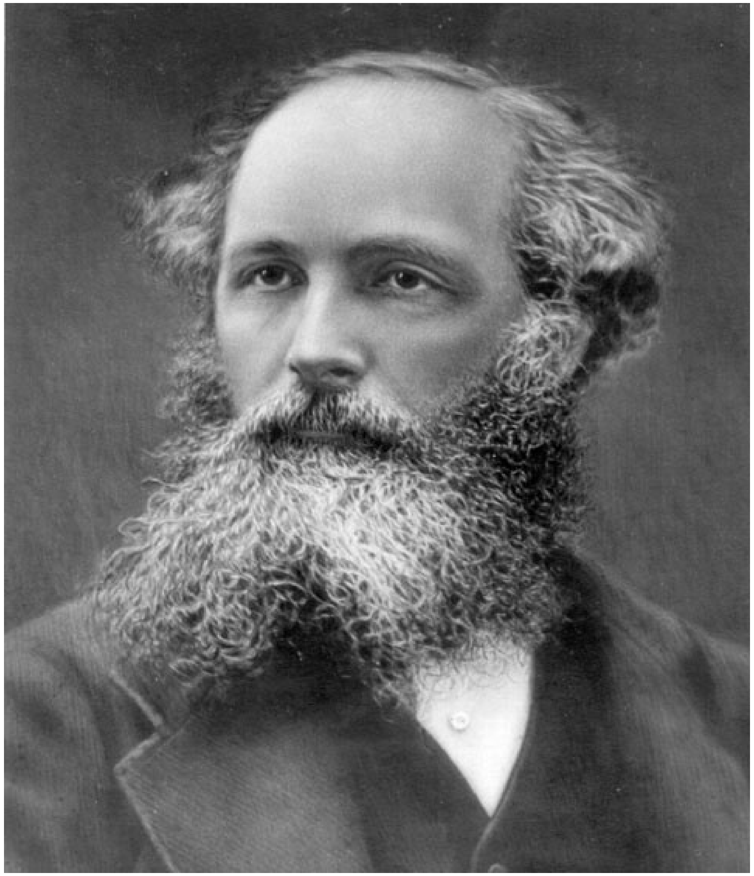
Review of Electromagnetism



This review is not meant to teach the subject, but to repeat and to refresh, at least partially, what you have learnt at university.

Maxwell's equations (in material)

$$\oint \vec{H}(\vec{r},t)\cdot d\vec{s} = \iint \vec{J}(\vec{r},t)\cdot d\vec{A} + \frac{d}{dt}\iint \vec{D}(\vec{r},t)\cdot d\vec{A}$$

$$\oint \vec{E}(\vec{r},t)\cdot d\vec{s} = -\frac{d}{dt}\iint \vec{B}(\vec{r},t)\cdot d\vec{A}$$

$$\oiint \vec{D}(\vec{r},t)\cdot d\vec{A} = \iiint \rho(\vec{r},t)\, dV$$

$$\oiint \vec{B}(\vec{r},t)\cdot d\vec{A} = 0$$

$\vec{E}, \vec{H}$ electric and magnetic field

$\vec{D}, \vec{B}$ electric displacement and magnetic induction

$\vec{J}$ electric current density

ρ electric charge density

$\iint \vec{J}(\vec{r},t)\cdot d\vec{A}$ stands for all currents going through the area A. It may consist of 3 parts

$$\vec{J}(\vec{r},t)=\vec{J}_c(\vec{r},t)+\vec{J}_{cv}(\vec{r},t)+\vec{J}_i(\vec{r},t)$$

$\vec{J}_c(\vec{r},t)=\kappa\vec{E}(\vec{r},t)$ conduction current (Ohm's law)

$\vec{J}_{cv}(\vec{r},t)=\rho(\vec{r},t)\vec{v}(\vec{r},t)$ convection current

$\vec{J}_i(\vec{r},t)$ impressed current

$\iiint \rho(\vec{r},t)\,dV$ stands for all charges in the volume V

Current and *charge may have different distributions*:
point, line, surface, volume

Maxwell's equations (in differential form)

With Stokes' theorem:

$$\oint \vec{E}\cdot d\vec{s} = \iint (\vec{\nabla}\times\vec{E})\cdot d\vec{A} = -\frac{d}{dt}\iint \vec{B}\cdot d\vec{A} = -\iint \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

$$\iint [\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t}]\cdot d\vec{A} = 0$$

since this is valid for any area:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (2)$$

correspondingly:

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad (1)$$

With Gauss' theorem:

$$\oint\!\!\oint \vec{D}\cdot d\vec{A} = \iiint \vec{\nabla}\cdot\vec{D}\,dV = \iiint \rho\,dV$$

$$\iiint [\vec{\nabla}\cdot\vec{D} - \rho]\,dV = 0$$

since this is valid for any volume: $$\vec{\nabla}\cdot\vec{D} = \rho \qquad (3)$$

correspondingly: $$\vec{\nabla}\cdot\vec{B} = 0 \qquad (4)$$

Time-harmonic fields

Time-harmonic fields can be written as complex quantities

$$\vec{E}(\vec{r},t)=\vec{E}_0(\vec{r})\cos(\omega t+\varphi)=\Re[\vec{E}_0(\vec{r})e^{i\varphi}e^{i\omega t}]=\Re[\tilde{\vec{E}}(\vec{r})e^{i\omega t}]$$

$\tilde{\vec{E}}(\vec{r})$ *is called phasor.*

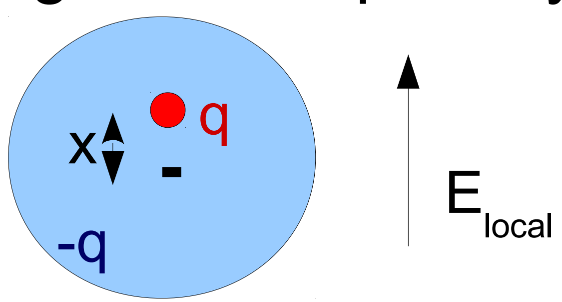
Advantages are:
- $\partial/\partial t \rightarrow i\omega$,
- phasors are vectors in a coordinate system rotating with ωt,
- $e^{i\omega t}$ cancels out in the equations

We will drop the tilde on following transparencies whenever the situation is sufficiently clear!

Fields in matter (time-harmonic case)

The effect of electric fields on matter can be described by a polarization field P, the effect of magnetic fields by a magnetization field M.

There are several electric reactions. E.g. a neutral atom changed to a dipole by a local field E_{local}



$$p_e = qx \quad \rightarrow \quad \vec{P} = n\,\vec{p}_e = \epsilon_0 \chi_e \vec{E}$$

n : *dipole density*

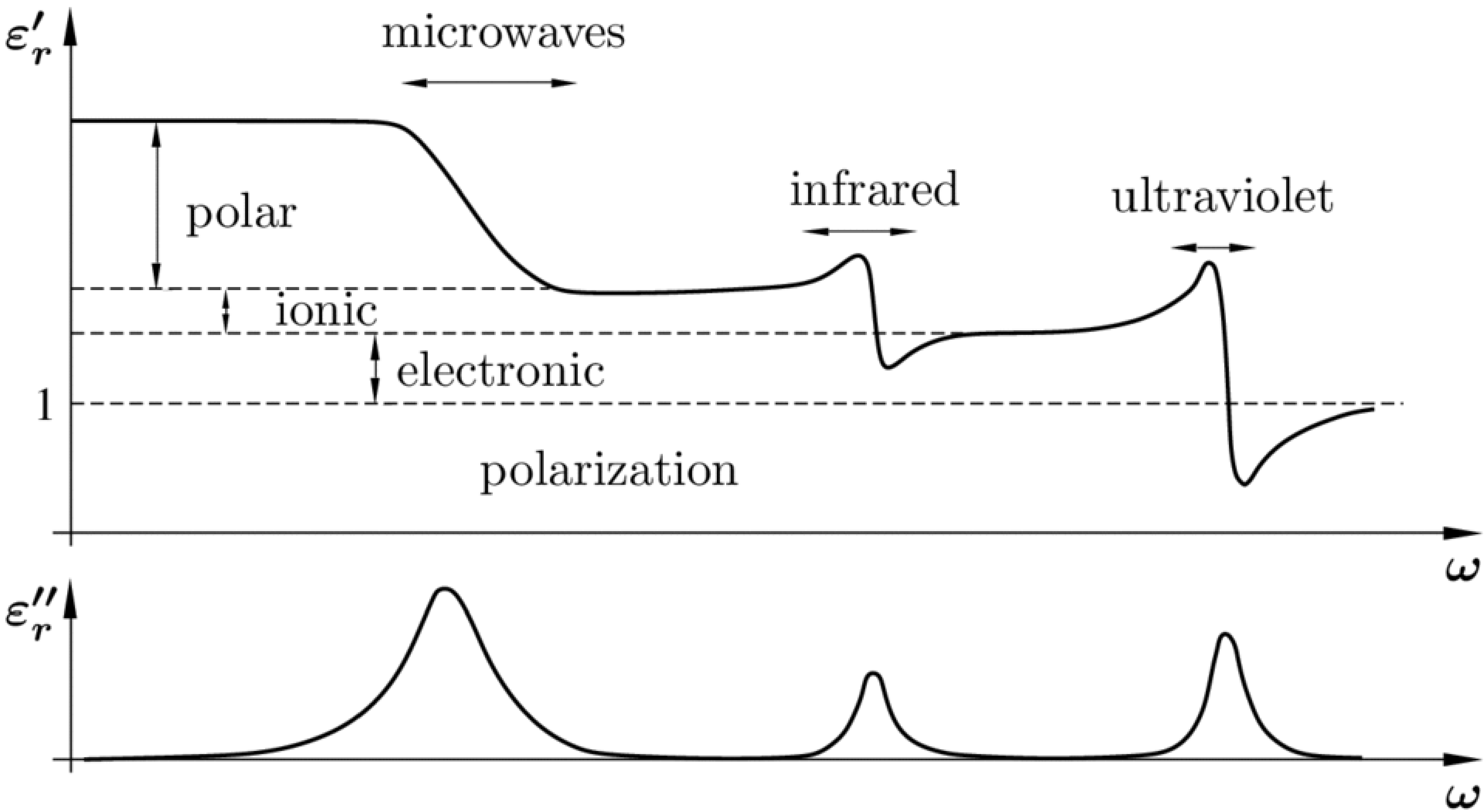
χ_e : *electric susceptibility*

Linear materials:

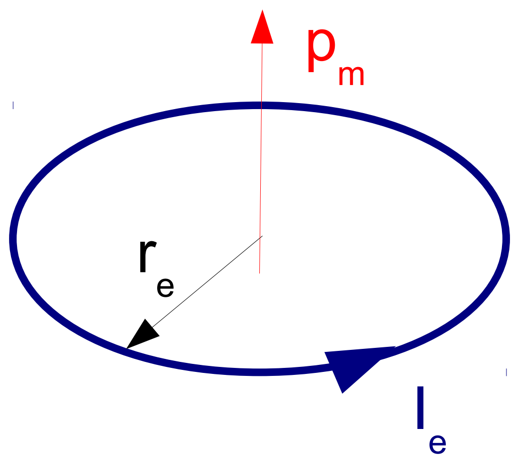
$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E} = \epsilon_r \epsilon_0 \vec{E} = \epsilon \vec{E}$$

$\epsilon_r = 1 + \chi_e$: *relative permittivity*

Dielectric behavior is a dynamic process, dependent on frequency ($\varepsilon_r = \varepsilon_r' - i\, \varepsilon_r''$, ε_r'' respresents the losses):



Magnetic reaction of material is due to particle spins (magnetic moments p_m). It can be described by means of magnetic dipoles, i.e. by circulating elementary currents:



$$p_m = \pi r_e^2 I_e \rightarrow \vec{M} = n \vec{p}_m = \chi_m \vec{H}$$

n : *dipole density*

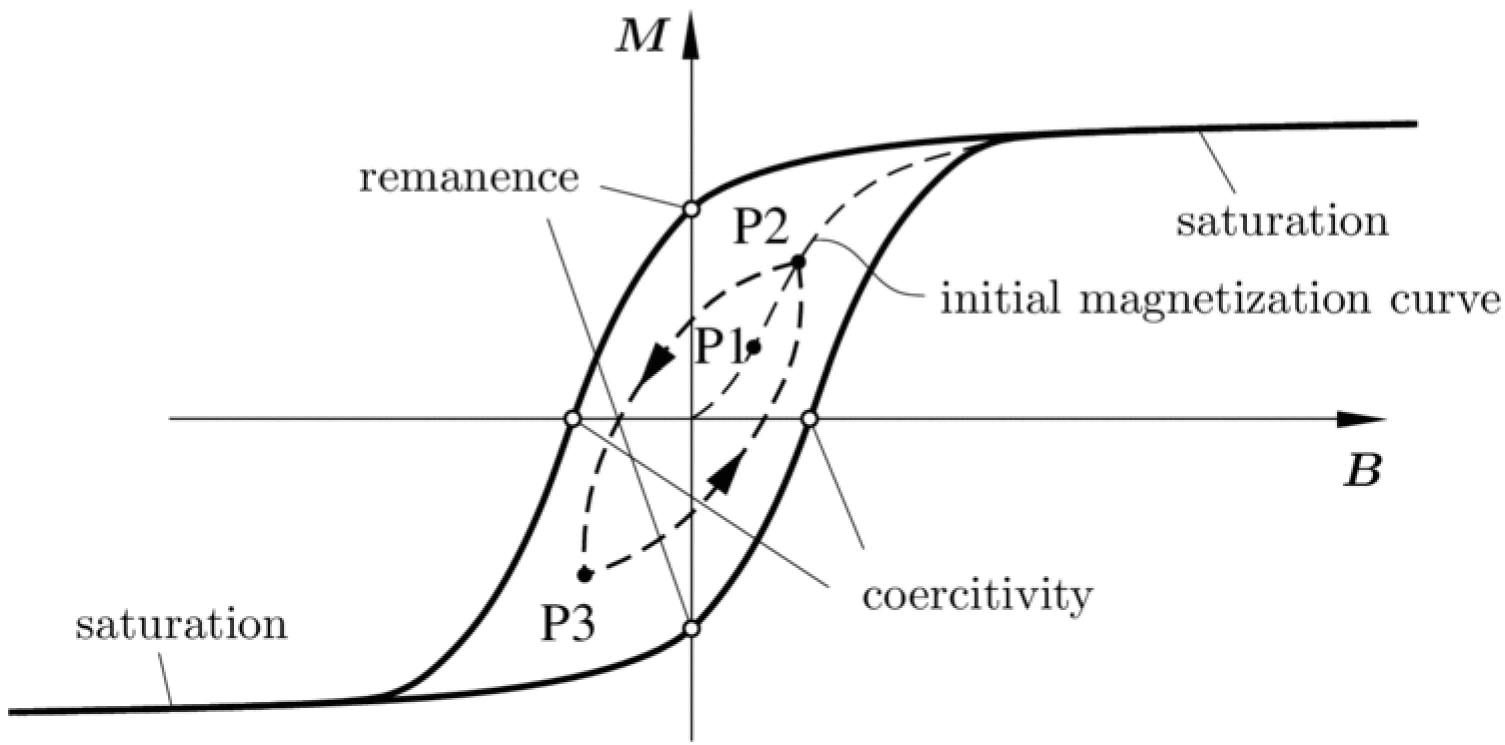
χ_m : *magnetic susceptibility*

Linear materials:

$$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M} = \mu_0 \vec{H} + \mu_0 \chi_m \vec{H} = \mu_r \mu_0 \vec{H} = \mu \vec{H}$$

$\mu_r = 1 + \chi_m$: *relative permeability*

For ferromagnetic materials the relation between the external field and the magnetization is non-linear and depends on the history of the material (hysteresis).



$$\vec{D}=\varepsilon_0\vec{E}+\varepsilon_0\chi\vec{P} \quad \text{and} \quad \vec{B}=\mu_0\vec{H}+\mu_0\vec{M}$$

take into account the reaction of the material due to fields E and H. The reaction is averaged over all atoms and/or molecules, i.e. over all elementary electric and magnetic dipoles.

