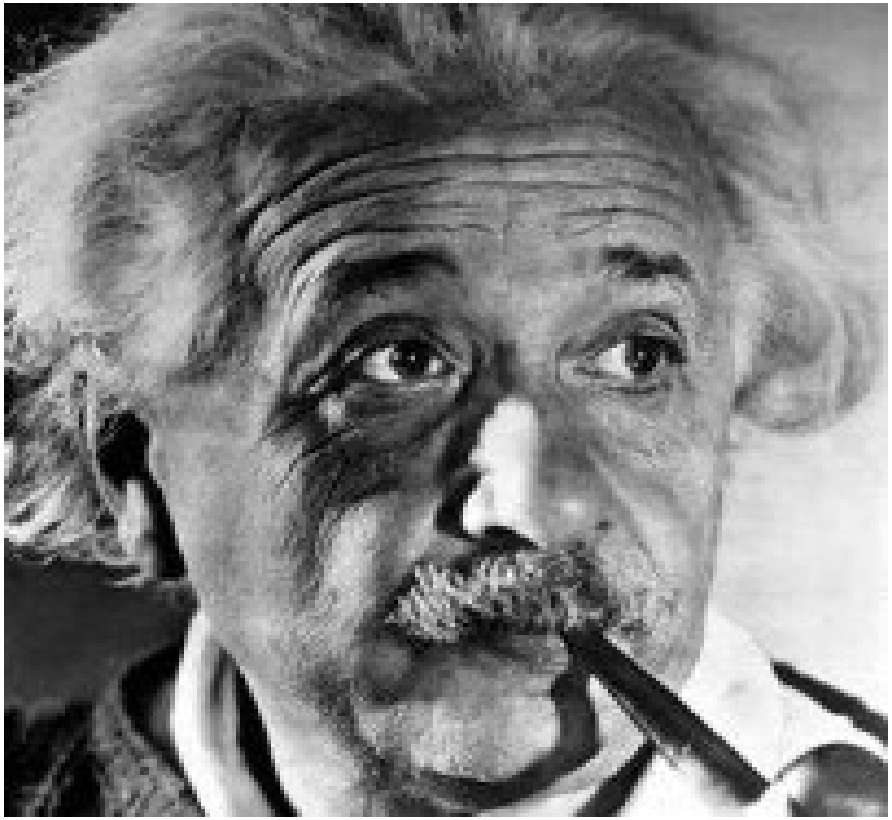
Review of Special Relativity



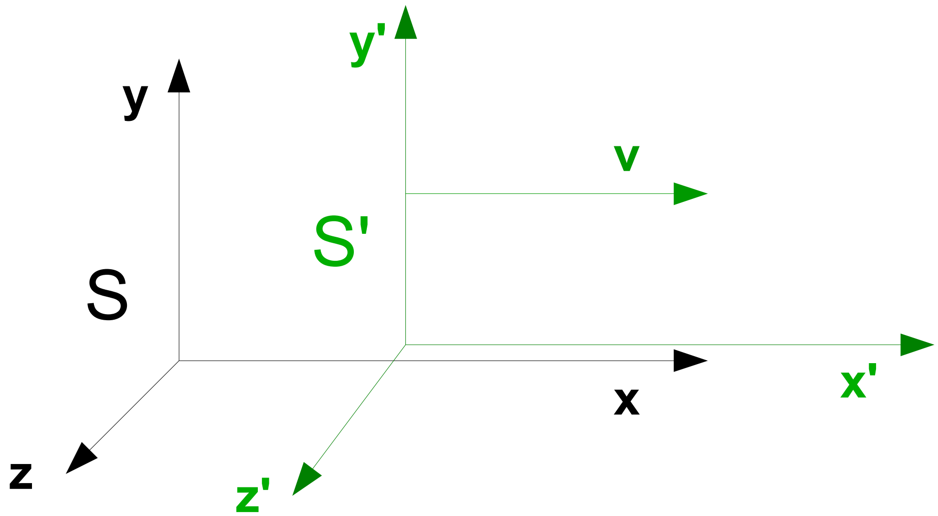
This review is not meant to teach the subject, but to repeat and to refresh, at least partially, what you have learnt at university.

Why was „Special Relativity“ needed?

Mechanical laws (Newton's laws) are the same for all inertial systems.

They are invariant under a Galilean transformation (G-T):

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t$$



Example: Man walking in train (S'), observer at rest (S).

El.-mag. laws are not invariant under a G-T,

take e.g. the wave equation $$[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}]\Phi = 0$$

it transforms (see appendix A1) to

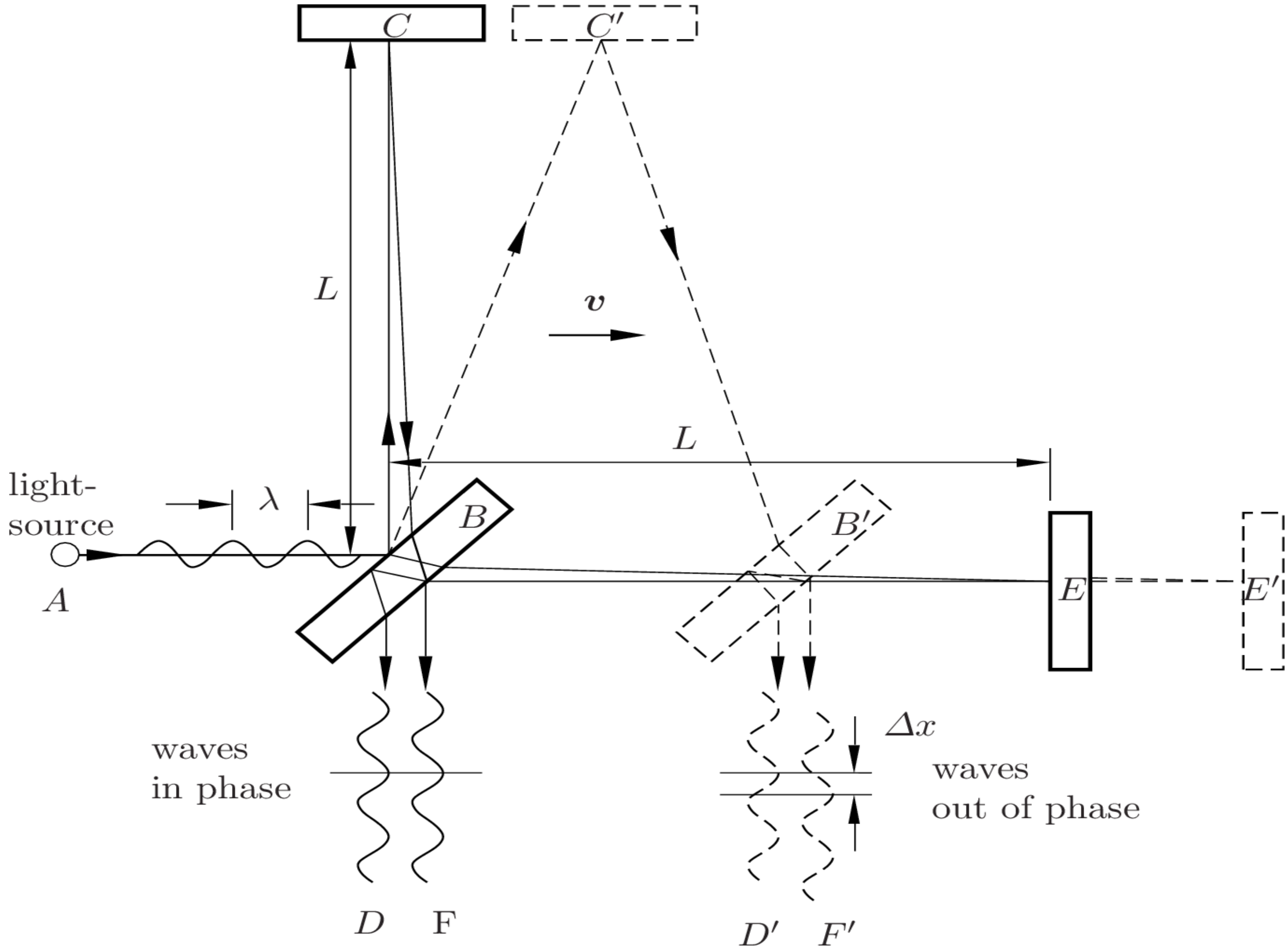
$$[\frac{\partial^2}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \frac{v^2}{c^2}\frac{\partial^2}{\partial x'^2} + 2\frac{v}{c^2}\frac{\partial^2}{\partial x'\partial t'}]\Phi = 0$$

Moreover, el.-mag. laws predict the speed of light as equal in all reference systems.
This contradicted the deep belief in a supporting media (ether) for the waves. If there were an ether, c would be different in different reference systems.

Many experiments tried to prove el.-mag. theory wrong.
They all failed!

Michelson-Morley interferometer experiment (1887)

showed that c is a constant and that there exists no „ether“.



travel time from B to E' $ct_1 = L + vt_1$

travel time from E' to B' $ct_2 = L - vt_2$

round trip travel time $t_{\parallel} = t_1 + t_2 = \gamma^2 2\frac{L}{c}, \quad \gamma = \frac{1}{\sqrt{1-(v/c)^2}}$

travel time B to C' $ct_3 = \sqrt{L^2 + (vt_3)^2}$

round trip travel time $t_{\perp} = 2t_3 = \gamma 2\frac{L}{c} = t_{\parallel}/\gamma$

Since $t_{\perp} = t_{\parallel}/\gamma$ there should be interference between D' and F'. But no interference was observed.

Conclusion: $L_{\parallel}$ appears shorter than $L_{\perp}$ by $1/\gamma$

→ Newton-Galileo concept of space and time had to be modified

Relativistic Kinematics

Einstein based his theory on two postulates:

1. All inertial frames are equivalent w.r.t. all laws of physics.
2. The speed of light is equal in all reference frames.

Consequence of 1st postulate:
Space is isotropic (all directions are equivalent)
Space is homogeneous (all points are equivalent)

Lorentz Transformation

Homogeneity of space and form-invariance of laws under transformation require a linear transformation:

$$
\begin{aligned}
ct' &= a_{00}\,ct + a_{01}\,x + a_{02}\,y + a_{03}\,z \\
x' &= a_{10}\,ct + a_{11}\,x + a_{12}\,y + a_{13}\,z \\
y' &= a_{20}\,ct + a_{21}\,x + a_{22}\,y + a_{23}\,z \\
z' &= a_{30}\,ct + a_{31}\,x + a_{32}\,y + a_{33}\,z
\end{aligned}
$$

Successive use of homogeneity and isotropy of space and the invariant speed of light determines all coefficients (see appendix A2). The result is the Lorentz-Transformation (L-T):

$$
ct' = \gamma(ct - \beta x) \qquad y' = y
$$

$$
x' = \gamma(x - \beta ct) \qquad z' = z, \qquad \beta = v/c, \qquad \gamma = 1/\sqrt{1-\beta^2}
$$

The inverse Transformation is obtained by replacing primed variables by unprimed and unprimed by primed and β by -β

$$
ct = \gamma(ct' + \beta x') \qquad y = y'
$$

$$
x = \gamma(x' + \beta ct') \qquad z = z'
$$

We write the L-T as

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$$\underline{\underline{L}} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\underline{\underline{L}}^{-1}$ is obtained by replacing β by -β.

L-T is an affine transformation. It preserves the rectilinearity and parallelisme of straight lines.

Consequences following from L-T

Time dilation:

Two events in S' *at* t_1', t_2' and *at location* $x_1' = x_2'$ *appear at* times t_1, t_2 in S

$$c(t_2 - t_1) = \gamma c(t_2' - t_1') + \gamma\beta(x_2' - x_1')$$

$$\rightarrow \quad \Delta t = \gamma \Delta t'$$

Length contraction:

A distance $x_2' - x_1'$ in S' *measured* in S *at* $t_1 = t_2$

$$x_2 - x_1 = \gamma(x_2' - x_1') + \gamma\beta c(t_2' - t_1') =$$

$$= \gamma(x_2' - x_1') + \gamma^2\beta c^2(t_2 - t_1) - \gamma^2\beta^2(x_2 - x_1) =$$

$$= \gamma(x_2' - x_1') - \gamma^2\beta^2(x_2 - x_1)$$

$$\rightarrow \quad \Delta x = \frac{1}{\gamma}\Delta x'$$

Time intervals and distances depend on the motion of the observer.

$$\Delta t \Leftarrow \gamma \Delta t' \quad \textit{and} \quad \Delta x \Leftarrow \frac{1}{\gamma} \Delta x'$$

$$(\textit{not standard equations!!})$$

Perpendicular dimensions remain: $\Delta y = \Delta y', \Delta z = \Delta z'$

Example: Length contraction, Michelson-Morley

Example: Muons created in upper atmosphere, v=0.994c, γ=9, travel a distance l.

