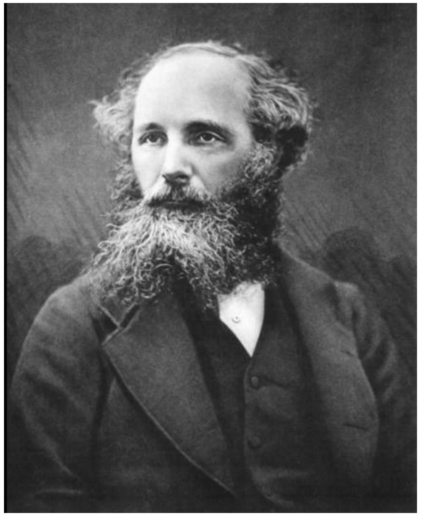
Introduction to RF

Andrea Mostacci
University of Rome “La Sapienza” and INFN, Italy



Goal of the lecture

Schedule 2020	Monday Feb 17	Tuesday Feb 18	Wednesday Feb 19	Thursday Feb 20	Friday Feb 21
09:00		**Introduction to RF** *A. Mostacci*	**Vacuum systems** *V. Baglin*	**Vacuum systems** *V. Baglin*	**RF Engineering** *F. Caspers*
10:00		**Coffee Break**	**Coffee Break**	**Coffee Break**	**RF Engineering** *F. Caspers / M. Wendt / M. Bozzolan*
10:15		**Introduction to RF** *A. Mostacci*	**Vacuum systems** *V. Baglin*	**Vacuum systems** *V. Baglin*	**Coffee Break**
11:15	**12:00 OFFICIAL OPENING (welcome & building visit)**	**Introduction to RF** *A. Mostacci*	**Vacuum systems** *V. Baglin / R. Kersevan*	**Vacuum systems** *V. Baglin / R. Kersevan*	***Bus leaves at 11:30 from JUAS***
12:15	**13:00 WELCOME LUNCH**	**BREAK**	**BREAK**	**BREAK**	***(Lunch at CERN, R2, offered by ESI)***
14:00	**14:00 Presentation of JUAS & Introduction of students** *P. Lebrun*	**RF Engineering** *F. Caspers*	**Vacuum systems** *V. Baglin*	**RF Engineering** *F. Caspers*	**VISIT AT CERN** ***AD / ELENA LINAC 4 Vacuum lab***
15:00	**Coffee Break**				
15:15	**Introduction to CERN practical days** *Magnet, Superconductivity, RF, Vacuum, CLEAR*	**RF Engineering** *F. Caspers*	**RF Engineering** *F. Caspers / M. Wendt / M. Bozzolan*	**RF Engineering** *F. Caspers / M. Wendt / M. Bozzolan*	
16:00		**Coffee Break**	**Coffee Break**	**Coffee Break**	
16:15		**RF Engineering** *F. Caspers*	**RF Engineering** *F. Caspers*	**RF Engineering** *F. Caspers*	***Bus leaves at 18:00 from CERN***
17:15	**CHECK-IN AT THE RESIDENCE & SHOPPING FOR GROCERIES**	**Particle accelerators, instruments of discovery in physics** - Seminar (incl. ESIPAP students) - *Ph. Lebrun*	**Accelerator driven system** Seminar (incl. ESIPAP students) *M. Baylac*		
18:15			**AFTER WORK AT ESI**		

Goal of the lecture

Show principles behind the practice discussed in the RF engineering module

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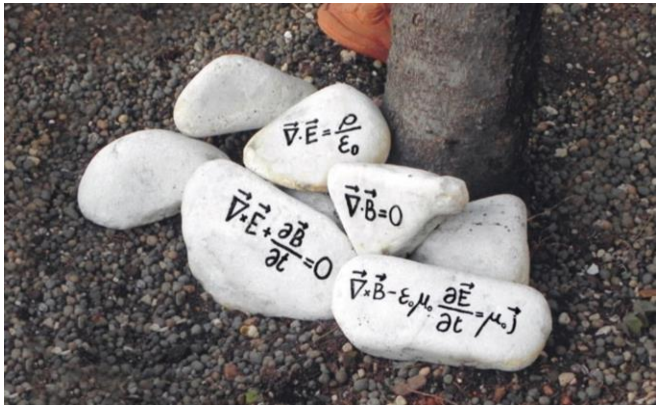
Outline

Goal of the lecture

Show principles behind the practice discussed in the RF engineering module

Maxwell equations



- General review
- The lumped element limit
- RF fields and particle accelerators
- The wave equation
- Energy conservation issue
- Maxwell equations for time harmonic fields
- Fields in media and complex permittivity
- Boundary conditions and materials
- Plane waves

Boundary value problems for metallic waveguides

- The concept of mode
- Maxwell equations and vector potentials
- Cylindrical waveguides: TM, TE and TEM modes
- Solving Maxwell Equations in metallic waveguides
- Rectangular waveguide (detailed example)
- Reading a simulation of a RF accelerating structure

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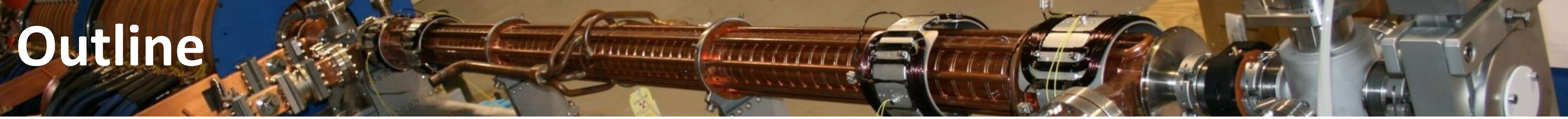

Outline

Maxwell equations
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- RF fields and particle accelerators
- The wave equation
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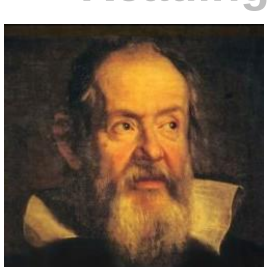

… The universe is written in the mathematical language and the letters are triangles, circles and other geometrical figures …

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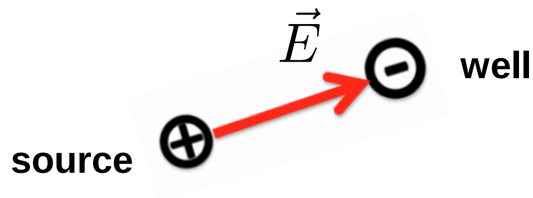
Classical electromagnetic theory (Maxwell equations)

1.
Charges are the sources of E-field.

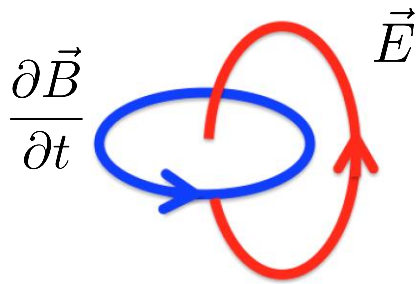


$$\nabla \cdot \vec{E} = \rho/\epsilon_0$$

2.
B-field has no sources.

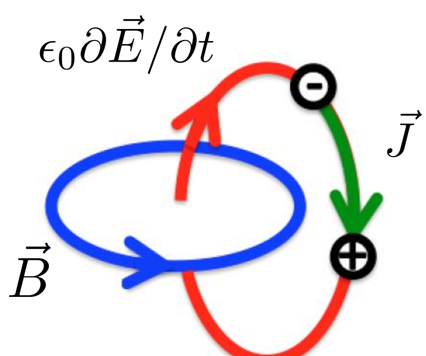
$$\nabla \cdot \vec{B} = 0$$

3.
Time varying E-field and B-field are chained.



$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

4.
B-field is chained to current.



$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$$

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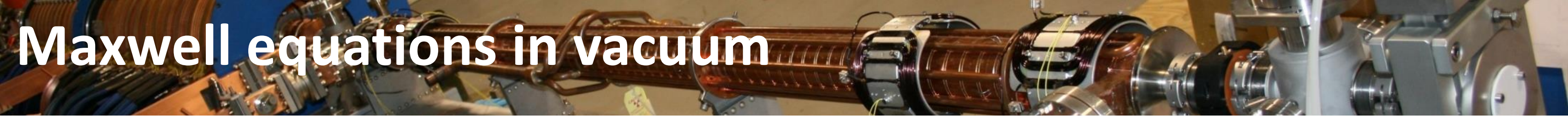


Maxwell equations in vacuum

$$\nabla \cdot \vec{E} = \rho/\epsilon_0$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \times \vec{B} = \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$$

$\vec{E}$	**Electric Field**	(V/m)	**fields**
$\vec{B}$	**Magnetic Flux Density**	(Wb/m^2)	
ρ	**Electric Charge Density**	(C/m^3)	**sources**
$\vec{J}$	**Electric Current Density**	(A/m^2)	

$\mu_0 = 4\pi\ 10^{-7}\ (H/m)$

Magnetic constant
(permeability of free space)

$\epsilon_0 = 1/c^2\mu_0 = 8.8542\ 10^{-12}\ (F/m)$

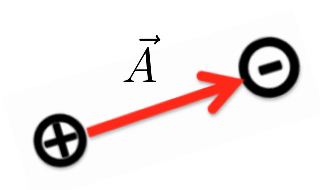
Electric constant
(permittivity of free space)

$c = 1/\sqrt{\mu_0\epsilon_0} = 299792458\ (m/s)$

Speed of light

Divergence operator

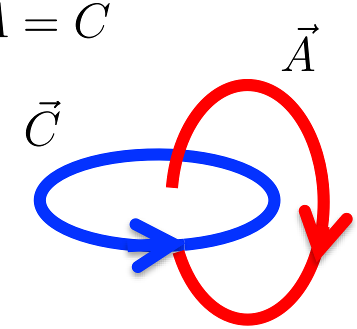
$$\nabla \cdot \vec{A} = \dots$$



The source of $\vec{A}$ is ...

Curl operator

$$\nabla \times \vec{A} = \vec{C}$$



$\vec{A}$ is chained to $\vec{C}$

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Some consequences of the IV equation

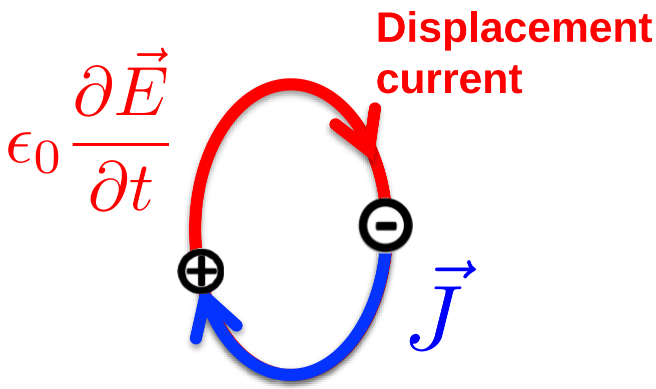
$$\nabla \times \vec{B} = \mu_0 \left(\epsilon_0 \frac{\partial \vec{E}}{\partial t} + \vec{J} \right)$$

$$0 = \nabla \cdot \nabla \times \vec{B} = \mu_0 \nabla \cdot \left(\epsilon_0 \frac{\partial \vec{E}}{\partial t} + \vec{J} \right) = 0$$

$$\nabla \cdot \vec{E} = \rho / \epsilon_0$$

The current density has closed lines.

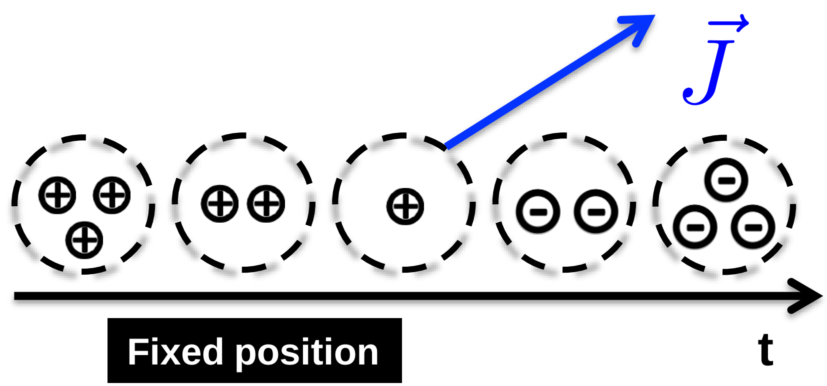
At a given position the source of J is the decrease of charge in time.



$$\nabla \cdot \left(\epsilon_0 \frac{\partial \vec{E}}{\partial t} + \vec{J} \right) = 0$$

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

Continuity equation



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Maxwell equations: the static limit

$$\frac{\partial}{\partial t} = 0$$

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} = 0$$

Ohm Law

→ **Kirchhoff Laws**

Lumped elements (electric networks)

$$\frac{\partial}{\partial t} \approx 0$$

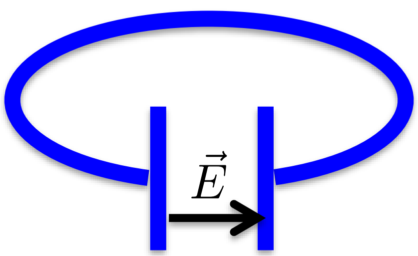
The lumped elements model for electric networks is used also when the field variation is negligible over the size of the network.

$$\frac{\partial}{\partial t} = 0$$

$$\nabla \times \vec{E} = 0$$

The E field is conservative.

The energy gain of a charge in closed circuit is zero.



No static, circular accelerators (RF instead!).

Electrostatics

$$\nabla \times \vec{E} = 0 \rightarrow \vec{E} = -\nabla V \xrightarrow[\text{free space}]{\nabla \cdot \vec{E} = 0} \nabla^2 V = 0$$

Laplace equation

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Charged particle interaction with time varying fields

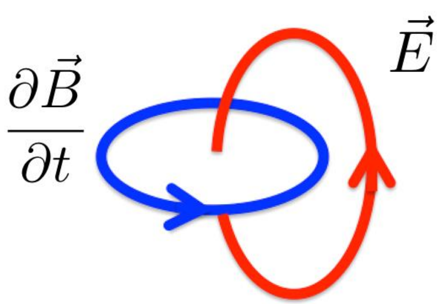
Beam manipulation

Particle acceleration, deflection …

External sources acting on the beam through EM fields.

RF devices

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$



Parasitic effects

Wakefields and coupling impedance

Extraction of beam energy

Beam Instabilities

Diagnostics

$$\nabla \cdot \vec{E} = \rho / \epsilon_0$$

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$$

$$\vec{J} = \rho \vec{v} = \frac{Q}{2\pi r} \delta(r) \delta(z - vt) \vec{v}$$

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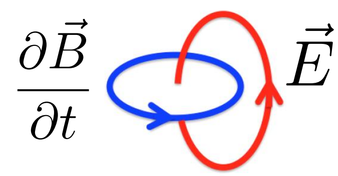
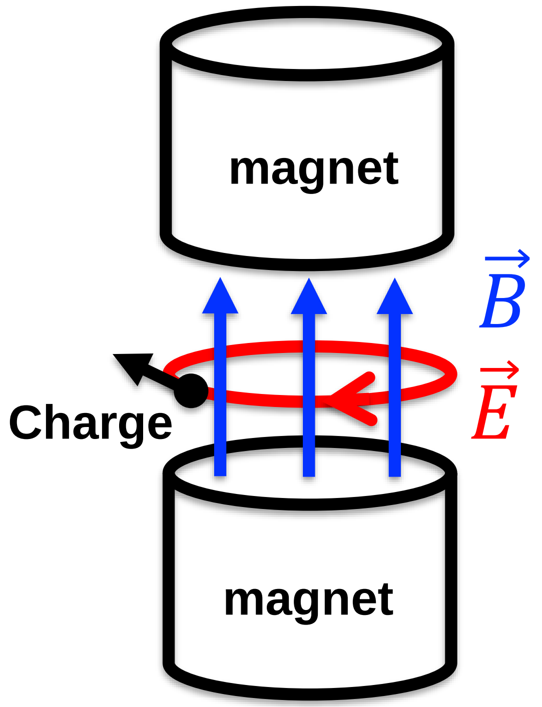
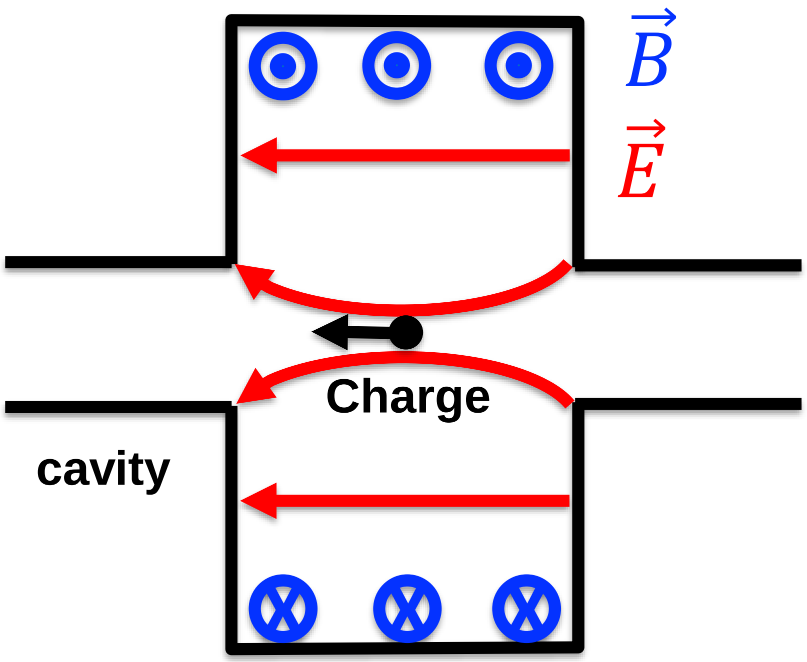
Particle acceleration by time varying fields

