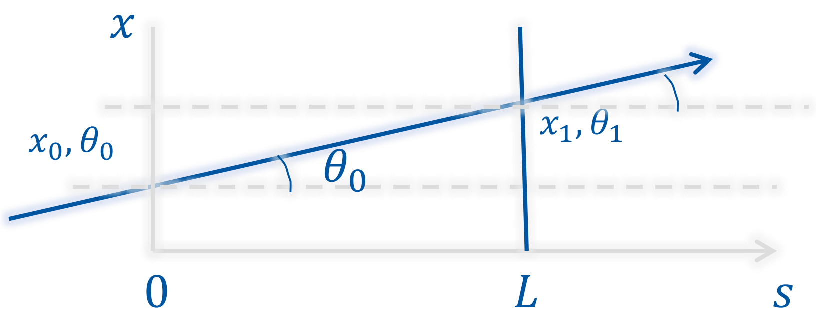
$$\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} x_0 \\ \theta_0 \end{pmatrix}$$

$M \equiv \frac{x_1}{x_0} = A$ magnification

$M_\theta \equiv \frac{\theta_1}{\theta_0} = D$ angular magnification

Drift space

Given a point on a wavefront of a ray, if we are in a homogeneous medium, it will just drift in space:



$$\begin{cases} \theta_1 = \theta_0 \\ x_1 = x_0 + L \sin\theta_0 = x_0 + L\theta_0 \end{cases}$$

(*paraxial approximation* $\sin\theta \simeq \theta$)

Matrix form

$$\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix} = M \begin{pmatrix} x_0 \\ \theta_0 \end{pmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_0 \\ \theta_0 \end{pmatrix}$$

N.B.: $\det(M) = 1$

Spherical mirror

Let us consider a spherical mirror (concave R>0, convex R<0).

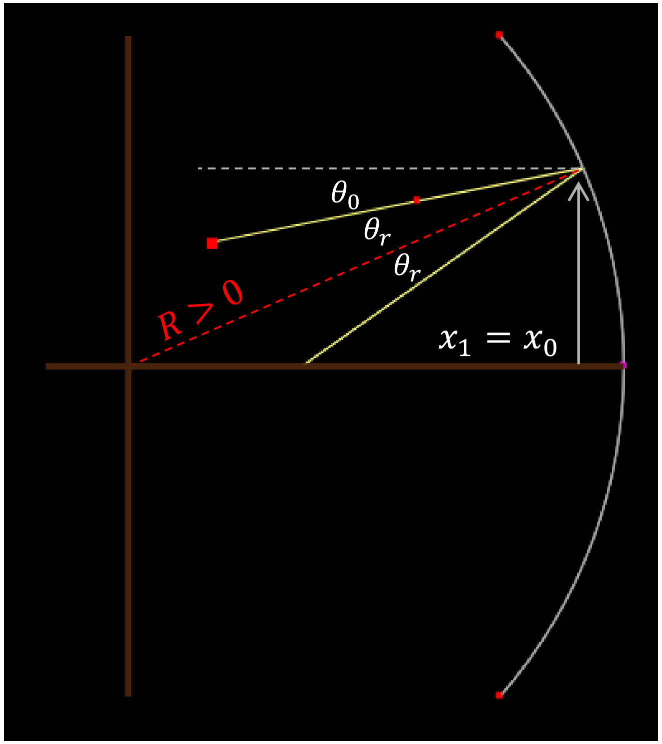
$$\begin{cases} x_1 = x_0 \\ \theta_1 = -(\theta_0 + 2\theta_r) \\ \theta_r + \theta_0 = \frac{x_1}{R} \end{cases}$$

↓

$$\theta_1 = -\frac{2x_1}{R} + \theta_0$$

$$\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -2/R & 1 \end{bmatrix} \begin{pmatrix} x_0 \\ \theta_0 \end{pmatrix}$$

N.B.: $\det(M) = 1$



Curved interface

Let us consider a curved interface (convex R>0, concave R<0)

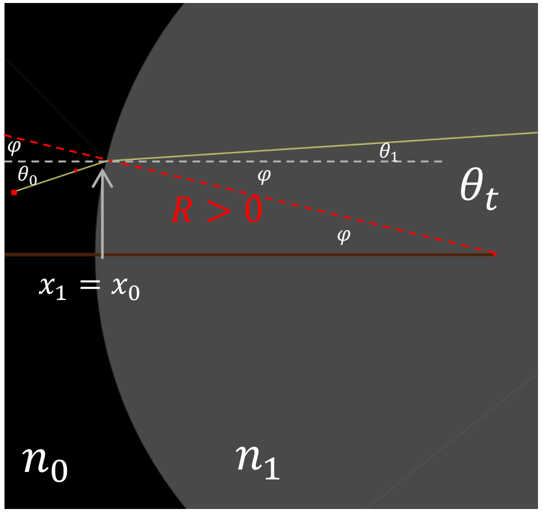
$$\begin{cases} x_1 = x_0 \\ n_0\theta_i = n_1\theta_t \text{ (Snell's law)} \\ \theta_i = \varphi + \theta_0 \\ \theta_t = \varphi + \theta_1 \\ \varphi = x_1/R \end{cases}$$

$$\downarrow$$

$$n_0(x_1/R + \theta_0) = n_1(x_1/R + \theta_1)$$

$$\frac{n_0}{n_1}\left(\frac{x_1}{R} + \theta_0\right) - x_1/R = \theta_1$$

θ_i



$$\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ \left(\frac{n_0}{n_1} - 1\right)/R & n_0/n_1 \end{bmatrix} \begin{pmatrix} x_0 \\ \theta_0 \end{pmatrix}$$

N.B.: $\det(M) = 1$ if $n_0 = n_1$: this is a result applicable to any optical system with same start/end refraction index.