Lifetime in restframe of muons

$$T'_{1/2} = 1.5 \mu s \quad \rightarrow \quad l' = 450\mathrm{m}$$

Lifetime on earth

$$T_{1/2} = \gamma T'_{1/2} = 13.5 \mu s \quad \rightarrow \quad l = 4\mathrm{km}$$

Transformation of velocity

A particle moving with velocity $\vec{u}\,'$ in $S\,'$ *has velocity* $\vec{u}$ in S

$$u_x = \frac{dx}{dt} = \frac{dx}{dt'}\frac{dt'}{dt} = \gamma\left(\frac{dx'}{dt'} + \beta c\right)\frac{dt'}{dt} = \gamma\left(u_x' + v\right)\frac{dt'}{dt}$$

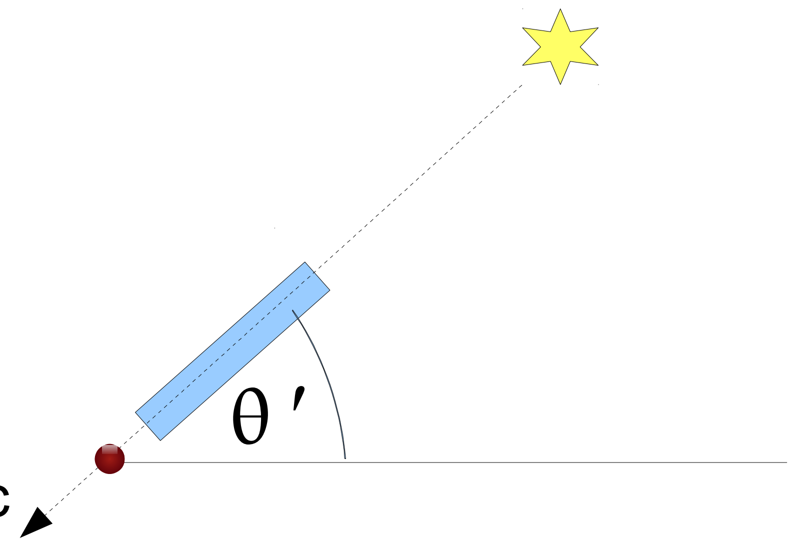
$$\frac{dt}{dt'} = \gamma\left(1 + \frac{\beta}{c}\frac{dx'}{dt'}\right) = \gamma\left(1 + \frac{v}{c^2}u_x'\right)$$

$$u_x = \frac{u_x' + v}{1 + vu_x'/c^2}, \quad u_y = \frac{u_y'}{\gamma\left(1 + vu_x'/c^2\right)}, \quad u_z = \frac{u_z'}{\gamma\left(1 + vu_x'/c^2\right)}$$

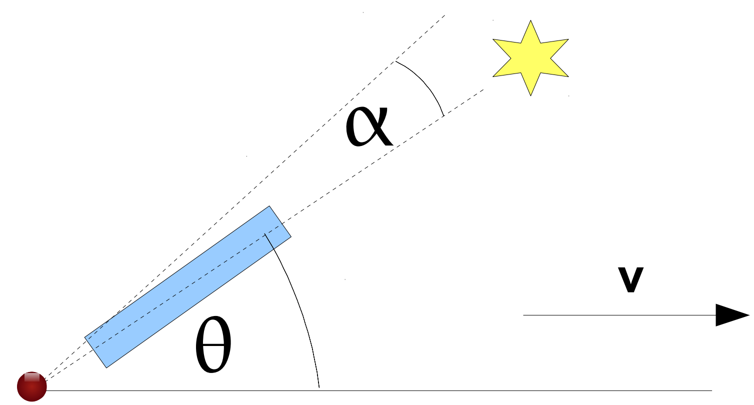
Example: Light aberration

A star appears on earth under an angle different from its real position.

Earth at rest:



Earth moving with v:



If the earth were at rest the light moves with

$$u'_x = -c\cos\vartheta', \qquad u'_y = -c\sin\vartheta'$$

For the moving earth the velocity transforms to

$$u_x = -c\cos\vartheta = \frac{-c\cos\vartheta' - v}{1+\beta\cos\vartheta'}$$

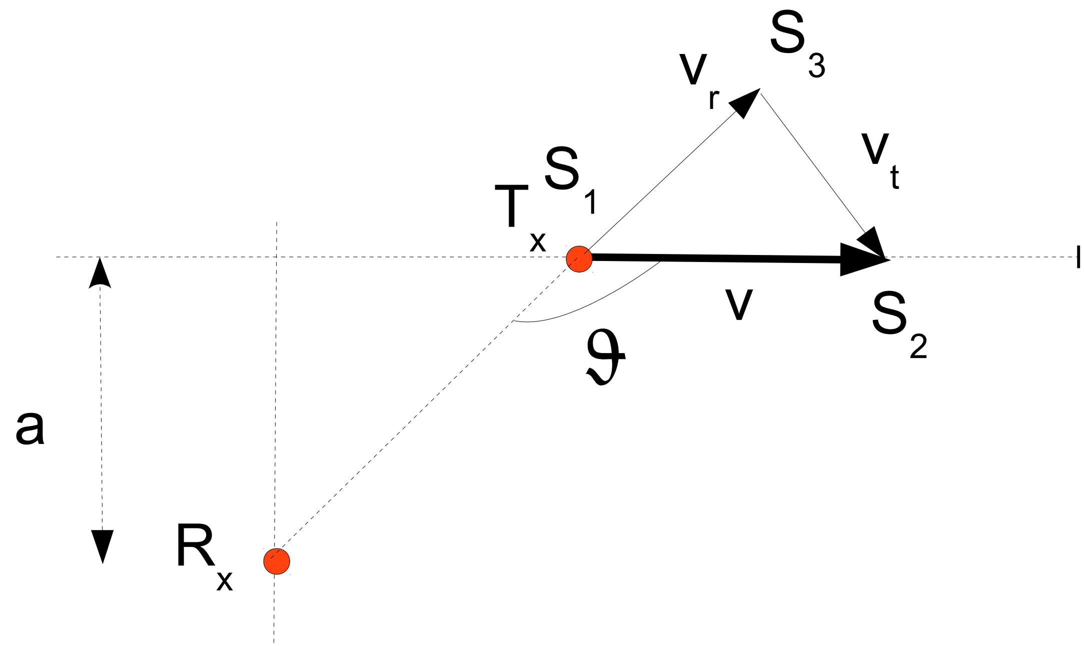
$$u_y = -c\sin\vartheta = \frac{-c\sin\vartheta'}{\gamma(1+\beta\cos\vartheta')}$$

Using $\tan\frac{\vartheta}{2} = \frac{\sin\vartheta}{1+\cos\vartheta}$ we obtain

$$\tan\frac{\vartheta}{2} = \frac{\sin\vartheta'}{\gamma(1+\beta)(1+\cos\vartheta')} = \sqrt{\frac{1-\beta}{1+\beta}}\tan\frac{\vartheta'}{2} < \tan\frac{\vartheta'}{2}$$

Example: Doppler effect

Emitter T_x is moving with v, receiver R_x at rest.



$$v_r = v\cos(\pi - \vartheta) = -v\cos(\vartheta)$$

A signal emitted at S_1 reaches R_x at time

$$t_1 = \frac{1}{c}\overline{R_x S_1} = \frac{a}{c\sin\vartheta}$$

T_x moves from S_1 to S_2 in an RF period T_0.

At S_2 it emitts a signal which reaches R_x at the time

$$t_2 = \gamma T_0 + \frac{1}{c}\overline{R_x S_3} = \gamma T_0 + \frac{1}{c}\left[\frac{a}{\sin\vartheta} - v\gamma T_0 \cos\vartheta\right]$$

where T_0 has been dilated by γ and
$\overline{R_x S_2} = \overline{R_x S_3}$ for $vT_0 \ll \overline{R_x S_1}$ has been used.

The RF period T experienced by R_x is

$$T = t_2 - t_1 = \gamma(1 - \beta\cos\vartheta)T_0$$

therefore

$$\frac{f}{f_0} = \frac{T_0}{T} = \frac{1}{\gamma(1-\beta\cos\vartheta)} = \frac{\sqrt{1-\beta^2}}{1-\beta\cos\vartheta}$$

$$\vartheta \to \pi: \qquad \frac{f}{f_0} \to \sqrt{\frac{1-\beta}{1+\beta}}$$ longitudinal Doppler effect

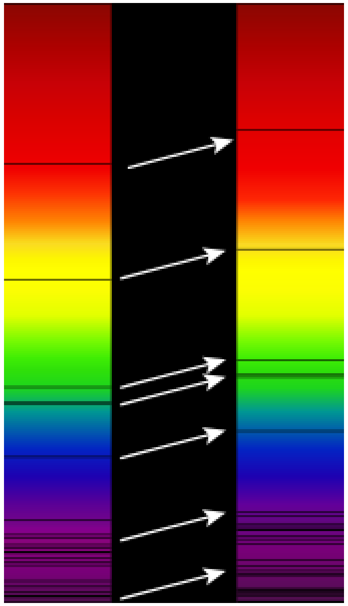
$$\vartheta = \frac{\pi}{2}: \qquad \frac{f}{f_0} = \sqrt{1-\beta^2}$$ transverse Doppler effect

Example: Astronomy

Transverse Doppler « longitudinal Doppler ($v \approx v_r$)

$$\frac{f_0}{f} = \frac{\lambda}{\lambda_0} = 1 + z = \sqrt{\frac{1+\beta}{1-\beta}}$$ → redshift: $z > 0$

e.g. $z = 3$: → $\beta = \frac{(1+z)^2 - 1}{(1+z)^2 + 1} = 0.882$



Transformation of acceleration

A particle moving with $\vec{u}\,'$ in S' and experiencing an acceleration $\vec{a}\,'$ has an acceleration $\vec{a}$ in S

$$a_x = \frac{du_x}{dt} = \frac{du_x}{dt'}\frac{dt'}{dt} = \frac{d}{dt'}\frac{u'_x + v}{1 + vu'_x/c^2}\frac{dt'}{dt} =$$

$$= \frac{a'_x}{\gamma^3 (1 + vu'_x/c^2)^3}$$

$$a_y = \frac{a'_y}{\gamma^2 (1 + vu'_x/c^2)^2} - \frac{(vu'_y/c^2)\,a'_x}{\gamma^2 (1 + vu'_x/c^2)^3}$$

$$a_z = \frac{a'_z}{\gamma^2 (1 + vu'_x/c^2)^2} - \frac{(vu'_z/c^2)\,a'_x}{\gamma^2 (1 + vu'_x/c^2)^3}$$

Acceleration in an inertial system is possible !!

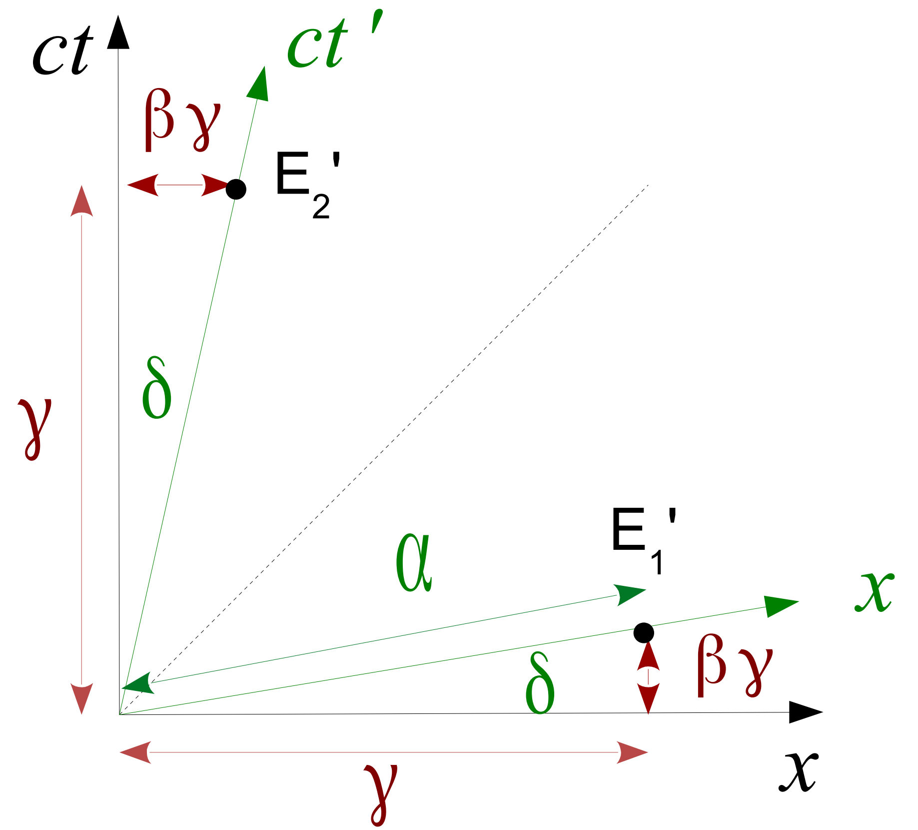
Minkowski diagram

Shows the coordinates in S' for an event E in S and vice versa.

$L-T:$
$ct=\gamma(ct'+\beta x')$
$x=\gamma(x'+\beta ct')$

Events E_1'(ct'=0,x'=1) and E_2'(ct'=1,x'=0) appear in S at (ct=βγ,x=γ) and (ct=γ,x=βγ)

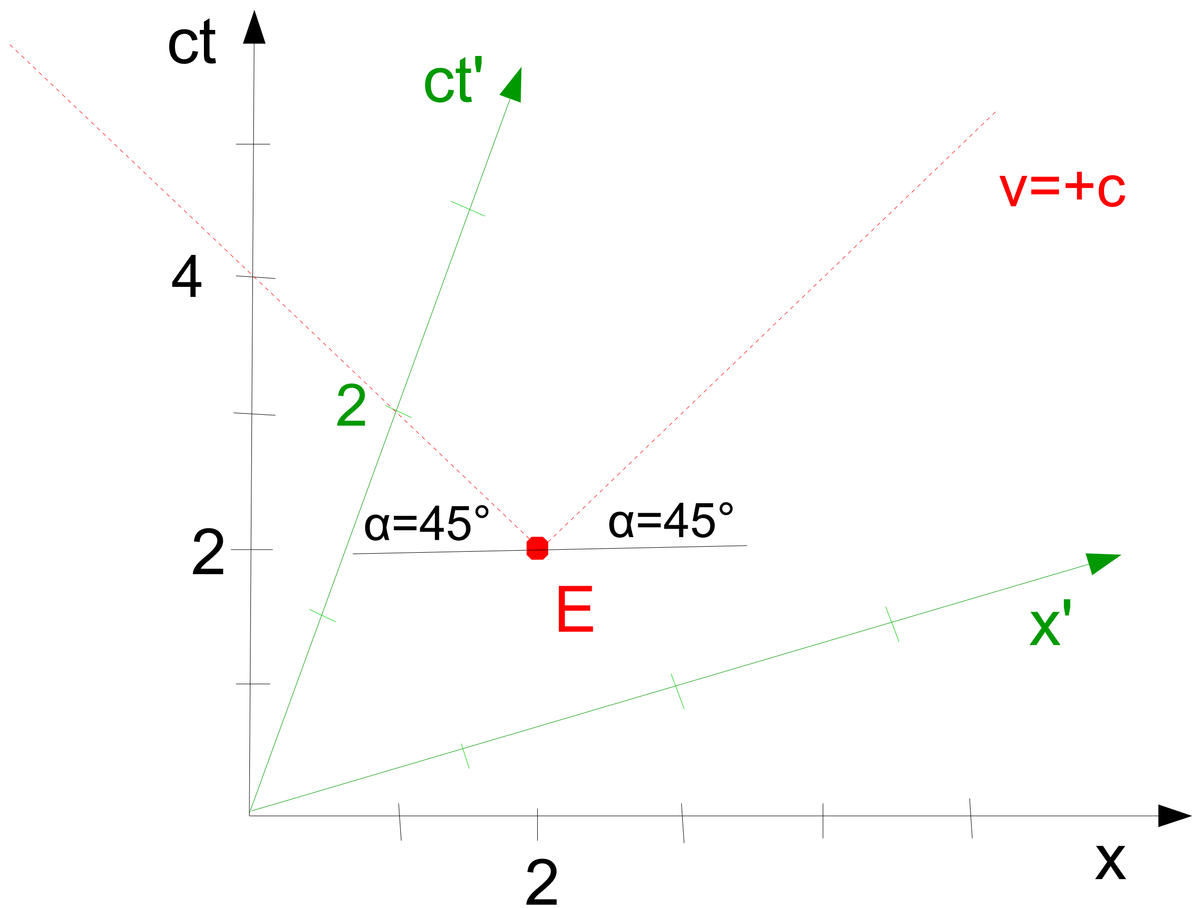
$$\tan(\delta)=\beta$$



Scale in S': $\alpha=\sqrt{\gamma^2+\beta^2\gamma^2}=\sqrt{\dfrac{1+\beta^2}{1-\beta^2}}$

Light pulse

Event E is a spherical light shell expanding in all directions from position x=2 at time ct=2.
The light reaches an observer at x=0 at the time ct=4 and an observer at x'=0 at time ct'=2.