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$





Betatron or
"unbunched" acceleration



Resonant or
"bunched" acceleration

Linear accelerator (LINAC)
Synchrotron

Courtesy of P. Bryant

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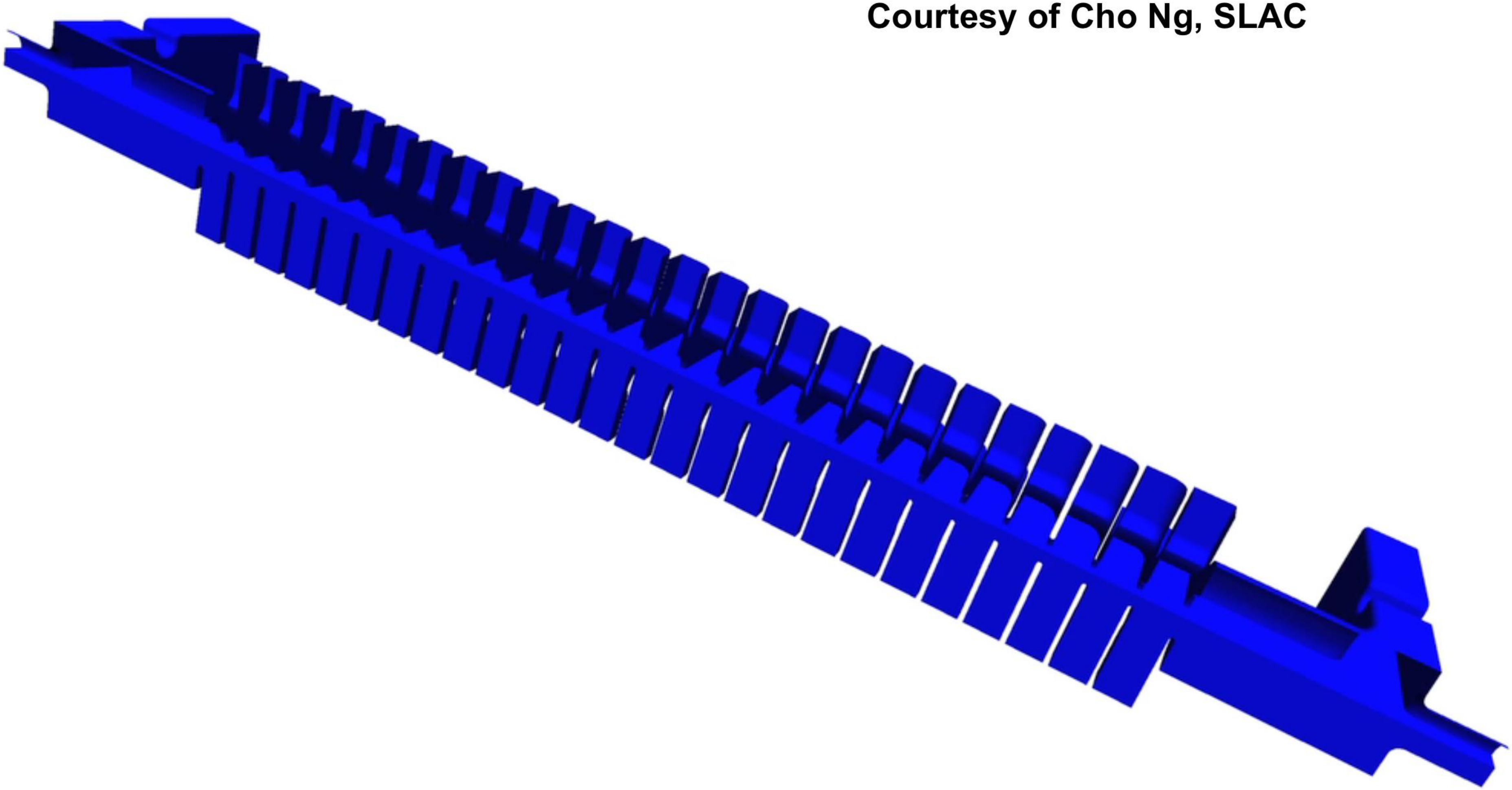
Parasitic effects: the wakefield

Courtesy of Cho Ng, SLAC



Particle in accelerators are charged, thus they are sources of EM fields …

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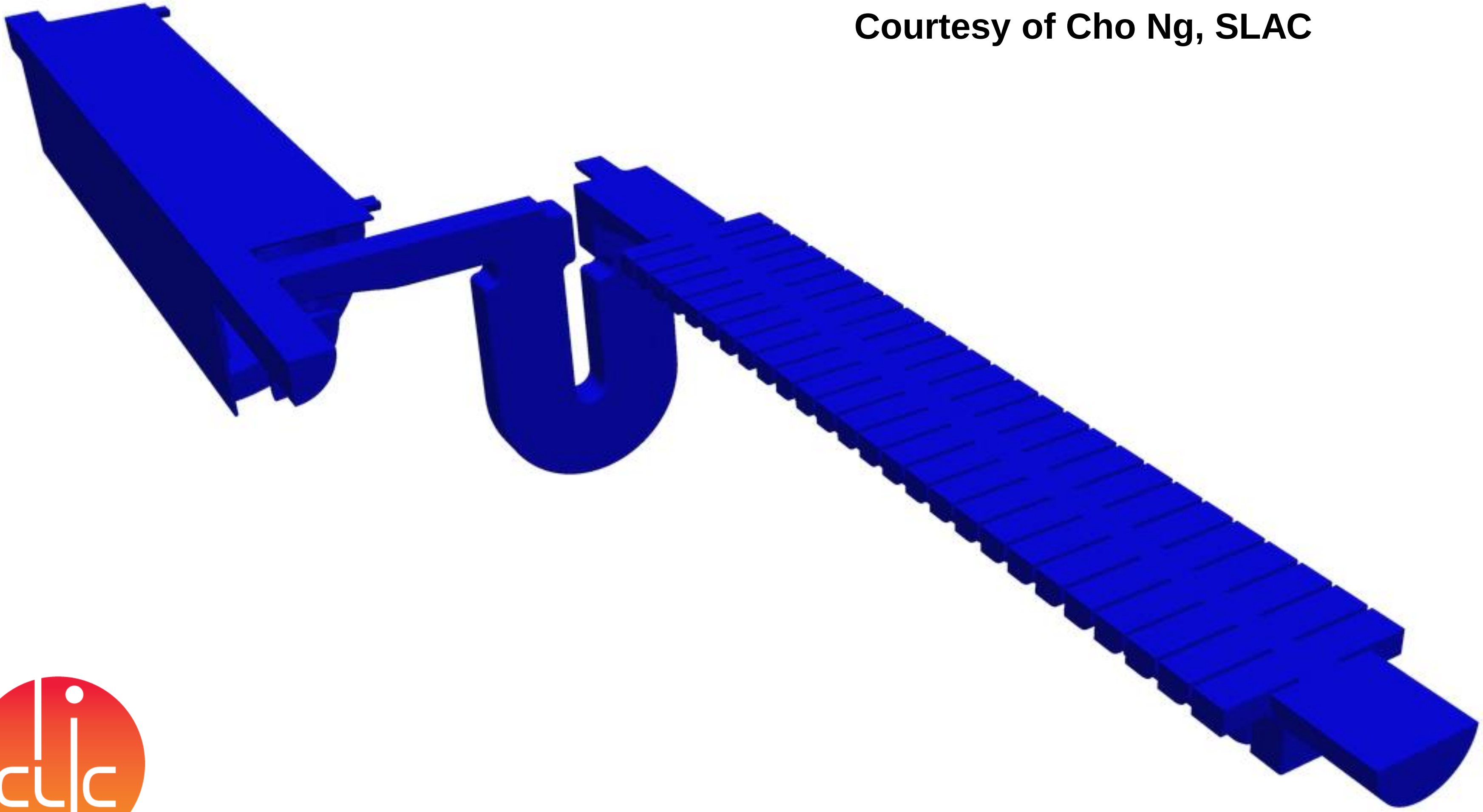
Wakefields extract beam energy to EM field

Courtesy of Cho Ng, SLAC



The principle is used in general purpose RF sources (e.g. klystrons) as well as in accelerators (e.g. particle wakefield accelerators)

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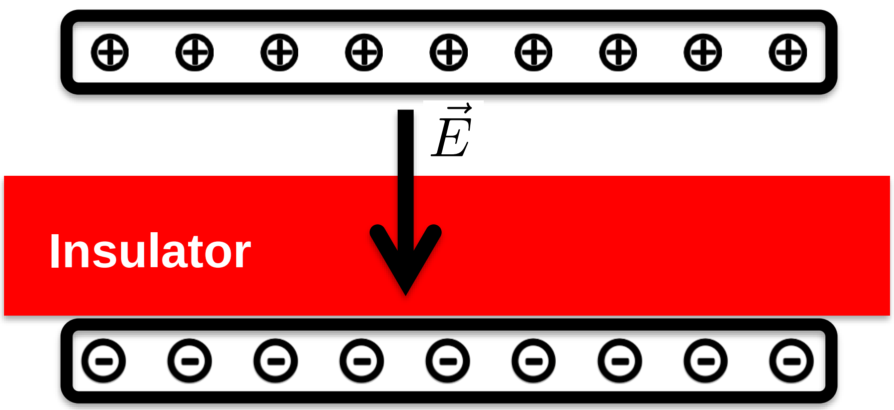
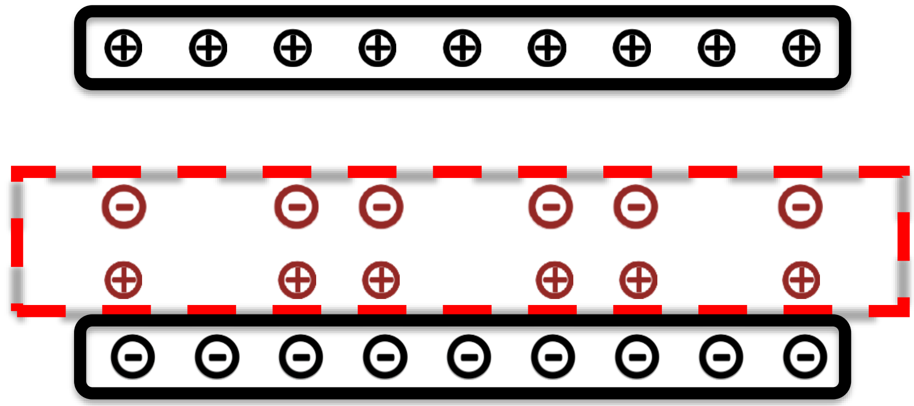
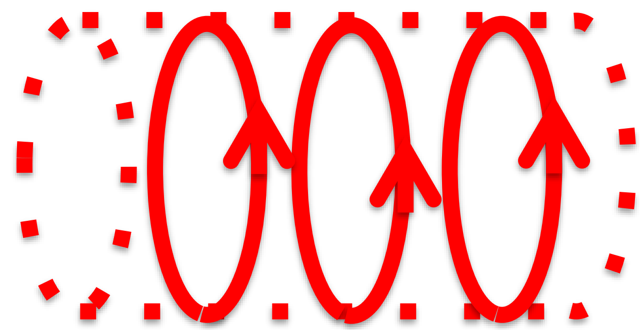
Maxwell equations in matter: the physical approach

The reality … | **… the model**





charges and currents IN VACUUM



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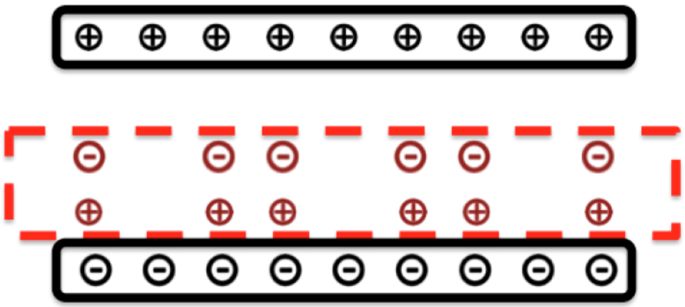
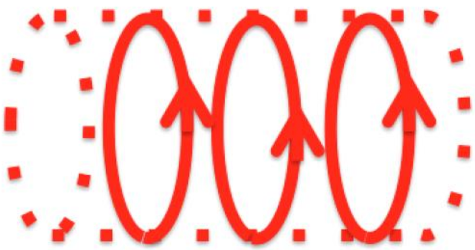
Maxwell equations in matter: the mathematics



Electric insulators (dielectric)



Magnetic materials (ferrite, superconductor)

Polarization charges

Magnetization currents

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

Constitutive relations

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

$\vec{D}$ **Electric Flux Density** (C/m^2)

$\vec{H}$ **Magnetic Field** (A/m)

$\vec{P}$ **Electric Polarization** (C/m^2)

$\vec{M}$ **Magnetization** (A/m)

Equivalence Principles in Electromagnetics Theory

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Maxwell equations: general expression and solution

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

fields

$\vec{E}$	**Electric Field**	(V/m)
$\vec{H}$	**Magnetic Field**	(A/m)
$\vec{B}$	**Magnetic Flux Density**	(Wb/m^2)
$\vec{D}$	**Electric Flux Density**	(C/m^2)

$$\vec{H} = \frac{\vec{B}}{\mu_0}$$

in vacuum

sources

ρ	**Electric Charge Density**	(C/m^3)
$\vec{J}$	**Electric Current Density**	(A/m^2)

Maxwell Equations: free space, no sources

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\nabla^2 \vec{E}$$

$$||$$

$$\nabla \times \nabla \times \vec{E}$$

$$||$$

$$-\mu_0 \frac{\partial}{\partial t}(\nabla \times \vec{H}) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t}$$

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{H} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2}$$

Wave equation

$$\frac{1}{v^2} = \mu_0 \epsilon_0 \Longrightarrow v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c$$

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Harmonic time dependence and phasors

Assuming sinusoidal electric field (Fourier)

Time dependence → $e^{j\omega t} = e^{j2\pi f\ t}$ → $\frac{\partial}{\partial t}\cdots = j\omega\ldots$

$$\vec{E}(\vec{r},t) = Re\left\{\vec{E}(\vec{r},\omega)e^{j\omega t}\right\}$$

Phasors are complex vectors

Power/Energy depend on time average of quadratic quantities

$$\left|\vec{E}(\vec{r},t)\right|^2_{average} = \frac{1}{T}\int_0^T \vec{E}(\vec{r},t)\cdot\vec{E}(\vec{r},t)dt = \cdots = \frac{1}{2}\vec{E}(\vec{r},\omega)\cdot\vec{E}^*(\vec{r},\omega) = \left|\vec{E}_{RMS}(\vec{r},\omega)\right|^2$$

$$\left|\vec{E}_{RMS}\right| = \left|\vec{E}\right|/\sqrt{2}$$

In the following we will use the same symbol for

Real vectors

$$\vec{E}(\vec{r},t), \vec{H}(\vec{r},t), \ldots$$

Complex vectors

$$\vec{E}\left(\vec{r},\omega\right), \vec{H}\left(\vec{r},\omega\right), \ldots$$

Note that, with phasors, a time animation is identical to phase rotation.

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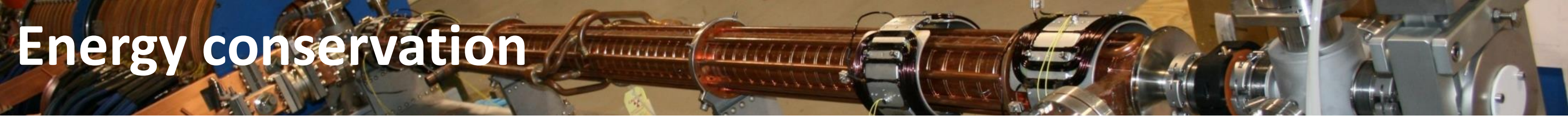
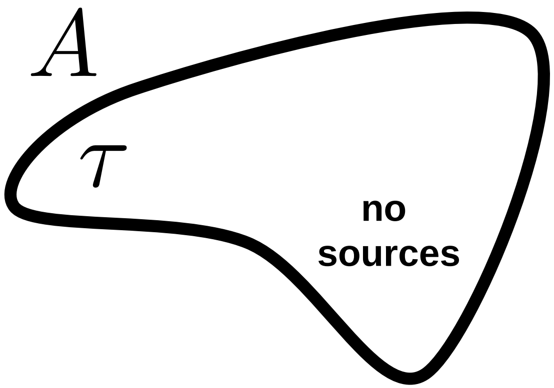


Energy conservation

bounding (closed) surface



Energy of the e.m. field in the volume τ

$$U = \int_\tau \frac{1}{2}\vec{E}\cdot\vec{D}\; d\tau + \int_\tau \frac{1}{2}\vec{H}\cdot\vec{B}\; d\tau$$

$\frac{1}{2}\vec{E}\cdot\vec{D} \rightarrow u_E$, $\frac{1}{2}\vec{H}\cdot\vec{B} \rightarrow u_H$

$$\frac{\partial U}{\partial t} = \int_\tau \left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right) d\tau = \frac{\partial}{\partial t}\left(U_E + U_H\right)$$

Using Maxwell equations, vector identities and divergence theorem

Rate of decrease of e.m. energy in τ →

$$-\frac{\partial U}{\partial t} = \int_\tau \vec{E}\cdot\vec{J} d\tau + \oint_A \vec{E}\times\vec{H}\cdot d\vec{A}$$

Energy per unit time transferred from electric field to moving charges in τ

Dissipated power (Joule effect) in τ

E.M. energy flowing through surface A per unit time (exiting from τ)

Radiated power through A

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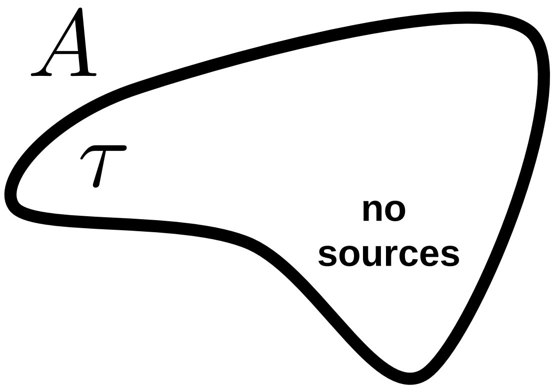
Energy conservation and Poynting theorem

bounding (closed) surface



Poynting theorem (conservation of energy)

$$-\frac{\partial U}{\partial t} = \int_\tau \vec{E}\cdot\vec{J}d\tau + \oint_A \vec{S}\cdot d\vec{A}$$

$$\vec{S}(\vec{r},t) = \vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)$$