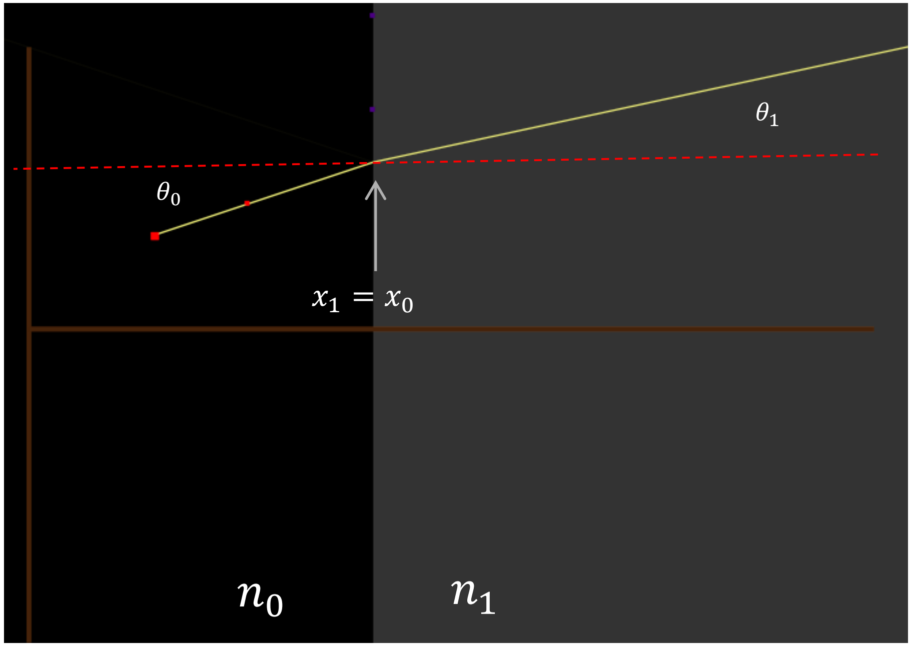
Curved interface: planar interface

For a planar interface we simply take the limit of large radius.

$$\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ \left(\frac{n_0}{n_1} - 1\right)/R & n_0/n_1 \end{bmatrix} \begin{pmatrix} x_0 \\ \theta_0 \end{pmatrix}$$

$$\downarrow \quad R \rightarrow \infty$$

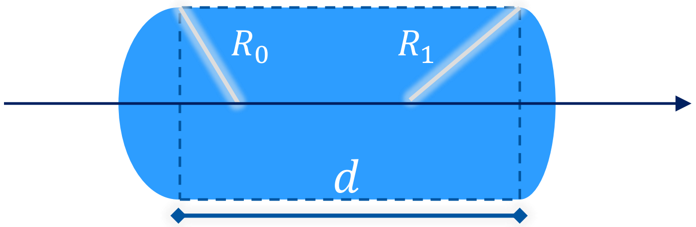
$$\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & n_0/n_1 \end{bmatrix} \begin{pmatrix} x_0 \\ \theta_0 \end{pmatrix}$$



Exercise 1: Thick lens

We have derived the basic elements that allow us to derive the matrices for more complex optical systems by simple matrix multiplication:
1) propagation through a region of uniform index,
2) reflection from a curved mirror
3) transmission through a curved interface of regions with different indices.

Classwork: Derive the ABCD matrix for a thick lens made of material $n_1 = n$ surrounded by air ($n_0 = 1$). Let the lens have curvatures R_0 and R_1 and thickness d.



What happens for $R_0, R_1 \to \infty$? Make a sketch.

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What happens for $R_0, R_1 \to \infty$? Make a sketch.

$$\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix} = \begin{bmatrix} 1 - \frac{d}{R_0}\left(1 - \frac{1}{n}\right) & \frac{d}{n} \\ -(n-1)\left(\frac{1}{R_0} - \frac{1}{R_1}\right) + \frac{d}{R_0 R_1}\left(2 - n - \frac{1}{n}\right) & 1 + \frac{d}{R_1}\left(1 - \frac{1}{n}\right) \end{bmatrix} \begin{pmatrix} x_0 \\ \theta_0 \end{pmatrix}$$

Thin lens

Considering a vanishing length d between the two lens surfaces we have:

$$\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -(n-1)\left(\frac{1}{R_0} - \frac{1}{R_1}\right) & 1 \end{bmatrix} \begin{pmatrix} x_0 \\ \theta_0 \end{pmatrix}$$

Which can be written as

$$\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{pmatrix} x_0 \\ \theta_0 \end{pmatrix}$$

Where f is the focal length given by:

$$1/f = (n-1)\left(\frac{1}{R_0} - \frac{1}{R_1}\right)$$

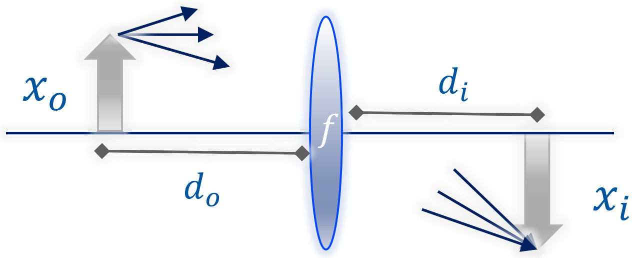
Lens maker's equation

When the radii are the same (in modulo): $\frac{1}{f} = (n-1)\frac{2}{R}$

Image formation

Given a transfer matrix, we would like to know if, placing an object in front of the optical system (d_o), an image is formed, and where (d_i).

$$\begin{pmatrix} x_i \\ \theta_i \end{pmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} x_o \\ \theta_o \end{pmatrix}$$



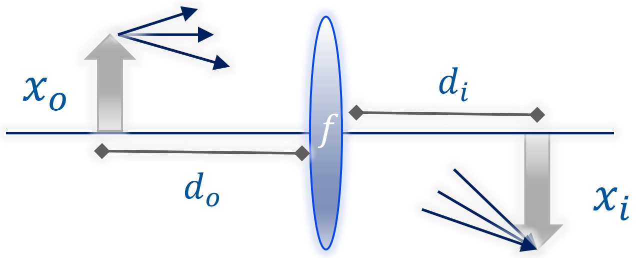
To have an image, all the rays should go to the same point regardless of the angle they start. A point x_o is mapped into a point x_i regardless of the angles θ_o the rays have which means:

$\boldsymbol{B = 0}$ (condition of image formation).

