Synchrotron Radiation — Exercises 1

Rasmus Ischebeck

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1 Future Circular Collider

Particle physicists are evaluating the potential of building a future circular collider, which aims at colliding two proton beams with 500 mA current each and 100 TeV center-of-mass collision energy (FCC-hh). *Note: this problem gives inconsistent results when using 100 TeV particle energy, as stated in the initial formulation.* The protons would be circulating in a storage ring with 100 km circumference, guided by superconducting magnets. The dipoles aim at a field of 16 T. Calculate

- The Lorentz factor γ
- The critical energy of the synchrotron radiation
- The total power emitted by both beams through synchrotron radiation

Answers

- The Lorentz factor γ can be calculated from the particle energy, which is half the center-of-mass energy $E_p = E_c/2$:

$$\gamma = \frac{E_p}{m_p c^2} = \frac{50\text{TeV}}{938\text{MeV}} = 5.33 \cdot 10^4$$

- The critical energy is:

$$E_{crit} = \frac{3}{2}\frac{\hbar c}{\rho}\gamma^3$$

We thus calculate the radius of curvature in the magnets. For ultra-relativistic particles, the momentum p is, to a good approximation, given by the E/c.

$$\rho = \frac{p}{eB} = \frac{50\text{TeV/c}}{e \cdot 16\text{T}} = \frac{50 \cdot 10^{12}\text{V}}{3 \cdot 10^8\text{m/s} \cdot 16\text{T}} = 10.4\text{km}$$

The critical energy is thus:

$$E_{crit} = 6.88 \cdot 10^{-16}\text{J} = 4.30\text{keV}$$

The total power emitted by one particle is (for ultrarelativistic particles, we can assume $\beta \approx 1$:

$$P = \frac{e^2 c}{6\pi\varepsilon_0}\frac{\beta^4\gamma^4}{\rho^2} = 3.42\text{nW}$$

To calculate the number of particles in each ring, we need to know the time that it takes for the particles to go around the circumference $\mathcal{C}$ of the ring (again, assuming $\beta \approx 1$:

$$t_{rev} = \frac{\mathcal{C}}{c} = \frac{100\text{km}}{3\cdot 10^8\text{m/s}} = 0.33\text{ms}$$

The current I is the charge per time interval, thus the number of protons N_p can be calculated:

$$I = \frac{Q}{t} = \frac{N_p e}{t_{rev}} \Rightarrow N_p = \frac{I t_{rev}}{e} = 1.04\cdot 10^{15}$$

The total radiated power from one beam is thus:

$$P_{beam} = 3.56\text{MW}$$

and for both beams:

$$P_{tot} = 7.12\text{MW}$$

2 Muon Storage Rings

Muons are considered as an alternative to electrons for a future circular lepton collider. Argue

- why they might be preferable to electrons, and
- what could be possible disadvantages.

Answers

- The synchrotron radiation is reduced, because the Lorentz factor γ is lower at the same center-of-mass collision energy. Since the synchrotron radiation power is proportional γ^4, the possible reduction is:

$$\frac{P_{SR}(\mu)}{P_{SR}(e)} = \left(\frac{m_e}{m_\mu}\right)^4 = 5.47\cdot 10^{-10}$$

- However, muons are unstable and have only a short lifetime (increased somewhat if they have relativistic energies. Furthermore, they are produced in small numbers and with a large emittance $\Rightarrow$ this requires a lot of accumulation and damping of the beam

3 Undulator Radiation

Assume an undulator of 15 mm period and 5 m length. The pole tip field is $B_t = 1.5$ T, and the gap can be varied between 8 and 16 mm.
This undulator is placed in a storage ring, with an electron beam energy of $E = 3.2$ GeV, and a beam current of 500 mA. The beam is focused to a waist of $\sigma_x = \sigma_y = 20$ µm inside the undulator.

- What range can be reached with the fundamental photon energy?
- What brilliance can be reached at the fundamental photon energy?
- Is there a significant flux higher harmonics?

Answers

- The energy of photons for the fundamental resonance ($n = 1$) can be calculated from the wavelength:

$$E_\gamma = \frac{hc}{\lambda}$$

This can be calculated from the resonance condition:

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)$$

We thus need to calculate the undulator parameter K:

$$K = \frac{eB_0\lambda_u}{2\pi m_e c}$$

...for which we need the magnetic field amplitude on axis, which can be calculated from the tip field B_{tip} and the gap g:

$$B_0 = \frac{B_{tip}}{\cosh\left(\frac{\pi g}{\lambda_u}\right)}$$

Thus, we calculate, for the two gaps of 8 and 6 mm, respectively:

$$B_0 = \begin{cases} 0.5426\text{T} & \text{(for 8 mm)} \\ 0.1050\text{T} & \text{(for 16 mm)} \end{cases}$$

$$K = \begin{cases} 0.7600 & \text{(for 8 mm)} \\ 0.1471 & \text{(for 16 mm)} \end{cases}$$

$$\lambda = \begin{cases} 2.465\,\text{Å} & \text{(for 8 mm)} \\ 1.933\,\text{Å} & \text{(for 16 mm)} \end{cases}$$

$$E_\gamma = \begin{cases} 5.0302\text{keV} & \text{(for 8 mm)} \\ 6.4135\text{keV} & \text{(for 16 mm)} \end{cases}$$