Poynting vector

$$(W/m^2)$$

$$\vec{E}(\vec{r},t) = Re\left\{\vec{E}(\vec{r},\omega)e^{j\omega t}\right\} = \frac{\vec{E}(\vec{r},\omega)e^{j\omega t} + \left[\vec{E}(\vec{r},\omega)e^{j\omega t}\right]^*}{2}$$

$$\vec{H}(\vec{r},t) = Re\left\{\vec{H}(\vec{r},\omega)e^{j\omega t}\right\} = \frac{\vec{H}(\vec{r},\omega)e^{j\omega t} + \left[\vec{H}(\vec{r},\omega)e^{j\omega t}\right]^*}{2}$$

$$\vec{S}(\vec{r},t) = \vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t) = Re\left\{\frac{\vec{E}(\vec{r},\omega)\times\vec{H}(\vec{r},\omega)^*}{2}\right\} + Re\left\{\frac{\vec{E}(\vec{r},\omega)\times\vec{H}(\vec{r},\omega)e^{j2\omega t}}{2}\right\}$$

$$\langle\vec{S}(\vec{r},t)\rangle_{\text{period}}$$

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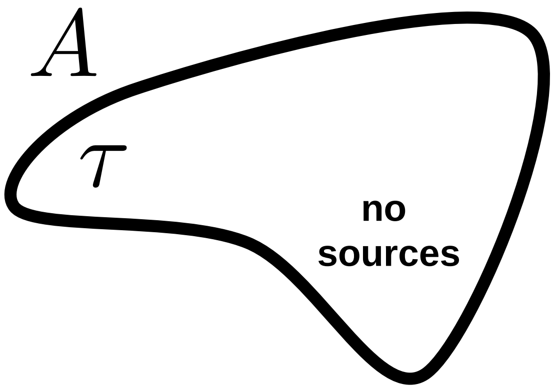
Energy conservation and Poynting theorem

bounding (closed) surface



Poynting theorem (conservation of energy)

$$-\frac{\partial U}{\partial t} = \int_{\tau} \vec{E} \cdot \vec{J} d\tau + \oint_{A} \vec{S} \cdot d\vec{A}$$

$$\vec{S}(\vec{r}, t) = \vec{E}(\vec{r}, t) \times \vec{H}(\vec{r}, t)$$

Poynting vector

$$(W/m^2)$$

$$\vec{S}(\vec{r}, t) = \vec{E}(\vec{r}, t) \times \vec{H}(\vec{r}, t) = Re\left\{\frac{\vec{E}(\vec{r}, \omega) \times \vec{H}(\vec{r}, \omega)^*}{2}\right\} + Re\left\{\frac{\vec{E}(\vec{r}, \omega) \times \vec{H}(\vec{r}, \omega)\, e^{j2\omega t}}{2}\right\}$$

$$\langle \vec{S}(\vec{r}, t) \rangle_{\text{period}}$$

$$\vec{S}(\vec{r}, \omega) = \vec{E}(\vec{r}, \omega) \times \vec{H}(\vec{r}, \omega)^*$$

Phasor of the Poynting vector

Real → **Active power**

Imag → **Reactive power**

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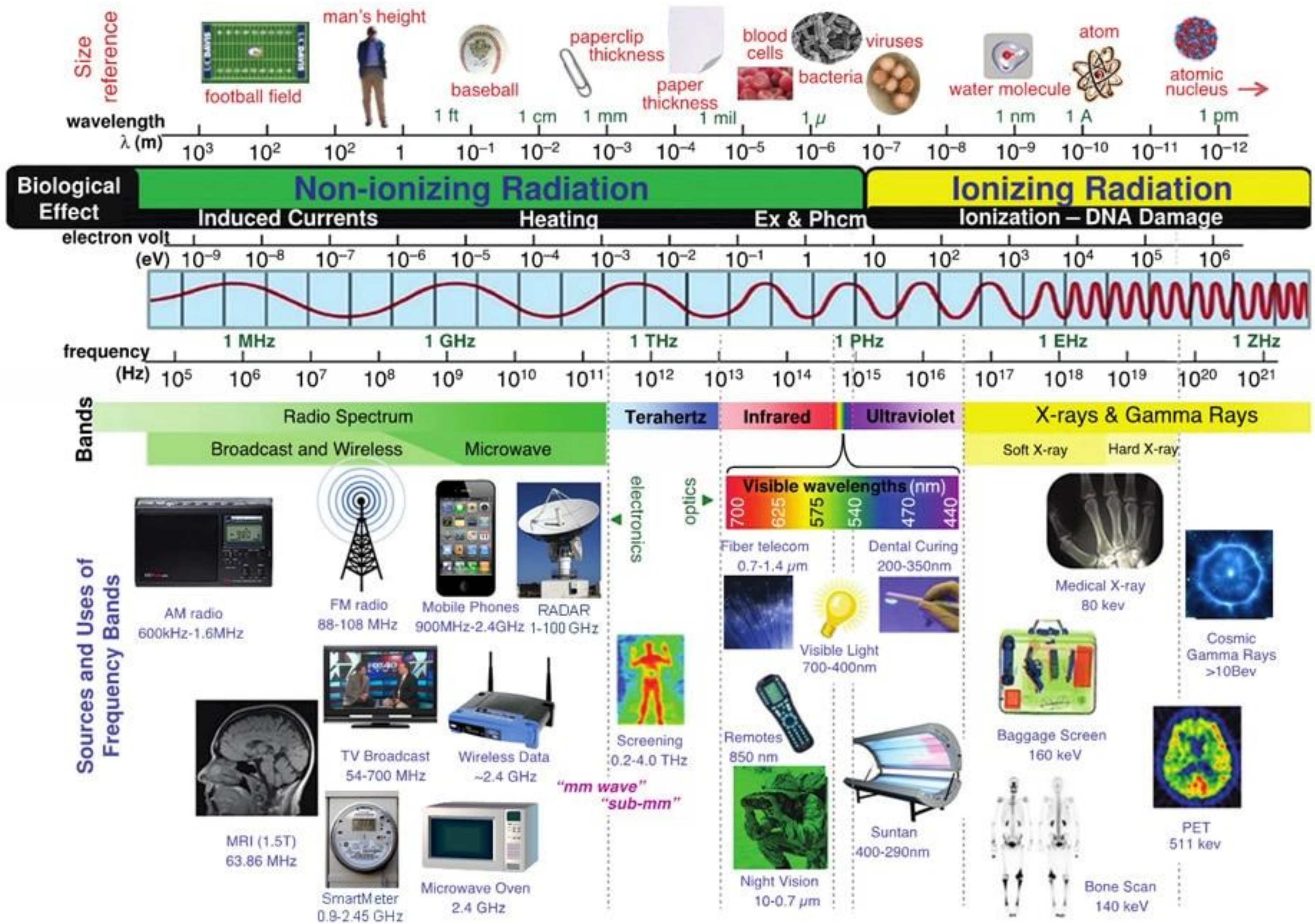
Electromagnetic radiation spectrum



Source: Common knowledge (Wikipedia)

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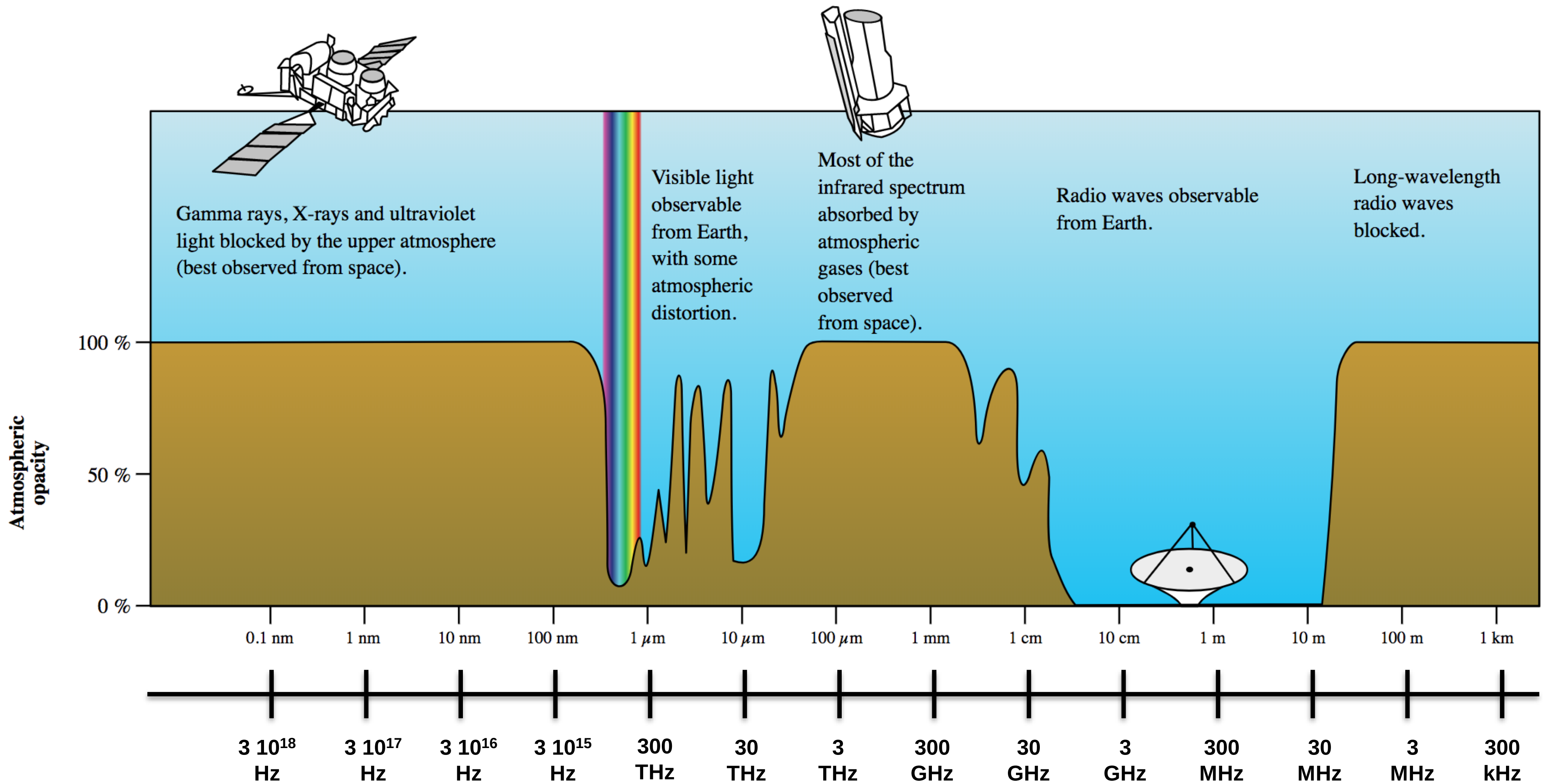
Electromagnetic radiation spectrum: users point of view



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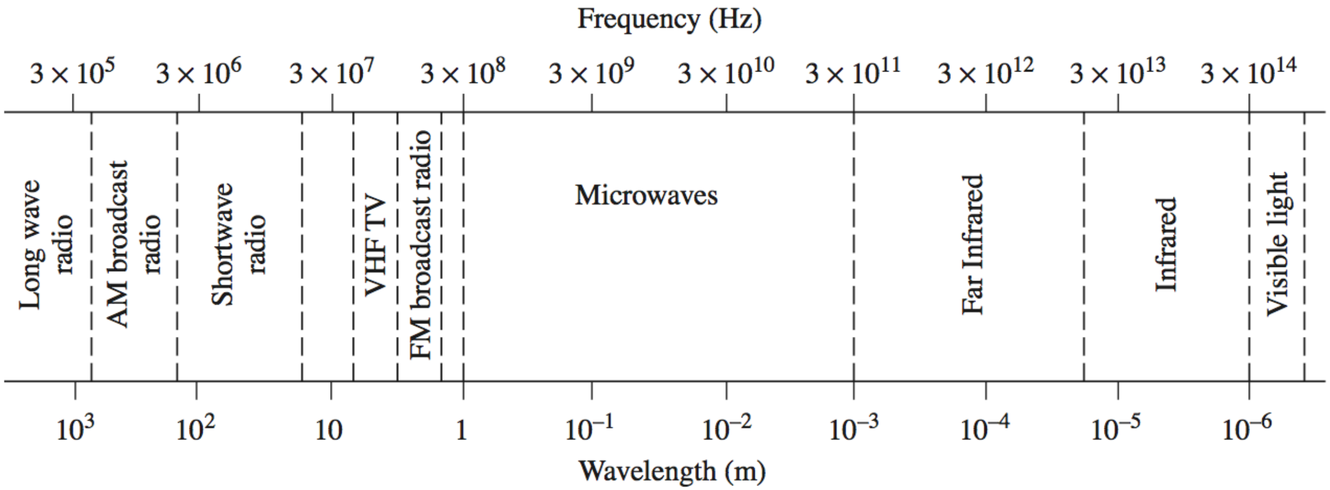
The electromagnetic spectrum for RF engineers



Typical Frequencies	
AM broadcast band	535–1605 kHz
Short wave radio band	3–30 MHz
FM broadcast band	88–108 MHz
VHF TV (2–4)	54–72 MHz
VHF TV (5–6)	76–88 MHz
UHF TV (7–13)	174–216 MHz
UHF TV (14–83)	470–890 MHz
US cellular telephone	824–849 MHz
	869–894 MHz
European GSM cellular	880–915 MHz
	925–960 MHz
GPS	1575.42 MHz
	1227.60 MHz
Microwave ovens	2.45 GHz
US DBS	11.7–12.5 GHz
US ISM bands	902–928 MHz
	2.400–2.484 GHz
	5.725–5.850 GHz
US UWB radio	3.1–10.6 GHz

Approximate Band Designations		
Medium frequency	300 kHz–3 MHz	
High frequency (HF)	3 MHz–30 MHz	
Very high frequency (VHF)	30 MHz–300 MHz	
Ultra high frequency (UHF)	300 MHz–3 GHz	
L band	1–2 GHz	today
S band	2–4 GHz	today
C band	4–8 GHz	today
X band	8–12 GHz	today
Ku band	12–18 GHz	near future
K band	18–26 GHz	near future
Ka band	26–40 GHz	near future
U band	40–60 GHz	
V band	50–75 GHz	
E band	60–90 GHz	
W band	75–110 GHz	
F band	90–140 GHz	

Source: Pozar, Microwave Engineering 4ed, 2012

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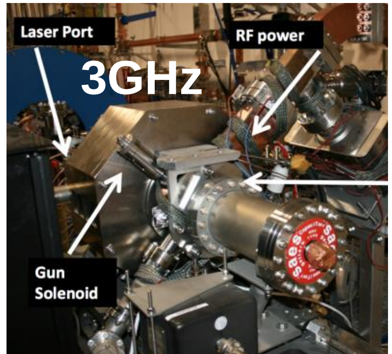
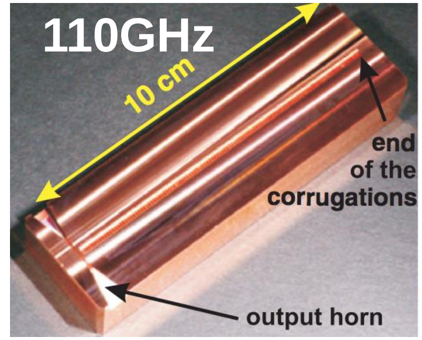
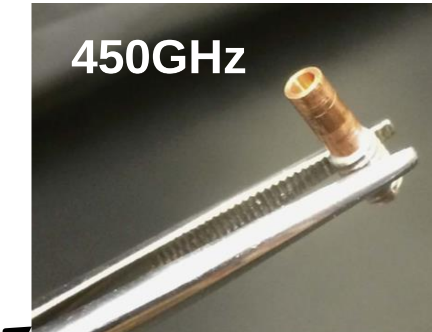
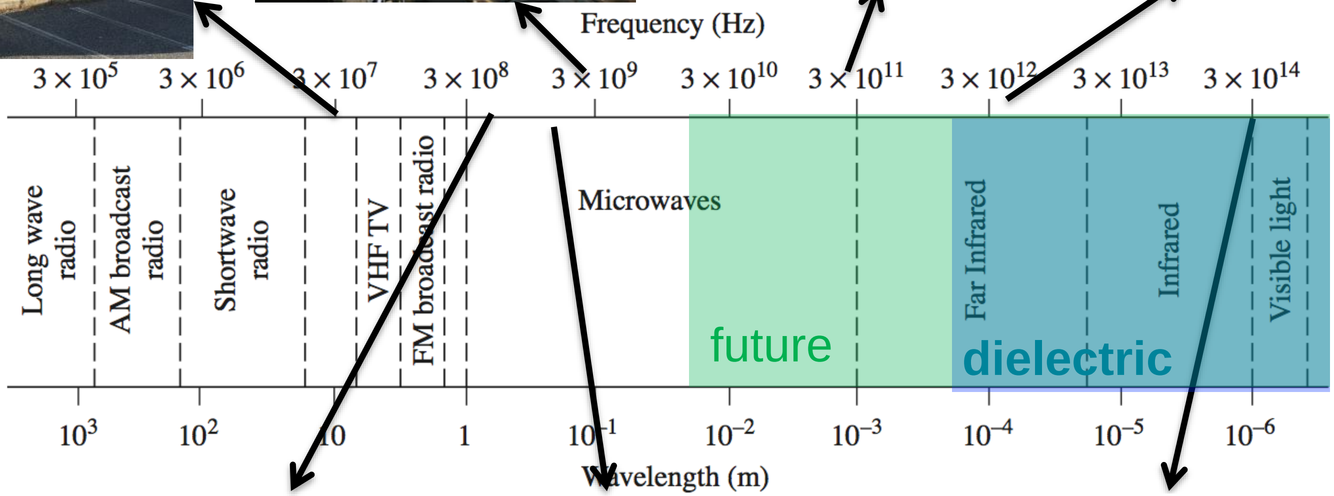
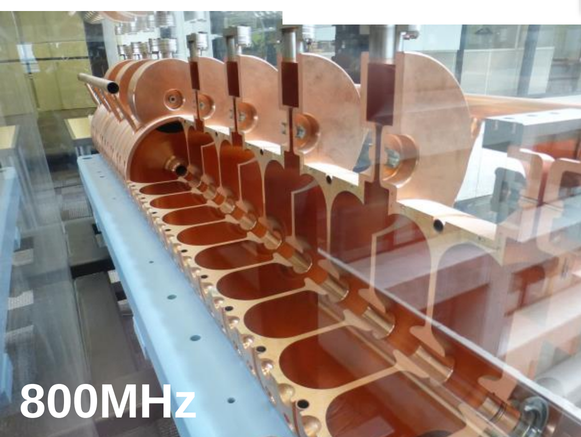
The RF spectrum and particle accelerator devices