Image formation

If the object is at distance d_o, for a thin lens we have:

$$\begin{pmatrix} x_i \\ \theta_i \end{pmatrix} = \begin{bmatrix} 1 & \boldsymbol{d_i} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & \boldsymbol{d_o} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_o \\ \theta_o \end{pmatrix}$$

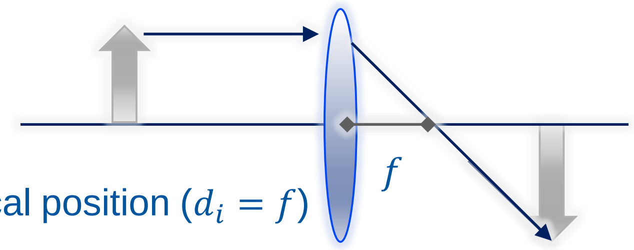


$$\begin{pmatrix} x_i \\ \theta_i \end{pmatrix} = \begin{bmatrix} 1 - \frac{d_i}{f} & d_o + d_i\left(-\frac{1}{f} d_o + 1\right) \\ -\frac{1}{f} & -\frac{1}{f} d_o + 1 \end{bmatrix} \begin{pmatrix} x_o \\ \theta_o \end{pmatrix}$$

$$\boldsymbol{B = 0} \rightarrow \begin{cases} \frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i} \\ M = A = -\frac{d_i}{d_o} \\ M_\theta = D = -\frac{d_o}{d_i} \end{cases} \longrightarrow \begin{pmatrix} x_i \\ \theta_i \end{pmatrix} = \begin{bmatrix} -\frac{d_i}{d_o} & 0 \\ -\frac{1}{f} & -\frac{d_o}{d_i} \end{bmatrix} \begin{pmatrix} x_o \\ \theta_o \end{pmatrix}$$

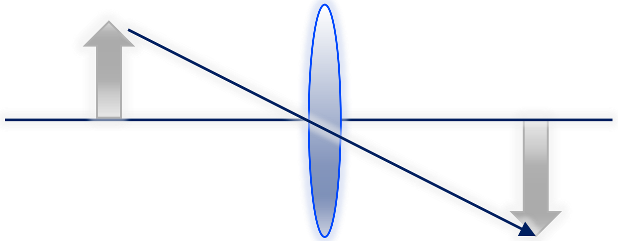
Practical drawing

1) A ray coming parallel to the lens is cross the axis at the focal point after the lens

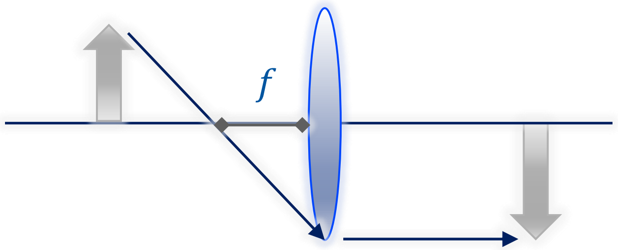


→ an image from far away is produced in the focal position ($d_i = f$)

2) A ray passing through the lens center is un-deflected



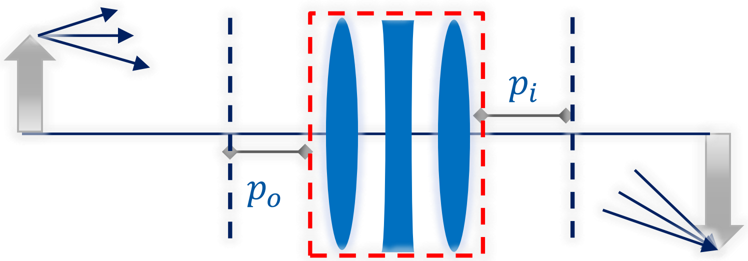
3) A ray passing through the focal point before the lens exits parallel.



Principal planes

For a general arrangement of optical elements starting and ending to the same refraction index there exist two planes such that the system allows image formation.

$$\begin{pmatrix} x_i \\ \theta_i \end{pmatrix} = \begin{bmatrix} 1 & p_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1 & p_o \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_o \\ \theta_o \end{pmatrix}$$



$$M = \begin{bmatrix} A + p_i C & A p_o + B + p_i p_o C + p_i D \\ C & C p_o + D \end{bmatrix}$$

$$\begin{cases} A + p_i C = 1 \ \rightarrow p_i = \dfrac{1 - A}{C} \\ D + p_o C = 1 \ \rightarrow p_o = \dfrac{1 - D}{C} \\ C \equiv -\dfrac{1}{f_{eff}} \end{cases}$$

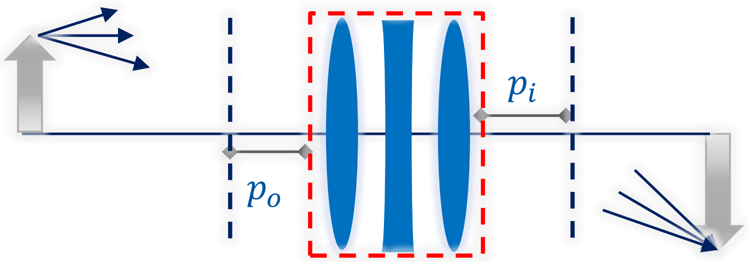
$$\rightarrow A p_o + B + p_i p_o C + p_i D = 1 - \det(M) = 0$$