In *many materials the relations* $\vec{P}=\vec{P}(\vec{E})$ and $\vec{M}=\vec{M}(\vec{H})$ *are linear* .

But in *general they are nonlinear , anisotropic , i.e. dependent on the direction of* $\vec{E}$ or $\vec{H}$, and *they are time* or *frequency dependent* .

They may also include losses .

There are losses due to radiation and interaction between electric and magnetic dipoles. Losses are responsible for the imaginary parts.

$$\epsilon=\epsilon'-i\epsilon''=\epsilon'(1-i\tan\delta_\epsilon)$$

$\tan\delta_\epsilon=\epsilon''/\epsilon',\qquad \delta_\epsilon$ electric loss angle

$$\mu=\mu'-i\mu''=\mu'(1-i\tan\delta_\mu)$$

$\tan\delta_\mu=\mu''/\mu',\qquad \delta_\mu$ magnetic loss angle

There are also losses due to collisions between free charges

$$\vec\nabla\times\vec H=\vec J+i\omega\epsilon\vec E=\kappa\vec E+i\omega\epsilon\vec E=i\omega\epsilon[1+\kappa/(i\omega\epsilon)]\vec E$$

$$\epsilon_c=\epsilon'-i\epsilon''=\epsilon[1-i\kappa/(\omega\epsilon)]$$

Most dielectrics: $\tan(\delta_\varepsilon)\ll 1,\ \tan(\delta_\mu)\approx 0$

Good conductors: $\kappa/\omega\varepsilon\gg 1\ \ (|\vec J|\gg|\partial\vec D/\partial t|)\ \rightarrow\ \varepsilon_c\approx\kappa/i\omega$

Boundary / continuity conditions

Maxwell's theory is a continuum theory. It requires continuous, double differentiable functions.

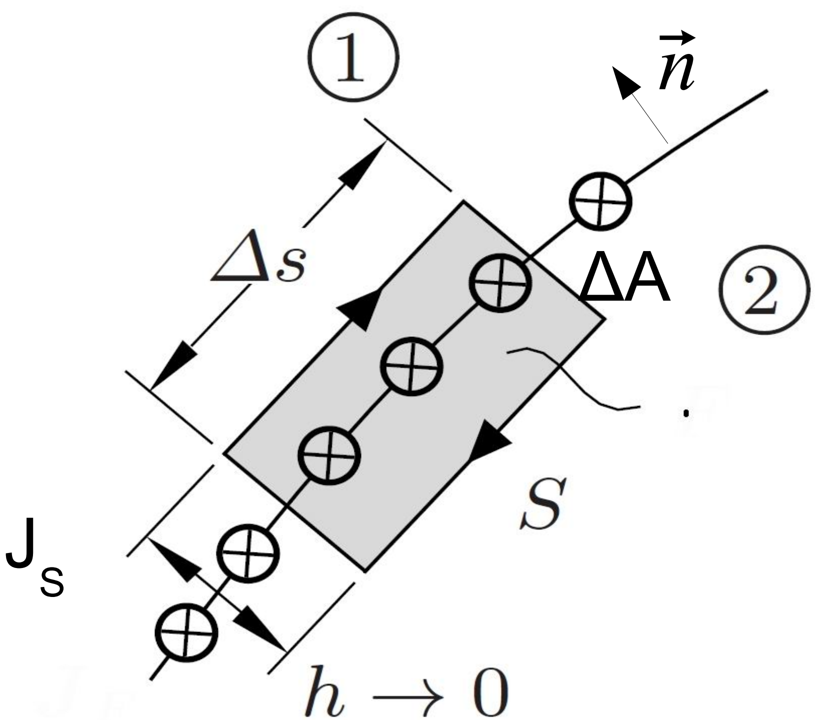
Solutions in different media have to be matched at the interface by boundary or continuity conditions.

Take Maxwell's equs. in integral form

$$\oint \vec{H}(\vec{r},t)\cdot d\vec{s} = \iint \vec{J}(\vec{r},t)\cdot d\vec{A} + \frac{d}{dt}\iint \vec{D}(\vec{r},t)\cdot d\vec{A}$$

$$\oint \vec{E}(\vec{r},t)\cdot d\vec{s} = -\frac{d}{dt}\iint \vec{B}(\vec{r},t)\cdot d\vec{A}$$

and make an intelligent choice for the integration area:



Δs is finite but small, such that the fields are constant, then

$$H_{t1}\Delta s - H_n h - H_{t2}\Delta s + H_n h =$$
$$= J_S \Delta s + \frac{\partial}{\partial t}\iint_{\Delta A} \vec{D}\cdot\Delta\vec{A}$$

for $h \to 0$ *it becomes*

$$H_{t1} - H_{t2} = J_S$$
$E_{t1} - E_{t2} = 0,$ correspondingly

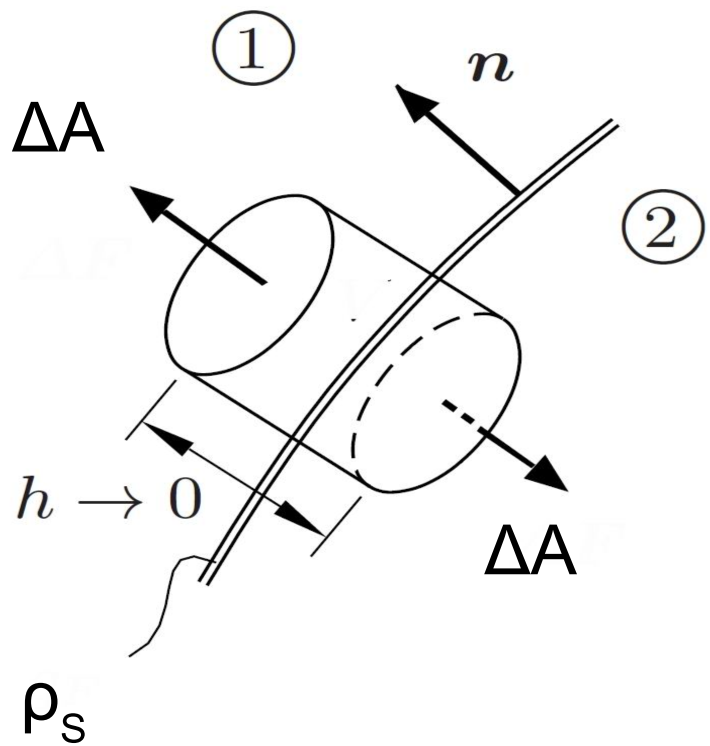
If medium 2 is perfectly electric conducting (pec) :

$$E_{t1} = 0, \quad H_{t1} = J_S$$

J_S is a surface current density.

An intelligent choice of the integration volume:

$$\oint\!\!\oint \vec{D}(\vec{r},t)\cdot d\vec{A} = \iiint \rho(\vec{r},t)\, dV$$
$$\oint\!\!\oint \vec{B}(\vec{r},t)\cdot d\vec{A} = 0$$



$$D_{n1}\Delta A - D_{n2}\Delta A + \iint_{\Delta A_{zyl}} \vec{D}\cdot d\vec{A} = \rho_S \Delta A$$

for $h \to 0$ *it becomes*

$$D_{n1} - D_{n2} = \rho_S$$
$$B_{n1} - B_{n2} = 0, \quad \text{correspondingly}$$

If medium 2 *is pec*: $D_{n1} = \rho_s, \quad B_{n1} = 0$

ρ_S is a surface charge density.

Application of Maxwell's equations

Electrostatic fields (H=0, δ/δt=0, ε=const.)

Maxwell's equations

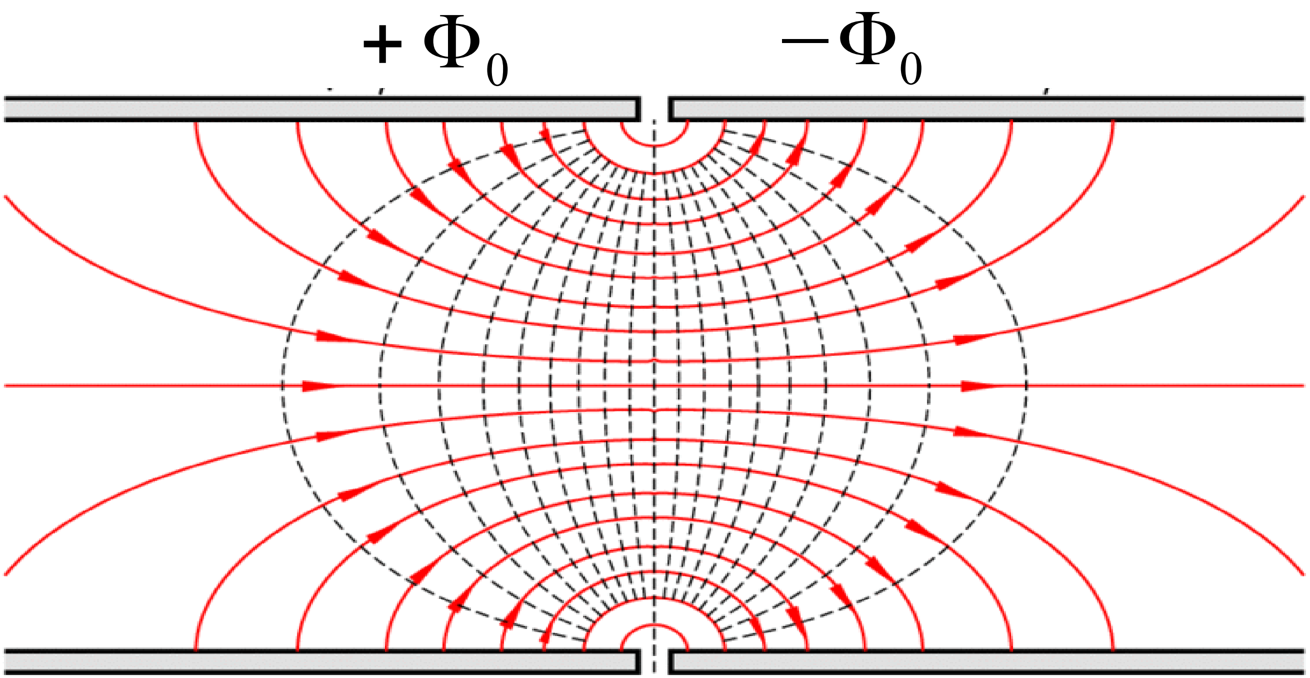
$$\vec{\nabla}\times\vec{E}=0 \rightarrow \vec{E}=-\vec{\nabla}\Phi \quad \text{since} \quad \vec{\nabla}\times\vec{\nabla}\Phi\equiv 0$$

$$\vec{\nabla}\cdot\vec{D}=\rho$$

Poisson equation:

$$\vec{\nabla}\cdot\vec{D}=\vec{\nabla}\cdot(\epsilon\vec{E})=\rho \rightarrow \vec{\nabla}^2\Phi=-\frac{\rho}{\epsilon} \qquad (1)$$

Example: Two round tubes forming an electrostatic lense



E-field pattern

(1) becomes circular symmetric Laplace equation

$$\vec{\nabla}^2 \Phi = \frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Phi}{\partial \rho} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \qquad (2)$$

Bernoulli ansatz

$$\Phi(\rho, z) = R(\rho) Z(z)$$

substituted in (2) and devided by RZ

$$\frac{1}{R}\frac{d^2 R}{d\rho^2} + \frac{1}{R\rho}\frac{dR}{d\rho} + \underbrace{\frac{1}{Z}\frac{d^2 Z}{dz^2}}_{k_z^2} = 0 \qquad (3)$$

Last term is independent of ρ and must be constant. It yields

$$\frac{d^2 Z}{dz^2} - k_z^2 Z = 0$$

with solutions

$$Z = \begin{cases} C_0 + D_0 z, & k_z = 0 \\ C e^{k_z z} + D e^{-k_z z}, & k_z \neq 0 \end{cases}$$

Condition at infinity

$$\Phi \;\; \textit{finite for} \;\; z=\pm\infty: \;\; C=D_0=0, \quad Z=\begin{cases} C_0, & k_z=0 \\ D\,\mathrm{e}^{-k_z|z|}, & k_z\neq 0 \end{cases}$$

The left over *equ.* (3) *is the Bessel differential equation*

$$\frac{d^2 R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}+k_z^2 R=0$$

with solutions

$$R=\begin{cases} A_0+B_0\ln(\rho/\rho_0), & k_z=0 \\ A\,J_0(k_z\rho)+B\,N_0(k_z\rho), & k_z\neq 0 \end{cases}$$

Condition at $\rho \to 0$

$$\Phi \;\; \textit{finite for} \;\; \rho=0: \qquad B_0=B=0$$

$$\Phi=A_0 C_0+A D\,J_0(k_z\rho)\,\mathrm{e}^{-k_z|z|}$$ for all k_z

Boundary conditions

$$\Phi = \begin{cases} -\Phi_0 & for \quad \rho = a\,,\ z > 0 \\ +\Phi_0 & for \quad \rho = a\,,\ z < 0 \end{cases} \qquad (4)$$

$$A_0 C_0 = -sign(z)\Phi_0, \quad J_0(k_z a) = 0 \rightarrow k_{zn} a = j_{0n}$$

Using above conditions Φ becomes

$$\Phi = sign(z)[-\Phi_0 + \sum_{n=1}^{\infty} A_n J_0(j_{0n}\frac{\rho}{a}) e^{-j_{0n}|z|/a}] \qquad (5)$$

and due to symmetry (4), Φ(z=0)=0, (5) becomes

$$\Phi_0 = \sum_{n=1}^{\infty} A_n J_0(j_{0n}\frac{\rho}{a}) \qquad (6)$$

To calculate the coefficients A_n we use a Fourier-Bessel expansion.