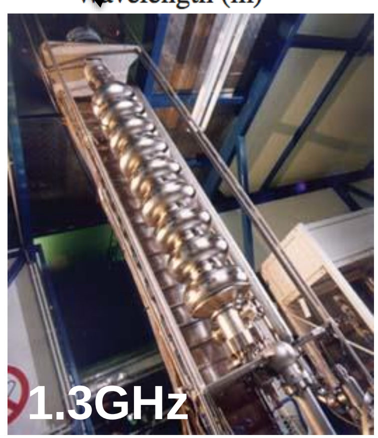


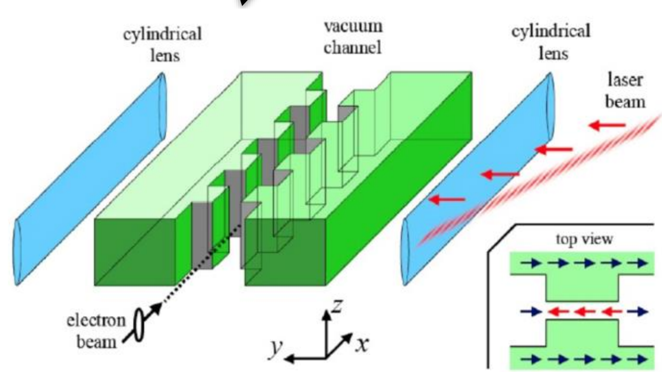












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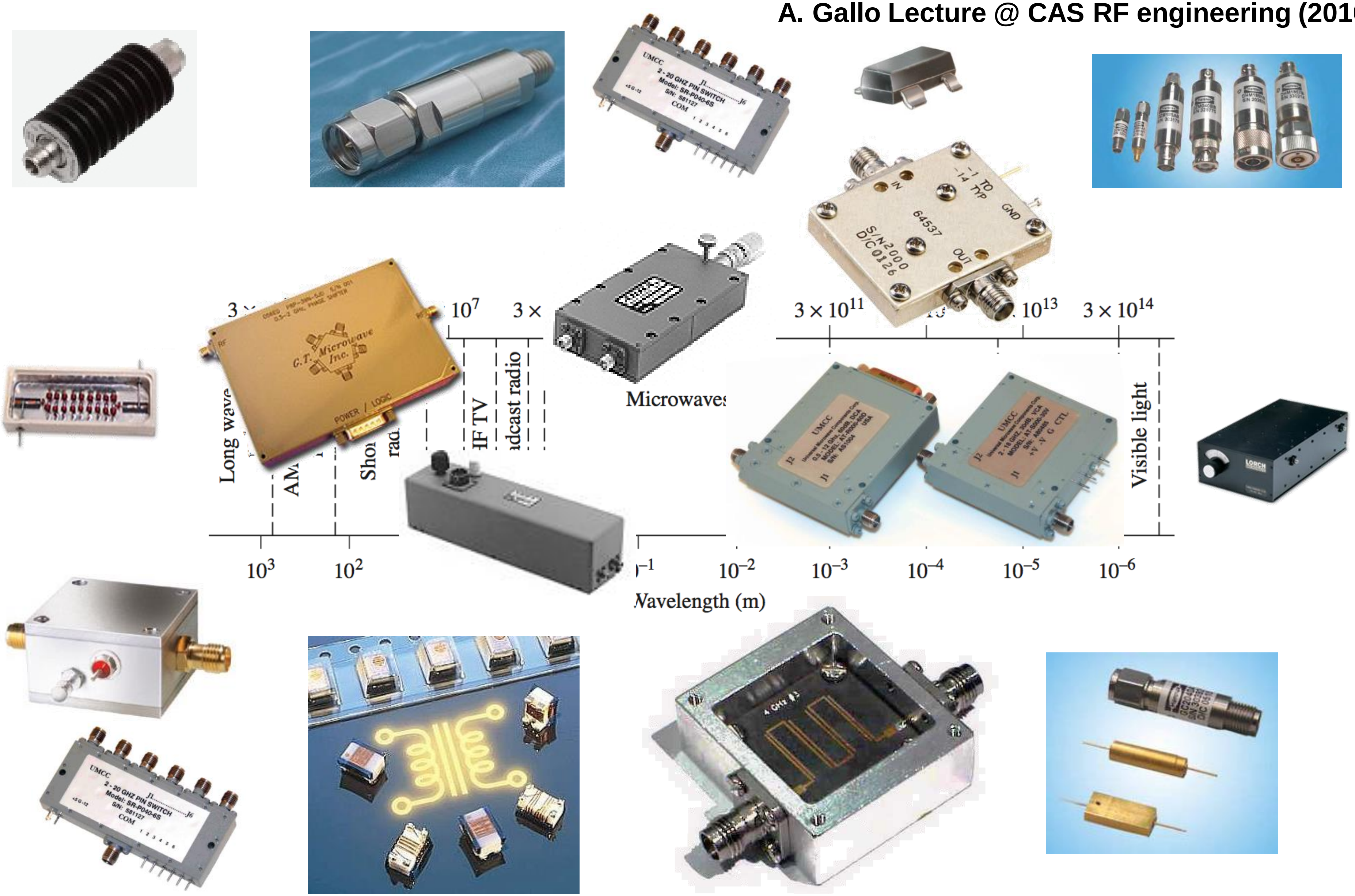
The RF spectrum and particle accelerator electronics

A. Gallo Lecture @ CAS RF engineering (2010)



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Harmonic fields in media: constitutive relations

Hyp: **Linear, Homogeneous, Isotropic and non Dispersive media**

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \qquad \vec{D} = \epsilon_c \vec{E} \qquad \epsilon_c = \epsilon' - j\epsilon''$$

complex permittivity

Losses (heat) due to damping of vibrating dipoles

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \qquad \vec{B} = \mu \vec{H} \qquad \mu = \mu' - j\mu''$$

complex permeability

Ohm Law $\vec{J}_c = \sigma \vec{E}$ $\quad \sigma$ **conductivity** (S/m)

Losses (heat) due to moving charges colliding with lattice

Material	Conductivity S/m (20°C)	Material	Conductivity S/m (20°C)
Aluminum	3.816×10^7	Nichrome	1.0×10^6
Brass	2.564×10^7	Nickel	1.449×10^7
Bronze	1.00×10^7	Platinum	9.52×10^6
Chromium	3.846×10^7	Sea water	3–5
Copper	5.813×10^7	Silicon	4.4×10^{-4}
Distilled water	2×10^{-4}	Silver	6.173×10^7
Germanium	2.2×10^6	Steel (silicon)	2×10^6
Gold	4.098×10^7	Steel (stainless)	1.1×10^6
Graphite	7.0×10^4	Solder	7.0×10^6
Iron	1.03×10^7	Tungsten	1.825×10^7
Mercury	1.04×10^6	Zinc	1.67×10^7
Lead	4.56×10^6		

Source: Pozar, Microwave Engineering 4ed, 2012

Harmonic fields in media: Maxwell Equations

Hyp: Linear, Homogeneous, Isotropic and non Dispersive media

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \qquad \vec{D} = \epsilon_c \vec{E} \qquad \epsilon_c = \epsilon' - j\epsilon''$$ **complex permittivity**

Losses (heat) due to damping of vibrating dipoles

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \qquad \vec{B} = \mu \vec{H} \qquad \mu = \mu' - j\mu''$$ **complex permeability**

Ohm Law $\vec{J}_c = \sigma \vec{E}$ $\quad\sigma$ **conductivity** (S/m)

Losses (heat) due to moving charges colliding with lattice

$$\frac{\partial}{\partial t} \cdots = j\omega \cdots$$

$$\nabla \cdot \vec{D} = \rho \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\nabla \times \vec{H} = j\omega\vec{D} + \vec{J}_c + \vec{J} = \cdots = j\omega\epsilon\vec{E} + \vec{J} \qquad \epsilon = \epsilon' - j\epsilon'' - j\frac{\sigma}{\omega}$$

$$\tan\delta = \frac{\omega\epsilon'' + \sigma}{\omega\epsilon'} = \frac{\text{Losses}}{\text{Displacement current}}$$

Loss tangent

$$\epsilon = \epsilon_r \epsilon_0 \left(1 - j\tan\delta\right)$$

Dielectric constant

$$\epsilon' = \epsilon_r \epsilon_0$$

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Harmonic fields in media: Maxwell Equations

DIELECTRIC CONSTANTS AND LOSS TANGENTS FOR SOME MATERIALS

Material	Frequency	ϵ_r	$\tan\delta$ (25°C)
Alumina (99.5%)	10 GHz	9.5–10.	0.0003
Barium tetratitanate	6 GHz	$37 \pm 5\%$	0.0005
Beeswax	10 GHz	2.35	0.005
Beryllia	10 GHz	6.4	0.0003
Ceramic (A-35)	3 GHz	5.60	0.0041
Fused quartz	10 GHz	3.78	0.0001
Gallium arsenide	10 GHz	13.0	0.006
Glass (pyrex)	3 GHz	4.82	0.0054
Glazed ceramic	10 GHz	7.2	0.008
Lucite	10 GHz	2.56	0.005
Nylon (610)	3 GHz	2.84	0.012
Parafin	10 GHz	2.24	0.0002
Plexiglass	3 GHz	2.60	0.0057
Polyethylene	10 GHz	2.25	0.0004
Polystyrene	10 GHz	2.54	0.00033
Porcelain (dry process)	100 MHz	5.04	0.0078
Rexolite (1422)	3 GHz	2.54	0.00048
Silicon	10 GHz	11.9	0.004
Styrofoam (103.7)	3 GHz	1.03	0.0001
Teflon	10 GHz	2.08	0.0004
Titania (D-100)	6 GHz	$96 \pm 5\%$	0.001
Vaseline	10 GHz	2.16	0.001
Water (distilled)	3 GHz	76.7	0.157

Source: Pozar, Microwave Engineering 4ed, 2012

ı Dispersive media

$''$ **complex permittivity**

ι'' **complex permeability**

ıctivity (S/m)

Losses (heat) due to moving charges colliding with lattice

$\vec{\cdot} + \vec{J}$ $\quad \epsilon = \epsilon' - j\epsilon'' - j\dfrac{\sigma}{\omega}$

Loss tangent

$$\epsilon = \epsilon_r \epsilon_0 \left(1 - j\tan\delta\right)$$

Dielectric constant

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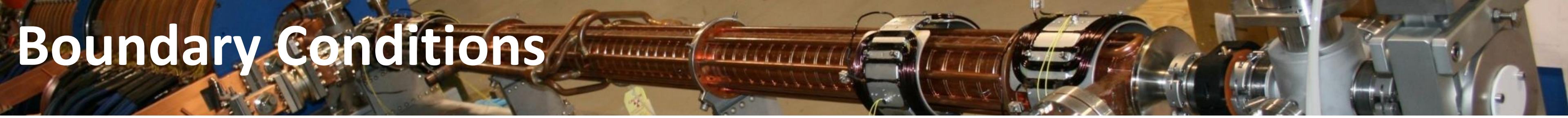
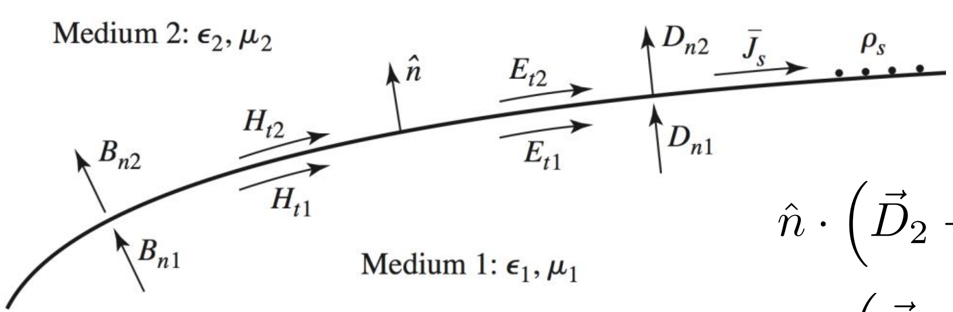
Boundary Conditions





ρ_s **Surface Charge Density** (C/m^2)

$\vec{J}_s$ **Surface Current Density** (A/m)

$$\hat{n}\cdot\left(\vec{D}_2-\vec{D}_1\right)=\rho_s \qquad \hat{n}\cdot\left(\vec{B}_2-\vec{B}_1\right)=0$$

$$\hat{n}\times\left(\vec{E}_2-\vec{E}_1\right)=0 \qquad \hat{n}\times\left(\vec{H}_2-\vec{H}_1\right)=\vec{J}_s$$

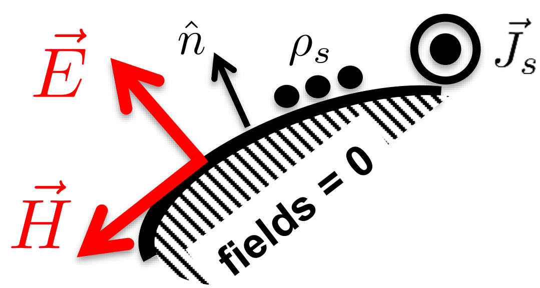
Fields at a lossless dielectric interface

$$\rho_s=0 \qquad \hat{n}\cdot\vec{D}_1=\hat{n}\cdot\vec{D}_2 \qquad \hat{n}\cdot\vec{B}_1=\hat{n}\cdot\vec{B}_2$$

$$\vec{J}_s=0 \qquad \hat{n}\times\vec{E}_1=\hat{n}\times\vec{E}_2 \qquad \hat{n}\times\vec{H}_1=\hat{n}\times\vec{H}_2$$

Perfect conductor (electric wall)

$\sigma\to\infty$

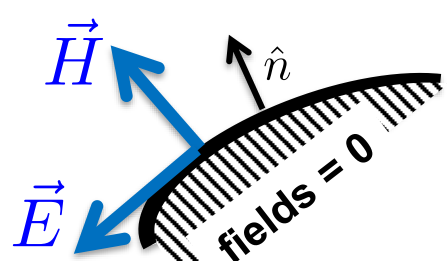


$$\hat{n}\cdot\vec{D}=\rho_s \qquad \hat{n}\cdot\vec{B}=0$$

$$\hat{n}\times\vec{E}=0 \qquad \hat{n}\times\vec{H}=\vec{J}_s$$

Magnetic Wall (dual of the E-wall)

approx.



$$\hat{n}\cdot\vec{D}=0 \qquad \hat{n}\cdot\vec{B}\neq 0$$

$$\hat{n}\times\vec{H}=0 \qquad \hat{n}\times\vec{E}\neq 0$$

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Helmotz equation and its simplest solution

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$e^{j\omega t} \rightarrow \frac{\partial^2}{\partial t^2} \cdots = -\omega^2 \ldots$$

Helmotz equation

$$\nabla^2 \vec{E} + \omega^2 \mu\epsilon \vec{E} = 0$$

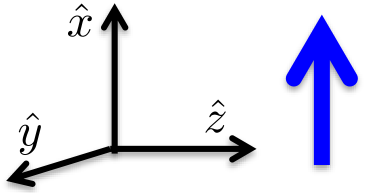
$$\nabla^2 \vec{H} + \omega^2 \mu\epsilon \vec{H} = 0$$

$$k = \omega\sqrt{\mu\epsilon} \qquad (1/m)$$

Propagation/phase constant

Wave number

The simples solution:
the wave with the plane wave-front



$$\vec{E} = E_x \hat{x}$$

Uniform in x, y

Lossless medium

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$$

$$\frac{d^2 E_x}{dz^2} + k^2 E_x = 0$$

$$E_x(z) = E^+ e^{-jkz} + E^- e^{jkz}$$

$$E_x(z,t) = Re\left\{E(z,\omega)e^{j\omega t}\right\} = E^+ \cos(\omega t - kz) + E^- \cos(\omega t + kz)$$

It is a wave, moving in the +z direction or –z direction

Phase velocity **Velocity at which a fixed phase point on the wave travels**

$$\omega t \mp kz = \text{const}$$

$$v_p = \frac{dz}{dt} = \frac{d}{dt}\left(\frac{\omega t \mp \text{const}}{k}\right) = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}$$

Speed of light

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Plane waves and Transverse Electro-Magnetic (TEM) waves

Wave length **Distance between two consecutive maxima (or minima or …)**

$$(\omega t - kz) - [\omega t - k(z+\lambda)] = 2\pi$$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi v_p}{\omega} = \frac{v_p}{f}$$

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

Compute H …

$$E_x(z) = E^+ e^{-jkz} + E^- e^{jkz}$$

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Wave length Distance between two consecutive maxima (or minima or …)

$$(\omega t - kz) - [\omega t - k(z+\lambda)] = 2\pi \qquad \lambda = \frac{2\pi}{k} = \frac{2\pi v_p}{\omega} = \frac{v_p}{f}$$