$= 1$ (same n at start/end)

Principal planes

For a general arrangement of optical elements starting and ending to the same refraction index there exist two planes such that the system allows image formation.

$$\begin{pmatrix} x_i \\ \theta_i \end{pmatrix} = \begin{bmatrix} 1 & p_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1 & p_o \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_o \\ \theta_o \end{pmatrix}$$



$$M = \begin{bmatrix} A + p_i C & Ap_o + B + p_i p_o C + p_i D \\ C & Cp_o + D \end{bmatrix}$$

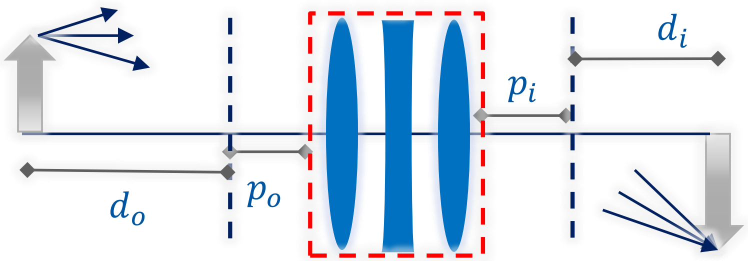
$$M = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_{eff}} & 1 \end{bmatrix}$$

The planes passing by p_o and p_i are called *principal planes*: from there any system will look like a thin lens system.

Principal planes

For a generic system with equal start/end refractive index, we can rewrite the condition for image formation:

$$\begin{cases} \frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f} & \textit{image condition} \\ M = -\frac{d_i}{d_o} & \textit{magnification} \\ M_\theta = -\frac{d_o}{d_i} & \textit{angular magnification} \end{cases}$$



These are the same result as of the thin lens case, with exception that now the length is taken with respect to the principal planes.

Practical experiment

Find the focal length of the given lens.

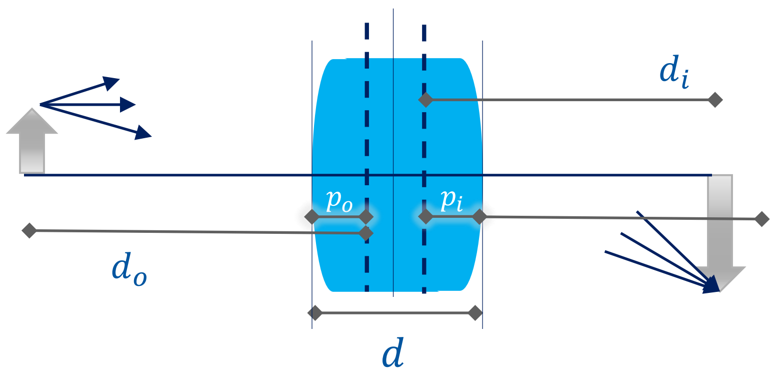
You can help yourself with a light ☺

The refraction index of glass is $n \sim 1.5$: what is the curvature radius of the lens in the thin lens assumption?

Exercise 2

Find the image position and magnification of an object placed 30 cm apart from a thick lens of $d = 1$ cm thickness, in/out radii of 20 cm, made of glass.

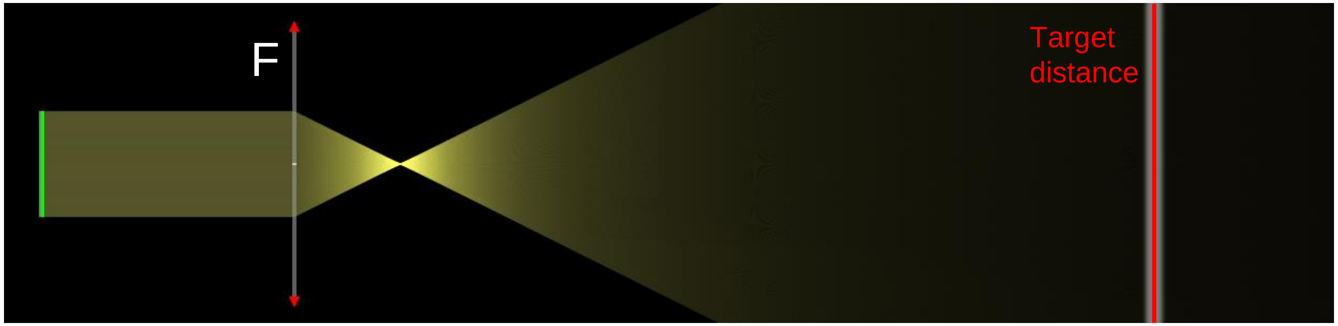
Reminder: a thick lens is a complex system, distances need to be computed from the principal planes.



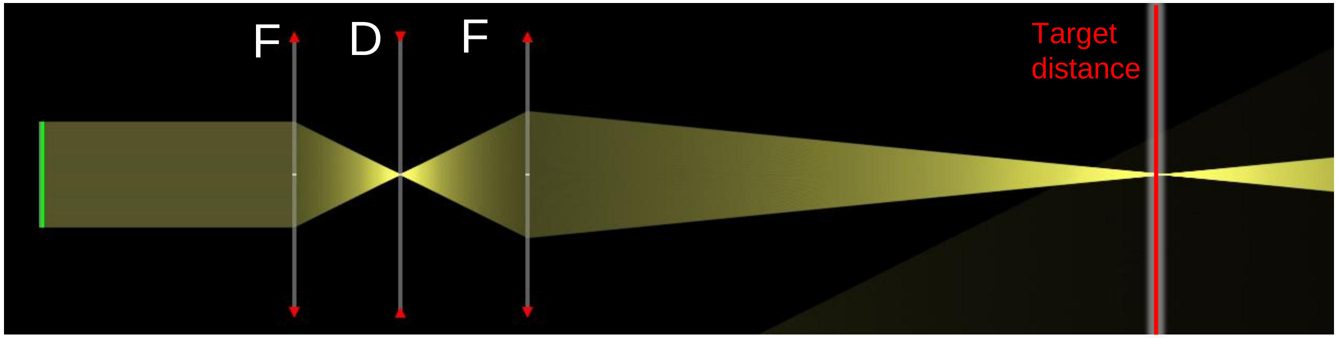
$$M_{thick} = \begin{bmatrix} 1 - \frac{d}{R_0}\left(1 - \frac{1}{n}\right) & \frac{d}{n} \\ -(n-1)\left(\frac{1}{R_0} - \frac{1}{R_1}\right) + \frac{d}{R_0 R_1}\left(2 - n - \frac{1}{n}\right) & 1 + \frac{d}{R_1}\left(1 - \frac{1}{n}\right) \end{bmatrix}$$