Multiplication of (6) with $\rho J_0(j_{0m}\rho/a)$ and integration over ρ

$$\Phi_0 \underbrace{\int_0^a J_0\left(j_{0m}\frac{\rho}{a}\right)\rho\, d\rho}_{\frac{a^2}{j_{0m}} J_1(j_{0m})} = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^a J_0\left(j_{0n}\frac{\rho}{a}\right) J_0\left(j_{0m}\frac{\rho}{a}\right)\rho\, d\rho}_{\delta_m^n \frac{a^2}{2} J_1^2(j_{0m})}$$

gives the final result

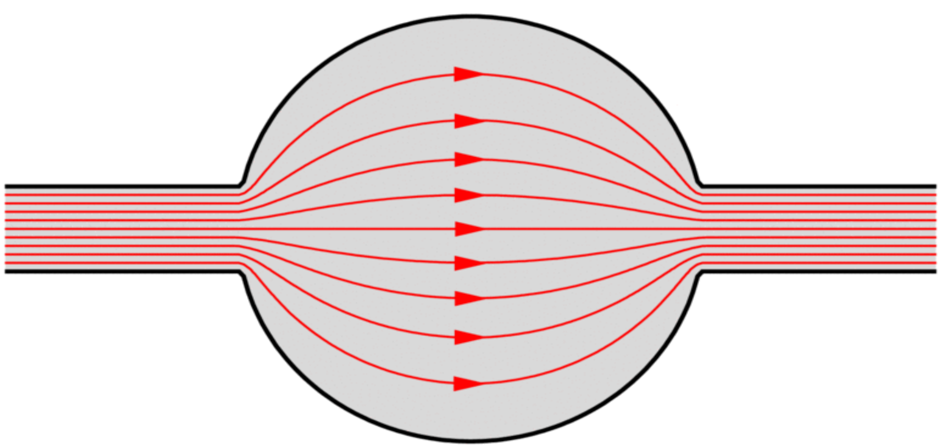
$$\Phi = sign(z)\,\Phi_0\left[-1 + 2\sum_{n=1}^{\infty} \frac{J_0\left(j_{0n}\frac{\rho}{a}\right)}{j_{0n} J_1(j_{0n})} e^{-j_{0n}|z|/a}\right]$$

Stationary currents (δ/δt=0, κ=const.)

Maxwell's equations: $\vec{\nabla}\times\vec{H}=J\,,\quad \vec{\nabla}\times\vec{E}=0 \rightarrow \vec{E}=-\vec{\nabla}\Phi$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H})=0=\vec{\nabla}\cdot\vec{J}=\vec{\nabla}\cdot(\kappa\vec{E}) \rightarrow \vec{\nabla}^2\Phi=0$$

similar to electrostatics but different boundary / continuity conditions: $J_n=\kappa\vec{E}_n=-\kappa d\Phi/dn=0.$



J-field lines

Magnetostatic fields

(E=0, δ/δt=0, μ=const.)

Maxwell's equations

$$\vec{\nabla}\cdot\vec{B}=0 \;\rightarrow\; \vec{B}=\vec{\nabla}\times\vec{A} \qquad \text{since } \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=0$$

$$\vec{\nabla}\times\vec{B}=\mu\vec{J} \;\rightarrow\; \vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\vec{\nabla}^2\vec{A}=\mu\vec{J}$$

Vectorpotential $\vec{A}$ is not fully determined. Substitution $\vec{A}\rightarrow\vec{A}+\vec{\nabla}\psi$ (gauge transformation) does not change $\vec{B}=\vec{\nabla}\times\vec{A}$. Gauge $\vec{\nabla}\cdot\vec{A}=0$ yields vectorial Poisson equ.

$$\vec{\nabla}^2\vec{A}=-\mu\vec{J}$$

The solution of which (see appendix A1) is

$$\vec{A}(\vec{r})=\frac{\mu}{4\pi}\iiint\frac{\vec{J}(\vec{r}\,')}{R}dV'$$

Quasi-stationary fields

$$|\vec{J}| = \kappa |\vec{E}| \gg \left|\frac{d\vec{D}}{dt}\right| = \omega \epsilon |\vec{E}| \quad \rightarrow \quad \frac{\epsilon}{\kappa} = T_r \ll \frac{1}{\omega} = \frac{T}{2\pi}$$

T_r is called relaxation time

Maxwell's equations

$$\vec{\nabla} \times \vec{B} = \mu \vec{J}, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \vec{\nabla} \cdot \vec{D} = \rho, \quad \vec{\nabla} \cdot \vec{B} = 0$$

Potentials

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \rightarrow \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \frac{\partial \vec{A}}{\partial t}, \quad \vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0 \quad \rightarrow \quad \vec{E} = -\vec{\nabla} \Phi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{\nabla}\times\vec{B}=\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\vec{\nabla}^2\vec{A}=$$

$$=\mu\vec{J}=\mu\kappa\vec{E}=-\vec{\nabla}(\mu\kappa\Phi)-\mu\kappa\frac{\partial\vec{A}}{\partial t}$$

$\vec{A}$ and Φ *are not fully determined. Substituting*

$$\vec{A}\ \rightarrow\ \vec{A}+\vec{\nabla}\psi, \qquad \Phi\ \rightarrow\ \Phi-\frac{\partial\psi}{\partial t}$$

does not change $\vec{B}=\vec{\nabla}\times\vec{A}, \quad \vec{E}=-\vec{\nabla}\Phi-\frac{\partial\vec{A}}{\partial t}$

So, use gauge: $\vec{\nabla}\cdot\vec{A}=-\mu\kappa\Phi$

$$\rightarrow\quad \vec{\nabla}^2\vec{A}-\mu\kappa\frac{\partial\vec{A}}{\partial t}=0$$

vectorial diffusion equation

Poynting's theorem

(ε, μ, κ=const. and real, J=κE, full set of Maxwell's equations)

If fields move a charge $\rho\, dV$ by a distance δs in the interval δt, the work done by the fields (dissipated power) is

$$d\frac{\delta W}{\delta t}=d\vec{f}\cdot\frac{\delta\vec{s}}{\delta t}=\rho\, dV\left(\vec{E}+\vec{v}\times\vec{B}\right)\cdot\vec{v}=\vec{E}\cdot\rho\,\vec{v}\,dV=\vec{E}\cdot\vec{J}\,dV$$

$$\textit{Express } \vec{E}\cdot\vec{J} \textit{ with the aid of Maxwell's equations}$$

$$\vec{E}\cdot \quad \vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$$

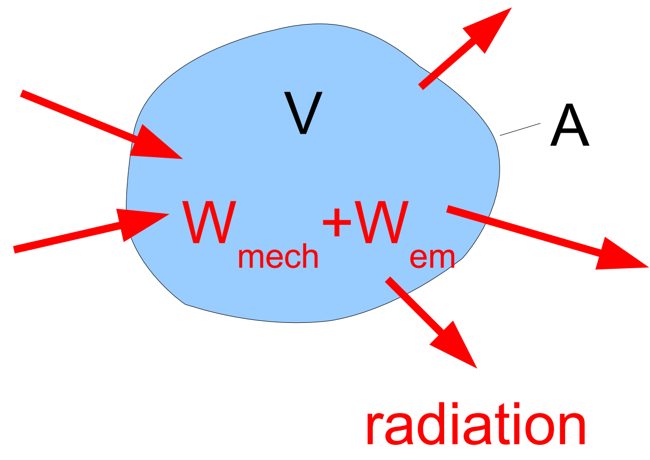
$$-\vec{H}\cdot \quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\rightarrow \quad -\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right)=\vec{E}\cdot\vec{J}+\frac{\partial}{\partial t}\left[\frac{1}{2}\vec{E}\cdot\vec{D}+\frac{1}{2}\vec{H}\cdot\vec{B}\right]$$

We get Poynting's theorem after integration over V and application of Gauss' law:

$$-\oiint (\vec{E}\times\vec{H})\cdot d\vec{A} =$$

$$= \iiint \vec{E}\cdot\vec{J}\,dV + \frac{\partial}{\partial t}\iiint \left(\frac{1}{2}\vec{E}\cdot\vec{D} + \frac{1}{2}\vec{H}\cdot\vec{B}\right) dV$$



Poynting vector (radiation flux) $\vec{S}=\vec{E}\times\vec{H}$

dissipated power density $p_d=\vec{E}\cdot\vec{J}$

electric energy density $w_e=(1/2)\vec{E}\cdot\vec{D}$

magnetic energy density $w_m=(1/2)\vec{H}\cdot\vec{B}$

Energy radiated into the volume V equals the dissipation plus the increase of stored electromagnetic energy in V.

Poynting's theorem for time-harmonic fields (see appendix 2)

For time-harmonic fields it is e.g. $\vec{E} = \Re[\tilde{\vec{E}} \, e^{i\omega t}]$

and Poyting's theorem becomes

$$-\oiint \vec{S}_c \cdot d\vec{A} = \iiint \bar{p}_d \, dV + i 2\omega \iiint (\bar{w}_m - \bar{w}_e) \, dV \qquad (1)$$

complex, time-averaged radiation flux: $\vec{S}_c = \frac{1}{2} \tilde{\vec{E}} \times \tilde{\vec{H}}^*$

time-averaged electric and magnetic energy density $\bar{w}_e = \frac{1}{4} \tilde{\vec{E}} \cdot \tilde{\vec{D}}^*, \quad \bar{w}_m = \frac{1}{4} \tilde{\vec{H}} \cdot \tilde{\vec{B}}^*$

time-averaged dissipated power density $\bar{p}_d = \frac{1}{2} \tilde{\vec{E}} \cdot \tilde{\vec{J}}^*$

Real part of (1) gives the time-averaged active power (dissipation) and the imaginary part the reactive power.

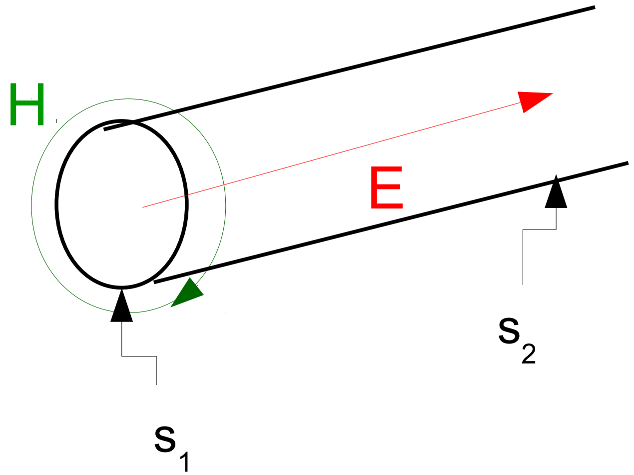
In good conductors is $W_m \gg W_e$ ($|E| \ll |H|$)

$$-\oiint \vec{S}_c \cdot d\vec{A} = \bar{P}_c = \bar{P}_d + i\,2\,\omega\,\bar{W}_m$$

This allows to calculate the resistance and internal inductance of a conductor. We define

$$I^* = \oint \tilde{\vec{H}}^* \cdot d\vec{s}$$

$$U = \int_{s_1}^{s_2} \tilde{\vec{E}} \cdot d\vec{s} = I\left(R + i\,\omega\,L_i\right)$$



and obtain

$$\bar{P}_c = \frac{1}{2} U I^* = \frac{1}{2} |I^2| \left(R + i\,\omega\,L_i\right) = \bar{P}_d + i\,2\,\omega\,\bar{W}_m$$

Electromagnetic waves

(ε, μ=const., ρ=J=0)

The simplest electromagnetic wave is a plane wave.
It depends only on one space variable (direction of propagation) and on the time.

$$\vec{E}=\vec{E}(z,t), \quad \vec{H}=\vec{H}(z,t)$$

First two Maxwell's eqs. $\vec{\nabla}\times\vec{H}=\epsilon\frac{\partial\vec{E}}{\partial t}, \quad \vec{\nabla}\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t}$

give two sets of uncoupled equations:

$$-\frac{\partial H_y}{\partial z}=\epsilon\frac{\partial E_x}{\partial t} \qquad \frac{\partial E_x}{\partial z}=-\mu\frac{\partial H_y}{\partial t}$$

$$\frac{\partial H_x}{\partial z}=\epsilon\frac{\partial E_y}{\partial t} \qquad -\frac{\partial E_y}{\partial z}=-\mu\frac{\partial H_x}{\partial t}$$

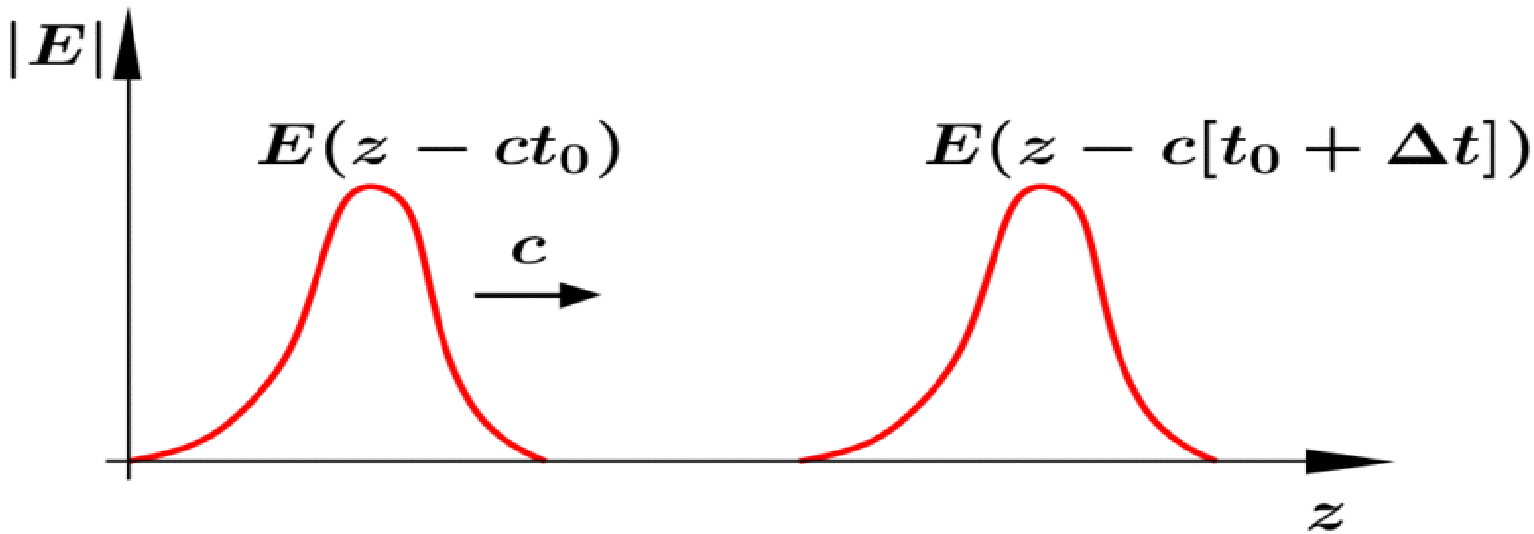
From the red set e.g. follows the wave equation

$$\frac{\partial^2 E_x}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} = 0, \qquad c = \frac{1}{\sqrt{\mu\epsilon}}$$

with d'Alembert's solution

$$E_x = f(z-ct) + g(z+ct) = E_x^+ + E_x^-$$

$$ZH_y = f(z-ct) - g(z+ct) = ZH_y^+ - ZH_y^-, \qquad Z = \sqrt{\frac{\mu}{\epsilon}}$$