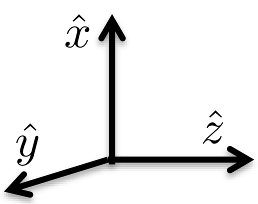
$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

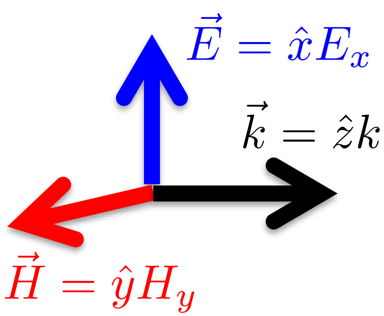
$$E_x(z) = E^+ e^{-jkz} + E^- e^{jkz}$$

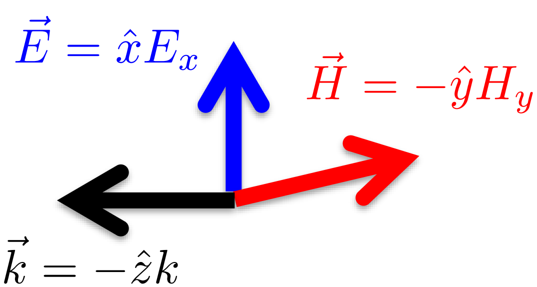
$$H_x = H_z = 0 \qquad H_y = \frac{j}{\omega\mu}\frac{\partial E_x}{\partial z} = \frac{1}{\eta}\left(E^+ e^{-jkz} - E^- e^{jkz}\right)$$

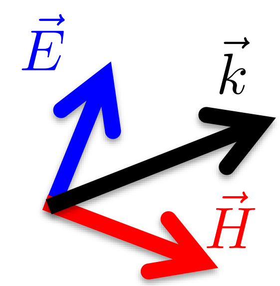
$$\eta = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\epsilon}}$$ **Intrinsic impedance of the medium** (Ω) $$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\ \Omega$$

The ratio of E and H component is an impedance called wave impedance









$$\vec{H} = \frac{1}{\eta}\hat{k} \times \vec{E}$$

TEM wave **E and H field are transverse to the direction of propagation.** $$Z_{TEM} = \eta$$

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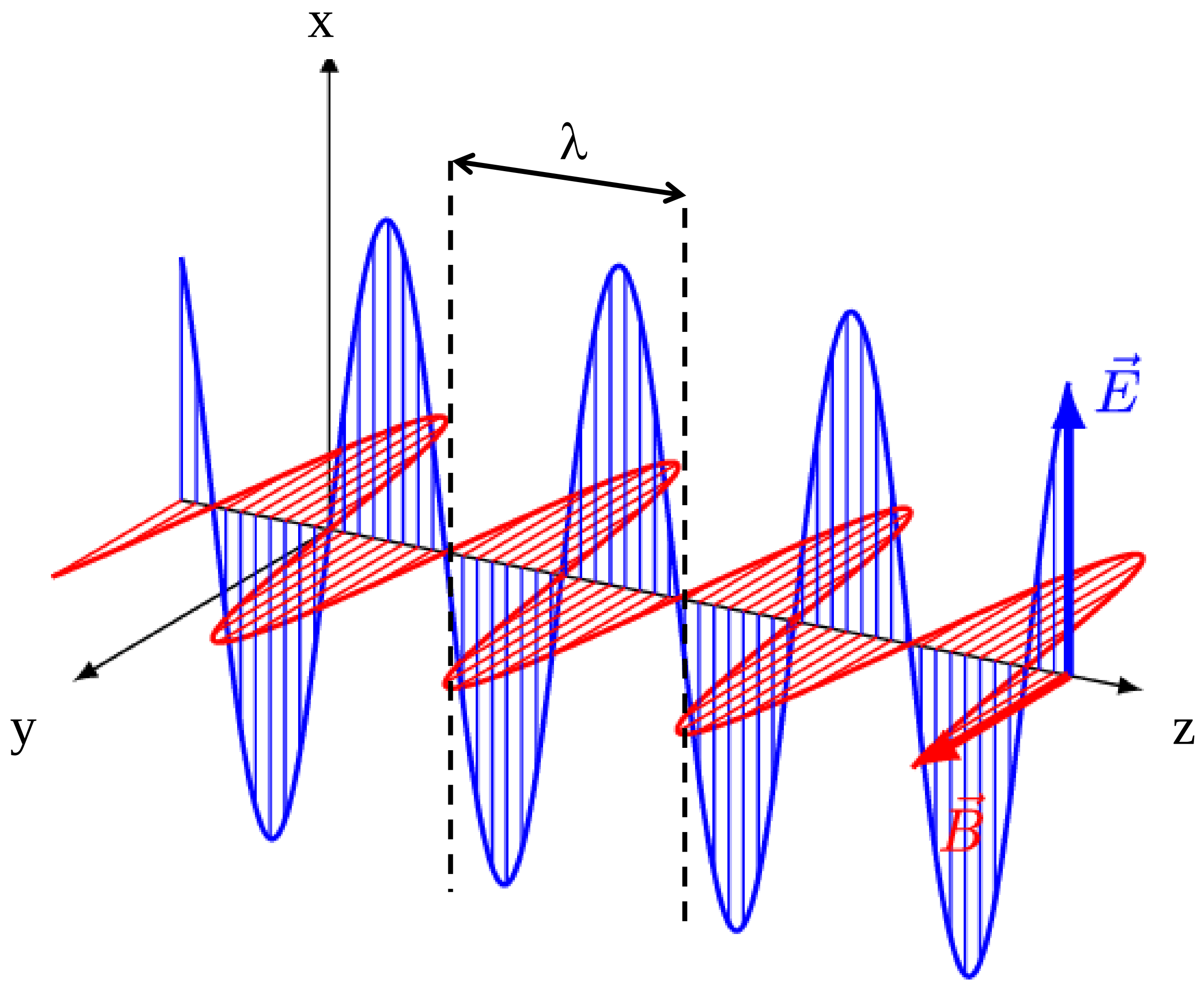
Plane waves and Transverse Electro-Magnetic (TEM) waves



Note that E and B fields oscillate with the same phase.

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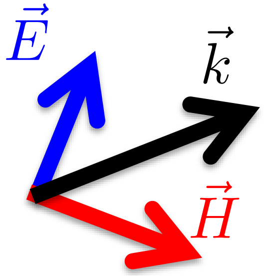
Poynting vector for a plane wave



$$\vec{H} = \frac{1}{\eta}\hat{k} \times \vec{E} \longrightarrow \frac{E}{H} = \eta = \sqrt{\frac{\mu}{\epsilon}} = \frac{1}{\epsilon v} = \mu v$$

$$\vec{S}(\vec{r},t) = \vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t)$$

$$\vec{S} = \frac{1}{\eta}\vec{E} \times \left(\hat{k} \times \vec{E}\right) = \frac{1}{\eta}\left[\left(\vec{E}\cdot\vec{E}\right)\hat{k} - \cancel{\left(\vec{E}\cdot\hat{k}\right)}\,\vec{E}\right] =$$

$$= \epsilon\left|\vec{E}\right|^2 \vec{v} = \frac{1}{2}\left(\epsilon\left|\vec{E}\right|^2 + \mu\left|\vec{H}\right|^2\right)\vec{v} = \left(u_E + u_H\right)\vec{v} = u\vec{v}$$

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Current density, Poynting vector and transport phenomena

charge

$$\vec{J} = \rho_e \ \vec{v} \qquad \left(\frac{A}{m^2}\right)$$

Current density

$$\nabla \cdot \vec{J} + \frac{\partial \rho_e}{\partial t} = 0$$

Continuity equation

$$\frac{dQ}{dt} = \int_A \vec{J} \cdot d\vec{A} = I$$

e.m. energy

$$\vec{S} = u \ \vec{v} \qquad \left(\frac{W}{m^2}\right)$$

Poynting vector

$$\nabla \cdot \vec{S} + \frac{\partial u}{\partial t} = 0$$

Poynting theorem

$$\frac{dU}{dt} = \int_A \vec{S} \cdot d\vec{A} = P_{\text{rad}}$$

mass

$$\vec{J}_m = \rho_m \ \vec{v} \qquad \left(\frac{kg/s}{m^2}\right)$$

Mass flux density

$$\nabla \cdot \vec{J}_m + \frac{\partial \rho_m}{\partial t} = 0$$

Continuity equation

$$\frac{dm}{dt} = \int_A \vec{J}_m \cdot d\vec{A}$$

probability

$$\vec{J} = Re\left\{\psi^* \frac{\hbar}{i\ m} \nabla \psi\right\} = = Re\left\{\psi^* \frac{\vec{p}}{m} \psi\right\} \qquad \left(\frac{1/s}{m^2}\right)$$

Probability current density

$$\frac{\partial}{\partial t} \int_\tau P\left(\vec{r}, t\right) d\tau = -\int_{\partial\tau} \vec{J} \cdot d\vec{A}$$

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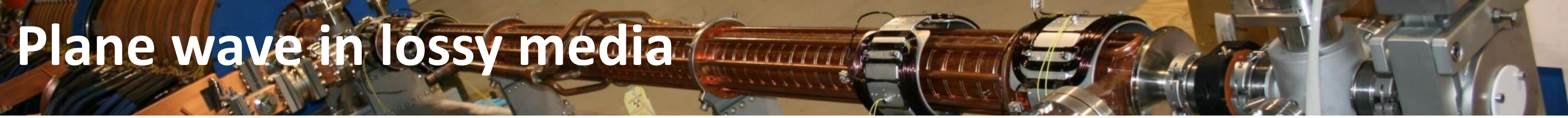
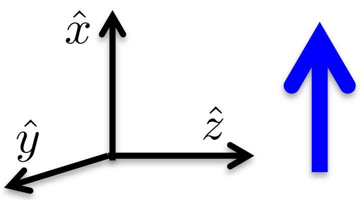


Plane wave in lossy media

$$\nabla^2 \vec{E} + \omega^2 \mu\epsilon \vec{E} = 0 \qquad \epsilon = \epsilon_r \epsilon_0 \left(1 - j\tan\delta\right) \qquad \tan\delta = \frac{\omega\epsilon'' + \sigma}{\omega\epsilon'}$$

Definition: $\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon} = j\omega\sqrt{\mu\epsilon_0\epsilon_r(1 - j\tan\delta)}$

Attenuation constant → α **Phase constant** → β



$\vec{E} = E_x \hat{x}$

Uniform in x, y

$$\frac{d^2 E_x}{dz^2} - \gamma^2 E_x = 0 \qquad E_x(z) = E^+ e^{-\gamma z} + E^- e^{\gamma z}$$

Positive z direction

$$e^{-\gamma z} = e^{-\alpha z} e^{-j\beta z}$$

— time →

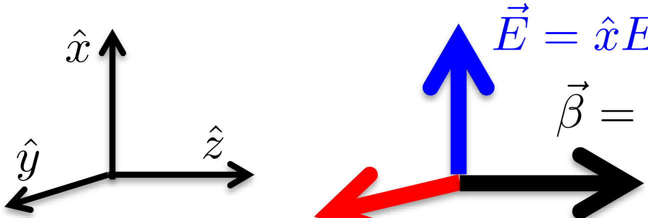
$$e^{-\alpha z}\cos\left(\omega t - \beta z\right)$$

$$v_p = \frac{\omega}{\beta} \qquad \lambda = \frac{2\pi}{\beta}$$

$$H_y = \frac{j}{\omega\mu}\frac{\partial E_x}{\partial z} = -\frac{j\gamma}{\omega\mu}\left(E^+ e^{-\gamma z} - E^- e^{\gamma z}\right) = \frac{1}{\eta}\left(E^+ e^{-\gamma z} - E^- e^{\gamma z}\right) \qquad \eta = \frac{j\omega\mu}{\gamma} \rightarrow \sqrt{\frac{\mu}{\epsilon}}$$

complex

$$Z_{TEM} = \eta$$



$\vec{E} = \hat{x}E_x$

$\vec{\beta} = \hat{z}\beta$

$\vec{H} = \hat{y}H_y$

$$\vec{H} = \frac{1}{\eta}\hat{\beta} \times \vec{E}$$

Attenuating TEM "wave" …

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Plane waves in good conductors

Good conductor

Conduction current >> displacement current

$$\sigma E \gg \omega\epsilon_c E$$

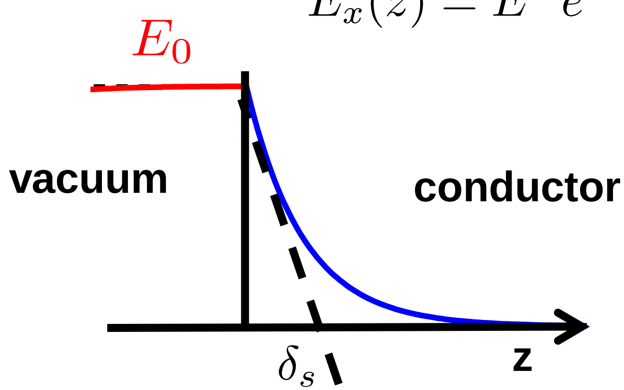
$$\tan\delta = \frac{\omega\epsilon'' + \sigma}{\omega\epsilon'} \approx \frac{\sigma}{\omega\epsilon_0\epsilon_r}$$

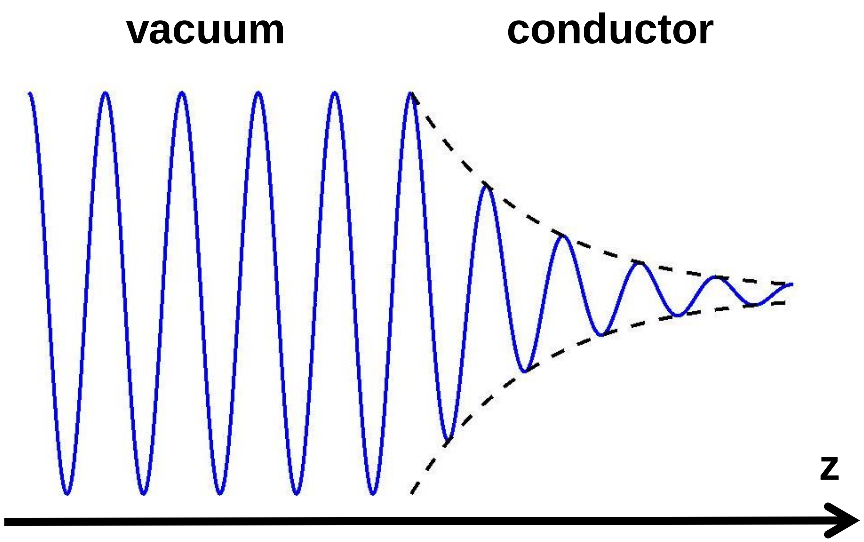
$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon} \simeq (1+j)\sqrt{\frac{\omega\mu\sigma}{2}}$$

Characteristic depth of penetration: skin depth

$$\delta_s = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}}$$

$$E_x(z) = E^+e^{-\gamma z} + E^-e^{\gamma z}$$





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Plane waves in good conductors

Good conductor

Conduction current >> displacement current

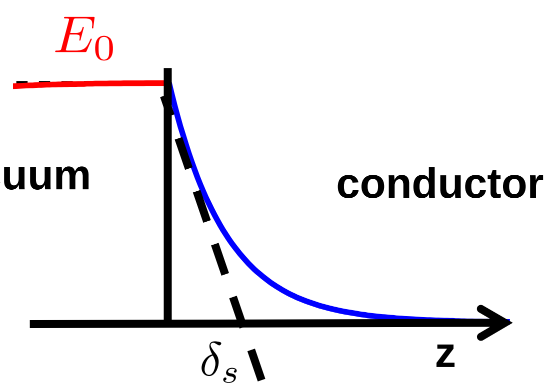
$$\sigma E \gg \omega\epsilon_c E$$

$$\tan\delta = \frac{\omega\epsilon'' + \sigma}{\omega\epsilon'} \approx \frac{\sigma}{\omega\epsilon_0\epsilon_r}$$

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon} \simeq (1+j)\sqrt{\frac{\omega\mu\sigma}{2}}$$

Characteristic depth of penetration: skin depth

$$\delta_s = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}}$$



Al	$\delta_s = 8.14\ 10^{-7}\ m$
Cu	$\delta_s = 6.60\ 10^{-7}\ m$
Au	$\delta_s = 7.86\ 10^{-7}\ m$
Ag	$\delta_s = 6.40\ 10^{-7}\ m$

@ 10 GHz

impedance of the medium

$$\eta = \frac{j\omega\mu}{\gamma} \simeq (1+j)\sqrt{\frac{\omega\mu}{2\sigma}} = (1+j)\frac{1}{\sigma\delta_s}$$

? Copper @ 100 MHz

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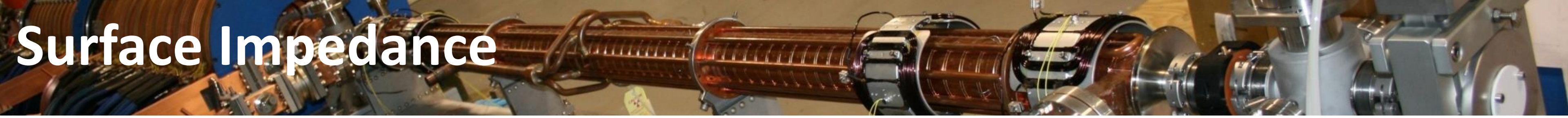
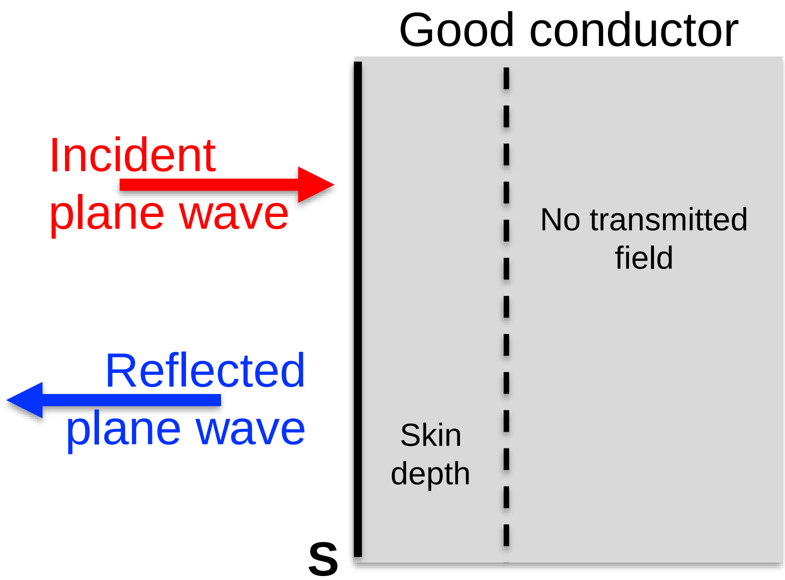


Surface Impedance



Goal: account for an imperfect conductor

The power that is transmitted into the conductor is dissipated as heat within a **very short distance** from the surface.