Light transport

Suppose we want to focus light at a *large* distance.
A single lens would not do a great job..



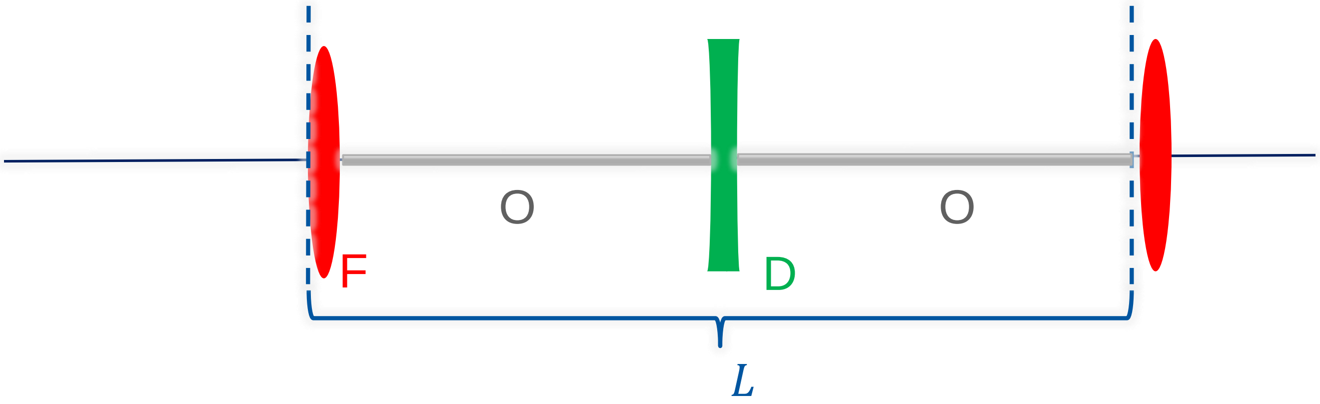
But a sequence of focusing-defocusing (F-D) elements could do it!



Light transport

Sequence of F-D lenses spaced by $L/2$ drift:

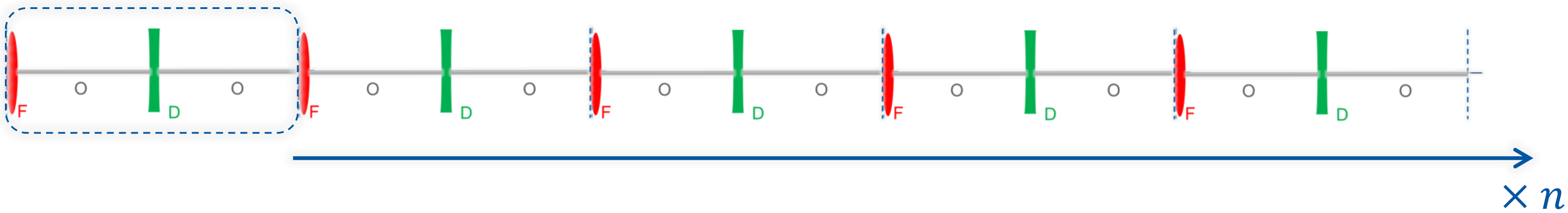
$$FODO = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & L/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & L/2 \\ 0 & 1 \end{bmatrix}$$



$$FODO = \begin{bmatrix} 1 + \frac{L}{2f} & L + \frac{L^2}{4f} \\ -\frac{L}{2f^2} & 1 - \frac{L}{2f} - \frac{L^2}{4f^2} \end{bmatrix}$$

Light transport

Repeating the FODO cell we can focus light from one point to another.



For n cells: $M = FODO^{\boldsymbol{n}} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{\boldsymbol{n}}$

Starting and ending to the same refraction index, $\det(M) = 1$.
We can apply Sylvester's theorem which states:

$$FODO^{\boldsymbol{n}} = \frac{1}{\sin\theta}\begin{bmatrix} A\sin N\theta - \sin(N-1)\theta & B\sin N\theta \\ C\sin N\theta & D\sin N\theta - \sin(N-1)\theta \end{bmatrix}$$

with $\cos\theta = \frac{1}{2}(A+D)$ $\xrightarrow{\theta \text{ is real} \leftrightarrow \text{stability}}$ $\left|\frac{1}{2}(A+D)\right| \leq 1$

Light transport

In a FODO cell:

$$FODO = \begin{bmatrix} 1+\frac{L}{2f} & L+\frac{L^2}{4f} \\ -\frac{L}{2f^2} & 1-\frac{L}{2f}-\frac{L^2}{4f^2} \end{bmatrix}$$

The terms A and D are:

$$\begin{cases} A = 1+\frac{L}{2f} \\ D = 1-\frac{L}{2f}-\frac{L^2}{4f^2} \end{cases}$$

The stability of the FODO transport system is ensured if:

$$\left|\frac{1}{2}(A+D)\right| \leq 1 \longrightarrow \left|\left(1-\frac{L^2}{8f^2}\right)\right| \leq 1 \longrightarrow \boldsymbol{L \leq 4f}$$

Stability for a FODO transport

Light transport

You will see the same stability condition for transverse particle optics in A.Latina's lecture next week!