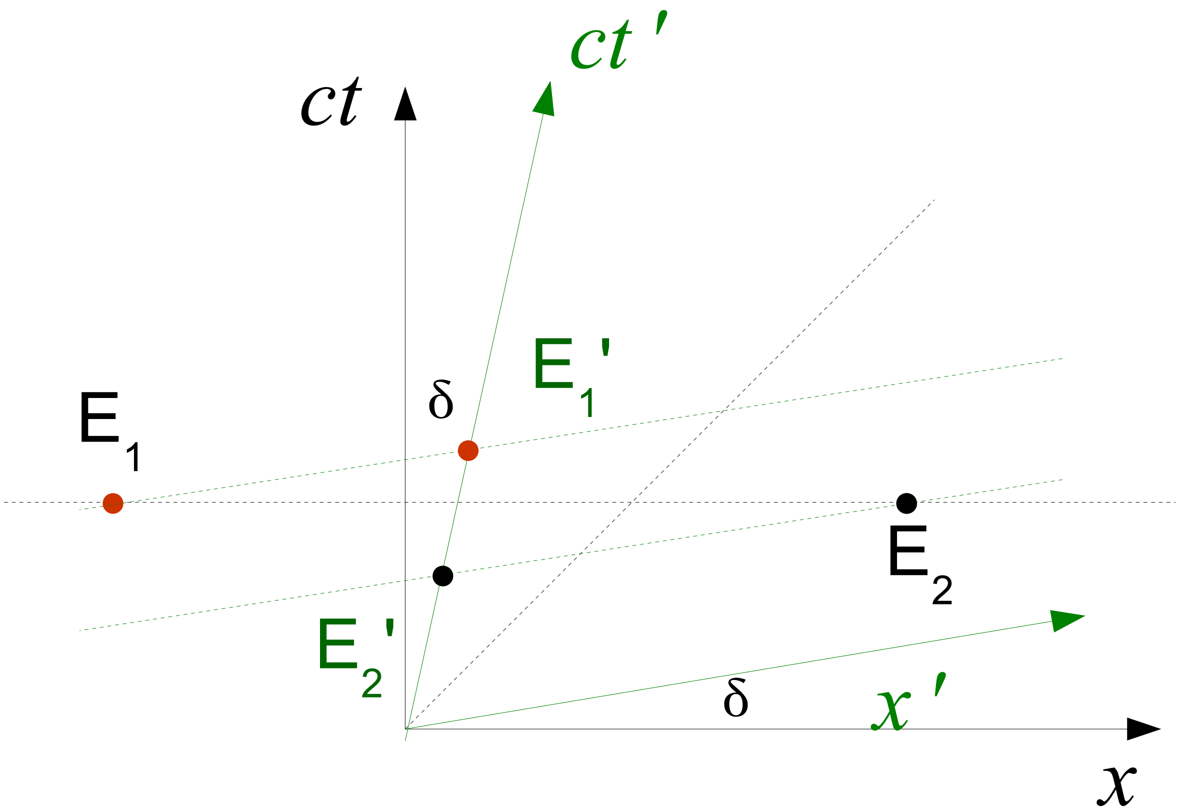
Light shell

$$x^2+y^2+z^2=(ct)^2$$

$$y=z=0 \rightarrow ct=\pm x$$

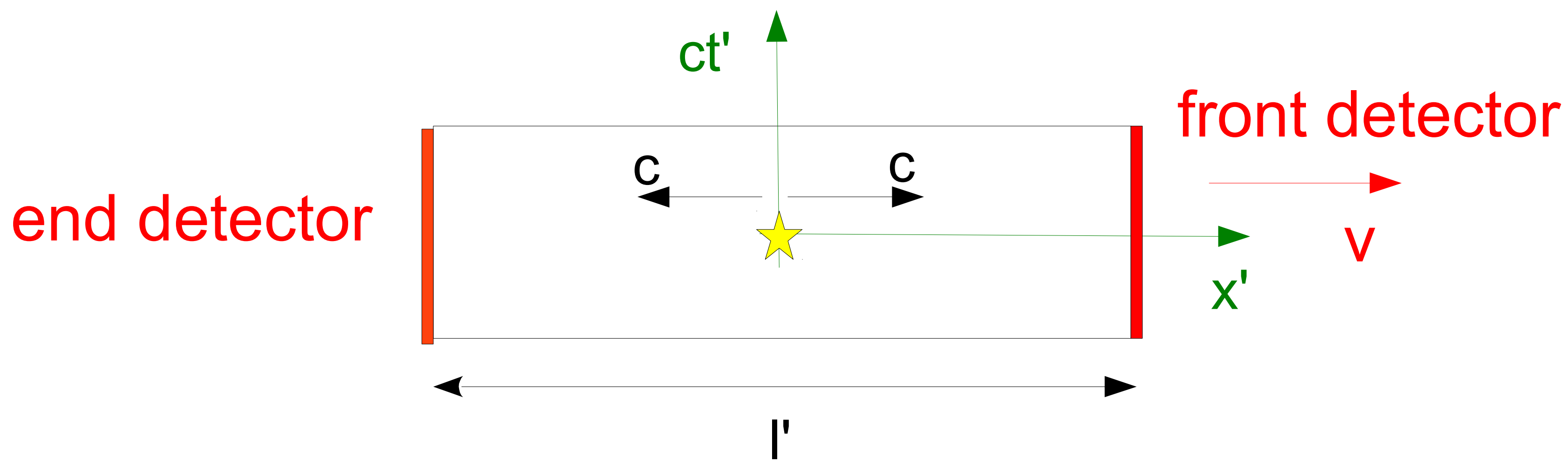
Simultaneity

Events E_1, E_2 which are simultaneous in S are not simultaneous in S'.



Example: Light flash in rocket

Rocket is moving with v (frame S'). Light flash is emitted at the center and reaches the front and end detector at the same time. In S the times are different.

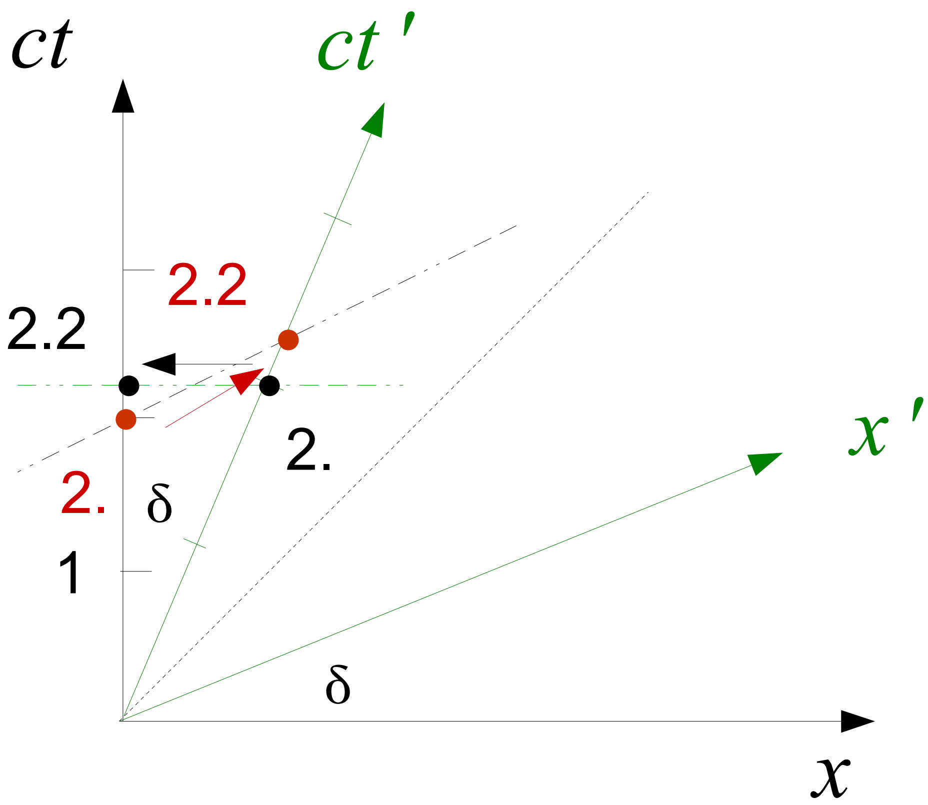


in space craft: $ct_f' = ct_e' = \frac{l'}{2}$

on earth: $ct_f = \frac{l}{2} + vt_f, \quad ct_e = \frac{l}{2} - vt_e, \quad t_f - t_e = \gamma^2 \beta \frac{l}{c} = \gamma \beta \frac{l'}{c}$

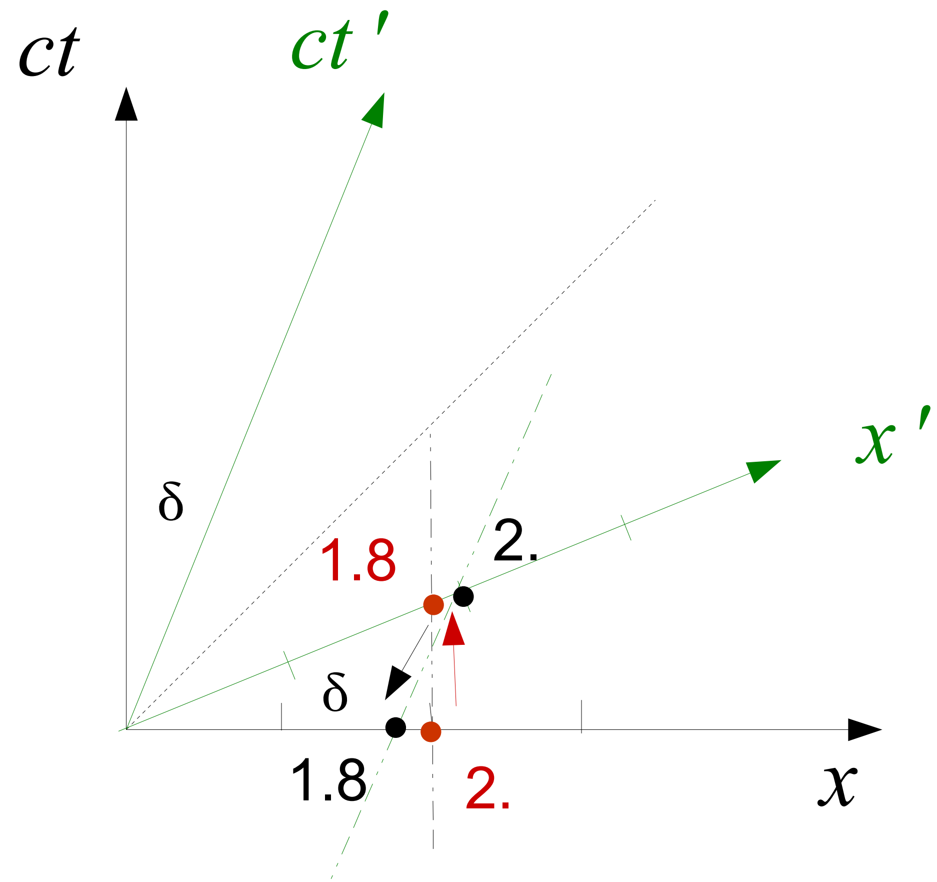
Time dilation:

β=0.42, δ=22.8°, γ=1.1, α=1.2

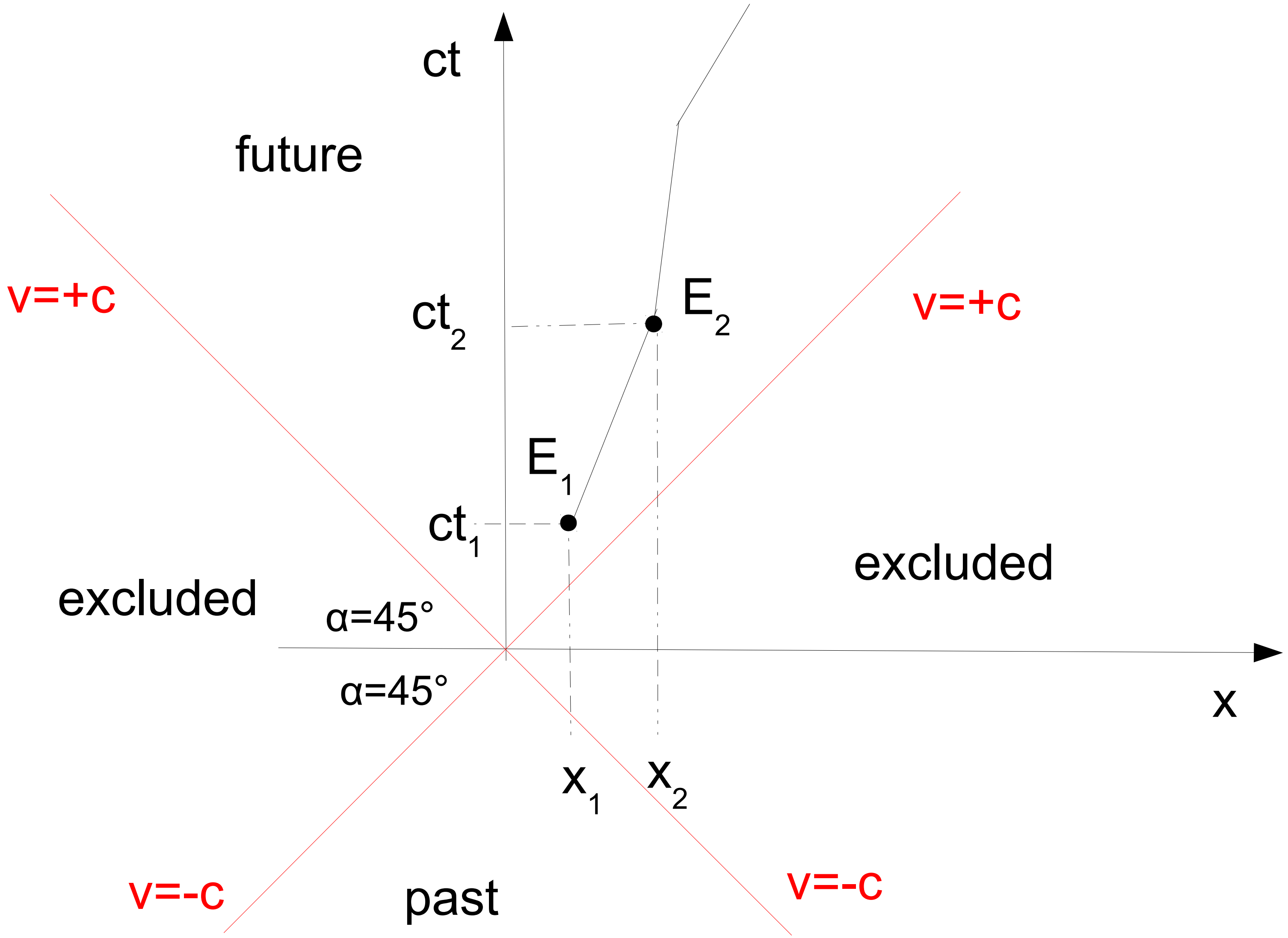


Length contraction:

β=0.42, δ=22.8°, γ=1.1, α=1.2



World-line (path-time diagram)



Relativistic Dynamics

Based on two principles:

1. Conservation of linear momentum   ($p=mu$)

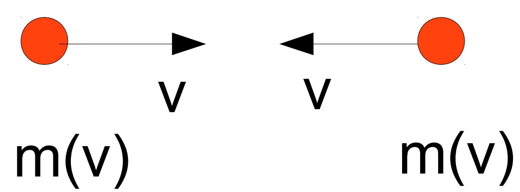
2. Conservation of energy   ($E=mc^2$)

Moving mass

Because of $E=mc^2$ we choose as ansatz $m=m(v)$ and calculate the function $m(v)$ by an experiment.

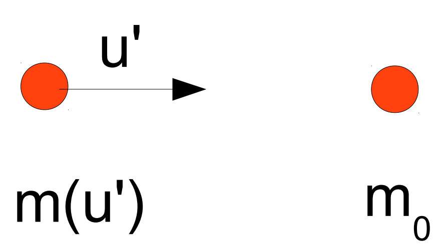
Inelastic collision between 2 identical particles:

In laboratory frame S:



Composite particle at rest after collision

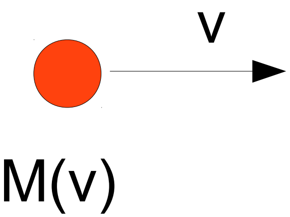
In rest frame S' of right particle:



S moves with v to the right w.r.t. S'.
The left particle has velocity u=v in the moving frame S and u' in the rest frame S'.
Therefore

$$u' = \frac{u + v}{1 + vu/c^2} \rightarrow u' = \frac{2v}{1+\beta^2}, \quad 1 - \left(\frac{u'}{c}\right)^2 = \left(\frac{1-\beta^2}{1+\beta^2}\right)^2 \qquad (1)$$

After collision composite particle moves with v in S'



Conservation of momentum $$m(u')u' = M(v)v \qquad (2)$$

Conservation of energy $$m(u')c^2 + m_0 c^2 = M(v)c^2 \qquad (3)$$

From (1), (2), (3) after eliminating M

$$m(u')=\frac{1+\beta^2}{1-\beta^2}m_0=\frac{m_0}{\sqrt{1-(u'/c)^2}}=\gamma_{u'}m_0 \qquad (4)$$

Mass

$$m(u')=\gamma_{u'}m_0$$

From (1), (2), (4)

$$M(v)=\frac{2m_0}{1-(v/c)^2}=\gamma M_0 \quad \rightarrow \quad M_0=2\gamma m_0$$

Rest mass not conserved: $M_0-2m_0=2m_0(\gamma-1)>0$

E_{kin} completely converted into mass:

$$2E_{kin}=2(E-E_0)=2(\gamma m_0c^2-m_0c^2)=(M_0-2m_0)c^2$$

Momentum $\vec{p}(u)=m(u)\vec{u}=\gamma_u m_0 \vec{u}$

Force $\vec{f}=\frac{d\vec{p}}{dt}=m_0\frac{d\gamma_u}{dt}\vec{u}+m_0\gamma_u\frac{d\vec{u}}{dt}$

with $\frac{d\gamma_u}{dt}=\frac{d}{dt}\frac{1}{\sqrt{1-\vec{u}\cdot\vec{u}/c^2}}=\frac{\gamma_u^3}{c^2}(\vec{u}\cdot\vec{a})$ we get

Force $\vec{f}=\gamma_u^3\frac{m_0}{c^2}(\vec{u}\cdot\vec{a})\vec{u}+\gamma_u m_0\vec{a}$

$\vec{f},\vec{u},\vec{a}$ are in general not collinear!

Using $\vec{f}\cdot\vec{u}=\gamma_u^3 m_0(\vec{u}\cdot\vec{a})$ one can solve for $\vec{a}$

$$\gamma_u m_0\vec{a}=\vec{f}-\frac{1}{c^2}(\vec{f}\cdot\vec{u})\vec{u}$$

Force: $\vec{f} = \gamma_u^3 \frac{m_0}{c^2}(\vec{u}\cdot\vec{a})\vec{u} + \gamma_u m_0 \vec{a}$

In *linear motion (linac)* with $\vec{u} = (u_x, 0, 0)$, $\vec{f} = (f_x, 0, 0)$, $\vec{a} = (a_x, 0, 0)$ it is

$$f_x = \gamma_u^3 m_0 a_x$$

and one speaks of

$$\text{longitudinal mass} = \gamma_u^3 m_0$$

In *circular motion (synchrotron)* with $\vec{f} \perp \vec{u}$ it is

$$\vec{f} = \gamma_u m_0 \vec{a}$$

and one speaks of

$$\text{transverse mass} = \gamma_u m_0$$

Energy

A particle moves with $\vec{u}=(u_x,0,0)$ and experiences a force f_x.

Work done at path dx is

$$dE_{kin}=f_x dx=\gamma_u^3 m_0 a_x dx=\gamma_u^3 m_0 \frac{du_x}{dt} dx=\gamma_u^3 m_0 u_x du_x$$

$$E_{kin}=m_0 c^2 \int_o^{\beta_u} \frac{\beta_u d\beta_u}{(1-\beta_u^2)^{3/2}}=\gamma_u m_0 c^2-m_0 c^2=E-E_0$$

Power absorbed by the particle

$$P=\frac{dE_{kin}}{dt}=\frac{d\left(m-m_0\right)}{dt}c^2=\frac{dm}{dt}c^2$$

$$\vec{f}=\frac{d\vec{p}}{dt}=\frac{dm}{dt}\vec{u}+m\frac{d\vec{u}}{dt}=\frac{1}{c^2}\frac{dE_{kin}}{dt}\vec{u}+\gamma_u m_0\vec{a}$$

before $\gamma_u m_0\vec{a}=\vec{f}-\frac{1}{c^2}\left(\vec{f}\cdot\vec{u}\right)\vec{u}$

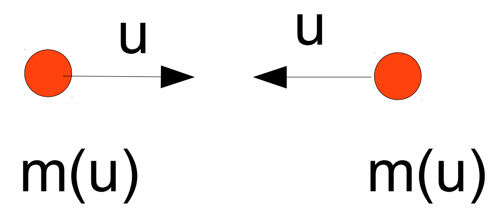
$$\rightarrow \quad P=\frac{dE_{kin}}{dt}=\vec{f}\cdot\vec{u}$$

The temporal change of E_{kin} of a body, or the power it absorbs, is the scalar product of $\vec{f}$ and $\vec{u}$, as in classical mechanics.

Example: Collider

1) 3.5 TeV head-on p-p collider

before collision



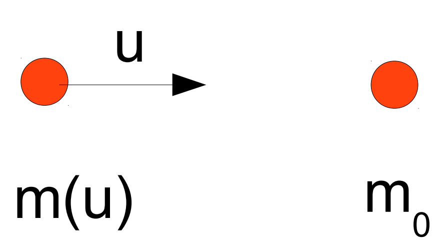
composite particle at rest after inelastic collision

transparency 27:

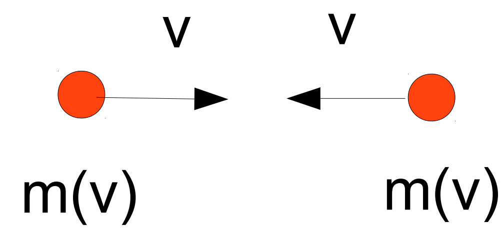
$$M_0 = 2\gamma_u m_0 \quad \rightarrow \quad E_{CM} = M_0 c^2 = 2\gamma_u m_0 c^2 = 7\,TeV$$

2) 3.5 TeV fixed target p-machine

In laboratory frame S before collision



Center of mass frame S' (Σp=0) moves to the right with v



While seen from S', S moves to the left with v.
The left particle has velocity v in S' and it is (transp. 11)

$$v = \frac{u - v}{1 - uv/c^2}. \quad \text{This yields}$$

$$\beta_v = \frac{1}{\beta_u}\left(1 - \sqrt{1 - \beta_u^2}\right), \qquad \gamma_v = \sqrt{\frac{1}{2}(1 + \gamma_u)}$$

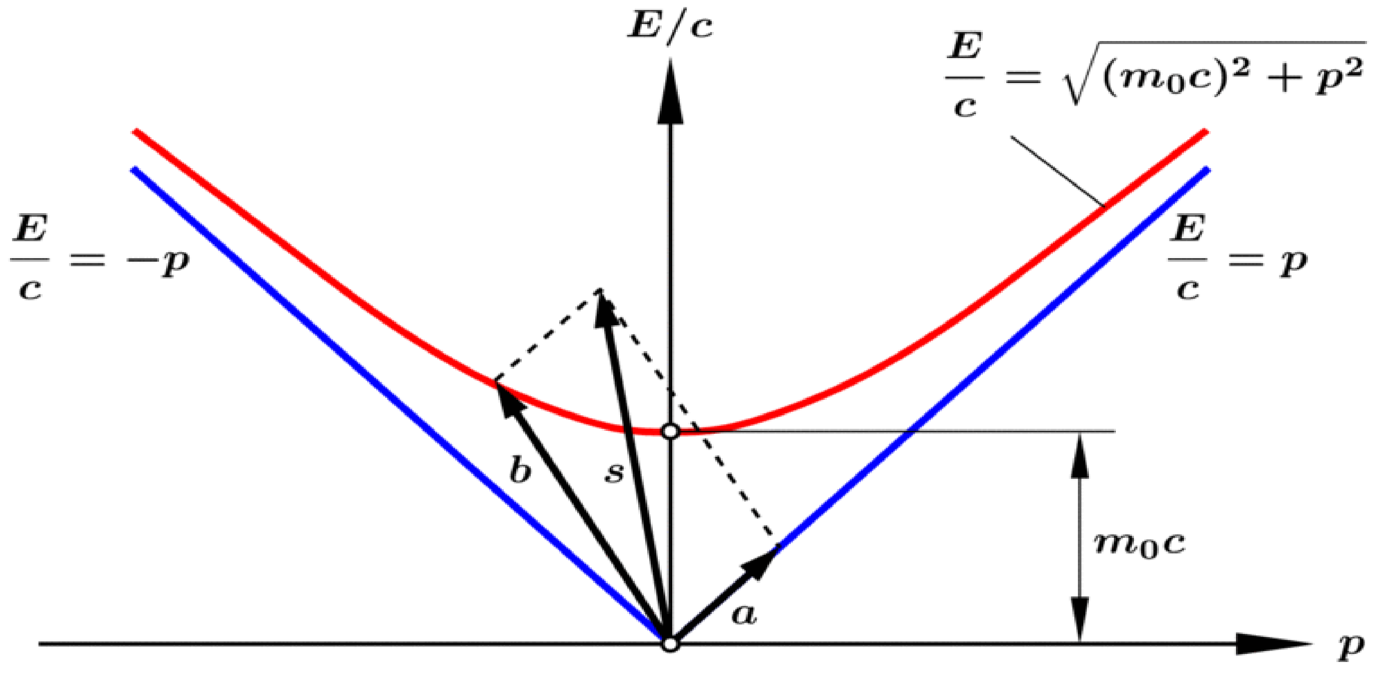
$$E_{CM} = 2\gamma_v E_0 = \sqrt{2(1 + \gamma_u)}\, E_0 =$$