Being $\vec{J}_S = \hat{n} \times \vec{H}\Big|_S$ when $\sigma \to \infty$

Approximation

Replace the exponentially decaying volume current volume with a uniform current extending a distance of one skin depth

$$\bar{J}_t = \begin{cases} \bar{J}_s/\delta_s & \text{for } 0 < z < \delta_s \\ 0 & \text{for } z > \delta_s, \end{cases}$$

$$\delta_s = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}}$$

Power loss

computed as if the metal were a perfect conductor

$$P_t = \frac{1}{2\sigma}\int_S\int_0^{\delta_S}\frac{|\vec{J}_S|^2}{\delta_S^2}dSdz = \frac{1}{2}\frac{1}{\sigma\delta_S}\int_S|\vec{J}_S|^2 dS = \frac{R_s}{2}\int_S|\hat{n}\times\vec{H}|^2 dS$$

valid if $\eta \ll \eta_0$

Surface resistance

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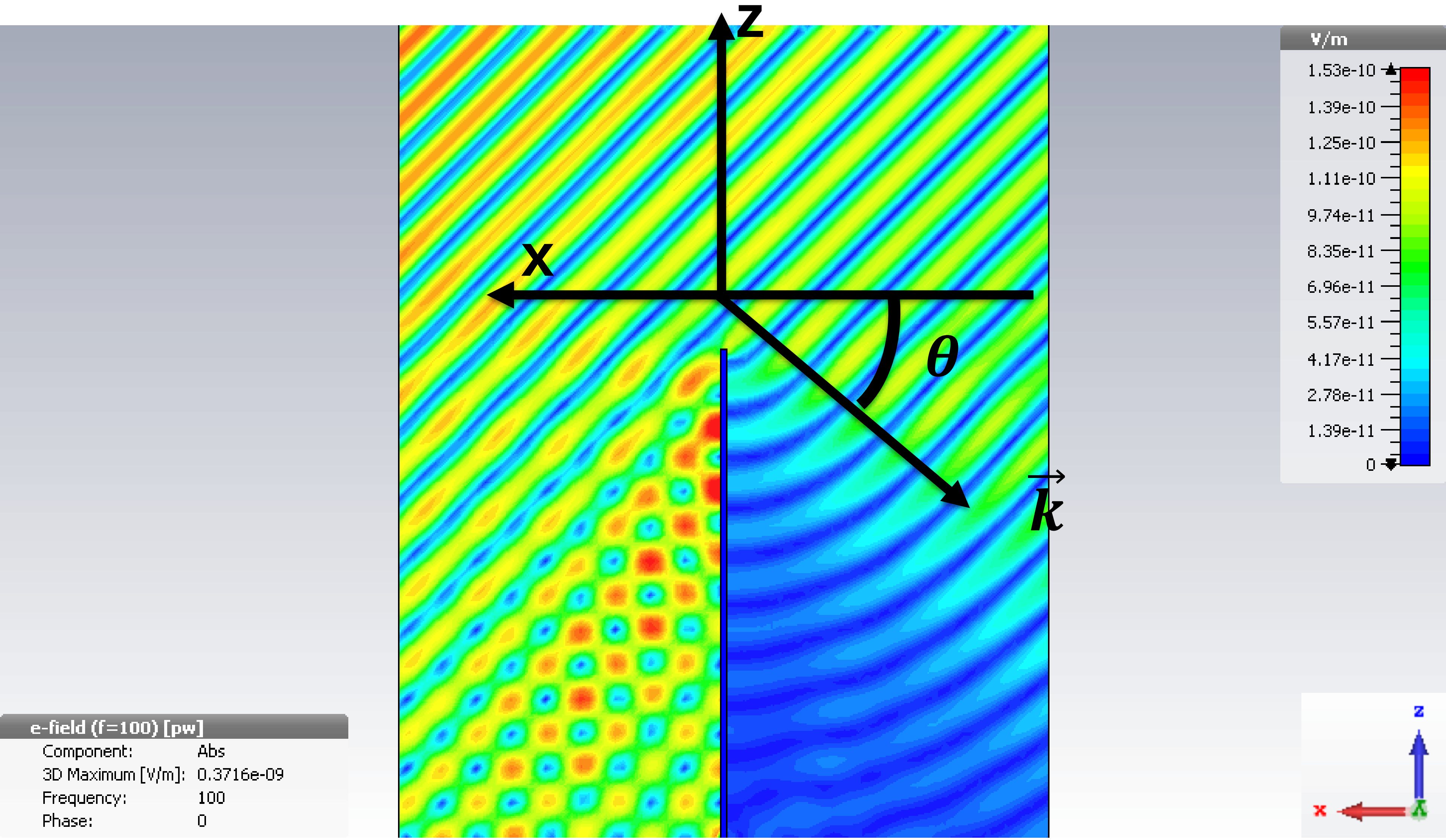
Reflection of plane waves (a first boundary value problem)

Simulations by
L. Ficcadenti, INFN

Courtesy of
M. Ferrario, INFN-LNF



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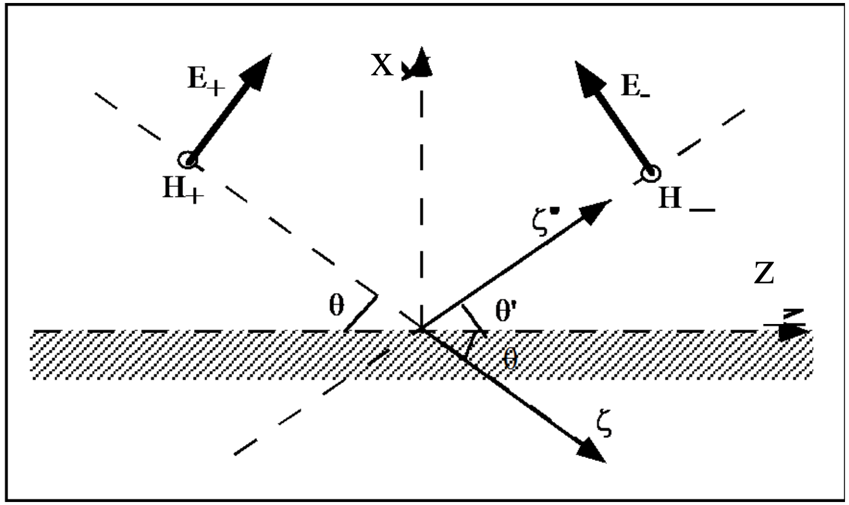
Plane wave reflected by a perfectly conducting plane

Courtesy of
M. Ferrario, INFN-LNF

$$\sigma = \infty$$



In the plane xz the field is given by the superposition of the incident and reflected wave:

$$E(x,z,t) = E_+(x_o,z_o,t_o)e^{i\omega t - ik\zeta} + E_-(x_o,z_o,t_o)e^{i\omega t - ik\zeta'}$$

$$\zeta = z\cos\theta - x\sin\theta \qquad \zeta' = z\cos\theta' + x\sin\theta'$$

And it has to fulfill the boundary conditions (no tangential E-field)

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Reflection of plane waves (a first boundary value problem)

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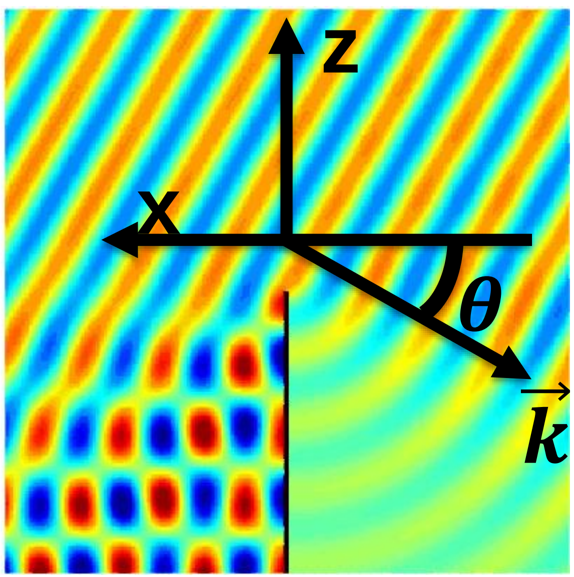
Taking into account the boundary conditions the longitudinal component of the field becomes:

$$E_z(x,z,t) = \left(E_+ \sin\theta\right) e^{i\omega t - ik\left(z\cos\theta - x\sin\theta\right)} - \left(E_+ \sin\theta\right) e^{i\omega t - ik\left(z\cos\theta + x\sin\theta\right)}$$

$$= 2iE_+ \sin\theta \sin\left(kx\sin\theta\right) e^{i\omega t - ikz\cos\theta}$$



Standing Wave pattern (along x)

Guided wave pattern (along z)

The phase velocity is given by

$$v_{fz} = \frac{\omega}{k_z} = \frac{\omega}{k\cos\theta} = \frac{c}{\cos\theta} > c$$

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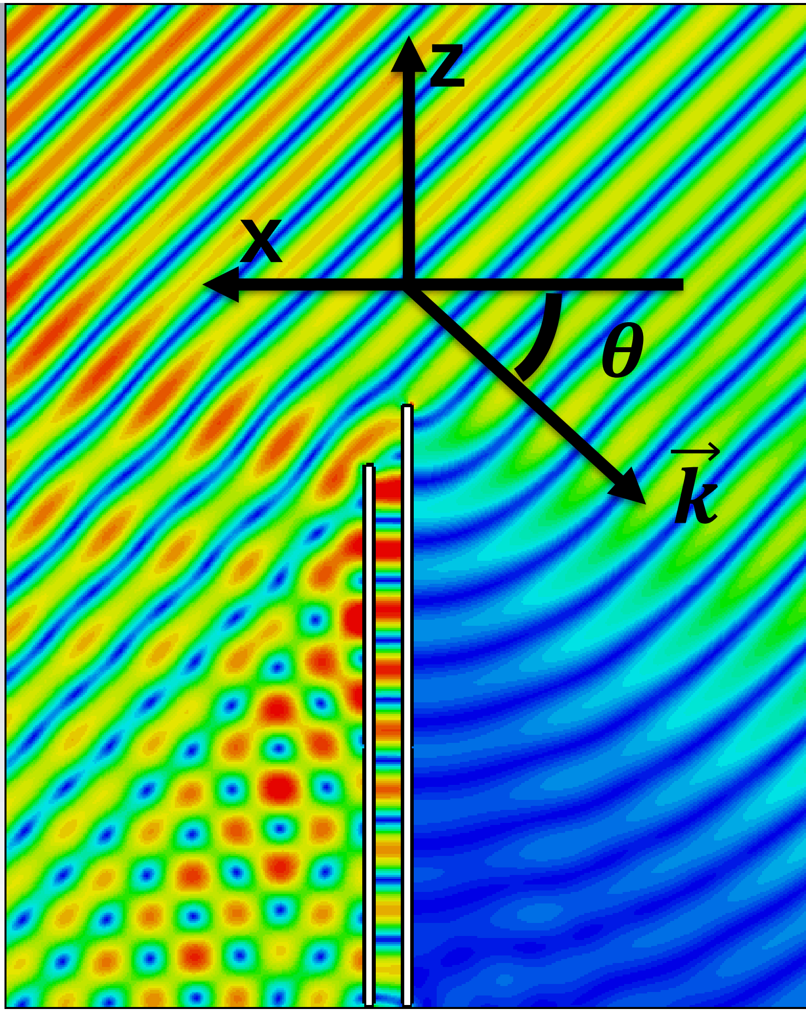
From reflections to waveguides



Simulations by
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Courtesy of
M. Ferrario, INFN-LNF

Put a metallic boundary parallel to the first wall (the E-field is normal).

Between the two walls there must be an integer number of half wavelengths (at least one).

For a given distance, there is a maximum wavelength, i.e. there is **cut-off frequency**.

$$v_{f_z} = \frac{\omega}{k_z} = \frac{\omega}{k\cos\theta} = \frac{c}{\cos\theta} > c$$

→ It can not be used as it is for particle acceleration/deflection

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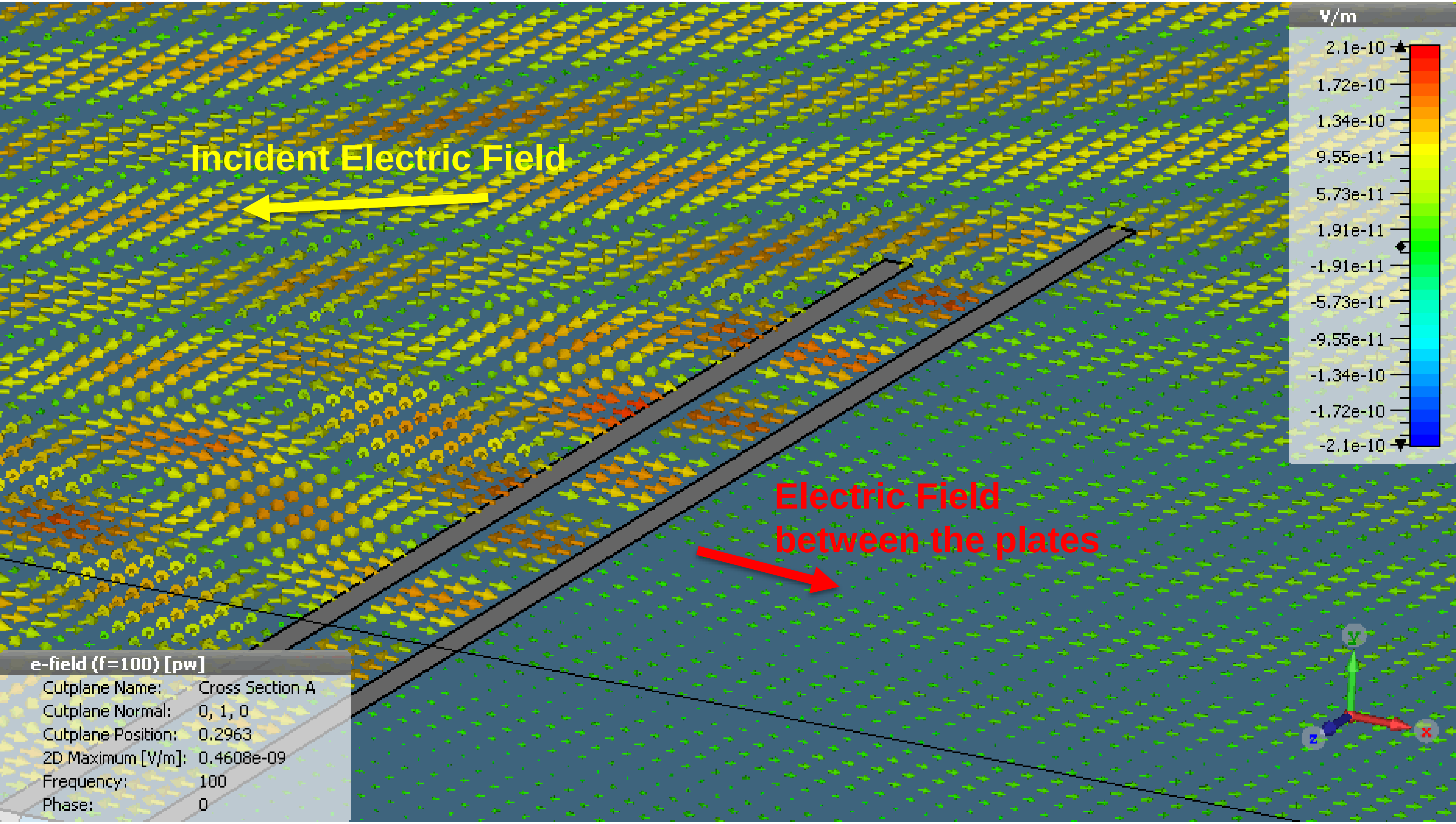

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Maxwell equations and boundary value problem

Maxwell equation with sources + boundary conditions = boundary value problem

Homogeneous medium

Sources

$\vec{J}, \ \rho$

$$\nabla \cdot \vec{E} = \rho/\epsilon \qquad \nabla \cdot \vec{H} = 0$$

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \qquad \nabla \times \vec{H} = +j\omega\epsilon\vec{E} + \vec{J}$$

Do you see asymmetries?