In a FODO cell:

$$FODO = \begin{bmatrix} 1+\frac{L}{2f} & L+\frac{L^2}{4f} \\ -\frac{L}{2f^2} & 1-\frac{L}{2f} \end{bmatrix}$$

The terms A and D are:

$$\begin{cases} A = 1+\frac{L}{2f} \\ D = 1-\frac{L}{2f}-\frac{L^2}{4f^2} \end{cases}$$

The stability of the FODO

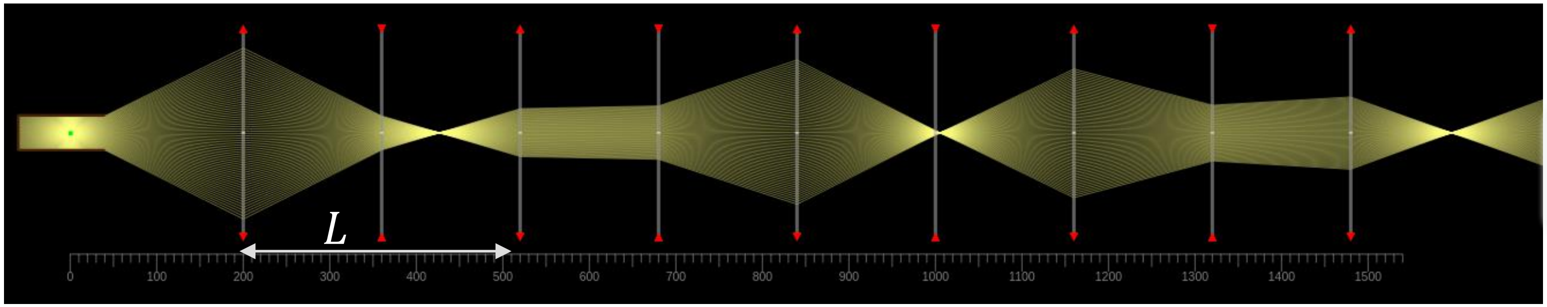
$$\left|\frac{1}{2}(A+D)\right| \le 1 \longrightarrow$$

juas

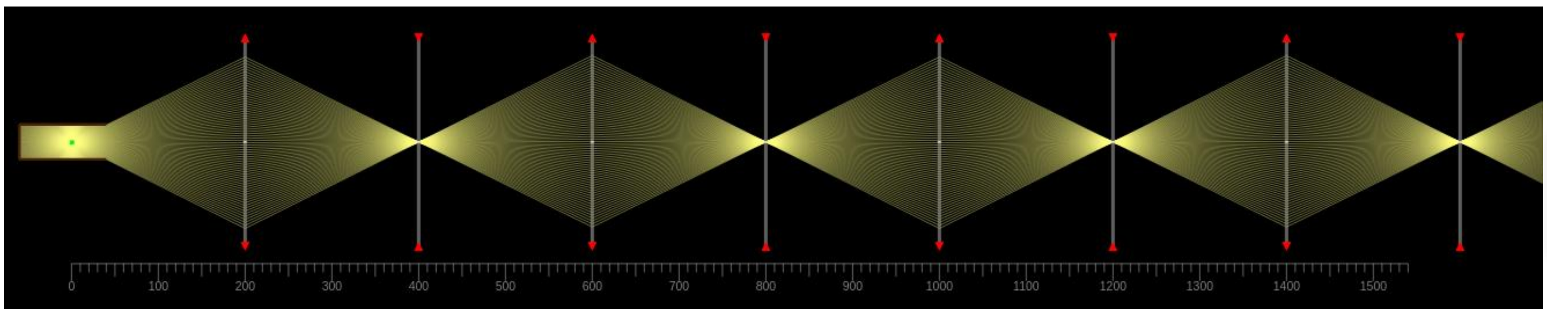
JUAS - TIMETABLE 2020 - WEEK 2

Schedule 2020	Monday Jan 20	Tuesday Jan 21	Wednesday Jan 22	Thursday Jan 23	Friday Jan 24
09:00	Transverse Dynamics *A. Latina*	Transverse Dynamics *A. Latina*	Transverse Dynamics *A. Latina*	Cyclotrons *B. Jacquot*	Linacs *D. Alesini*
10:00 10:15	Coffee Break	Coffee Break	Coffee Break	Coffee Break	Coffee Break
	Transverse Dynamics *A. Latina*	Transverse Dynamics *A. Latina*	Transverse Dynamics *A. Latina*	Cyclotrons *B. Jacquot*	Linacs *D. Alesini*
11:15	Transverse Dynamics *A. Latina*	Transverse Dynamics *A. Latina*	Transverse Dynamics *A. Latina*	Cyclotrons *B. Jacquot*	Linacs *D. Alesini*
12:15	WORKING LUNCH	BREAK	BREAK	BREAK	BREAK
14:00	Intro. to MAD-X *G. Sterbini*	Transverse Dynamics *A. Latina*	Cyclotrons *B. Jacquot*	Linacs *D. Alesini*	Transverse Dynamics *A. Latina*
15:00	MADX *N. Fuster Martinez / H. Garcia Morales / A. Latina / G. Sterbini*	MADX *N. Fuster Martinez / H. Garcia Morales / A. Latina / G. Sterbini*	Cyclotrons *B. Jacquot*	Linacs *D. Alesini*	Transverse Dynamics *A. Latina*
16:00 16:15	Coffee Break	Coffee Break	Coffee Break	Coffee Break	Coffee Break
	MADX *N. Fuster Martinez / H. Garcia Morales / A. Latina / G. Sterbini*	MADX *N. Fuster Martinez / H. Garcia Morales / A. Latina / G. Sterbini*	Cyclotrons *B. Jacquot*	Linacs *D. Alesini*	MADX *N. Fuster Martinez / H. Garcia Morales / A. Latina / G. Sterbini*
17:15	Hybrid collisions in the LHC *J. Jowett*		European Projects for Collaborative Accelerator R&D Seminar *M. Vretenar*		MADX *N. Fuster Martinez / H. Garcia Morales / A. Latina / G. Sterbini*
18:15			AFTER WORK AT ESI		