Similar solutions follow from the blue set with E_y and H_x.

velocity of light : $c = \frac{1}{\sqrt{\mu \epsilon}}$

wave impedance : $Z = \sqrt{\frac{\mu}{\epsilon}}$

$\approx 377\,\Omega$ in *free space*

field properties :

$\vec{E} \perp \vec{H}$

$\vec{S} = \vec{E} \times \vec{H}$ → *direction of propagation*

$\vec{E}, \vec{H}$ are $\perp$ to *direction of propagation*

$E^{+}/H^{+} = -E^{-}/H^{-} = Z$

Time-harmonic plane wave

$$\left(\frac{\partial}{\partial t}=i\omega\,,\ \epsilon_r=\epsilon_r{}'-i\epsilon_r{}''\right)$$

Wave equation becomes Helmholtz equation:

$$\frac{\partial^2 E_x}{\partial z^2}+k^2 E_x=0\,,\qquad k=\omega\sqrt{\mu\epsilon}$$

$$E_x=A\,e^{i(\omega t-kz)}+B\,e^{i(\omega t+kz)}\qquad=E_x^{+}+E_x^{-}$$

$$ZH_y=A\,e^{i(\omega t-kz)}-B\,e^{i(\omega t+kz)}=ZH_y^{+}-ZH_y^{-}$$

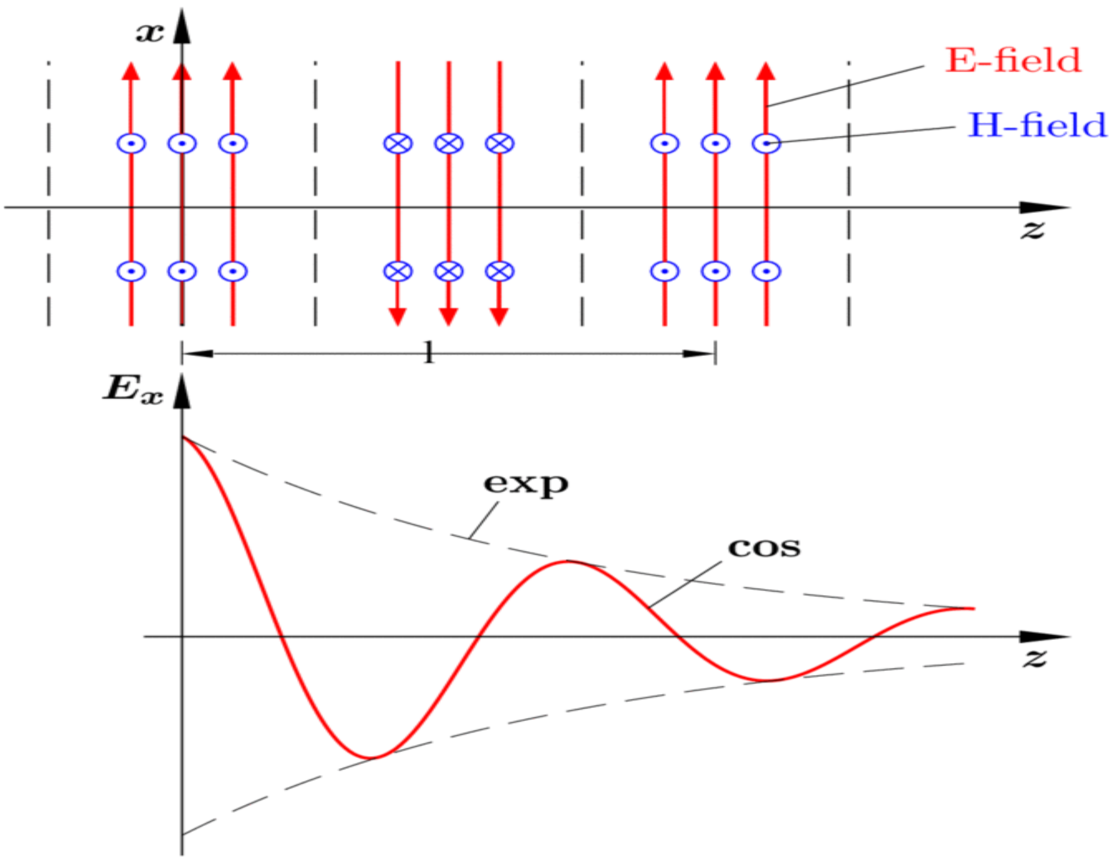
loss-free material: $k=\omega/c=2\pi/\lambda$

lossy dielectric: $k=\omega\sqrt{\mu\epsilon_r\epsilon_0}=\beta-i\alpha$

α: *attenuation constant*, β: *phase constant*

$$\frac{\beta}{k_0}=\sqrt{\frac{\epsilon_r'}{2}+\frac{\epsilon_r'}{2}\sqrt{1+\left(\frac{\epsilon_r''}{\epsilon_r'}\right)^2}}\,,\quad \frac{\alpha}{k_0}=\sqrt{-\frac{\epsilon_r'}{2}+\frac{\epsilon_r'}{2}\sqrt{1+\left(\frac{\epsilon_r''}{\epsilon_r'}\right)^2}}$$

real physical field: $E_x^+=\Re A e^{i(\omega t-kz)}=A\cos(\omega t-\beta z)e^{-\alpha z}$



Low-loss dielectrics: $\varepsilon_r'' \ll \varepsilon_r'$

$$\beta \approx \sqrt{\epsilon_r'}\, k_0, \quad \alpha \approx \frac{1}{2}\frac{\epsilon_r''}{\sqrt{\epsilon_r'}} k_0, \quad Z \approx \frac{Z_0}{\sqrt{\epsilon_r'}}\left(1 + \frac{i}{2}\frac{\epsilon_r''}{\epsilon_r'}\right)$$

Example: Polyamide (nylon), $\kappa = 10^{-8}\ \Omega^{-1}\mathrm{m}^{-1}$, $\varepsilon_r = 3$, $f = 10\mathrm{MHz}$

11% attenuation in 100km, arc $Z \approx 10^{-4}\,°$

Very good conductors (metallic): $\varepsilon_r'' \approx -i\kappa/\omega \gg \varepsilon_r'$

$$\beta \approx \alpha \approx \sqrt{\frac{\omega\mu\kappa}{2}} = \frac{1}{\delta_S}, \quad Z \approx (1+i)\frac{\alpha}{\kappa}, \quad arc\, Z = 45\,°$$

Skin depth ($z = \delta_s$):

$$e^{-\alpha\delta_S} = \frac{1}{e} \rightarrow \alpha\delta_S = 1$$

In general, β is a function of ω and is called dispersion relation. Developing β around ω_0

$$\beta(\omega)=\beta(\omega_0)+\left(\frac{d\beta}{d\omega}\right)_{\omega_0} d\omega+O[(d\omega)^2]$$

Phase velocity

$$\phi=\omega t\mp\beta z=const. \quad \rightarrow \quad \frac{d\phi}{dt}=\omega\mp\beta\frac{dz}{dt}=\omega\mp\beta v_{ph}=0$$

$$v_{ph}=\pm\frac{\omega}{\beta(\omega_0)}$$

v_{ph} has no physical importance. Monochromatic waves carry no information.

Example: Water waves at shore

Group velocity (velocity with which a signal propagates)

As an example take two plane waves with ω_1 and ω_2

$$\omega_1 = \omega_0 + \delta\omega, \qquad \omega_2 = \omega_0 - \delta\omega$$

$$\beta_1 = \beta_0 + \delta\beta, \qquad \beta_2 = \beta_0 - \delta\beta$$

$$\Re\left[e^{i(\omega_1 t - \beta_1 z)} + e^{i(\omega_2 t - \beta_2 z)}\right] = 2\cos(\delta\omega t - \delta\beta z)\cos(\omega_0 t - \beta_0 z)$$

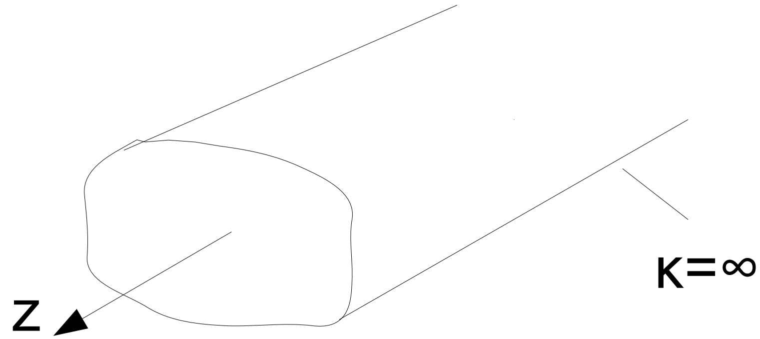
$$v_g = \frac{\delta\omega}{\delta\beta} \rightarrow v_g = \left(\frac{d\omega}{d\beta}\right)_{\omega_0}$$

v_g is the velocity with which the envelope propagates.
Signals with small bandwidth $2\delta\omega$ propagate with v_g.

Large bandwidth signals require higher order terms $O((\delta\omega)^2)$.

Energy velocity, see appendix 3.

Cylindrical, ideal conducting waveguides



Substituting one of the 2 first Maxwell's equ. into the other gives a 2nd order diff. equ., which requires 2 independent functions. The 3d and 4th equ. are additional conditions.
These conditions and the required independent functions are fulfilled by

$$\vec{\nabla}\cdot\vec{E}=0 \quad \rightarrow \quad \vec{E}^{TE}=\vec{\nabla}\times\vec{A}^{TE}, \quad \vec{A}^{TE}=A^{TE}\vec{e}_z, \quad TE-waves$$

$$\vec{\nabla}\cdot\vec{H}=0 \quad \rightarrow \quad \vec{H}^{TM}=\vec{\nabla}\times\vec{A}^{TM}, \quad \vec{A}^{TM}=A^{TM}\vec{e}_z, \quad TM-waves$$

With the vector potentials A one gets e.g. for TE-waves

$$\vec{\nabla}\times\vec{H}=\epsilon\frac{\partial\vec{E}}{\partial t}=\epsilon\frac{\partial}{\partial t}\vec{\nabla}\times\vec{A} \quad\rightarrow\quad \vec{\nabla}\times(\vec{H}-\epsilon\frac{\partial\vec{A}}{\partial t})$$

$$\rightarrow \quad \vec{H}=\vec{\nabla}\Phi+\epsilon\frac{\partial\vec{A}}{\partial t}$$

and from Maxwell's 2nd equ.

$$\vec{\nabla}\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t} \quad\rightarrow$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\vec{\nabla}^2\vec{A}=-\vec{\nabla}(\mu\frac{\partial\Phi}{\partial t})-\mu\epsilon\frac{\partial^2\vec{A}}{\partial t^2}$$

$\vec{A},\Phi$ are not fully determined. Substituting

$$\vec{A}\rightarrow\vec{A}+\vec{\nabla}\psi, \qquad \Phi\rightarrow\Phi-\epsilon\partial\psi/\partial t$$

yields the same $\vec{E}$, $\vec{H}$.

One can make a gauge-transformation and choose e.g. the Lorenz gauge

$$\vec{\nabla}\cdot\vec{A}=-\mu\frac{\partial\Phi}{\partial t}$$

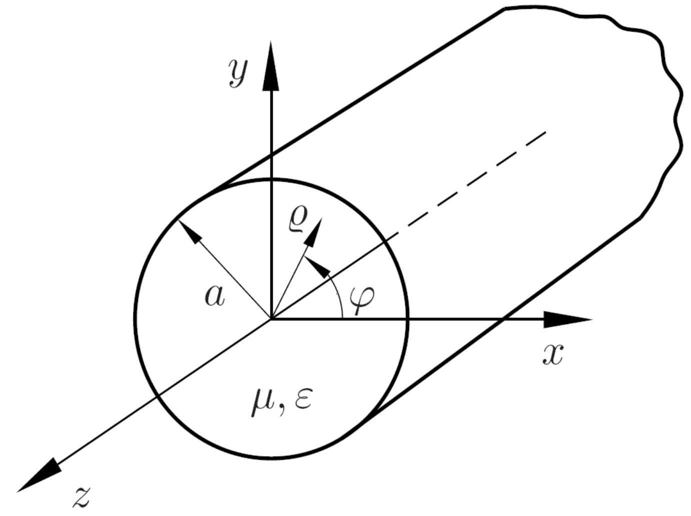
which results in a vectorial wave equ.

$$\vec{\nabla}^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=0$$

Similarily, we proceed for the TM-case and obtain the same equ.. Since A has only a cartesian component, the vectorial wave equ. becomes a scalar one and in case of time-harmonic fields a scalar Helmholtz equ.

$$\vec{\nabla}^2 A^p+k^2 A^p=0,\quad k=\frac{\omega}{c}=\omega\sqrt{\mu\epsilon},\quad p=\begin{Bmatrix}TE\\TM\end{Bmatrix}$$

Circular waveguide



Helmholtz equ. for circular cylinder coordinates:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 A}{\partial\varphi^2}+\frac{\partial^2 A}{\partial z^2}+k^2 A=0 \qquad (1)$$

Bernoulli ansatz: $A=R(\rho)\Phi(\varphi)Z(z)$

Substituted in (1) and devision by RΦZ

$$\frac{1}{\rho R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)+\frac{1}{\rho^2\Phi}\frac{\partial^2\Phi}{\partial\varphi^2}+\underbrace{\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}}_{-k_z^2}+k^2=0 \qquad (2)$$

$$\frac{d^2 Z}{dz^2} + k_z^2 Z = 0 \rightarrow Z = C_1 e^{-ik_z z} + C_2 e^{ik_z z} \rightarrow C_1 e^{-ik_z z}$$

for waves propagating in +z-direction

(2) becomes with k_z

$$\frac{\rho}{R}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial R}{\partial \rho}\right) + \underbrace{\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \varphi^2}}_{-k_\mu^2} + \rho^2 \left(k^2 - k_z^2\right) = 0 \qquad (3)$$

$$\frac{d^2 \Phi}{d\varphi^2} + k_\mu^2 \Phi = 0 \rightarrow \Phi = C_3 \cos(k_\mu \varphi) + C_4 \sin(k_\mu \varphi)$$

$$\rightarrow \Phi = C_3 \cos(m \varphi)$$

because of 2π-periodicity and free choice of origin

With m and k_z (3) becomes Bessel's equ.