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Maxwell equations and boundary value problem

Maxwell equation with sources + boundary conditions = boundary value problem

Homogeneous medium

Sources

$$\nabla \cdot \vec{E} = \rho/\epsilon \qquad \nabla \cdot \vec{H} = \rho_m/\mu$$

$\vec{J},\ \rho$ — **Actual or equivalent**

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} - \vec{J}_m \qquad \nabla \times \vec{H} = +j\omega\epsilon\vec{E} + \vec{J}$$

$\vec{J}_m,\ \rho_m$ — **equivalent**

Vector Helmotz Equation

$$\nabla^2\vec{E} + k^2\vec{E} = \nabla \times \vec{J}_m + j\omega\mu\vec{J} + \frac{1}{\epsilon}\nabla\rho$$

$$\nabla^2\vec{H} + k^2\vec{H} = -\nabla \times \vec{J} + j\omega\epsilon\vec{J}_m + \frac{1}{\mu}\nabla\rho_m$$

$$k^2 = \omega^2\mu\epsilon$$

Solution

Step 1 **Source free region** $\vec{J} = \vec{J}_m = \rho_m = \rho = 0$ **Homogeneous problem**

Step 2 $$\text{Solution} = \sum_k C_k\left(\vec{J}, \vec{J}_m, \rho_m, \rho\right) \text{Solution-Homogeneous-Problem}_k$$

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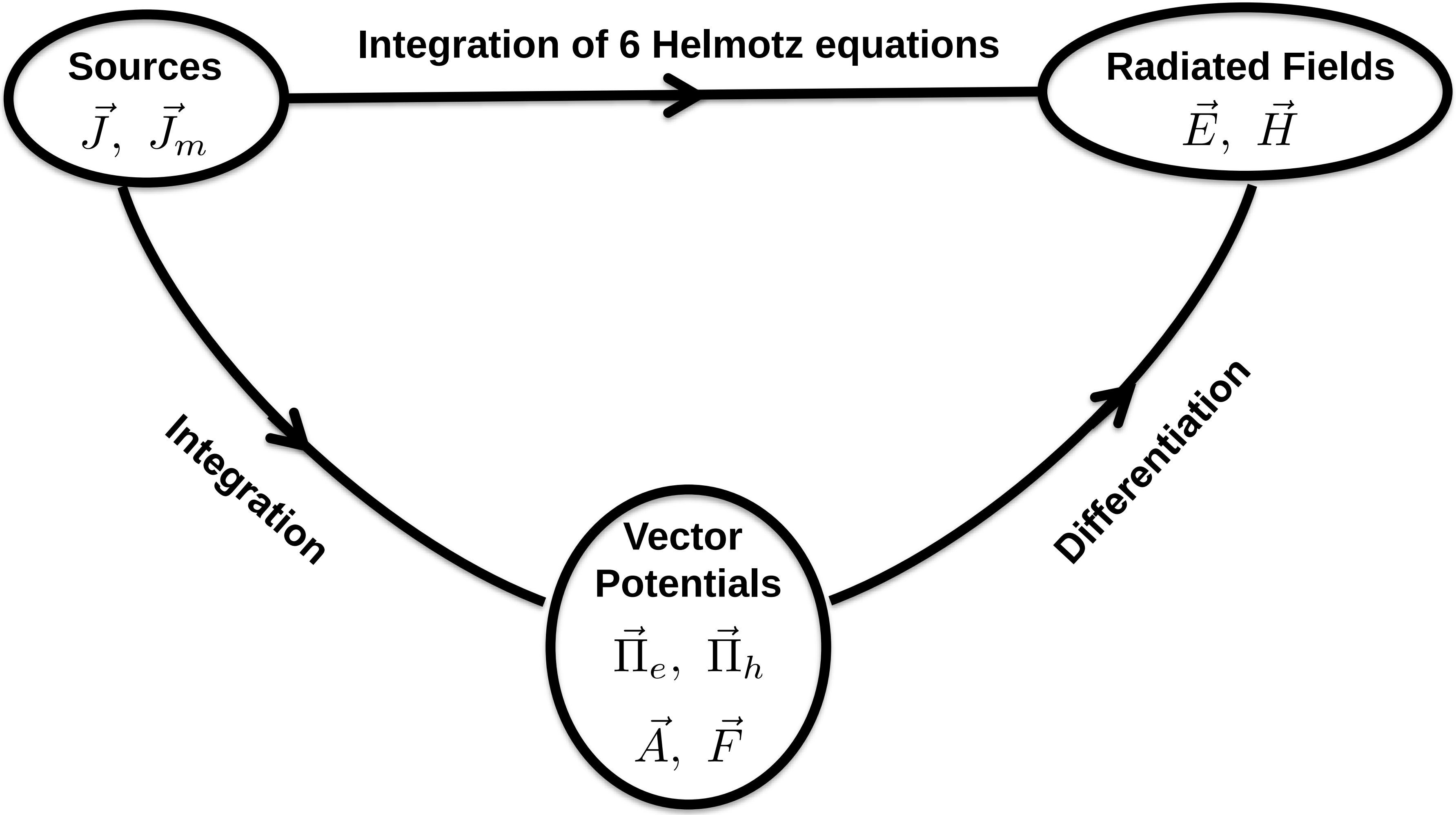
Method of solution of Helmotz equations



Solution of the homogeneous equation → Shape of radiated field → **MODES**

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Solution of Helmotz equations using potentials

Sources

$\vec{J},\ \vec{J}_m$

Integration of 6 Helmotz equations →

Radiated Fields

$\vec{E},\ \vec{H}$

Integration of Helmotz equations

if $\nabla \cdot \vec{H} = 0$

$$\nabla^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J}$$

if $\nabla \cdot \vec{E} = 0$

$$\nabla^2 \vec{F} + k^2 \vec{F} = -\mu \vec{J}_m$$

$$k^2 = \omega^2 \mu \epsilon$$

Vector Potentials

$\vec{A},\ \vec{F}$

Differentiation

$$\vec{H}_A = \frac{1}{\mu} \nabla \times \vec{A}$$

$$\vec{E}_A = -j\omega \vec{A} - \frac{j}{\omega\mu\epsilon} \nabla \left(\nabla \cdot \vec{A} \right)$$

$$\vec{E}_F = -\frac{1}{\epsilon} \nabla \times \vec{F}$$

$$\vec{H}_F = -j\omega \vec{F} - \frac{j}{\omega\mu\epsilon} \nabla \left(\nabla \cdot \vec{F} \right)$$

$$\vec{E} = \vec{E}_A + \vec{E}_F$$

$$\vec{H} = \vec{H}_A + \vec{H}_F$$

Why/when is it convenient?

MODES → **Solution of the homogeneous equations** →

$$\nabla^2 \vec{A} + k^2 \vec{A} = 0$$

$$\nabla^2 \vec{F} + k^2 \vec{F} = 0$$

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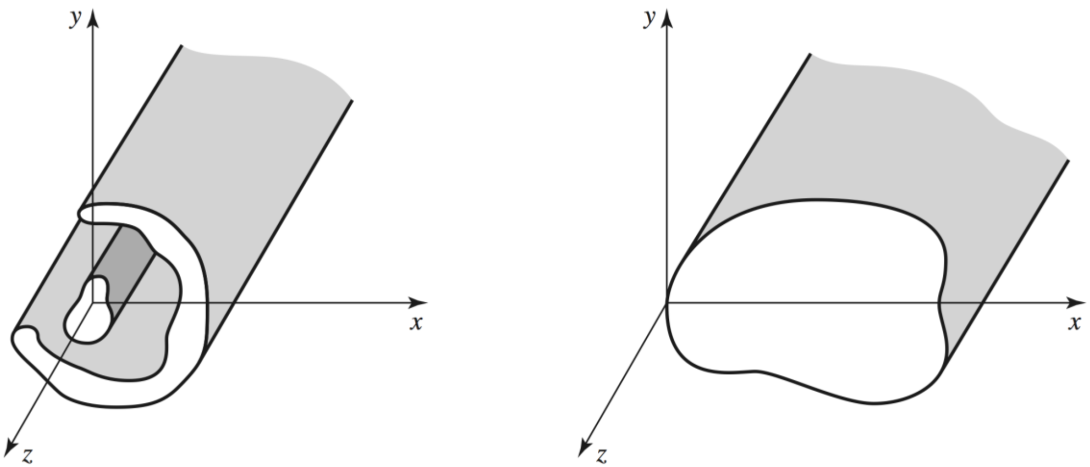
Modes of cylindrical waveguides: propagating field



Field propagating in the positive z direction

$$\vec{A} = \hat{z}\ A_z(x,y)\ e^{-j\beta z} = \hat{z}\ A$$

$$\vec{F} = \hat{z}\ F_z(x,y)\ e^{-j\beta z} = \hat{z}\ F$$

$$\nabla^2 = \nabla_t^2 + \frac{\partial^2}{\partial z^2} \longrightarrow \begin{aligned} &\nabla_t^2 A_z + \left(k^2 - \beta^2\right) A_z = 0 \\ &\nabla_t^2 F_z + \left(k^2 - \beta^2\right) F_z = 0 \end{aligned}$$

2 Helmotz equations
(transverse coordinates)

$$\vec{H}_A = \frac{1}{\mu}\nabla \times (\hat{z}A) \longrightarrow \vec{H}_A = \vec{h}_t\ e^{-j\beta z}$$

$$\vec{E}_A = -j\omega A\hat{z} - \frac{\beta}{\omega\mu\epsilon}\nabla A \longrightarrow \vec{E}_A = [\vec{e}_t + \hat{z}\ e_z]\, e^{-j\beta z}$$

Only E field along z
E-mode
Transverse Magnetic (TM)

$$\vec{E}_F = -\frac{1}{\epsilon}\nabla \times (\hat{z}F) \longrightarrow \vec{E}_F = \vec{e}_t\ e^{-j\beta z}$$

$$\vec{H}_F = -j\omega F\hat{z} - \frac{\beta}{\omega\mu\epsilon}\nabla F \longrightarrow \vec{H}_F = \left[\vec{h}_t + \hat{z}\ h_z\right] e^{-j\beta z}$$

Only H field along z
H-mode
Transverse Electric (TE)

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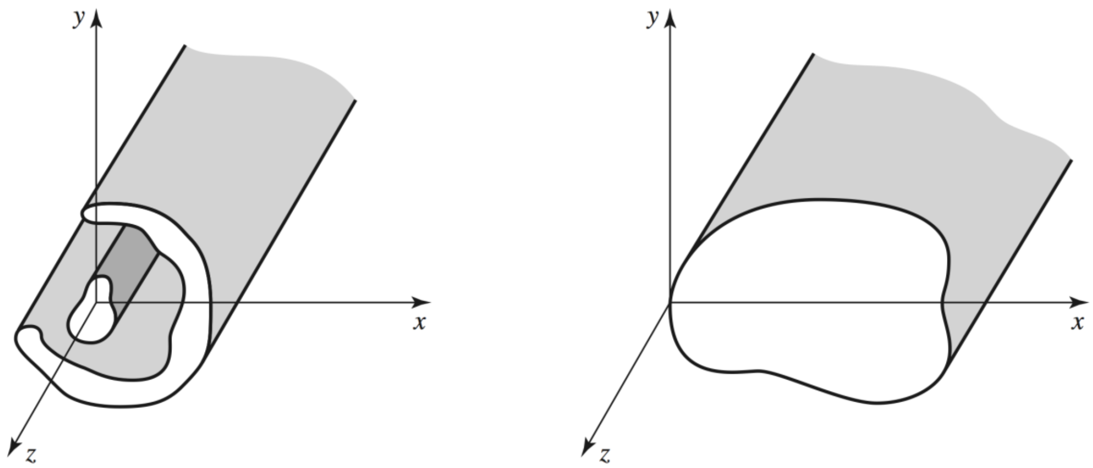
Modes of cylindrical waveguides: propagating field



Field propagating in the positive z direction

$$\vec{A} = \hat{z}\ A_z(x,y)\ e^{-j\beta z} = \hat{z}\ A$$

$$\vec{F} = \hat{z}\ F_z(x,y)\ e^{-j\beta z} = \hat{z}\ F$$

$$\vec{H}_A = \frac{1}{\mu}\nabla \times (\hat{z}A) \longrightarrow \vec{H}_A = \vec{h}_t\ e^{-j\beta z}$$

$$\vec{E}_A = -j\omega A\hat{z} - \frac{\beta}{\omega\mu\epsilon}\nabla A \longrightarrow \vec{E}_A = [\vec{e}_t + \hat{z}\ e_z]\, e^{-j\beta z}$$

Only E field along z
E-mode
Transverse Magnetic (TM)

$$\vec{E}_F = -\frac{1}{\epsilon}\nabla \times (\hat{z}F) \longrightarrow \vec{E}_F = \vec{e}_t\ e^{-j\beta z}$$

$$\vec{H}_F = -j\omega F\hat{z} - \frac{\beta}{\omega\mu\epsilon}\nabla F \longrightarrow \vec{H}_F = \left[\vec{h}_t + \hat{z}\ h_z\right] e^{-j\beta z}$$

Only H field along z
H-mode
Transverse Electric (TE)

$$\vec{E} = \vec{E}_A + \vec{E}_F \qquad \vec{H} = \vec{H}_A + \vec{H}_F \longrightarrow$$

TM modes + **TE modes**

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Transverse Electric Magnetic mode

Example

Look for a Transverse Electric Magnetic mode $E_z = H_z = 0$

$$\vec{E},\ \vec{H},\ v_p?$$

Hint 1 **Start from a TM mode (vector potential A)** $H_z = 0$

$$\nabla = \nabla_t + \hat{z}\frac{\partial}{\partial z} \qquad \vec{A} = \hat{z}\ A_z(x,y)\ e^{-j\beta z} = \hat{z}\ A \qquad \nabla \cdot \vec{A} = \cdots$$

Hint 2 $\vec{E}_A = \cdots$

Solution

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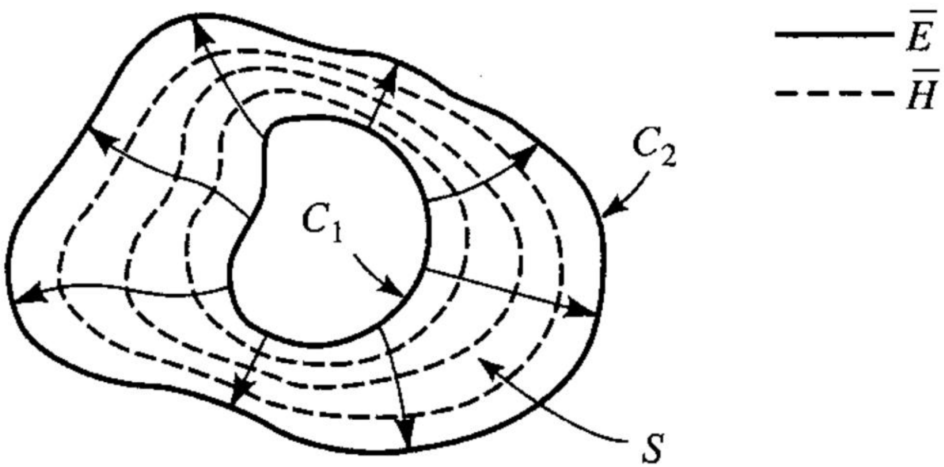
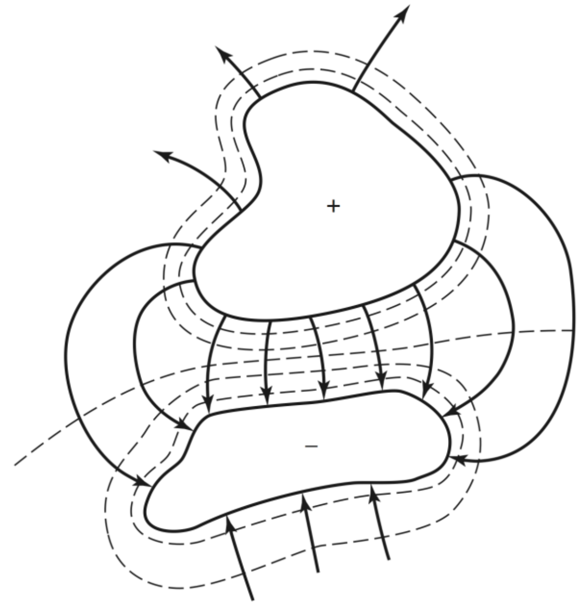
Example

Solution **For a given** A_z $\vec{H} = \frac{1}{\mu}\nabla_t \times (\hat{z}A_z)\, e^{-j\omega\sqrt{\mu\epsilon}z}$ $\vec{E} = -\frac{1}{\sqrt{\mu\epsilon}}\,\nabla_t A_z\; e^{-j\omega\sqrt{\mu\epsilon}z}$

3. **TEM waves are possible only if there are at least two conductors.**





4. **The plane wave is a TEM wave of two infinitely large plates separated to infinity**