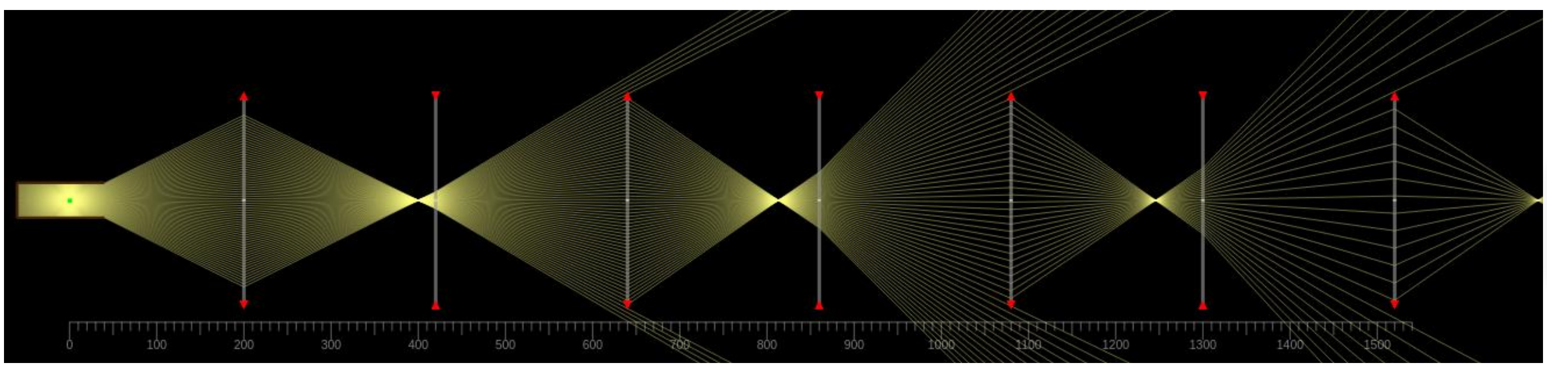
Example of light transport between focusing/defocusing lenses with $f = 100$.



$L < 4f$



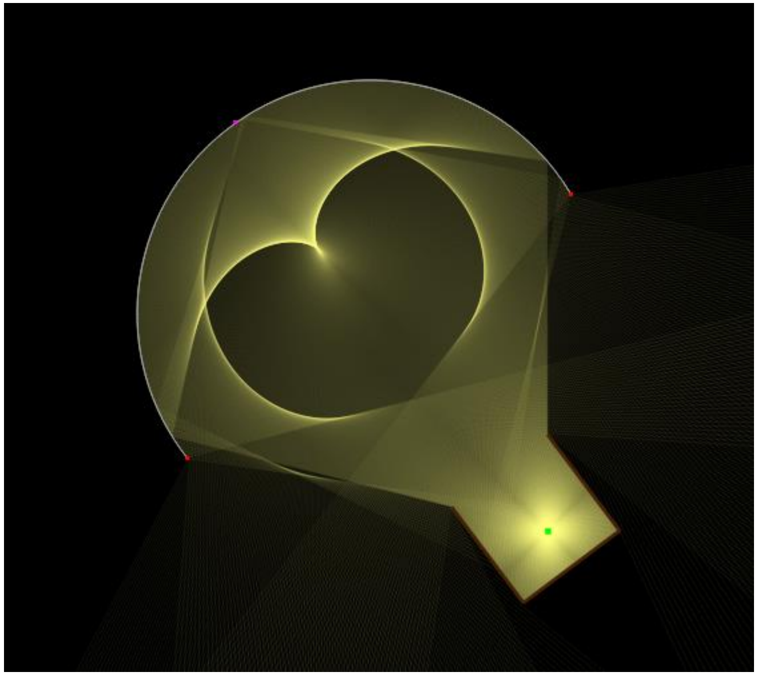
$L = 4f$



$L > 4f$

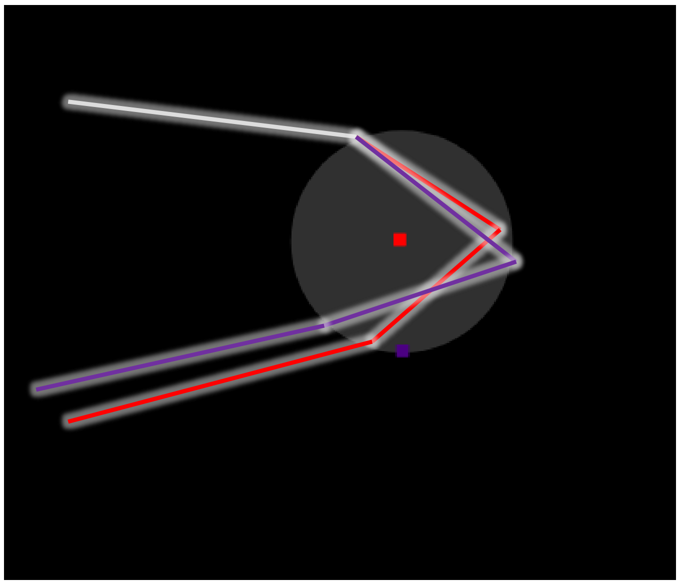
The not so ideal world…

Spherical aberrations



- In circular mirrors different focal length for large angles smears the focus in a so-called *caustic* line.
- We can use parabolic mirror or additional lens corrections to prevent this.