$$\frac{\partial^2 R}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial R}{\partial \rho}+[k_c^2-\frac{m^2}{\rho^2}]R=0, \qquad k_c=\sqrt{k^2-k_z^2}$$

$$R=C_5 J_m(k_c\rho)+C_6 N_m(k_c\rho) \quad \rightarrow \quad R=C_5 J_m(k_c\rho)$$

because Neumann's function is infinite at ρ=0

Vector potential:

$$A=C_m\cos(m\varphi)J_m(k_c\rho)\mathrm{e}^{-ik_z z}$$

TE−waves: $\vec{E}=\vec{\nabla}\times(A\vec{e}_z)$

$$E_\varphi=-\partial A/\partial\rho \sim J_m'(k_c\rho)$$

$$E_\varphi(\rho=a)=0 \quad \rightarrow \quad k_{cmn}a=j'_{mn}$$

j'_{mn}: n^{th} non vanishing zero of J'_m

$$E_\rho = \frac{1}{\rho}\frac{\partial A}{\partial \phi} = -\frac{m}{\rho} C_{mn} \sin(m\varphi) J_m\left(j'_{mn}\frac{\rho}{a}\right) \mathrm{e}^{-ik_z z}$$

$$E_\varphi = -\frac{\partial A}{\partial \rho} = -\frac{j'_{mn}}{a} C_{mn} \cos(m\varphi) J'_m\left(j'_{mn}\frac{\rho}{a}\right) \mathrm{e}^{-ik_z z}$$

$$\vec{\nabla} \times \vec{E} = -i\omega\mu \vec{H}:$$

$$H_\rho = \frac{k_z}{\omega\mu}\frac{j'_{mn}}{a} C_{mn} \cos(m\varphi) J'_m\left(j'_{mn}\frac{\rho}{a}\right) \mathrm{e}^{-ik_z z}$$

$$H_\varphi = -\frac{k_z}{\omega\mu}\frac{m}{\rho} C_{mn} \sin(m\varphi) J_m\left(j'_{mn}\frac{\rho}{a}\right) \mathrm{e}^{-ik_z z}$$

$$H_z = \frac{-1}{i\omega\mu}\left(\frac{j'_{mn}}{a}\right)^2 C_{mn} \cos(m\varphi) J_m\left(j'_{mn}\frac{\rho}{a}\right) \mathrm{e}^{-ik_z z}$$

$TM-waves: \quad \vec{H}=\vec{\nabla}\times(A\vec{e}_z), \quad \vec{\nabla}\times\vec{H}=i\omega\epsilon\vec{E}$

$$E_z=\frac{k_c^2}{i\omega\epsilon}A \sim J_m(k_c\rho), \qquad E_z(\rho=a)=0 \rightarrow k_{cmn}a=j_{mn}$$

$$H_\rho=-\frac{m}{\rho}D_{mn}\sin(m\varphi)J_m(j_{mn}\frac{\rho}{a})e^{-ik_z z}$$

$$H_\varphi=-\frac{j_{mn}}{a}D_{mn}\cos(m\varphi)J'_m(j_{mn}\frac{\rho}{a})e^{-ik_z z}, \qquad H_z=0$$

$$E_\rho=-\frac{k_z}{\omega\epsilon}\frac{j_{mn}}{a}D_{mn}\cos(m\varphi)J'_m(j_{mn}\frac{\rho}{a})e^{-ik_z z}$$

$$E_\varphi=\frac{k_z}{\omega\epsilon}\frac{m}{\rho}D_{mn}\sin(m\varphi)J_m(j_{mn}\frac{\rho}{a})e^{-ik_z z}$$

$$E_z=\frac{1}{i\omega\epsilon}\left(\frac{j_{mn}}{a}\right)^2 D_{mn}\cos(m\varphi)J_m(j_{mn}\frac{\rho}{a})e^{-ik_z z}$$

The ratio of the transverse field components is the field (wave) impedance

$$Z_F = \frac{E_\rho}{H_\varphi} = -\frac{E_\varphi}{H_\rho} = \begin{cases} Z_F^{TE} = \dfrac{\omega\mu}{k_z} \\ Z_F^{TM} = \dfrac{k_z}{\omega\epsilon} \end{cases}$$

The dependence of the propagation constant k_z on frequency is the dispersion relation

$$k_{cmn}^2 = k^2 - k_{zmn}^2 \quad \rightarrow \quad k_{zmn} = \sqrt{k^2 - k_{cmn}^2}$$

$$k_{zmn} = \begin{cases} \text{real} & & k > k_{cmn} & \textit{propagation} \\ 0 & \textit{for} & k = k_{cmn} & \\ \text{imaginary} & & k < k_{cmn} & \textit{attenuation} \end{cases}$$

critical wavenumber : $k_{cmn} = \begin{Bmatrix} j_{mn}'/a & \text{for} & TE \\ j_{mn}/a & & TM \end{Bmatrix}$

cutoff frequency : $f_{cmn} = c\, k_{cmn} / 2\pi$

cutoff wavelength : $\lambda_{cmn} = 2\pi / k_{cmn}$

guide wavelength : $\lambda_{zmn} = 2\pi / k_{zmn} = \dfrac{\lambda}{\sqrt{1-(\lambda/\lambda_{cmn})^2}}$

free space wavelength λ

energy flux density $S_{cz} = \frac{1}{2}(\vec{E} \times \vec{H}^*)_z = \frac{1}{2} Z_F (|H_\rho|^2 + |H_\varphi|^2)$

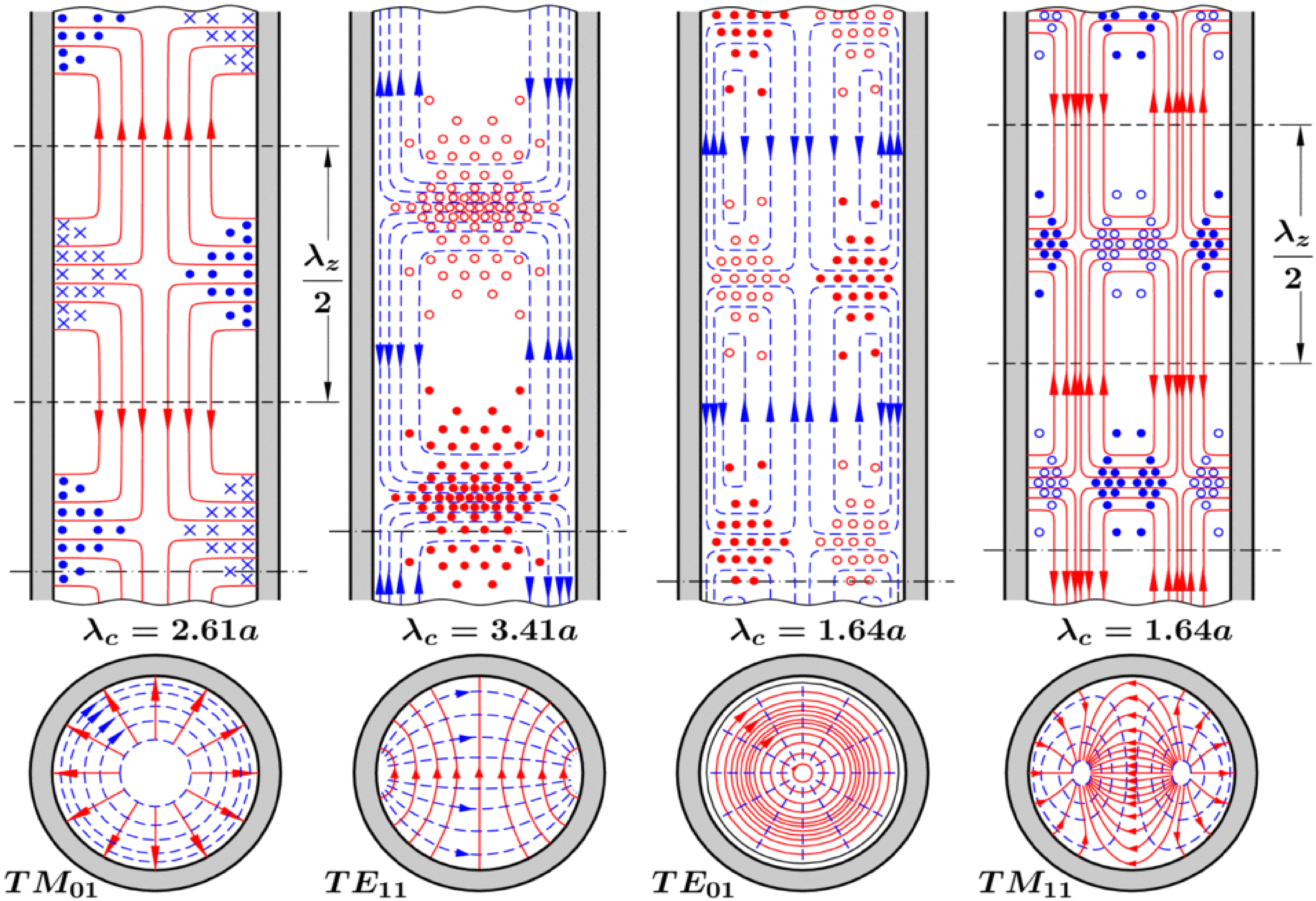
$$= \begin{Bmatrix} \text{imaginary} & & k < k_c \\ 0 & \text{for} & k = k_c \\ \text{real} & & k > k_c \end{Bmatrix}$$

Each mn defines a certain (eigen-) mode. The general solution is the linear combination of all modes

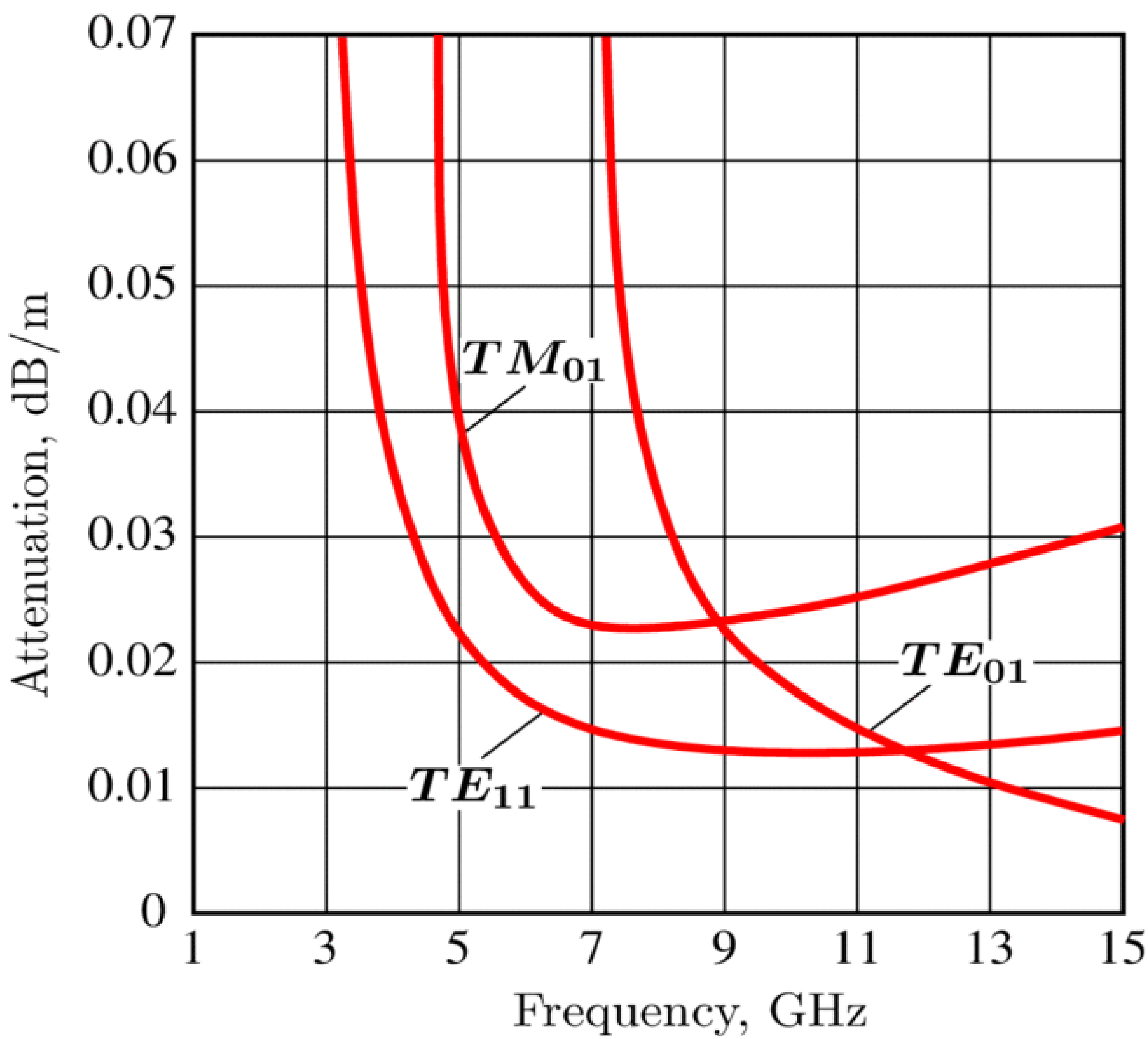
$$\vec{E} = \sum_m \sum_n \left(\vec{E}_{mn}^{TE} + \vec{E}_{mn}^{TM} \right), \qquad \vec{H} = \sum_m \sum_n \left(\vec{H}_{mn}^{TE} + \vec{H}_{mn}^{TM} \right)$$

Modes are normally sorted refering to their cutoff frequency:

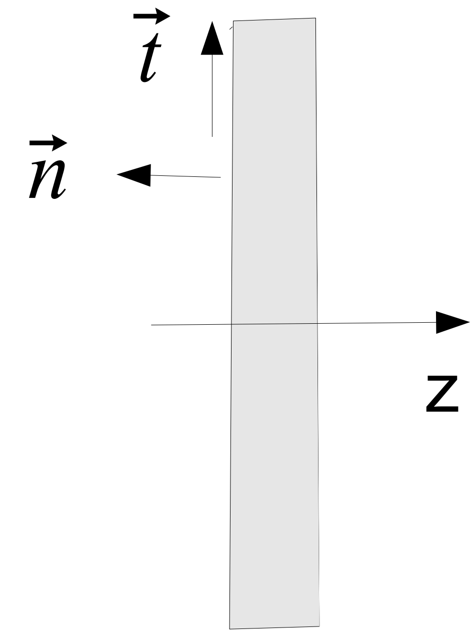
type	m	n	$(f_c$ / GHz)(a/cm)
TE	1	1	8.78
TM	0	1	11.46
TE	2	1	14.56
TE/TM	0/1	1/1	18.29
TE	3	1	20.05