- The brilliance of an undulator can be calculated as follows:

$$\mathcal{B} = \frac{\dot{N}}{4\pi^2 \sigma_{x,\mathrm{eff}} \sigma_{y,\mathrm{eff}} \sigma_{x',\mathrm{eff}} \sigma_{y',\mathrm{eff}}}$$

  For the fundamental wavelength, the number of photons is:

$$\dot{N} = 1.43 \cdot 10^{14} \cdot N_{\mathrm{periods}} \cdot I_{\mathrm{beam}} \cdot Q_1(K)$$

  where $Q_1(K)$ can be calculated from the undulator parameter. This can be calculated with the following MATLAB function:

```
function Q = Q(n,K)
Q = (1+K.^2/2)/n .* F(n,K);

function F = F(n,K)
F = n^2*K.^2./(1+K.^2/2).^2 .* ...
    ( besselj((n+1)/2,Y(n,K)) - besselj((n-1)/2,Y(n,K)) ).^2;

function Y = Y(n,K)
Y = n*K.^2 ./ (4*(1+K.^2/2));
```

  Or, alternatively in PYTHON:

```
@mfunction("Q")
def Q(n=None, K=None):
    Q = (1 + K**2 / 2) / n * F(n, K)

@mfunction("F")
def F(n=None, K=None):
    F = n**2 * K**2 / (1 + K**2 / 2)**2 *\
    (besselj((n+1)/2, Y(n, K)) - besselj((n-1)/2, Y(n, K)))**2

@mfunction("Y")
def Y(n=None, K=None):
    Y = n * K**2 / (4 * (1 + K**2 / 2))
```

  With K from above, we calculate:

$$Q_1(K) = \begin{cases} 0.3968 & \text{(for 8 mm)} \\ 0.0213 & \text{(for 16 mm)} \end{cases}$$

$$\dot{N} = \begin{cases} 9.447 \cdot 10^{15} \gamma/\mathrm{s} & \text{(for 8 mm)} \\ 5.068 \cdot 10^{14} \gamma/\mathrm{s} & \text{(for 16 mm)} \end{cases}$$

  To calculate the effective source size $\sigma_{x,\mathrm{eff}}$ and $\sigma_{y,\mathrm{eff}}$, we use

$$\sigma_{x,\mathrm{eff}} = \sqrt{\sigma_{x,\mathrm{beam}}^2 + \sigma_r^2}$$

The diffraction limited source size σ_r can be calculated from the wavelength λ of the radiation and the length of the undulator L_u:

$$\sigma_r = \frac{1}{4\pi}\sqrt{\lambda \cdot L_u} = \begin{cases} 2.794\mu\text{m} & \text{(for 8 mm)} \\ 2.474\mu\text{m} & \text{(for 16 mm)} \end{cases}$$

The effective source size is thus:

$$\sigma_{x,\text{eff}} = \begin{cases} 20.19\mu\text{m} & \text{(for 8 mm)} \\ 20.15\mu\text{m} & \text{(for 16 mm)} \end{cases}$$

Since the beam is assumed to be symmetric, $\sigma_{x,\text{eff}} = \sigma_{y,\text{eff}}$ For the effective divergence, let's just use the diffraction limit of the radiation and ignore the divergence of the beam (i.e. use a zero-emittance beam). In this case,

$$\sigma_{x',\text{eff}} = \sigma_{y',\text{eff}} = \sigma_{r'} = \sqrt{\frac{\lambda}{L_u}} = \begin{cases} 7.0\mu\text{rad} & \text{(for 8 mm)} \\ 6.4\mu\text{rad} & \text{(for 16 mm)} \end{cases}$$

The brilliance is thus:

$$\mathcal{B} = \begin{cases} 1.19 \cdot 10^{22} \\ 8.18 \cdot 10^{20} \end{cases} \frac{\gamma}{\text{s} \cdot \text{mm}^2 \cdot \text{mrad}^2 (0.1\% BW)}$$

- The flux at higher harmonics is negligible, in particular at the gap of 16 mm. To find this, we calculate $Q_3(K)$ and compare to $Q_1(K)$:

$$Q_3(K) = \begin{cases} 0.0309 & \text{(for 8 mm)} \\ 3.06 \cdot 10^{-6} & \text{(for 16 mm)} \end{cases} \qquad Q_1(K) = \begin{cases} 0.3968 & \text{(for 8 mm)} \\ 0.0193 & \text{(for 16 mm)} \end{cases}$$

  The flux at the third harmonic is thus reduced by a factor 12.8 and 6307, respectively, in comparison to the fundamental wavelength.