$$= \sqrt{2(E_0 + E)E_0} = 81\, GeV \qquad (E_{p0} = 938\, MeV)$$

Energy-momentum equation and diagram

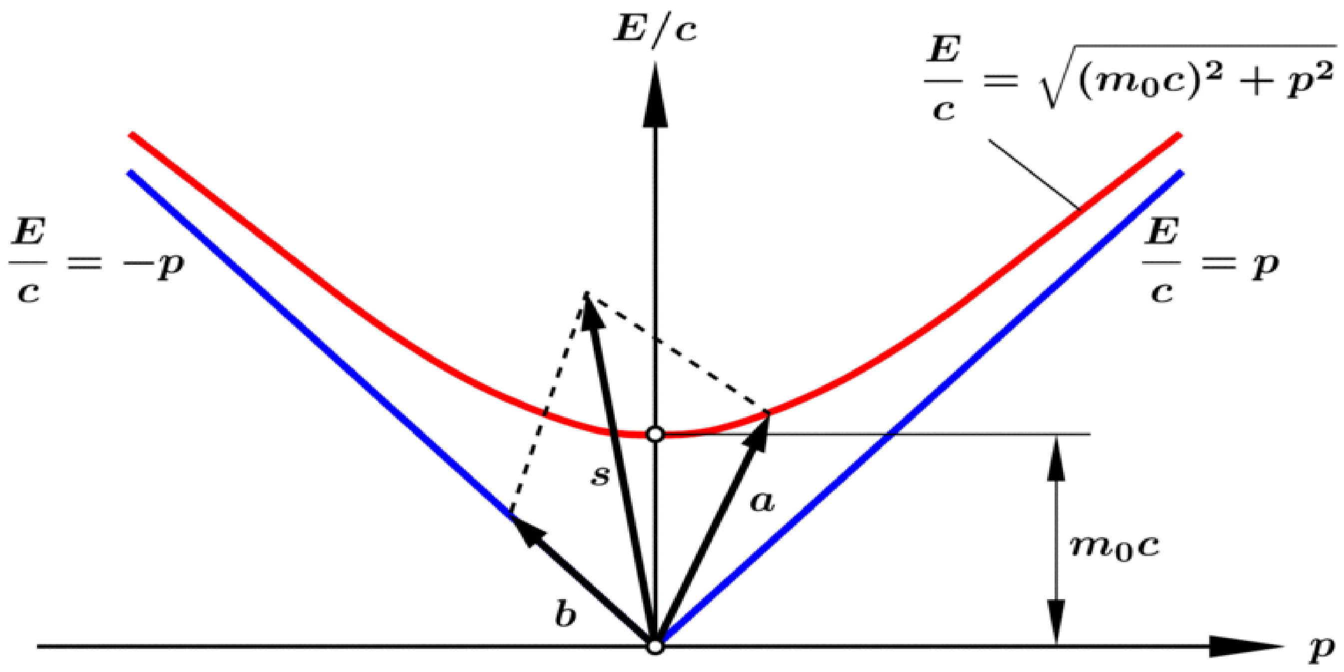
$$E^2=(mc^2)^2=(m_0c^2)^2\gamma^2=(m_0c^2)^2\frac{(1-\beta^2)+\beta^2}{1-\beta^2}=E_0^2+(pc)^2$$

$$\frac{E}{c}=\sqrt{(m_0c)^2+p^2},\qquad mass-less\ particle:\quad E=|p|c$$

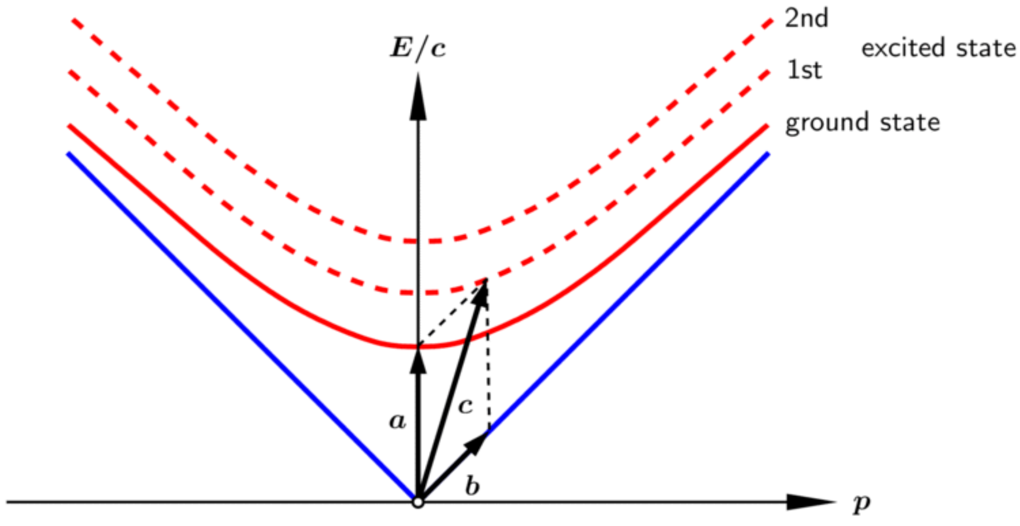


Since conserved quantities are plotted, arrows can be added like vectors.

All interactions are allowed in which energy-momentum vectors a, b after interaction add to vector s.

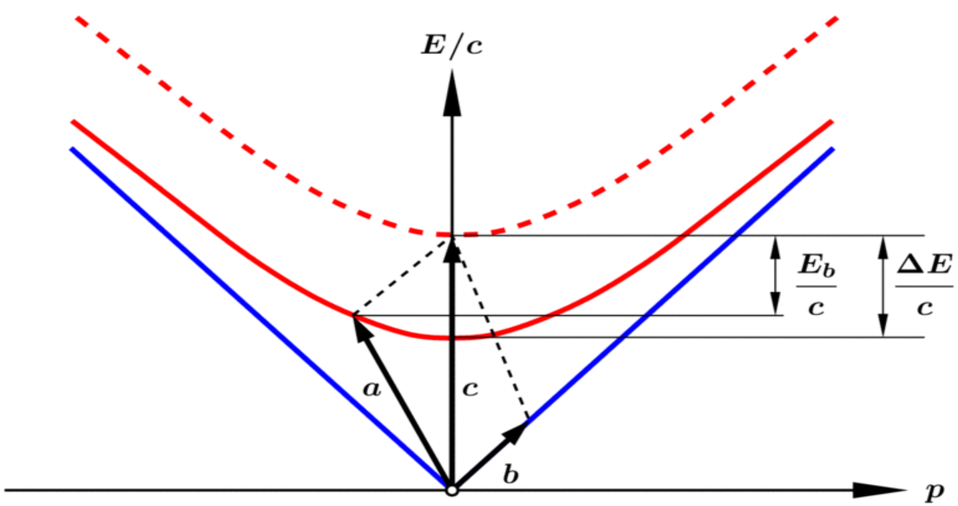


Example: Photon absorption by a particle at rest



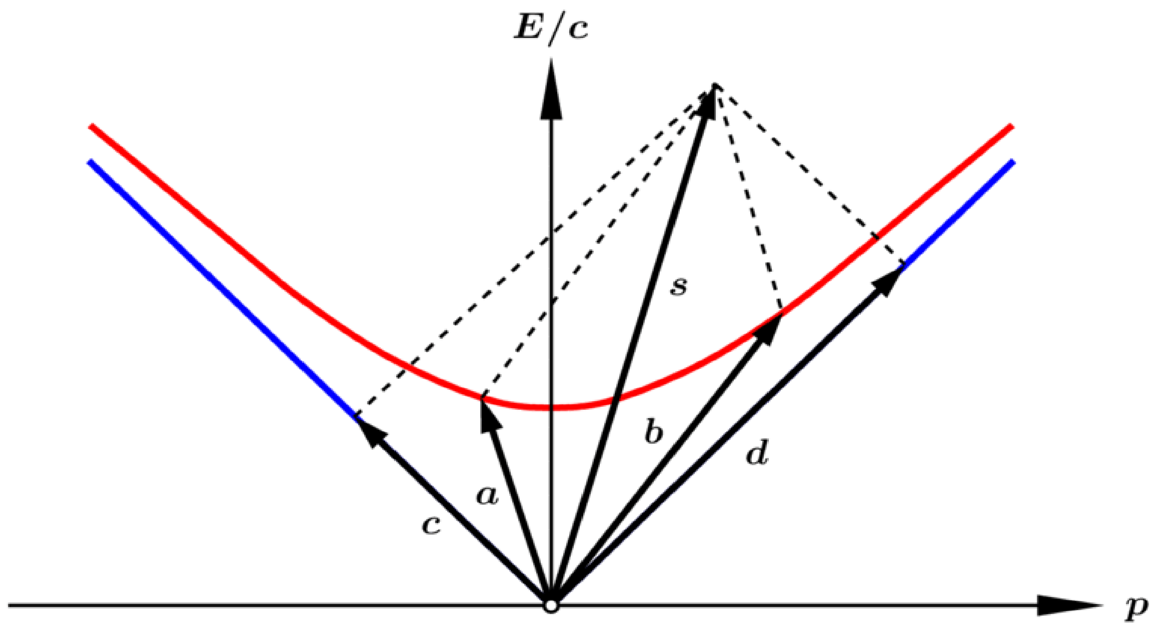
Absorption only for composite particles with excited states.

Example: Photon emission by a composite particle at rest



$\Delta E > E_b$
difference in recoil of particle

Example: Pair annihilation



4-Vectors

Normal, 3-dimensional vectors are defined by a linear transformation. They are invariant against translation and rotation of the coordinate system.

Similar we define 4-vectors by the Lorentz-Transformation:
Any quadruple which transforms with an L-T is a 4-vector.

Let us define a contra- and a covariant 4-vector:

contravariant $X^{\mu}=(X^0, X^1, X^2, X^3)$

covariant $X_{\mu}=(X_0, -X_1, -X_2, -X_3)$

Using Einsteins summation rule the scalar product is

$$X^{\mu} X_{\mu} = X^0 X_0 - X^1 X_1 - X^2 X_2 - X^3 X_3 = X'^{\mu} X'_{\mu}$$

It is invariant under an L-T.

In general:

The scalar product of any two 4-vectors is L-T invariant.

$$A^{\mu} B_{\mu} = A'^{\mu} B'_{\mu}$$

Position 4-vector

The quadruple

$X^{\mu}=(ct, x, y, z)$

transforms with an L-T and is called position 4-vector

Velocity 4-vector

$U^{\mu}=\frac{dX^{\mu}}{dt}$ is not a 4-vector, since dt is not invariant. Try to find a quantity with dimension of time and which is invariant.

An event which moves by (dx, dy, dz) in dt has the Lorentz invariant space-time interval ds

$$ds^2=dX^{\mu}dX_{\mu}=(cdt)^2-dx^2-dy^2-dz^2=dX'^{\mu}dX'_{\mu}$$

We write

$$ds=cdt\sqrt{1-\frac{1}{c^2}\left(\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}\right)}=c\,dt\sqrt{1-\left(\frac{v}{c}\right)^2}=$$

$$=c\frac{dt}{\gamma}=c\,d\tau \quad \rightarrow \quad dt=\gamma\,d\tau$$

Now, $d\tau$ is the time interval an observer moving with v would measure. τ is called proper time and is Lorentz invariant (Lorentz scalar). Using τ instead of t, we get the velocity 4-vector

$$U^\mu = \frac{dX^\mu}{d\tau} = \frac{dX^\mu}{dt}\frac{dt}{d\tau} = \left(c, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)\frac{dt}{d\tau} =$$

$$= \gamma\left(c, v_x, v_y, v_z\right) \quad \rightarrow$$

$$U^\mu U_\mu = \gamma^2 c^2 \left[1 - \frac{1}{c^2}\left(v_x^2 + v_y^2 + v_z^2\right)\right] = c^2$$

U^μ is not a measurable quantity. dX^μ is space-time distance between two events in one frame, while $d\tau$ is time increment in a different frame in which both events take place at the same location.

But U^μ helps to facilitate calculations.

Example: Man in aircraft and on ground

Energy-momentum 4-vector

A particle with m_0 moves in S with $\vec{u}=(u_x,0,0)$

$$E=\gamma_u m_0 c^2, \qquad p_x=\gamma_u m_0 u_x=E\frac{u_x}{c^2} \qquad (1)$$

In S' it's velocity (transp. 11), energy and momentum are together with (1)

$$u_x'=\frac{u_x-v}{1-u_x v/c^2} \quad \rightarrow \quad \gamma_{u'}=\frac{1}{\sqrt{1-(u_x'/c)^2}}=\gamma_u\gamma(1-\beta_u\beta)$$

$$\frac{E'}{c}=\gamma_{u'}m_0 c=\gamma\left(\frac{E}{c}-\beta p_x\right)$$

$$p_x'=\gamma_{u'}m_0 u_x'=\gamma\left(p_x-\beta\frac{E}{c}\right), \qquad p_y'=p_y=0, \qquad p_z'=p_z=0$$

Since E/c and p_x transform with L-T, they form the energy-momentum 4-vector $P^\mu=(E/c, p_x, p_y, p_z)$.