λ_z
$\frac{\lambda_z}{2}$
$\lambda_c = 2.61a$
$\lambda_c = 3.41a$
$\lambda_c = 1.64a$
$\lambda_c = 1.64a$
TM_{01}
TE_{11}
TE_{01}
TM_{11}

Copper waveguide with a=2.5 cm



Impedance boundary condition on good conductors

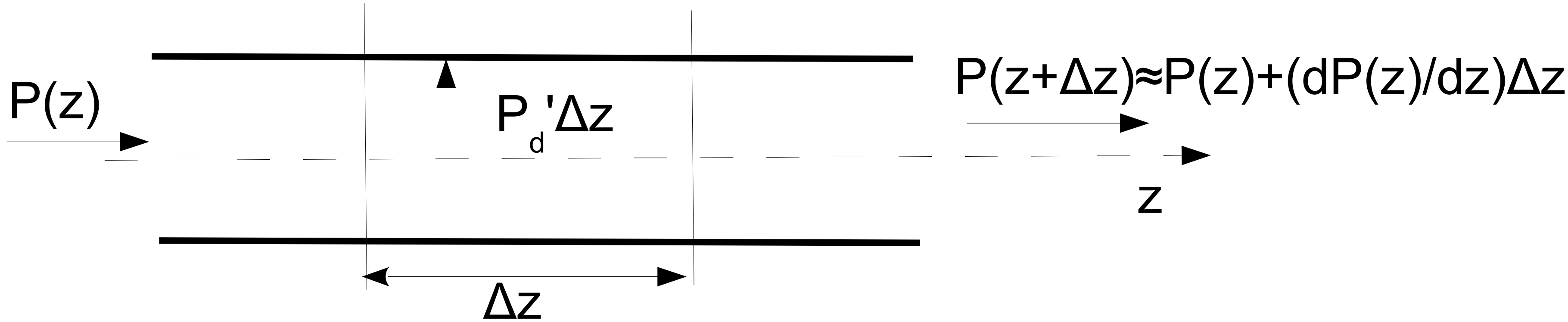


1. In metals (high conductivity κ) we can neglect the displacement current compared to the conduction current, ($|\partial D/\partial t| \ll |J|$).
2. On metallic surfaces is approximately $E_\perp$, $H_\parallel$.
3. Tangential to the surface the typical length of change is λ_0.

   Normal to the surface, in the metal, the typical length of change is $\delta_s \ll \lambda_0$.
4. Assumptions 1 through 3 allow for the derivation of a very good approximation for the tangential surface fields (see appendix A2)

$$\vec{E}_{t0} \approx Z_W (\vec{n} \times \vec{H}_{t0}), \qquad Z_W = \frac{1+i}{\kappa \delta_s}, \quad \text{wall impedance}$$

Attenuation in waveguides (power-loss method)



conservation of power : $\dfrac{dP(z)}{dz} = -P_d'$

$$\vec{E},\ \vec{H} \sim e^{-\alpha z},\ P(z) \sim e^{-2\alpha z} \rightarrow \frac{dP(z)}{dz} = -2\alpha P(z) = -P_d'$$

dissipation per waveguide surface area:

$$\frac{\Delta P_d}{\Delta A} = -\vec{n}\cdot\Re(\vec{S}_c) = -\frac{1}{2}\Re(\vec{n}\cdot(\vec{E}_{t0}\times\vec{H}_{t0}^*)) = \frac{1}{2}\Re(Z_W)|\vec{H}_{t0}|^2$$

$$\frac{\Delta P_d}{\Delta A} = \frac{1}{2\kappa\delta_s}\left|\vec{H}_{t0}\right|^2$$

dissipation per waveguide length:

$$P_d' = \frac{1}{2\kappa\delta_S}\oint\left|\vec{H}_{t0}\right|^2 ds$$

transported active power:

$$P(z) = \iint \Re(\vec{S}_c)\cdot d\vec{A} = \frac{1}{2}\iint \Re(\vec{E}\times\vec{H}^*)\cdot\vec{e}_z\, dA =$$

$$= \frac{1}{2}\iint \Re(\vec{E}_{transv}\times\vec{H}^*_{transv})_z\, dA = \frac{1}{2} Z_F \iint \left|\vec{H}_{transv}\right|^2 dA$$

attenuation: $\alpha = \frac{1}{2}\frac{P_d'}{P(z)}$

Resonant cavities

Example: Cylindrical cavity, radius a, length g, TM-modes

Superposition of forward and backward traveling waves (see transp. 45, 47)

$$E_\varphi = \frac{k_z}{\omega \epsilon} \frac{m}{\rho} D_{mn} \sin(m\varphi) J_m(k_{cmn}\rho) [e^{-ik_z z} - r_{mn} e^{ik_z z}]$$

Boundary conditions fix r and k_z

$$E_\varphi(z=0)=0 \quad \rightarrow \quad r_{mn}=1, \qquad E_\varphi \sim \sin(k_z z)$$

$$E_\varphi(z=g)=0 \quad \rightarrow \quad k_{zp} g = p\pi, \qquad p=0,1,2,\ldots$$

Now, the other field components can be calculated from the vector potential (see appendix A3).

Example: TM_{010}-resonator (m=0, n=1, p=0)

$$H_\varphi = 2\frac{j_{01}}{a} D_{010} J_1\left(j_{01}\frac{\rho}{a}\right)$$

$$E_z = -i\frac{2}{\omega\epsilon}\left(\frac{j_{01}}{a}\right)^2 D_{010} J_0\left(j_{01}\frac{\rho}{a}\right)$$

Resonance frequency

$$k_{010} = \frac{\omega_{010}}{c_0} = k_{c01} = \frac{j_{01}}{a}$$

$$f_{010} = \frac{\omega_{010}}{2\pi} = \frac{j_{01} c_0}{2\pi a}$$

Stored energy

$$\bar{W} = \bar{W}_e + \bar{W}_m = 2\bar{W}_e = \frac{1}{2}\iiint \vec{E}\cdot\vec{D}^* dV = \frac{\epsilon}{2}\iiint |E_z|^2 dV$$

$$\bar{W} = \frac{2\pi g}{\omega_{010}^2 \epsilon} \frac{j_{01}^4}{a^2} |D_{010}|^2 J_1^2(j_{01})$$

Dissipation per unit area

$$\bar{P}_d'' = \frac{1}{2\kappa\delta_s} |\vec{H}_{t0}|^2$$

total dissipation

$$\bar{P}_d = \oiint \bar{P}_d'' dA = \frac{4\pi}{\kappa\delta_s} j_{01}^2 \left(1 + \frac{g}{a}\right) |D_{010}|^2 J_1^2(j_{01})$$

Quality factor (Q-value)

$$Q_0 = \frac{\omega_{010} \bar{W}}{\bar{P}_d} = \frac{1}{\delta_s} \frac{g}{1 + g/a} \quad \rightarrow \quad \delta_s Q_0 = 2\frac{V}{S} \sim \frac{Volume}{Surface}$$

Q_0 gives the decay rate of the stored energy or the time T_f to fill the cavity.

From *power conservation*

$$-\frac{d\bar{W}}{dt}=\bar{P}_d=\frac{\omega_{010}}{Q_0}\bar{W} \quad \rightarrow \quad \bar{W}=\bar{W}_0 e^{-2t/T_f}, \quad T_f=2\frac{Q_0}{\omega_{010}}$$

Example: 3 Ghz copper cavity, $g=\lambda_{010}/2=5$ cm

$j_{01}=2.405$, $J_1(j_{01})=0.5191$, $\kappa=58\cdot 10^6\ \Omega^{-1}\text{m}^{-1}$

$a=3.83$ cm, $\delta_s=1.21$ μm, $Q_0=17963$, $T_f=1.9$μs

Resonance behaviour of a cavity mode

Instead of lossy walls assume ideal conducting walls and lossy dielectric filling. That preserves the ideal mode but allows for studying losses.

The cavity is driven by a current J passing through it. J splits into a conduction current $J_c = \kappa E$, responsible for the losses in the dielectric, and in an enforced current J_0 as driving term:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \vec{\nabla}^2\vec{E} = -\mu\frac{\partial}{\partial t}\vec{\nabla}\times\vec{H} =$$

$$= -\mu\frac{\partial}{\partial t}(\vec{J}_0 + \kappa\vec{E} + \epsilon\frac{\partial\vec{E}}{\partial t})$$

$$\vec{\nabla}^2\vec{E} - \mu\kappa\frac{\partial\vec{E}}{\partial t} - \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} = \mu\frac{\partial\vec{J}_0}{\partial t} \quad \text{with gauge} \quad \vec{\nabla}\cdot\vec{E} = 0 \qquad (1)$$

We expand E in (eigen-)modes

$$\vec{E} = \sum_r a_r(t)\vec{e}_r(x, y, z), \quad \text{r goes over all m,n,p} \qquad (2)$$

where $\vec{\nabla}^2 \vec{e}_r + k_r^2 \vec{e}_r = 0$

$\vec{\nabla} \cdot \vec{e}_r = 0$ in volume, $\quad \vec{n} \times \vec{e}_r = 0$ on walls

$$\iiint \vec{e}_r \cdot \vec{e}_s \, dV = \delta_r^s$$

Substituting (2) in (1) and deviding by -με

$$\sum_r \left[\frac{d^2 a_r}{dt^2} + \frac{\kappa}{\epsilon}\frac{da_r}{dt} + \frac{k_r^2}{\mu\epsilon} a_r\right]\vec{e}_r = -\frac{1}{\epsilon}\frac{\partial \vec{J}_0}{\partial t}. \qquad (3)$$

Multiplying (3) with e_s and integrating over V

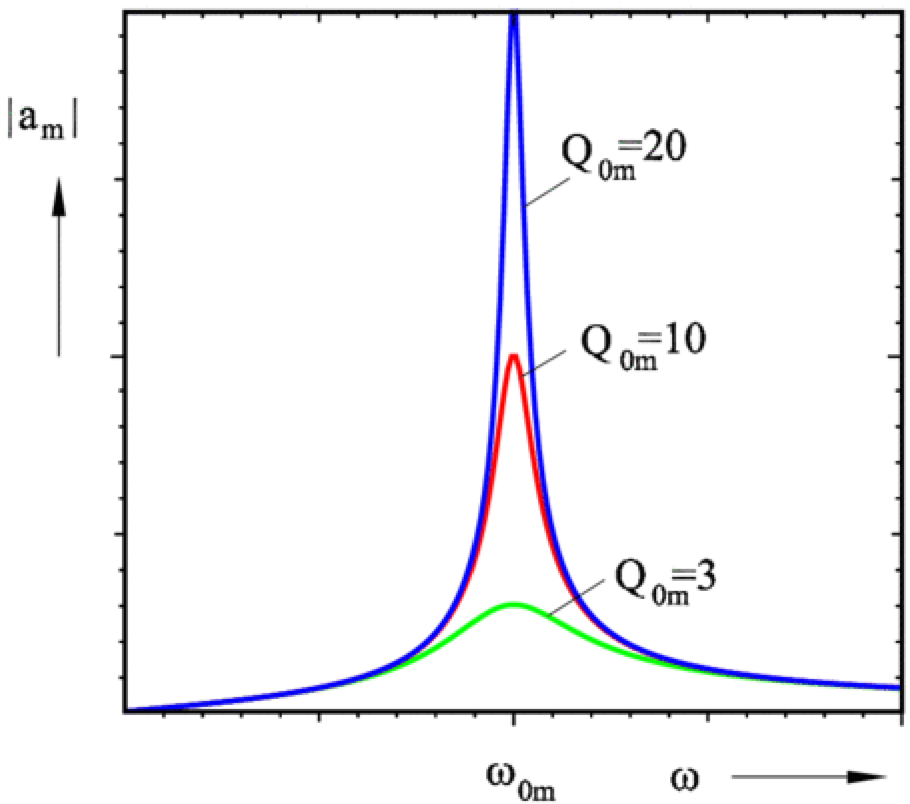
$$\frac{d^2 a_s}{dt^2}+\frac{\kappa}{\epsilon}\frac{da_s}{dt}+\frac{k_s^2}{\mu\epsilon}a_s=-\frac{1}{\epsilon}\iiint\frac{\partial\vec{J}_0}{\partial t}\cdot\vec{e}_s dV=\frac{\partial f_s}{\partial t}. \qquad (4)$$

In case of time-harmonic excitation (4) becomes

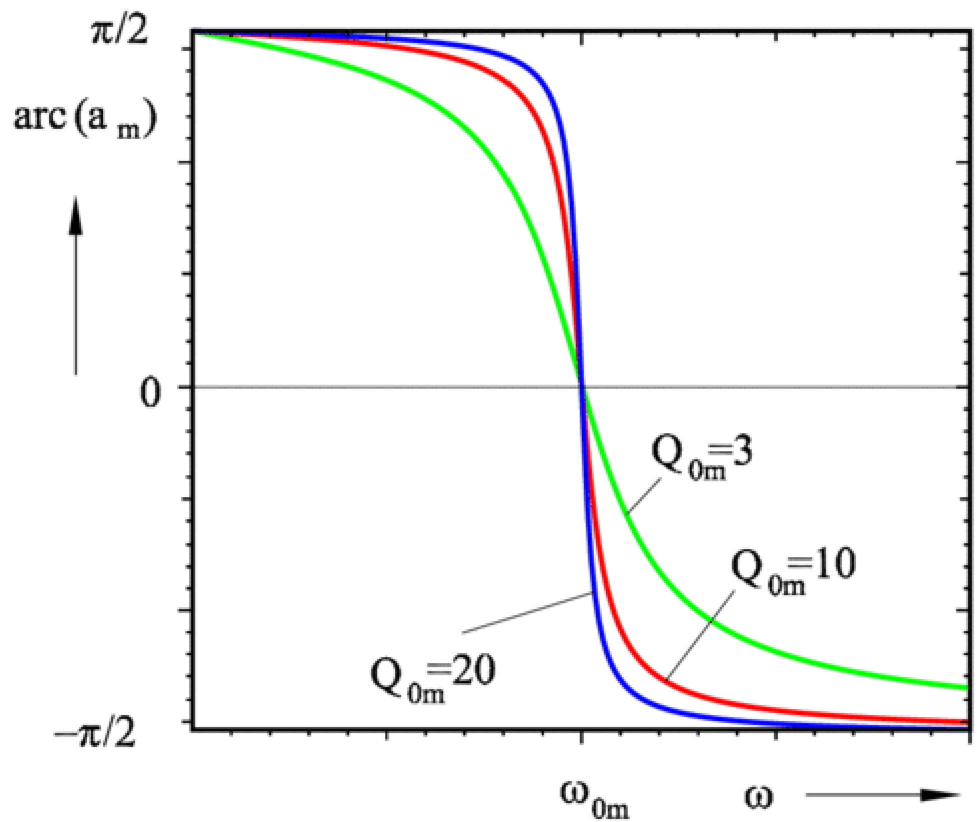
$$[-\omega^2+i\frac{\kappa}{\epsilon}\omega+\frac{k_s^2}{\mu\epsilon}]a_s=i\omega f_s$$

$$a_s=\frac{Q_s}{\omega_s}\frac{f_s}{1+iQ_s[\frac{\omega}{\omega_s}-\frac{\omega_s}{\omega}]}, \qquad \omega_s=ck_s, \quad Q_s=\frac{\epsilon\omega_s}{\kappa}.$$

Now replace Q_s by Q_0 as calculated with impedance-boundary-condition and ω_s by the resonance frequency ω_{mnp}.