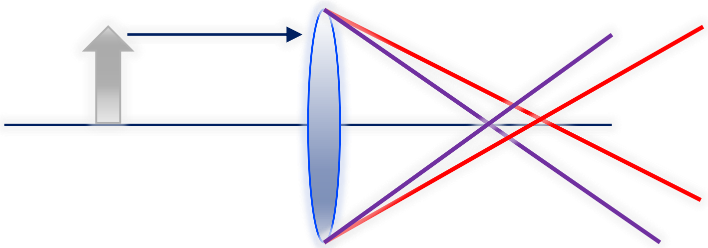
Chromatic aberrations I



Refractive index depends on wavelength → nicely decomposed in rainbows (primary and secondary depending on number of reflection in rain drops).

Chromatic aberrations II

Also a thin lens exhibit different focusing depending on wavelength ("chromaticity" of the lens).



Keep "chromaticity" word in mind → key parameter in accelerator design and control.

Appendix

Recall of some vector relations

Nabla operator in cartesian coordinates:

$$\nabla = \left[\frac{\mathrm{d}}{\mathrm{dx}}\hat{x} + \frac{\mathrm{d}}{\mathrm{dy}}\hat{y} + \frac{\mathrm{d}}{\mathrm{dz}}\hat{z}\right]$$

If S is a vector field and ϕ a scalar function we have:

$$\nabla \cdot (\phi S) = \phi \nabla \cdot S + \mathrm{S}^{\mathrm{T}} \nabla \phi$$

The gradient of a vector field is a dyadic:

$$\nabla S = [\nabla S_x \quad \nabla S_y \quad \nabla S_z]$$

Taking the divergence we get back a vector:

$$\nabla \cdot (\nabla f) = \left[\frac{\mathrm{d}}{\mathrm{dx}}\hat{x} + \frac{\mathrm{d}}{\mathrm{dy}}\hat{y} + \frac{\mathrm{d}}{\mathrm{dz}}\hat{z}\right] \cdot [\nabla S_x \quad \nabla S_y \quad \nabla S_z] = \begin{pmatrix} \nabla \cdot \nabla S_x \\ same\ with\ \nabla S_y \\ same\ with\ \nabla S_z \end{pmatrix}$$

Eikonal equation derivation

$$\frac{1}{k_0^2}\nabla^2\left(\bar{E}_0 e^{-j\left(k_0 R(\bar{r})\right)}\right) + n(\bar{r})^2 \bar{E}_0 e^{-j\left(k_0 R(\bar{r})\right)} = 0$$

We need to compute $\nabla^2 \bar{f} = \nabla \cdot (\nabla \bar{f})$ with $\bar{f} = \bar{E}_0 e^{-j\left(k_0 R(\bar{r})\right)}$

The $\nabla \bar{f}$ is a dyadic:

$$\nabla\left(\bar{E}_0 e^{-j\left(k_0 R(\bar{r})\right)}\right) = \left(-jk_0 \nabla R(\bar{r})\bar{E}_0 + \nabla \bar{E}_0\right) e^{-j\left(k_0 R(\bar{r})\right)}$$

$$= \begin{pmatrix} -jk_0 \nabla R \cdot E_{0,x} + \nabla E_{0,x} \\ same\ with\ \nabla E_{0,y} \\ same\ with\ \nabla E_{0,z} \end{pmatrix} e^{-j\left(k_0 R(\bar{r})\right)}$$

$\nabla \cdot (\nabla \bar{f})$ is then given by

$$\begin{pmatrix} \nabla \cdot \left(\left(-jk_0 \nabla R \cdot E_{0,x} + \nabla E_{0,x}\right) e^{-j\left(k_0 R(\bar{r})\right)}\right) \\ same\ with\ \nabla E_{0,y} \\ same\ with\ \nabla E_{0,z} \end{pmatrix}$$

Eikonal equation derivation

For the x component we have:

$$\nabla \cdot \left(-jk_0 \nabla R(\bar{r}) E_{0,x} e^{-j\left(k_0 R(\bar{r})\right)} + \nabla E_{0,x} e^{-j\left(k_0 R(\bar{r})\right)}\right) = e^{-j\left(k_0 R(\bar{r})\right)} \cdot$$

$$\cdot \left(-jk_0\, \nabla^2 R(\bar{r}) E_{0,x} - jk_0 \nabla E_{0,x} \cdot \nabla R(\bar{r}) - k_0^2 \nabla R(\bar{r}) E_{0,x} \nabla R(\bar{r}) + \nabla^2 E_{0,x} - jk_0 \nabla E_{0,x} \cdot \nabla R(\bar{r})\right)$$

And for all three components:

$$\left[-jk_0\, \nabla^2 R(\bar{r}) E_{0,x} - 2jk_0 \nabla E_{0,x} \cdot \nabla R(\bar{r}) + \nabla^2 E_{0,x} - k_0^2 E_{0,x} \nabla R(\bar{r}) \cdot \nabla R(\bar{r})\right] e^{-j\left(k_0 R(\bar{r})\right)}\ \hat{\boldsymbol{x}}$$

$$\left[-jk_0\, \nabla^2 R(\bar{r}) E_{0,y} - 2jk_0 \nabla E_{0,y} \cdot \nabla R(\bar{r}) + \nabla^2 E_{0,y} - k_0^2 E_{0,y} \nabla R(\bar{r}) \cdot \nabla R(\bar{r})\right] e^{-j\left(k_0 R(\bar{r})\right)}\ \hat{\boldsymbol{y}}$$

$$\left[-jk_0\, \nabla^2 R(\bar{r}) E_{0,z} - 2jk_0 \nabla E_{0,z} \cdot \nabla R(\bar{r}) + \nabla^2 E_{0,z} - k_0^2 E_{0,z} \nabla R(\bar{r}) \cdot \nabla R(\bar{r})\right] e^{-j\left(k_0 R(\bar{r})\right)}\ \hat{\boldsymbol{z}}$$