1. Derivation of energy momentum equation:

In *a frame where momentum does not vanish*:

$$P^\mu P_\mu = (E/c)^2 - p_x^2 - p_y^2 - p_z^2 = (E/c)^2 - p^2$$

In *a frame with vanishing momentum*:

$$P'^\mu P'_\mu = (E_0/c)^2$$

$$P^\mu P_\mu = P'^\mu P'_\mu \quad \rightarrow \quad E^2 = E_0^2 + (pc)^2$$

2. Derivation of Planck's hypothesis E=hν

A photon with energy E' in S' travels in -x' direction

$$p_x' = -\frac{E'}{c}, \qquad \text{L-T} \rightarrow \frac{E}{c} = \gamma\left(\frac{E'}{c} + \beta p_x'\right) = \sqrt{\frac{1-\beta}{1+\beta}}\frac{E'}{c} \qquad (1)$$

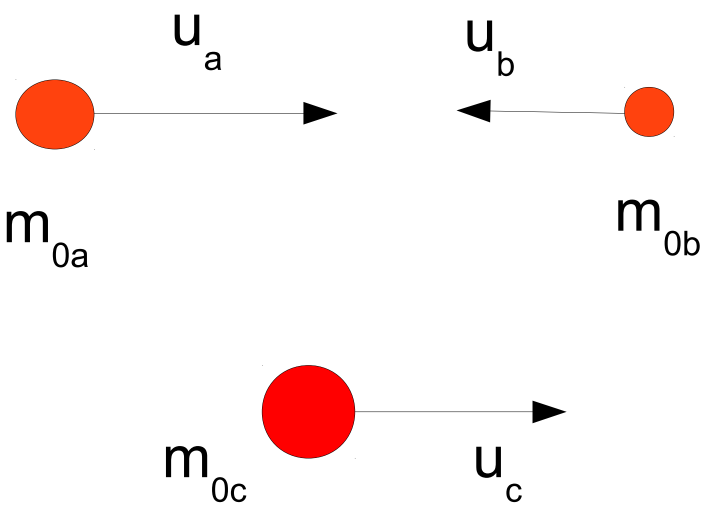
Frequ. Doppler shift (transp. 17, θ=180°): $\nu = \sqrt{\frac{1-\beta}{1+\beta}}\nu' \quad (2)$

(1) devided by (2): $\frac{E}{\nu} = \frac{E'}{\nu'} = const. = h$

3. Inelastic collision:

before collision

after collision



Energy, momentum conservation:

$$E_a + E_b = E_c, \quad \vec{p}_a + \vec{p}_b = \vec{p}_c$$

$$\rightarrow \quad P_a^\mu + P_b^\mu = P_c^\mu \qquad \cdot (P_{a\mu} + P_{b\mu} = P_{c\mu})$$

$$P_a^\mu P_{a\mu} + 2P_a^\mu P_{b\mu} + P_b^\mu P_{b\mu} = P_c^\mu P_{c\mu} \qquad (1)$$

Rest frames for (a), (b), (c) $(E/c = m_o c)$:

$$P_a^\mu P_{a\mu} = (m_{0a} c)^2, \qquad P_b^\mu P_{b\mu} = (m_{0b} c)^2$$

$$P_c^\mu P_{c\mu} = (m_{0c} c)^2 \qquad (2)$$

Laboratory frame:

$$P_a^\mu = (\gamma_a m_{0a} c, \gamma_a m_{0a} u_a, 0, 0)$$

$$P_b^\mu = (\gamma_b m_{0b} c, -\gamma_b m_{0b} u_b, 0, 0)$$

$$2 P_a^\mu P_{b\mu} = 2 \gamma_a \gamma_b m_{0a} m_{0b} (c^2 + u_a u_b) \qquad (3)$$

(2), (3) substituted in (1)

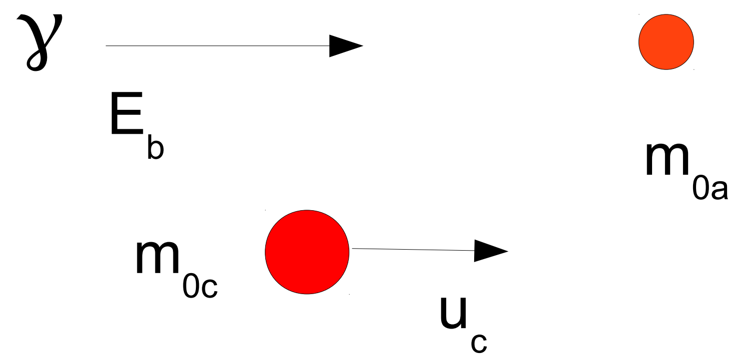
$$m_{0c} = \sqrt{m_{0a}^2 + m_{0b}^2 + 2 m_{0a} m_{0b} \gamma_a \gamma_b \left(1 + \frac{u_a u_b}{c^2}\right)} \geq m_{0a} + m_{0b}$$

Rest mass of particle (c) is larger than the rest masses of (a) and (b).

4. Absorption of a photon by an atom at rest

before absorption

after absorption



see example 3: $$P_a^{\mu} P_{a\mu} + 2P_a^{\mu} P_{b\mu} + P_b^{\mu} P_{b\mu} = P_c^{\mu} P_{c\mu} \qquad (1)$$

rest frame of (a) and (c)

$$P_a^{\mu} = (m_{0a} c, 0, 0, 0), \qquad P_c^{\mu} = (m_{0c} c, 0, 0, 0)$$

photon (b) frame $$P_b^{\mu} = \left(\frac{E_b}{c}, p_{bx}, 0, 0\right) = \left(\frac{h\nu}{c}, \frac{h\nu}{c}, 0, 0\right)$$

$$P_a^\mu P_{a\mu} = (m_{0a} c)^2, \qquad P_b^\mu P_{b\mu} = 0$$
$$P_c^\mu P_{c\mu} = (m_{0c} c)^2, \qquad P_a^\mu P_{b\mu} = m_{0a} h\nu \qquad (2)$$

(2) substituted in (1):

$$(m_{0a} c)^2 + 2\mathrm{m}_{0a} c \frac{h\nu}{c} + 0 = (m_{0c} c)^2$$

$$m_{oc} = \sqrt{m_{0a}^2 + 2\mathrm{m}_{0a} \frac{h\nu}{c^2}} = m_{0a} \sqrt{1 + 2\frac{h\nu}{m_{0a} c^2}}$$

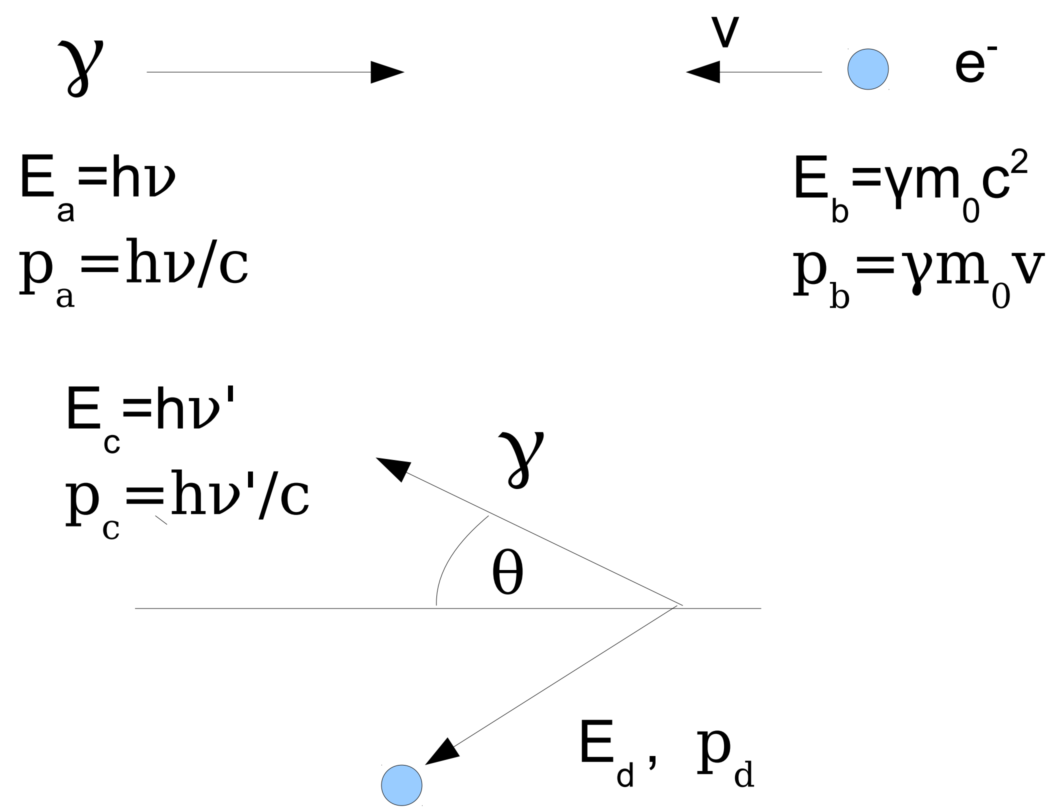
$$\textit{If } m_{0a} c^2 \gg h\nu \quad \rightarrow \quad m_{oc} c^2 \approx m_{oa} c^2 + h\nu$$

Rest energy of particle (c) equals rest energy of particle (a) plus photon energy.

5. Compton effect (photon scattered at electron)

before collision

after collision



Energy, momentum conservation:

$$P_a^{\mu} + P_b^{\mu} = P_c^{\mu} + P_d^{\mu} \qquad (1)$$

Scalar product of (1) with itself $\rightarrow$

$$P_a^\mu P_{a\mu} + 2P_a^\mu P_{b\mu} + P_b^\mu P_{b\mu} = P_c^\mu P_{c\mu} + 2P_c^\mu P_{d\mu} + P_d^\mu P_{d\mu} \quad (2)$$

for photons: $P_a^\mu P_{a\mu} = P_c^\mu P_{c\mu} = 0$

in rest frames of (b), (d): $P_b^\mu P_{b\mu} = P_d^\mu P_{d\mu}$

substituted in (2): $P_a^\mu P_{b\mu} = P_c^\mu P_{d\mu} \quad (3)$

multiplication of (1) with $P_{c\mu}$ and use of (3):

$$P_a^\mu P_{c\mu} + P_b^\mu P_{c\mu} = P_c^\mu P_{c\mu} + P_d^\mu P_{c\mu} = P_a^\mu P_{b\mu} \quad (4)$$

in laboratory frame:

$$P_a^\mu = \left(\frac{h\nu}{c}, \frac{h\nu}{c}, 0, 0\right)$$

$$P_b^\mu = \left(\gamma m_0 c, -\gamma m_0 v, 0, 0\right)$$

$$P_c^\mu = \left(\frac{h\nu'}{c}, -\frac{h\nu'}{c}\cos\vartheta, \frac{h\nu'}{c}\sin\vartheta, 0\right)$$

substituted in (4):

$$\left(\frac{h}{c}\right)^2 \nu\nu'(1+\cos\vartheta)+\gamma m_0 h\nu'(1-\beta\cos\vartheta)=$$
$$=\gamma m_0 h\nu(1+\beta)$$

$$\frac{\nu'}{\nu}=\frac{1+\beta}{1-\beta\cos\vartheta+(1+\cos\vartheta)E_a/E_b}$$

Electron at rest, β=0, θ=180°-φ (forward scattering):

$$\frac{\nu'}{\nu}=\frac{1}{1+(1-\cos\varphi)h\nu/m_0c^2}$$ *use of* $\nu=c/\lambda$ *yields*

$$\lambda'-\lambda=\frac{h}{m_0c}(1-\cos\varphi)$$ *Compton equation*

$$\frac{h}{m_oc}=2{,}42.10^{-12}\,m$$ *Compton wavelength*

Astronomy: Microwave background radiation with $E_a \approx 10^{-3}$ eV is scattered at high energy electrons $\gamma \gg 10^8$

$$\vartheta \approx 0, \quad 1+\beta \approx 2,$$

$$\beta = \sqrt{1-\frac{1}{\gamma^2}} \rightarrow 1-\beta \approx \frac{1}{2\gamma^2}$$

$$\frac{\nu'}{\nu} \approx \frac{4\gamma^2}{1+4\gamma^2 E_a/E_b} \approx \frac{E_b}{E_a}, \quad E_a = h\nu, \quad E_b = \gamma E_{e0}$$

$$\rightarrow \quad E_c = h\nu' \approx \gamma E_{e0} = \gamma\, 511\, keV$$

Dramatic increase of photon energy !

Acceleration 4-vector

We use again the proper time τ and $d\gamma_u/dt$ from transp. 28.

Then the acceleration 4-vector is the derivative of the velocity 4-vector with respect to τ

$$A^\mu = \frac{dU^\mu}{d\tau} = \frac{dU^\mu}{dt}\frac{dt}{d\tau} = \gamma_u \left[\frac{d\gamma_u}{dt}(c,\vec{u}) + \gamma_u \frac{d}{dt}(c,\vec{u})\right]$$

$$= \frac{\gamma_u^4}{c^2}(\vec{u}\cdot\vec{a})(c,\vec{u}) + \gamma_u^2(0,\vec{a})$$

$$A^\mu A_\mu = -\frac{\gamma_u^6}{c^2}(\vec{u}\cdot\vec{a})^2 - \gamma_u^4 a^2$$

U^μ and A^μ are perpendicular → $U^\mu A_\mu = 0$

For A^μ the same remarks are valid as for U^μ. But it is useful to calculate the proper acceleration α.

$$A^\mu = \frac{\gamma_u^4}{c^2}(\vec{u}\cdot\vec{a})(c, \vec{u}) + \gamma_u^2(0, \vec{a}), \quad A^\mu A_\mu = -\frac{\gamma_u^6}{c^2}(\vec{u}\cdot\vec{a})^2 - \gamma_u^4 a^2 \quad (1)$$

In an instantaneous rest frame S' of a particle the acceleration 4-vector defines the proper acceleration α

$$u' = 0, \quad \gamma_{u'} = 1, \quad A'^\mu = (0, \vec{\alpha}), \quad A'^\mu A'_\mu = -\alpha^2 \quad (2)$$

In case of *linear acceleration* $\vec{u} \| \vec{a}$, *follows* from (1), (2)

$$-\alpha^2 = -\gamma_u^6 \beta_u^2 a^2 - \gamma_u^4 a^2 = -\gamma_u^6 a^2 \quad \rightarrow \quad \alpha = \gamma_u^3 a$$

the same result as in transp. 17 with $u'_x = 0$, v=u

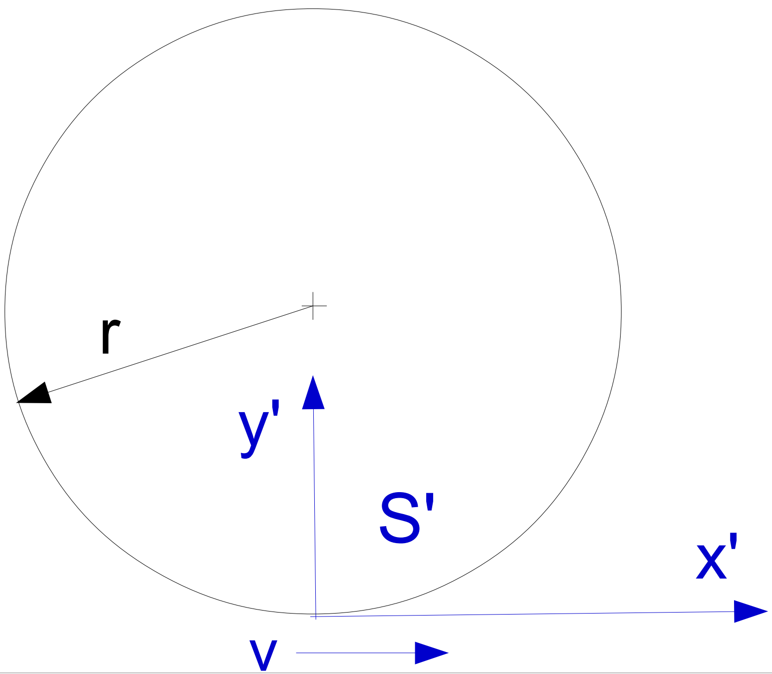
$$a_x = \frac{a'_x}{\gamma_u^3} \quad \rightarrow \quad a'_x = \alpha = \gamma_u^3 a_x$$