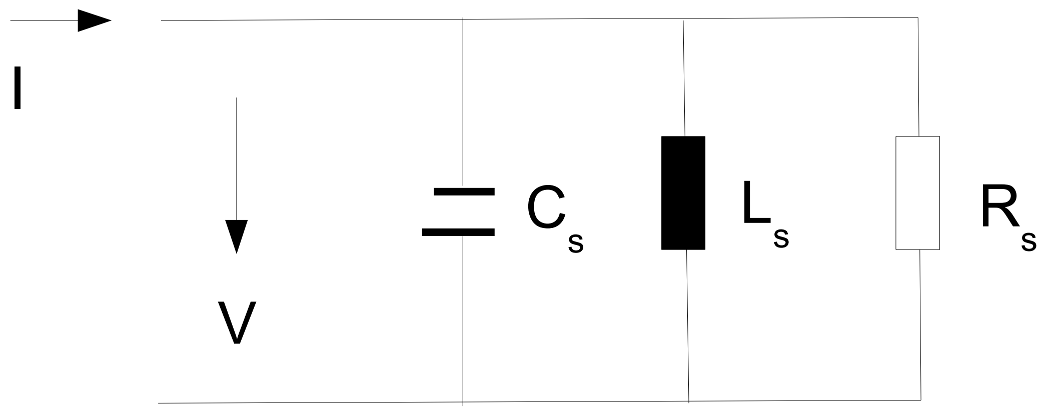


|a_m|
Q_0m=20
Q_0m=10
Q_0m=3
ω_0m
ω



π/2
arc (a_m)
0
−π/2
Q_0m=3
Q_0m=10
Q_0m=20
ω_0m
ω

Well seperated modes can be represented by a lumped element resonator



$$\omega_s = \frac{1}{\sqrt{L_s C_s}}, \quad Q_s = \frac{\omega_s W_s}{P_{ds}} = \omega_s R_s C_s$$

Bandwidth $$B_s = \frac{(\omega_s + \delta\omega) - (\omega_s - \delta\omega)}{\omega_s} = 2\frac{\delta\omega}{\omega_s} = \frac{1}{Q_s}$$

Filling time $$T_{fs} = 2\frac{Q_s}{\omega_s} = \frac{1}{\delta\omega}$$

Accelerating voltage for a particle passing the cavity on-axis with velocity v

$$V_s = \left| \int_0^g a_s \vec{e}_s \cdot \vec{e}_z \, e^{i\omega t} dz \right|, \qquad z = vt$$

Shunt impedance (available V_s for given P_{ds})

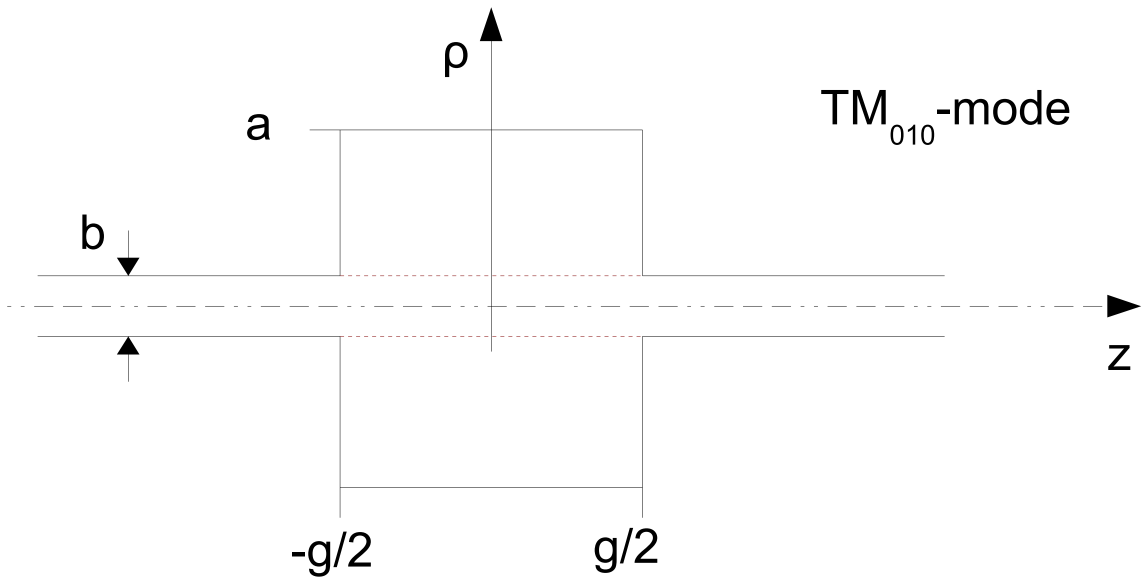
$$R_{shs} = \frac{V_s^2}{P_{ds}} = 2 R_s$$

R-upon-Q (available V_s for given W_s, geometrical quantity, independent of losses)

$$\frac{R_{shs}}{Q_s} = \frac{V_s^2}{\omega_s W_s} = \frac{2}{\omega_s C_s}$$

ω_s, Q_s and R_{shs}/Q_s define R_s, L_s, C_s .

Influence of beam pipe on voltage gain



Spectral expansion of field in tube region, 0≤ρ≤b

$$E_z(\rho,z)=\int_{-\infty}^{\infty} A(k_z)\, I_0(K\rho)\, e^{-ik_z z}\, dk_z\,, \qquad K=\sqrt{k_z^2-k^2} \qquad (1)$$

$$A(k_z)\, I_0(K\rho)=\frac{1}{2\pi}\int_{-\infty}^{\infty} E_z(\rho,z)\, e^{ik_z z}\, dz$$

Approximation: $$E_z(\rho=b,z)=\begin{cases} E_0\,, & -g/2\leqslant z\leqslant g/2 \\ 0\,, & |z|>g/2 \end{cases}$$

$$A(k_z)I_0(Kb)=\frac{E_0}{2\pi}\int_{-g/2}^{g/2} e^{ik_z z}dz = \frac{E_0 g}{2\pi}\frac{\sin(k_z g/2)}{k_z g/2} \qquad (2)$$

Substituting (2) into (1) yields for the voltage gain

$$V(\rho)=\int_{-\infty}^{\infty} E_z(\rho,z)e^{i\omega t}dz=\int_{-\infty}^{\infty} E_z(\rho,z)e^{ikz/\beta}dz, \quad vt=z, \;\; \beta=v/c$$

$$=\frac{E_0 g}{2\pi}\int_{-\infty}^{\infty} dk_z \frac{\sin(k_z g/2)}{k_z g/2}\frac{I_0(K\rho)}{I_0(Kb)}\int_{-\infty}^{\infty} dz\, e^{i(k/\beta-k_z)z}=E_0 g\, T\, F(\rho,b)$$

Transit time factor: $T=\frac{\sin(kg/2\beta)}{kg/2\beta}$

Reduction factor due to beam pipe: $F(\rho,b)=\frac{I_0(k\rho/\beta\gamma)}{I_0(kb/\beta\gamma)}, \quad \gamma=1/\sqrt{1-\beta^2}$

$$\frac{V(\rho=0)}{V(\rho=b)}=\frac{1}{I_0(kb/\beta\gamma)}\approx\sqrt{2\pi kb/\beta\gamma}\; e^{-kb/\beta\gamma} \quad \text{for} \quad kb/\beta\gamma>2$$

Appendix

A1 Solution of the vectorial diffusion equation

We decompose the vectorial equ.

$$\vec{\nabla}^2 \vec{A} = -\mu \vec{J}$$

into cartesian components

$$\vec{\nabla}^2 A_i = -\mu J_i , \qquad i = x, y, z \tag{1}$$

Coulomb's law gives the field and scalar potential of a point charge q:

$$\vec{E} = \frac{q}{4\pi\epsilon r^2} \vec{e}_r = -\vec{\nabla}\Phi \quad \rightarrow \quad E_r = -\frac{d\Phi}{dr} \quad \rightarrow \quad \Phi = \frac{q}{4\pi\epsilon r}$$

Φ is solution of the inhomogeneous Poisson equ.

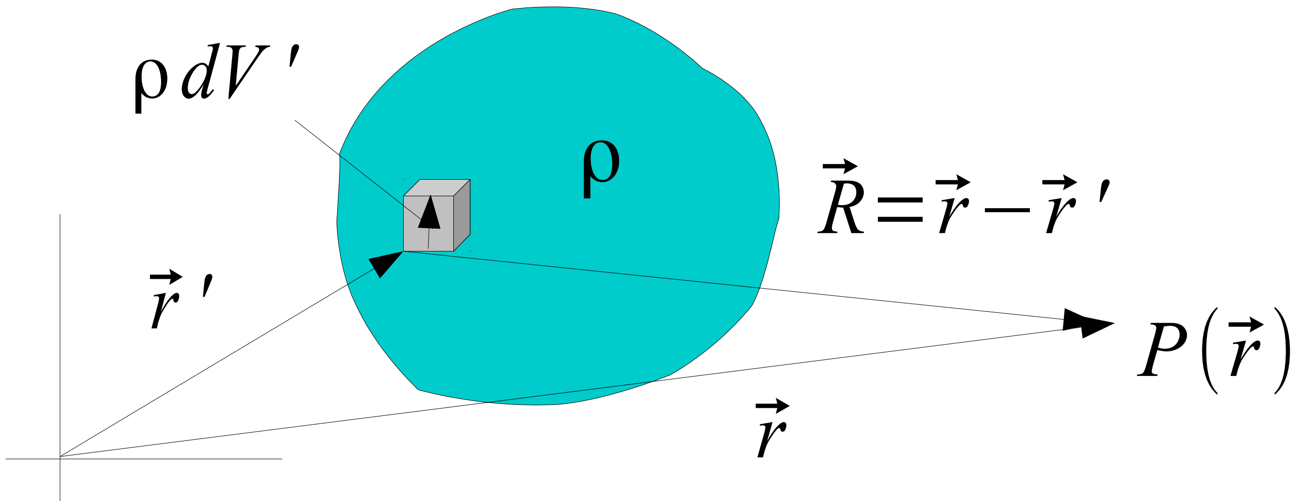
$$\vec{\nabla}^2 \Phi = -\frac{q}{\epsilon} \delta(r) \qquad (2)$$

Comparing (1) and (2), we see that (1) follows from (2) by substituting

$$\Phi \rightarrow A_i, \quad \frac{1}{\epsilon} \rightarrow \mu, \quad q \rightarrow J_i \qquad (3)$$

Next we use the solution Φ of (2) as a „Green's function“ to calculate the potential of a charge distribution. This yields the Coulomb integral

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\vec{r}\,')}{R} dV' \qquad (4)$$



Using the substitution (3) in (4) we get the solution of (1). The vectorial form is then

$$\vec{A}(\vec{r})=\frac{\mu}{4\pi}\iiint\frac{\vec{J}(\vec{r}\,')}{R}\,dV'$$

A2

Poynting's theorem for time-harmonic fields

decompose e.g. $\vec{E}=\Re[\tilde{\vec{E}}\,\mathrm{e}^{i\omega t}]=\frac{1}{2}[\tilde{\vec{E}}\,\mathrm{e}^{i\omega t}+\tilde{\vec{E}}^*\,\mathrm{e}^{-i\omega t}]$

$$w_e=\frac{1}{2}\vec{E}\cdot\vec{D}=\frac{1}{8}[\tilde{\vec{E}}\cdot\tilde{\vec{D}}\,\mathrm{e}^{i2\omega t}+\tilde{\vec{E}}^*\cdot\tilde{\vec{D}}^*\,\mathrm{e}^{-i2\omega t}]+\frac{1}{8}[\tilde{\vec{E}}\cdot\tilde{\vec{D}}^*+\tilde{\vec{E}}^*\cdot\tilde{\vec{D}}]$$

$$=\frac{1}{4}\Re[\tilde{\vec{E}}\cdot\tilde{\vec{D}}\,\mathrm{e}^{i2\omega t}]+\frac{1}{4}\tilde{\vec{E}}\cdot\tilde{\vec{D}}^*$$

and after time-averaging $\bar{w}_e=\frac{1}{4}\tilde{\vec{E}}\cdot\tilde{\vec{D}}^*$

correspondingly: $\bar{w}_m=\frac{1}{4}\tilde{\vec{H}}\cdot\tilde{\vec{B}}^*,\quad \bar{p}_d=\frac{1}{2}\tilde{\vec{E}}\cdot\tilde{\vec{J}}^*$

$\vec{S}_c=\frac{1}{2}\tilde{\vec{E}}\times\tilde{\vec{H}}^*$ complex, time-averaged radiation flux

Using Maxwell's equations

$$(1/2)\tilde{\vec{E}}\cdot \quad \vec{\nabla}\times\tilde{\vec{H}}^{*}=\tilde{\vec{J}}^{*}-i\omega\tilde{\vec{D}}^{*}$$

$$-(1/2)\tilde{\vec{H}}^{*}\cdot \quad \vec{\nabla}\times\tilde{\vec{E}}=-i\omega\tilde{\vec{B}}$$

$$\rightarrow \quad -\vec{\nabla}\cdot\left(\frac{1}{2}\tilde{\vec{E}}\times\tilde{\vec{H}}^{*}\right)=\frac{1}{2}\tilde{\vec{E}}\cdot\tilde{\vec{J}}^{*}+i2\omega\left(\frac{1}{4}\tilde{\vec{H}}^{*}\cdot\tilde{\vec{B}}-\frac{1}{4}\tilde{\vec{E}}\cdot\tilde{\vec{D}}^{*}\right)$$

we get Poynting's theorem after integration over V and application of Gauss' law:

$$-\oint\!\!\oint \vec{S}_c\cdot d\vec{A}=\iiint \bar{p}_d\,dV+i2\omega\iiint(\bar{w}_m-\bar{w}_e)\,dV$$

Active power (time-averaged Joulean heat, dissipation)