4 Preparation for an Upgrade

Petra-III is a 2.3 km circumference light source at 6 GeV and 1 nm horizontal emittance, located at DESY in Hamburg. An upgrade based on multi-bend achromats will decrease the emittance to 10 pm. Before the upgrade, the DESY team wants to test instrumentation for the new ring at low emittance.
Suggest a way to lower the emittance at the existing machine in order to test the instrumentation. What are some issues with your suggestion?

Answers

There are two easy possibilities to reduce the (horizontal) emittance to test instrumentation at an existing storage ring:

- produce a round beam with a skew quadrupole: this will shift emittance from the horizontal to the vertical plane; the beam size will be reduced by a factor $\sqrt{2}$ from the additional damping
- lower the ring energy; note: Petra-III decreased their energy in Summer 2019 to 3 GeV for testing instrumentation, achieving an emittance down to 200 pm

Issues include a much larger vertical emittance in the first case, and a significantly reduced flux, and significantly reduced photon energies in the second case.

5 New Storage Ring

You have been nominated as director for the new Mexican Light Source. The aim of this synchrotron is protein crystallography. What are your considerations? Give at least one advantage and one disadvantage for each of the two following aspects:

- Higher beam energy
- Large circumference

Answers

Higher beam energies:

$\oplus$ higher flux

$\oplus$ higher photon energies

$\oplus$ higher brilliance for short wavelengths

$\ominus$ higher cost, due to higher fields in the magnets, and due to the requirement of higher RF power to restore the synchrotron losses

$\ominus$ increased radiation damage

$\ominus$ increased natural emittance

Large circumference:

$\oplus$ lower fields in the dipoles

$\oplus$ higher beam energy possible

$\oplus$ smaller emittance

$\oplus$ less RF power needed

$\oplus$ more space for beamlines

$\ominus$ higher cost for land, building and vacuum system

6 Cosmic Electron

A cosmic electron with an energy of 1 GeV enters an interstellar region with a magnetic field of 1 nT. Calculate

- The radius of curvature
- The critical energy of the emitted synchrotron radiation
- The energy emitted in one turn

How would you measure this radiation?

Answers

- The radius of curvature is:

$$\rho = \frac{p}{eB} = \frac{1\mathrm{GeV/c}}{e \cdot 1\mathrm{nT}} = 3.33 \times 10^{9}\,\mathrm{m}$$

- The Lorentz factor is

$$\gamma = \frac{1\,\mathrm{GeV}}{511\,\mathrm{keV}} = 1957$$

The critical energy of the emitted synchrotron radiation is:

$$E_{crit} = \frac{3}{2}\frac{\hbar c}{\rho}\gamma^3 = 1.08 \times 10^{-25}\,\mathrm{J} = 0.67\,\mu\mathrm{eV}$$

- The energy emitted in one turn is:

$$U_0 = \frac{e^2\beta^4\gamma^4}{3\varepsilon_0\rho} = 4.29 \times 10^{-24}\,\mathrm{J}$$

To get a better feeling for the photon energy, let's convert to frequency:

$$\nu = \frac{E_gamm}{h} = 160\,\mathrm{MHz}$$

This is a frequency similarly to the ones used in FM radios ⇒ use an antenna to detect these photons

7 Superconducting Undulators

What is the advantage of using undulators made with superconducting coils, in comparison to permanent-magnet arrays? What are drawbacks?

Answers

- higher fields ⇒ higher K, even for small λ_u
- cryogenic cooling required

8 Emittance and Energy Spread

The equilibrium emittance of an electron bunch in a storage ring occurs when factors increasing ε are compensated by those reducing ε.

- Which effect increases the horizontal emittance ε_x?
- Which effect decreases the horizontal emittance ε_x?
- Which effect increases the vertical emittance ε_y?
- Which effect decreases the vertical emittance ε_y?

Answers

- SR emission in dispersive regions
- radiation damping
- skew components ⇒ coupling between x and y; space charge / intra-beam scattering
- radiation damping

9 Fundamental Limits

The ESRF-EBF (European Synchrotron Radiation Facility – Extremely Brilliant Source) has a circumference of 843.977 m. Electrons with an energy of 6 GeV circulate around the ring. The horizontal and vertical emittances are 110 pm and 5 pm, respectively. How far is this away from the de Broglie emittance, i.e. the minimum emittance given by the uncertainty principle?

Answers

The de Broglie emittance is given by

$$\varepsilon_{dB} = \frac{\lambda_{dB}}{4\pi} \qquad \text{where} \qquad \lambda_{dB} = \frac{h}{p}$$

With $p = 6\,\mathrm{GeV}/c$, $\lambda_{dB} = 2 \times 10^{-16}\,\mathrm{m}$, and the de Broglie emittance $\varepsilon_{dB} = 1.6 \times 10^{-17}\,\mathrm{m}$ is still a factor 310^5 from the actual emittance in the vertical plane.