In *case of circular motion*, $\vec{u} \perp \vec{a}$, *it follows* from $(1),(2)$

$$-\alpha^2 = -\gamma_u^4 a^2 \quad \rightarrow \quad \alpha = \gamma_u^2 a$$

Which is identical to the result from transp. 17 for $u_x' = u_y' = 0$ and v=u, i.e. a'_y in an instantaneous system S'

$$a_y = \frac{a'_y}{\gamma_u^2} \quad \rightarrow \quad a'_y = \alpha = \gamma_u^2 a_y$$



Frequency-wavenumber 4-vector

Plane wave: $\vec{E}=\vec{E}_0 \sin(\omega t-\vec{k}\cdot\vec{r}), \quad k=\frac{2\pi}{\lambda}=\frac{\omega}{c}$

Phase at a fixed position must be the same for all reference systems (c=c'):

$$\Phi=\omega t-\vec{k}\cdot\vec{r}=\omega t-(k_x x+k_y y+k_z z)=\Phi'$$

Φ can be written as

$$K^\mu X_\mu=K'^\mu X'_\mu$$

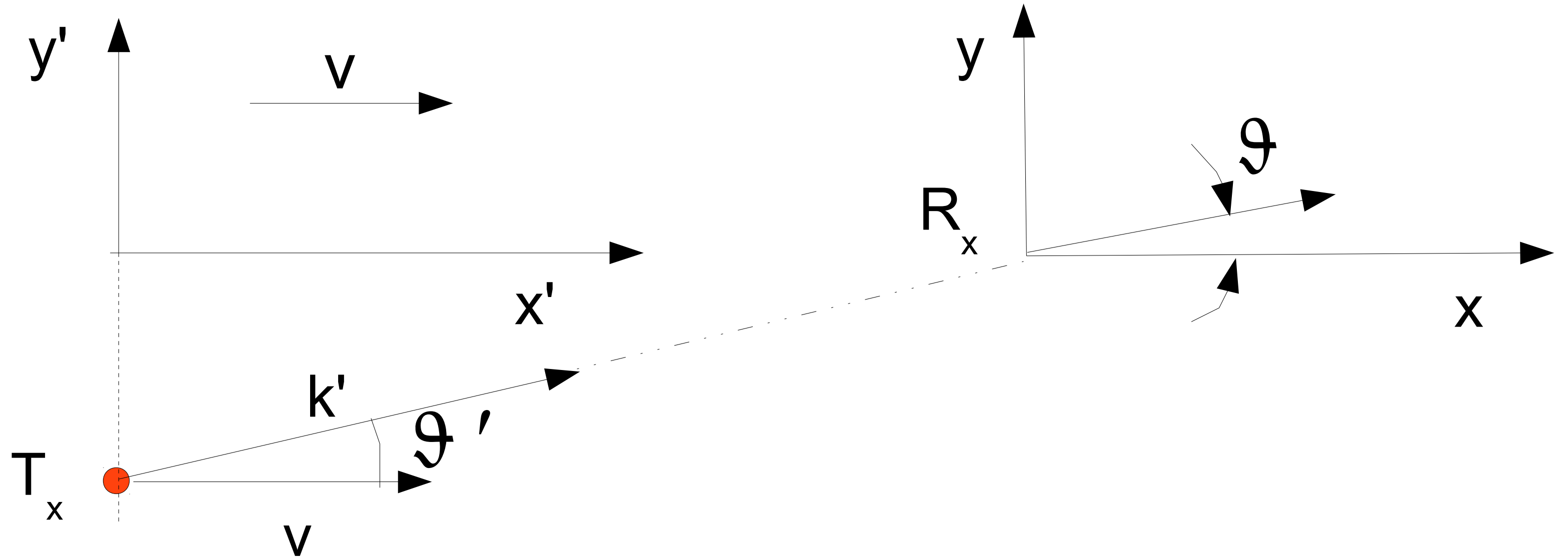
with the frequency-wavenumber 4-vector K^μ and X^μ

$$K^\mu=(\frac{\omega}{c},k_x,k_y,k_z), \quad X^\mu=(ct,x,y,z)$$

Since E=hν=ħω and E=pc for photons, it is p=ħω/c=ħk and

$$P^\mu=(\frac{E}{c},p_x,p_y,p_z)=\hbar K^\mu$$

6.Doppler effect



$$K^{\mu}=\left(\frac{\omega}{c},k_x,k_y,k_z\right)=\left(\frac{\omega}{c},\frac{\omega}{c}\cos\vartheta,\frac{\omega}{c}\sin\vartheta,0\right)$$

L-T of K^{μ} yields frequency in S'

$$\frac{\omega'}{c}=\gamma(1-\beta\cos\vartheta)\frac{\omega}{c} \quad\rightarrow\quad \omega=\frac{\omega'}{\gamma(1-\beta\cos\vartheta)} \qquad (1)$$

and the wave number in S'

$$k_x' = \frac{\omega'}{c}\cos\vartheta' = \gamma(-\beta + \cos\vartheta)\frac{\omega}{c}$$

$$k_y' = \frac{\omega'}{c}\sin\vartheta' = \frac{\omega}{c}\sin\vartheta, \qquad k_z' = 0 \qquad (2)$$

using $\tan\frac{\vartheta}{2} = \frac{\sin\vartheta}{1+\cos\vartheta}$ together with (1) and (2):

$$\tan\frac{\vartheta'}{2} = \frac{\sin\vartheta}{\gamma(1-\beta)(1+\cos\vartheta)} = \sqrt{\frac{1+\beta}{1-\beta}}\tan\frac{\vartheta}{2} > \tan\frac{\vartheta}{2}$$

The wave appears under a smaller angle in S than in S', see transp. 13.

Charge-current 4-vector

Going from S to S' charge must be conserved

$$\rho_0 dx\, dy\, dz = \rho' dx' dy' dz', \quad dx' = \frac{dx}{\gamma_u}, \quad dy' = dy, \quad dz' = dz$$

$$\rightarrow \quad \rho' = \gamma_u \rho_0$$

Moving charge density: $\rho = \gamma_u \rho_0$

Current density: $\vec{j} = \rho \vec{u} = \gamma_u \rho_0 \vec{u}$

Then

$$J^\mu = (\rho c, j_x, j_y, j_z) = \gamma_u \rho_0 (c, u_x, u_y, u_z) = \rho_0 U^\mu$$

is the charge-current 4-vector, since ρ_0 is a Lorentz scalar and U^μ a 4-vector.

Power-force 4-vector (Minkowski force)

Ansatz with proper time τ for a particle moving with $\vec{u}$:

$$F^{\mu}=\frac{dP^{\mu}}{d\tau}=\frac{dP^{\mu}}{dt}\frac{dt}{d\tau}=\gamma_u\left(\frac{1}{c}\frac{dE}{dt},\frac{dp_x}{dt},\frac{dp_y}{dt},\frac{dp_z}{dt}\right)$$

$$\frac{dE}{dt}=\frac{dE_{kin}}{dt}=\vec{f}\cdot\vec{u}$$

$$F^{\mu}=\gamma_u\left(\frac{1}{c}\vec{f}\cdot\vec{u},f_x,f_y,f_z\right)$$

F^{μ} is the power-force 4-vector, since P^{μ} is a 4-vector and τ a Lorentz scalar.

With F^{μ} and A^{μ} one gets the relativistic Newton's 2nd law:

$$F^{\mu}=m_0 A^{\mu}$$

Remarks

Again, for linear and circular motion Newton's relativistic law gives the results we had on the transps. 29, 53, 54 (see appendix A3).

Transformation of electromagnetic fields

Lorentz force $\vec{f} = q(\vec{E} + \vec{u} \times \vec{B})$

Power-force 4-vector

$$F^{\mu} = \gamma_u \left(\frac{1}{c}\vec{f}\cdot\vec{u}, f_x, f_y, f_z\right) = \gamma_u q \left(\frac{1}{c}\vec{E}\cdot\vec{u}, \vec{E} + \vec{u} \times \vec{B}\right)$$

Power-force 4-vector in components

$$\begin{bmatrix} F^0 \\ F^1 \\ F^2 \\ F^3 \end{bmatrix} = \frac{q}{c} \begin{bmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{bmatrix} \begin{bmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{bmatrix} = \frac{q}{c} \underline{T} U^\mu$$

With the Lorentz-transformation from S to S'
and the inverse transformation

$$\underline{L} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \underline{L}^{-1} = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we get

$$F'^{\mu} = \underline{L} F^{\mu} = \frac{q}{c} \underline{L}\,\underline{T}\, U^{\mu} = \frac{q}{c} \underline{L}\,\underline{T}\,\underline{L}^{-1} U'^{\mu} = \frac{q}{c} \underline{T}' U'^{\mu}$$

$$\underline{T}' = \begin{bmatrix} 0 & E_x & \gamma(E_y - vB_z) & \gamma(E_z + vB_y) \\ E_x & 0 & \gamma(cB_z - \beta E_y) & -\gamma(cB_y + \beta E_z) \\ \gamma(E_y - vB_z) & -\gamma(cB_z - \beta E_y) & 0 & cB_x \\ \gamma(E_z + vB_y) & \gamma(cB_y + \beta E_z) & -cB_x & 0 \end{bmatrix} \quad (1)$$

$$\begin{bmatrix} F^0{}' \\ F^1{}' \\ F^2{}' \\ F^3{}' \end{bmatrix} = \frac{q}{c} \begin{bmatrix} 0 & E_x' & E_y' & E_z' \\ E_x' & 0 & cB_z' & -cB_y' \\ E_y' & -cB_z' & 0 & cB_x' \\ E_z' & cB_y' & -cB_x' & 0 \end{bmatrix} \begin{bmatrix} \gamma_{u'} c \\ \gamma_{u'} u_x' \\ \gamma_{u'} u_y' \\ \gamma_{u'} u_z' \end{bmatrix} = \frac{q}{c} \underline{T}' U'^{\mu} \quad (2)$$

Comparing (1) with (2) we obtain:

$$E'_x = E_x \qquad B'_x = B_x$$

$$E'_y = \gamma(E_y - vB_z) \qquad B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right)$$

$$E'_z = \gamma(E_z + vB_y) \qquad B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

which can be written as

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel} \qquad \vec{B}'_{\parallel} = \vec{B}_{\parallel}$$

$$\vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{v} \times \vec{B}_{\perp}) \qquad \vec{B}'_{\perp} = \gamma\left(\vec{B}_{\perp} - \frac{1}{c^2}\vec{v} \times \vec{E}_{\perp}\right)$$

From the transformation of the electro-magnetic fields (transp. 63) follows that the two scalar expressions

$$\vec{E}'\cdot\vec{B}'=\vec{E}\cdot\vec{B} \qquad E'^2-c^2B'^2=E^2-c^2B^2$$

are invariant (exercise 8).

For plane waves the expressions are particular simple. We know that E is perpendicular to B and that E=cB. The expressions are zero and they are therefore zero in all frames. A plane wave remains a plane wave in all frames. Only the amplitudes change.

Let us assume a plane wave traveling into x-direction

$$E_x=E_z=0, \quad E_y=E_0, \quad B_z=E_0/c, \quad B_x=B_y=0$$

The transformed fields are

$$E_x'=E_z'=B_x'=B_y'=0$$

$$E_y'=E_0\sqrt{\frac{1-\beta}{1+\beta}}, \qquad B_z'=\frac{E_0}{c}\sqrt{\frac{1-\beta}{1+\beta}}$$

In a cotraveling frame with β=1 the wave vanishes !!