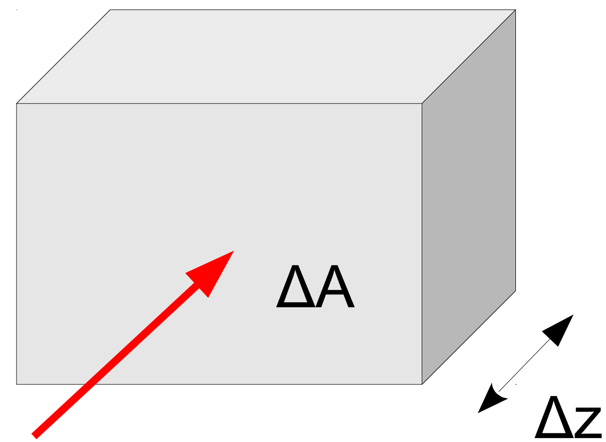
$$\bar{P}_{act}=-\oint\!\!\oint \Re[\vec{S}_c]\cdot d\vec{A}=\iiint \bar{p}_d\,dV=\bar{P}_d$$

Reactive power

$$\bar{P}_{react}=-\oint\!\!\oint \Im[\vec{S}_c]\cdot d\vec{A}=2\omega\iiint(\bar{w}_m-\bar{w}_e)\,dV=2\omega(\bar{W}_m-\bar{W}_e)$$

A3

Energy velocity



Energy transported by Δz in time Δt:

$$\frac{\bar{w}\,\Delta A\,\Delta z}{\Delta t} = S_c \Delta A \quad \rightarrow \quad v_e = \frac{\Delta z}{\Delta t} = \frac{S_c}{\bar{w}}$$

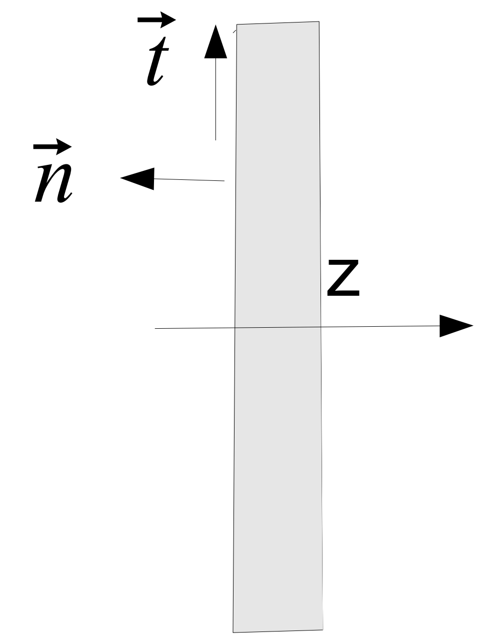
for plane waves

$$S_c = \frac{1}{2}\left(\vec{E}\times\vec{H}^*\right) = \frac{|E_0|^2}{2Z}, \qquad \bar{w} = \frac{1}{4}\vec{E}\cdot\vec{D}^* + \frac{1}{4}\vec{H}\cdot\vec{B}^* = \frac{1}{2}\epsilon|E_0|^2$$

$$v_e = \frac{1}{Z\epsilon} = \frac{1}{\sqrt{\mu\epsilon}} = c$$

A4

Impedance boundary condition on good conductors



On ideal conducting surfaces is the E-field perpendicular and the H-field tangential. On good metallic conductors we expect similar behaviour.

We decompose fields and nabla operator into tangential and normal components

$$\vec{E}=\vec{E}_t+E_z\vec{e}_z,\quad \vec{H}=\vec{H}_t+H_z\vec{e}_z,\quad \vec{\nabla}=\vec{\nabla}_t+\vec{e}_z\frac{\partial}{\partial z}$$

and subsequently also Maxwell's equs., where we neglect the displacement current as compared to the conduction current, IδD/δtI<<IJI:

$$\vec{\nabla}\times\vec{H}=\kappa\vec{E}: \qquad \vec{E}_t=-\frac{1}{\kappa}\vec{e}_z\times\vec{\nabla}_t H_z+\frac{1}{\kappa}\vec{e}_z\times\frac{\partial\vec{H}_t}{\partial z}$$

$$E_z\vec{e}_z=\frac{1}{\kappa}\vec{\nabla}_t\times\vec{H}_t$$

$$\vec{\nabla}\times\vec{E}=-i\omega\mu_0\vec{H}: \quad \vec{H}_t=-\frac{i}{\omega\mu_0}\vec{e}_z\times\vec{\nabla}_t E_z+\frac{i}{\omega\mu_0}\vec{e}_z\times\frac{\partial\vec{E}_t}{\partial z}$$

$$H_z\vec{e}_z=\frac{i}{\omega\mu_0}\vec{\nabla}_t\times\vec{E}_t$$

Tangential to the surface the typical length of change is λ_0.
Normal to the surface, in metal, the typical length of change *is* $\delta_S \ll \lambda_0$.
With an order of magnitude approximation $|\vec{\nabla}_t| \approx 1/\lambda_0$
one gets for the magnitude of E_z and H_z

$$|E_z| = \left|\frac{1}{\kappa}\vec{\nabla}_t \times \vec{H}_t\right| \approx \frac{1}{\kappa\lambda_0}|\vec{H}_t| = \pi\left(\frac{\delta_s}{\lambda_0}\right)^2 Z_0|\vec{H}_t|$$

$$Z_0|H_z| = \left|\frac{i}{\omega\mu_0}\vec{\nabla}_t \times \vec{E}_t\right| \approx \frac{1}{\omega\mu_0}\frac{Z_0}{\lambda_0}|\vec{E}_t| = \frac{1}{2\pi}|\vec{E}_t|.$$

With that we estimate the green terms

$$\left|\frac{1}{\kappa}\vec{e}_z \times \vec{\nabla}_t H_z\right| \approx \frac{1}{\kappa\lambda_0}|H_z| = \pi\left(\frac{\delta_s}{\lambda_0}\right)^2 Z_0|H_z| \approx \frac{1}{2}\left(\frac{\delta_s}{\lambda_0}\right)^2 |\vec{E}_t|$$

$$\left|\frac{i}{\omega\mu_0}\vec{e}_z \times \vec{\nabla}_t E_z\right| \approx \frac{1}{\omega\mu_0\lambda_0}|E_z| = \frac{1}{2\pi Z_0}|E_z| \approx \frac{1}{2}\left(\frac{\delta_s}{\lambda_0}\right)^2 |\vec{H}_t|.$$

One finds that they can be neglected compared to E_t, H_t .
So, the tangential parts of Maxwell's equs. are simplified to

$$\kappa \vec{E}_t \approx \vec{e}_z \times \frac{\partial \vec{H}_t}{\partial z} \qquad (1)$$

$$i\omega\mu_0 \vec{H}_t \approx -\vec{e}_z \times \frac{\partial \vec{E}_t}{\partial z}.$$

Eliminating $\vec{E}_t$ one gets an equ. for $\vec{H}_t$

$$\frac{\partial^2 \vec{H}_t}{\partial z^2} - i\omega\mu_0\kappa \vec{H}_t = 0$$

with the solution

$$\vec{H}_t = \vec{H}_{t0}\, \mathrm{e}^{-(1+i)z/\delta_s}. \qquad (2)$$

(2) substituted into (1) gives a boundary condition at real (non-ideal) metallic surfaces

$$\vec{E}_{t0} \approx Z_W (\vec{n} \times \vec{H}_{t0}), \qquad Z_W = \frac{1+i}{\kappa\delta_s} \quad \text{wall impedance}$$

A5 Cylindrical cavity, radius a, length g, TM-modes

Superposition of forward and backward traveling waves (see transp. 49) gives for the E_φ component

$$E_\varphi = \frac{k_z}{\omega \epsilon} \frac{m}{\rho} D_{mn} \sin(m\varphi) J_m(k_{cmn}\rho)[\mathrm{e}^{-ik_z z} - r_{mn}\mathrm{e}^{ik_z z}]$$

Boundary conditions fix r and k_z

$$E_\varphi(z=0)=0 \quad \rightarrow \quad r_{mn}=1, \qquad E_\varphi \sim \sin(k_z z)$$

$$E_\varphi(z=g)=0 \quad \rightarrow \quad k_{zp} g = p\pi, \quad p=0,1,2,\ldots$$

and the vector potential can be written as

$$A = 2\mathrm{D}_{mnp} \cos(m\varphi) \cos(k_{zp} z) J_m(k_{cmn}\rho), \quad k_{cmn} = j_{mn}/a$$

TM-modes follow from $\vec{H}=\vec{\nabla}\times(A\vec{e}_z)$, $i\omega\epsilon\vec{E}=\vec{\nabla}\times\vec{H}$ as

$$H_\rho=-2\frac{m}{\rho}D_{mnp}\sin(m\varphi)\cos(k_{zp}z)J_m(k_{cmn}\rho)$$

$$H_\varphi=-2k_{cmn}D_{mnp}\cos(m\varphi)\cos(k_{zp}z)J_m'(k_{cmn}\rho),\ H_z=0$$

$$E_\rho=i2\frac{k_{zp}}{\omega\epsilon}k_{cmn}D_{mnp}\cos(m\varphi)\sin(k_{zp}z)J_m'(k_{cmn}\rho)$$

$$E_\varphi=-i2\frac{k_{zp}}{\omega\epsilon}\frac{m}{\rho}D_{mnp}\sin(m\varphi)\sin(k_{zp}z)J_m(k_{cmn}\rho)$$

$$E_z=-i2\frac{k_{cmn}^2}{\omega\epsilon}D_{mnp}\cos(m\varphi)\cos(k_{zp}z)J_m(k_{cmn}\rho)$$

$$k_{cmn}=\sqrt{k^2-k_{zp}^2}=\frac{j_{mn}}{a}\quad\rightarrow\quad k_{mnp}=\frac{\omega_{mnp}}{c_0}=\left(\frac{j_{mn}}{a}\right)^2+k_{zp}^2$$

Literature:

David K. Cheng, Field and wave electromagnetics.
Addison-Wesley 1990

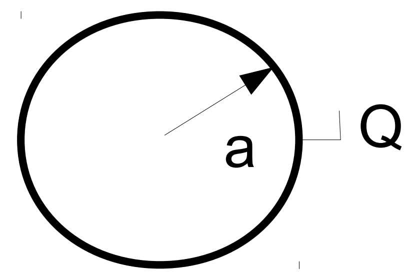
David J. Griffiths, Introduction to electrodynamics.
Prentice Hall 1999

J. D. Jackson, Classical electrodynamics.
John Wiley & Sons 1975

Exercise 1:

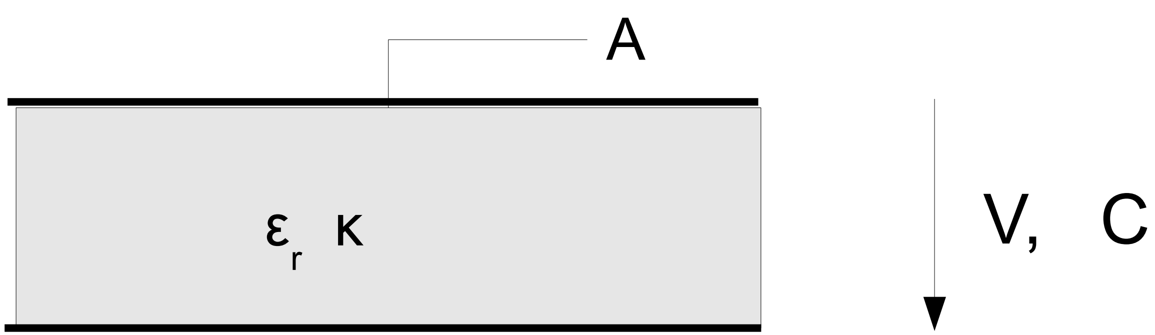
Given is a conducting hollow sphere carrying a charge Q. What is the field inside and outside and what is the stored energy?

If the sphere is a model for an electron (E_{0e}= 511keV) what is then the classical electron radius r_e=a ?



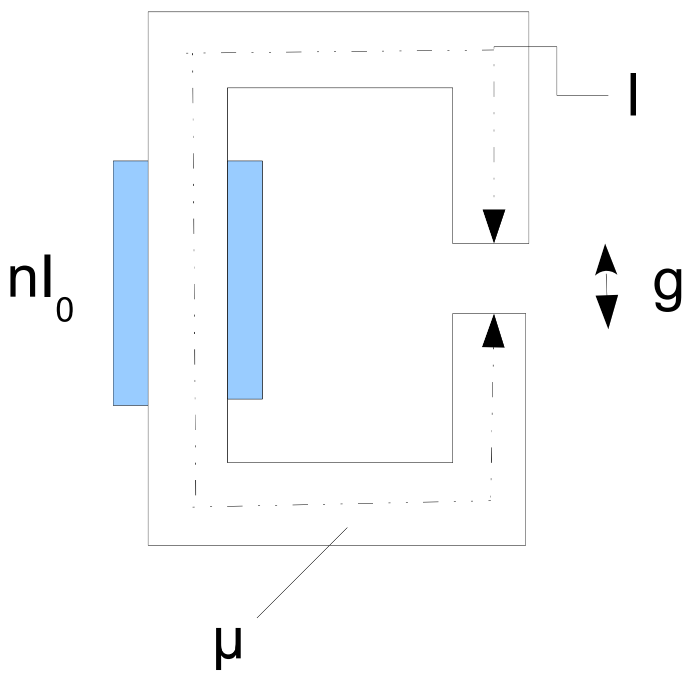
Exercise 2:

A capacotor is filled with a lossy dielectric and charged to a voltage V. What is the time constant for discharge?



Exercise 3:

A long dipole magnet is excited by a coil with n windings and current I_0. Calculate the magnetic field in the air gap.



Exercise 4:

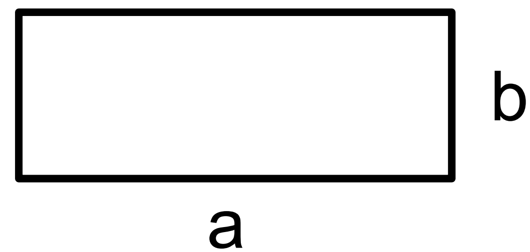
Derive the multi-poles for a static 2-dimensional circular magnetic field.
Remark: Solve the magnetic potential equation in circular cylindrical coordinates and free-space.

Exercise 5:

Give the E- and H-field of a z-polarized plane wave which propagates in x-direction.
What is the time-averaged radiated power density?

Exercise 6:

Derive the longitudinal vector potential for TM-waves in a rectangular waveguide.
What is the equation for the separation constants?



Exercise 7:

Give the guide wavelength and phase and group velocity of a TM_{11}-mode in a rectangular waveguide.

Exercise 8:

Calculate the accelerating voltage, shunt impedance and R-upon-Q of a TM_{110}-mode in a rectangular cavity resonator with dimensions a,b,g.