7. Uniformly moving charge

Point charge q at rest in origin of S'

$$\vec{E}' = \frac{q}{4\pi\epsilon_0 (x'^2 + y'^2 + z'^2)^{3/2}} (x', y', z'), \qquad \vec{B}' = 0$$

The point P=(0,a,0) in S has coordinates P'=(-vt',a,0) in S', yielding

$$\vec{E}'_P(t') = \frac{q}{4\pi\epsilon_0 (v^2 t'^2 + a^2)^{3/2}} (-vt', a, o)$$

Transformation of t' : $ct' = \gamma(ct - \beta x) = \gamma ct \quad \text{for} \quad x = 0$

$$\vec{E}'_P(t) = \frac{q}{4\pi\epsilon_0 (v^2 \gamma^2 t^2 + a^2)^{3/2}} (-v\gamma t, a, o)$$

Transformation of fields

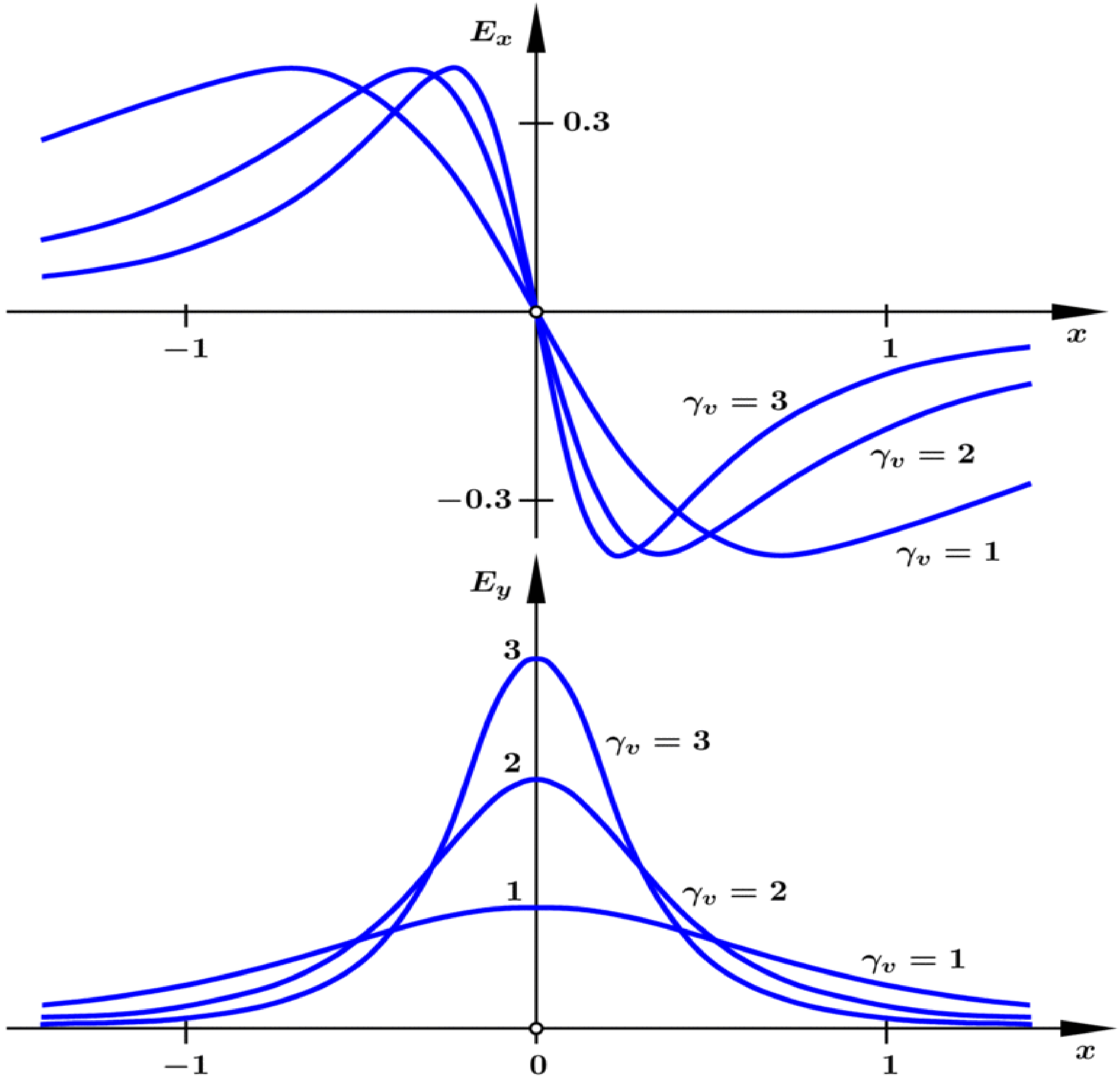
$$E_{Px} = E_{Px}', \qquad B_{Px} = 0$$

$$E_{Py} = \gamma E_{Py}', \qquad B_{Py} = -\gamma \frac{v}{c^2} E_{Pz}'$$

$$E_{Pz} = \gamma E_{Pz}', \qquad B_{Pz} = +\gamma \frac{v}{c^2} E_{Py}'$$

$$\vec{E}_P(t) = \frac{q}{4\pi\epsilon_0 (\gamma^2 v^2 t^2 + a^2)^{3/2}} (-\gamma v t, \gamma a, 0)$$

$$\vec{B}_P(t) = \frac{q}{4\pi\epsilon_0 (\gamma^2 v^2 t^2 + a^2)^{3/2}} \left(0, 0, \gamma \frac{v}{c^2} a\right)$$


E_x
0.3
−0.3
−1
1
x
$\gamma_v = 3$
$\gamma_v = 2$
$\gamma_v = 1$
E_y
3
2
1
$\gamma_v = 3$
$\gamma_v = 2$
$\gamma_v = 1$
−1
0
1
x

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A. P. French: Special relativity. W. W. Norton & Company, 1966

J. Freund: Special relativity for beginners. World Scientific, 2008

Appendix

A1

Variables which depend on several other variables transform as

$$\frac{\partial}{\partial x_i}=\sum_j \frac{\partial x_j'}{\partial x_i}\frac{\partial}{\partial x_j'}$$

this yields for the G-T of the wave equation, transp. 3

$$\frac{\partial}{\partial x}=\frac{\partial x'}{\partial x}\frac{\partial}{\partial x'}+\frac{\partial t'}{\partial x}\frac{\partial}{\partial t'}=\frac{\partial}{\partial x'}, \qquad \frac{\partial^2}{\partial x^2}=\frac{\partial^2}{\partial x'^2}$$

$$\frac{\partial}{\partial t}=\frac{\partial x'}{\partial t}\frac{\partial}{\partial x'}+\frac{\partial t'}{\partial t}\frac{\partial}{\partial t'}=-v\frac{\partial}{\partial x'}+\frac{\partial}{\partial t'}$$

$$\frac{\partial^2}{\partial t^2}=v^2\frac{\partial^2}{\partial x'^2}-2v\frac{\partial^2}{\partial x'\partial t'}+\frac{\partial^2}{\partial t'^2}$$

$$\left[\frac{\partial^2}{\partial x'^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}-\frac{v^2}{c^2}\frac{\partial^2}{\partial x'^2}+2\frac{v}{c^2}\frac{\partial^2}{\partial x'\partial t'}\right]\Phi=0$$

A2

Lorentz- Transformation

$$\begin{aligned} ct' &= a_{00}ct + a_{01}x + a_{02}y + a_{03}z \\ x' &= a_{10}ct + a_{11}x + a_{12}y + a_{13}z \\ y' &= a_{20}ct + a_{21}x + a_{22}y + a_{23}z \\ z' &= a_{30}ct + a_{31}x + a_{32}y + a_{33}z \end{aligned} \qquad (1)$$

Successive use of homogeneity and isotropy of space:

$a_{02} = a_{03} = 0$ events at $y = \pm y_0$ or $z = \pm z_0$ have to take place at equal times in S'

$a_{20} = a_{30} = 0$ origin, $x = y = z = 0$, has to stay on x'-axis

$a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0$

x-,x'- and y-,y'- and z-, z'-axis should stay parallel

$a_{22} = a_{33}$ because cylindrical symmetry requires equal transformation of y and z

Due to the relativistic principle the inverse L-T follows from replacing the primed by unprimed variables, the unprimed by primed variables and v by -v. Therefore

$$y' = a_{22}(v)\,y, \qquad y = a_{22}(-v)\,y' \qquad \text{with} \qquad a_{22}(v) = a_{22}(-v) = 1$$

since y=y' for v → 0. (1) is thus simplified to

$$ct' = a_{00}ct + a_{01}x, \qquad x' = a_{10}ct + a_{11}x$$
$$y' = y, \qquad z' = z \qquad (2)$$

A light pulse propagates on a spherical shell and the space-time interval

$$(ct')^2 - x'^2 - y'^2 - z'^2 = (ct)^2 - x^2 - y^2 - z^2 \qquad (3)$$

is invariant. Substituting (2) into (3) and comparing coefficients gives

$$1 - a_{00}^2 + a_{10}^2 = 0, \qquad 1 + a_{01}^2 - a_{11}^2 = 0, \qquad a_{10}a_{11} - a_{00}a_{01} = 0 \qquad (4)$$

These are 3 equations for 4 unknowns. The 4th equ. follows from the motion of the origin (x=y=z=0) in S', x'=-v t', and using (2)

$$ct' = a_{00}\,ct\,, \qquad -vt' = a_{10}\,ct \quad \rightarrow \quad a_{10} = -\frac{v}{c}a_{00} \qquad (5)$$

Inserting (5) into (4) determines all 4 unknowns and the final solution is

$$ct' = \gamma(ct - \beta x), \qquad x' = \gamma(x - \beta ct), \qquad y' = y\,, \qquad z' = z$$

A3

Linear and circular motion derived by means of power-force 4-vector and the relativistic 2nd Newton's law.

Linear motion, $\vec{u} = (u_x, 0, 0)$, $\vec{f} = (f_x, 0, 0)$ of a particle:

$$F^{\mu} = \gamma_u\left(\frac{1}{c}f_x u_x, f_x, 0, 0\right)$$

The rest frame of the particle, $u_x' = 0$, moves with $v = u_x$ and one obtains

$$F^1 = \gamma f_x, \qquad F^{1\prime} = f'_x$$

With the acceleration 4-vector

$$A^1{}'=\alpha_x, \qquad A^1=\gamma^4 a_x$$

it follows from the relativistic Newton's law

$$f_x{}'=m_0\alpha_x, \qquad f_x=\gamma^3 m_0 a_x$$

In circular motion, $\vec{f}\cdot\vec{u}=0$, with the instantaneous, tangential velocity $\vec{u}=(u_x,0,0)$ and acceleration a_y

$$F^\mu=\gamma(0,0,f_y,0)$$

Now we have

$$F^2=\gamma f_y, \qquad F^2{}'=f_y{}'$$

and for the acceleration

$$A^1{}'=\alpha_y, \qquad A^1=\gamma^2 a_y$$

Again using the relativistic Newton's law

$$f_y{}'=m_0\alpha_y, \qquad f_y=\gamma m_0 a_y$$

Both results correspond to the ones on transp. 29, 53, 54.

Obligatory homeworks

Experience has shown that not all students have learned „Special Relativity“ at university. Therefore, you receive the lecture notes ahead of the course, so that you can prepare yourself. The exercises are obligatory and will be collected at the beginning of the course.

If nothing else is stated, S is the fixed lab-frame and S' moves with v in x-direction.

Exercise 1

Prove that the scalar product of any two 4-vectors A^μ and B^μ is Lorentz invariant.

Exercise 2

Prove the relativistic 2nd law of Newton $F^\mu = m_0 A^\mu$ by comparing the components of the left and right side.

Exercise 3

A space craft travels away from earth with $\beta=0.8$. At a distance $d= 2.16\bullet10^{8}$ km a radio signal from earth is transmitted to the space craft. How long does the signal need to reach the space craft in the system of the earth? Solve it by using a 2-dimensional (ct,x) diagram.

Exercise 4

A charge q is at rest. At t=0 an electric field E_x is turned on.

Calculate the velocity in two different ways:

1) In the laboratory frame S.
2) By using the constant acceleration $\alpha=q\,E_x/m_0$ in the instantaneous rest frame S'.

Exercise 5

A particle moves in S with velocity $\vec{u}=(0,u,0)$ and experiences a force $\vec{f}=(0,f,0)$. What is the force $\vec{f}\,'$ in S' ? Use the transformation of u and L-T of F^{μ}.

Exercise 6

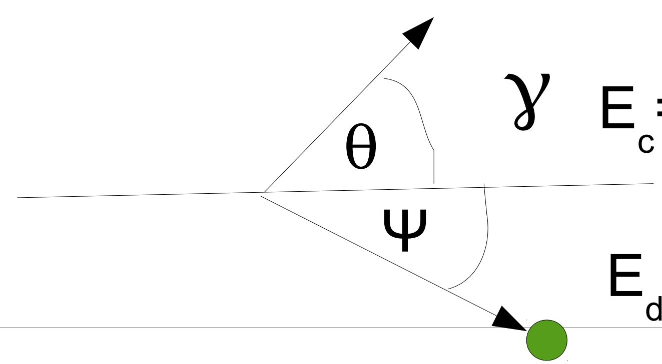
An electron with velocity u collides with a proton at rest. After the collision a photon (bremsstrahlung) is emitted and the electron and proton are moving together with u'. What is the energy of the emitted photon?

before collision

e^- → u p^+

$E_a=\gamma_u m_{0e}c^2$, $p_a=\gamma_u m_{0e}u$ $E_b=m_{0p}c^2$, $p_b=0$

after collision

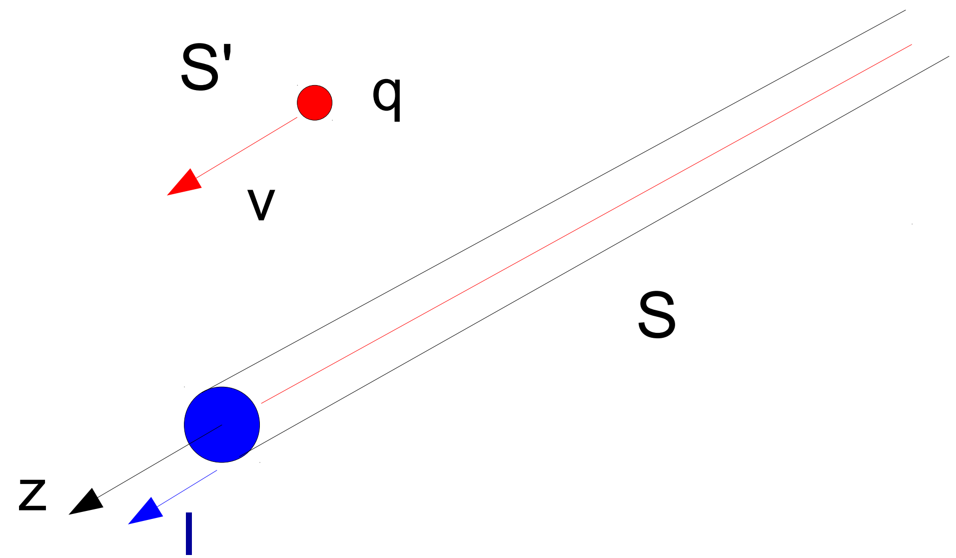


γ $E_c=h\nu$, $p_c c=h\nu/c$

$E_d=\gamma_{u'}(m_{0e}+m_{0p})c^2$, p_d

Exercise 7

A point charge q moves with velocity v parallel to a current carrying wire. Calculate the force on the charge in its rest frame S' by transforming the e.-m. fields.



Exercise 8

Prove that the two expressions are invariant

$$\vec{E}'\cdot\vec{B}'=\vec{E}\cdot\vec{B}, \qquad E'^2-c^2B'^2=E^2-c^2B^2$$