5. **Electrostatic problem with boundary conditions**

$\vec{e}_t$ $\longrightarrow$ $\vec{h}_t = \frac{1}{Z_{TEM}}\hat{z}\times\vec{e}_t$ $\longrightarrow$

$$\vec{E} = \vec{e}_t\; e^{-j\omega\sqrt{\mu\epsilon}z}$$

$$\vec{H} = \vec{h}_t\; e^{-j\omega\sqrt{\mu\epsilon}z}$$

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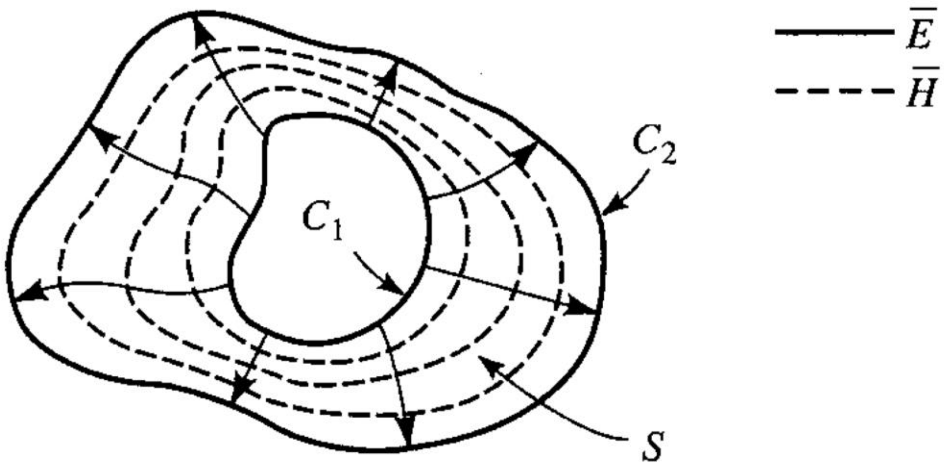
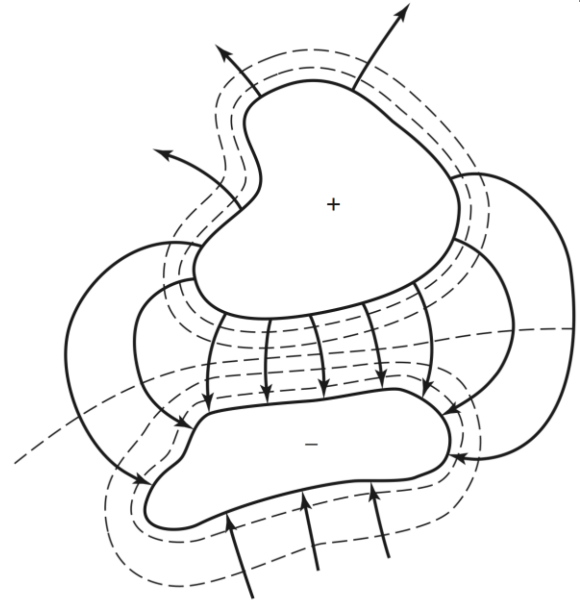
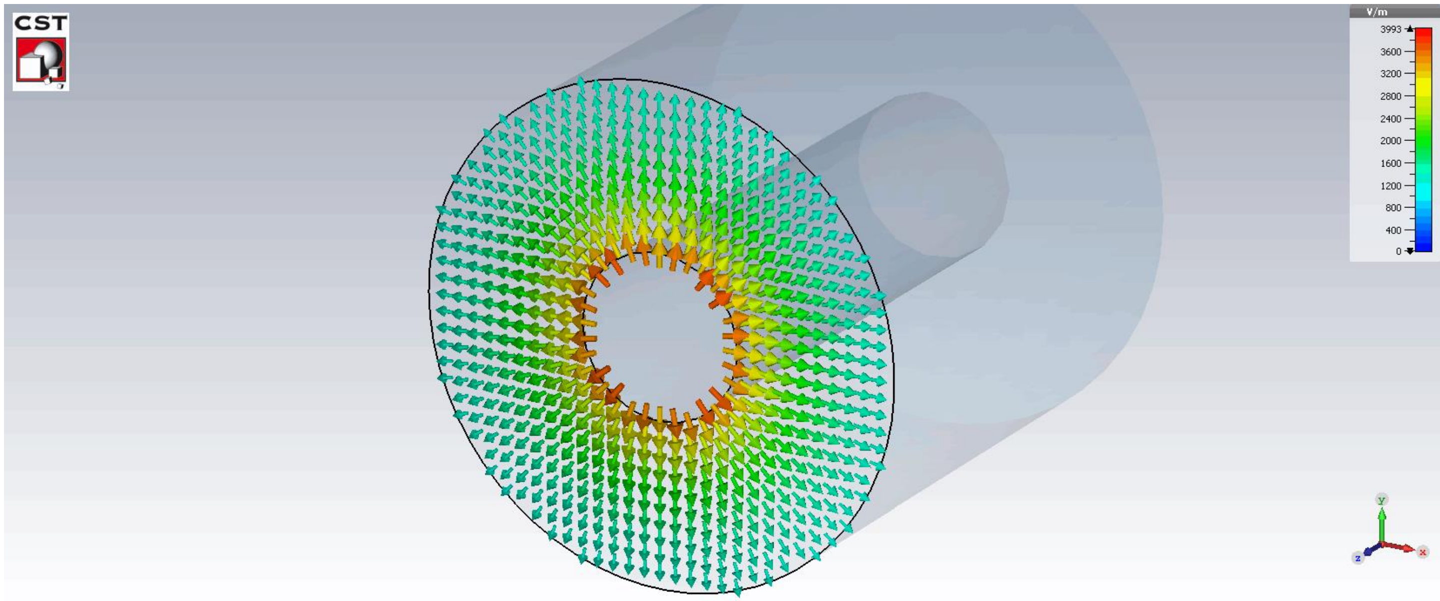
Common TEM waveguides







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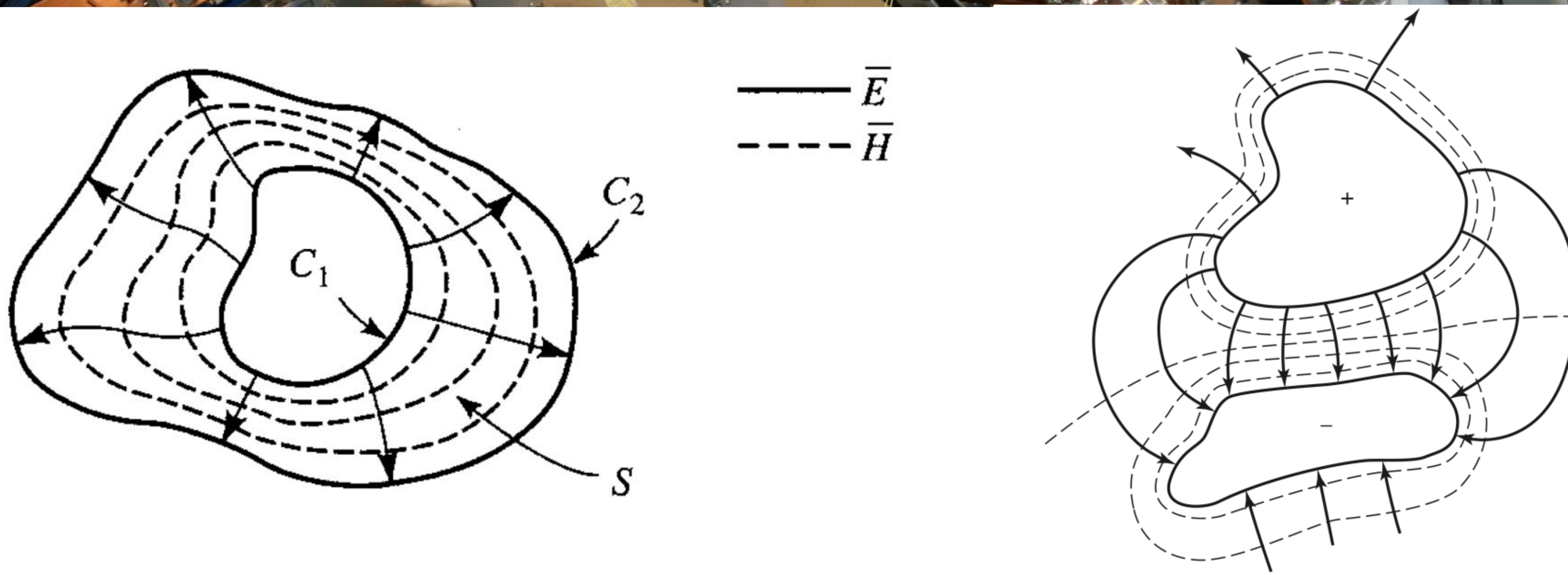
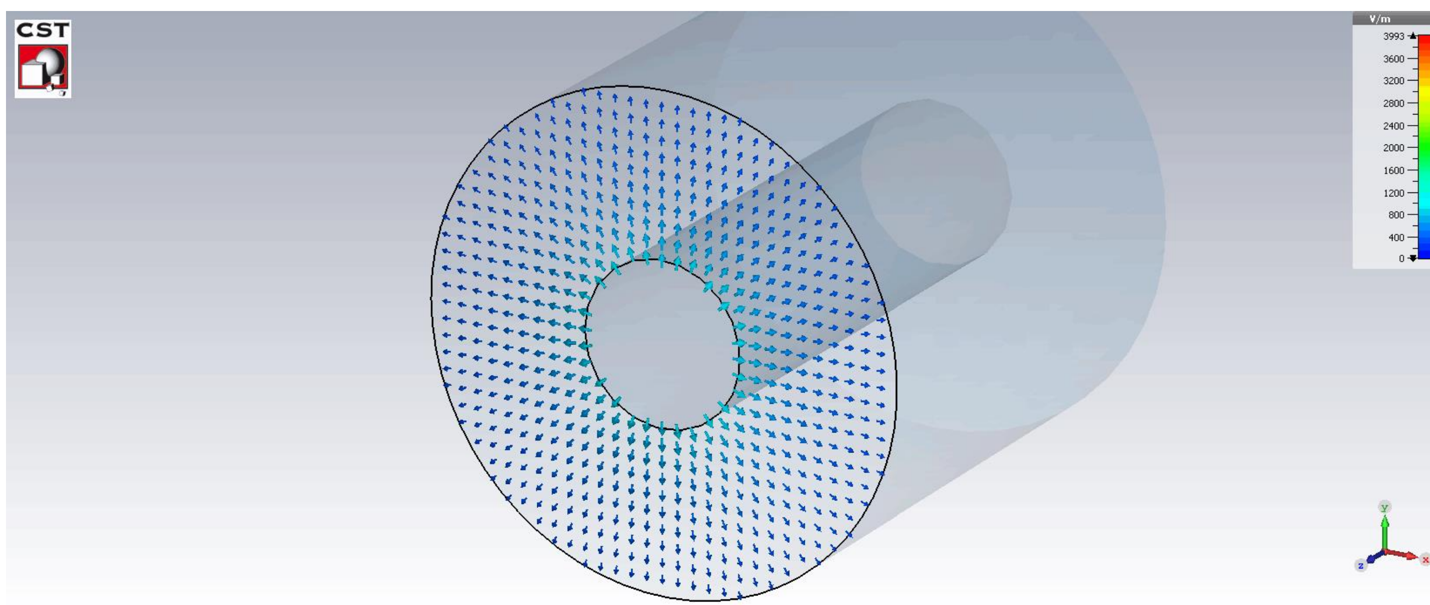
Common TEM waveguides





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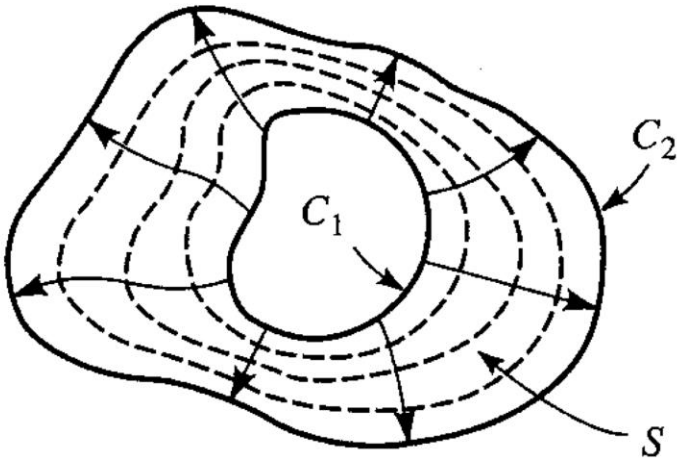
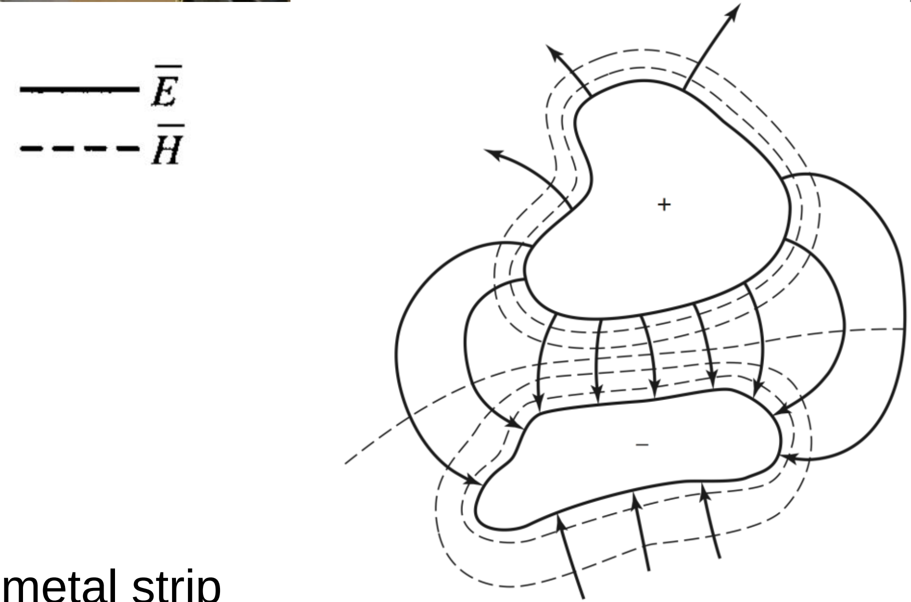
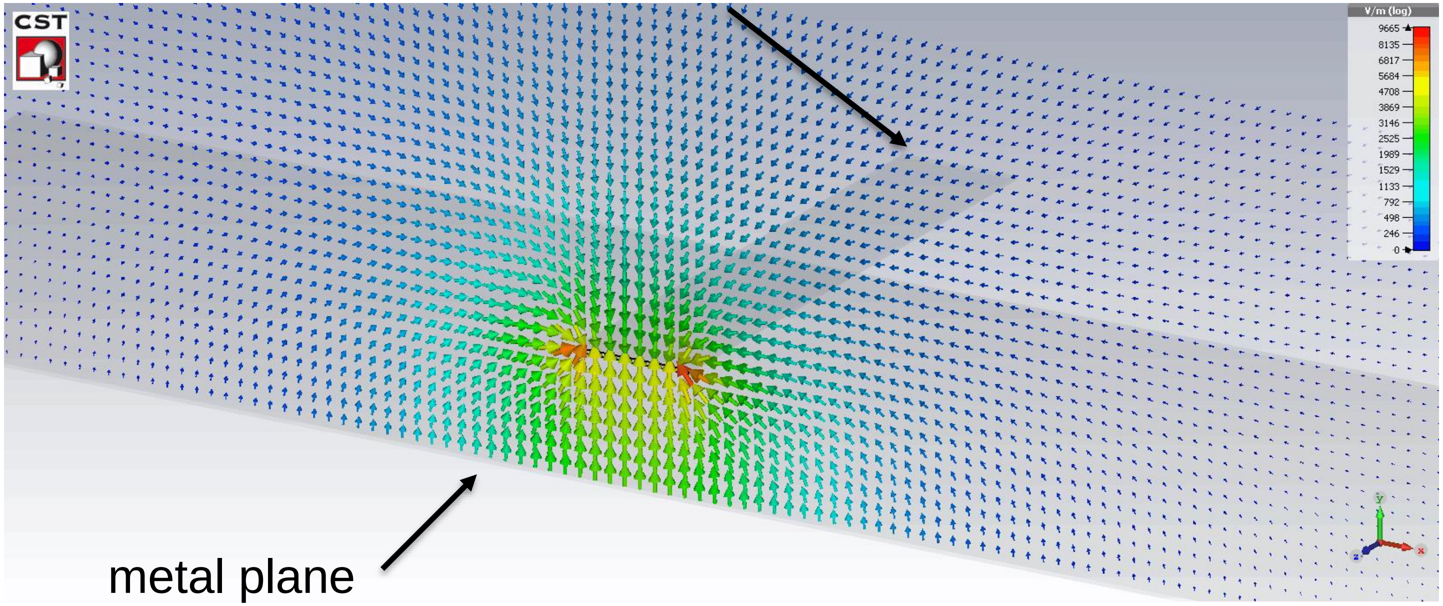
Common TEM waveguides





metal strip



metal plane

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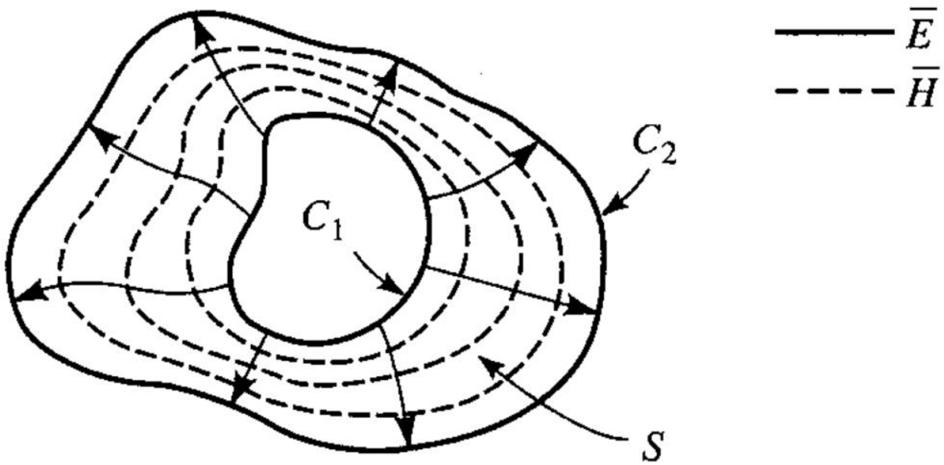
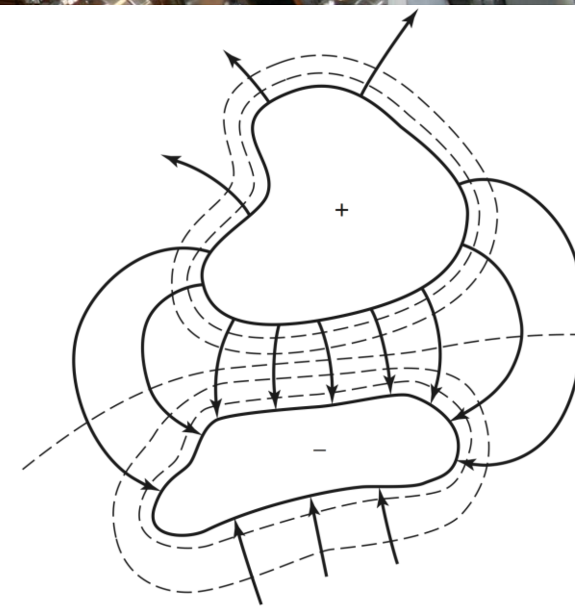
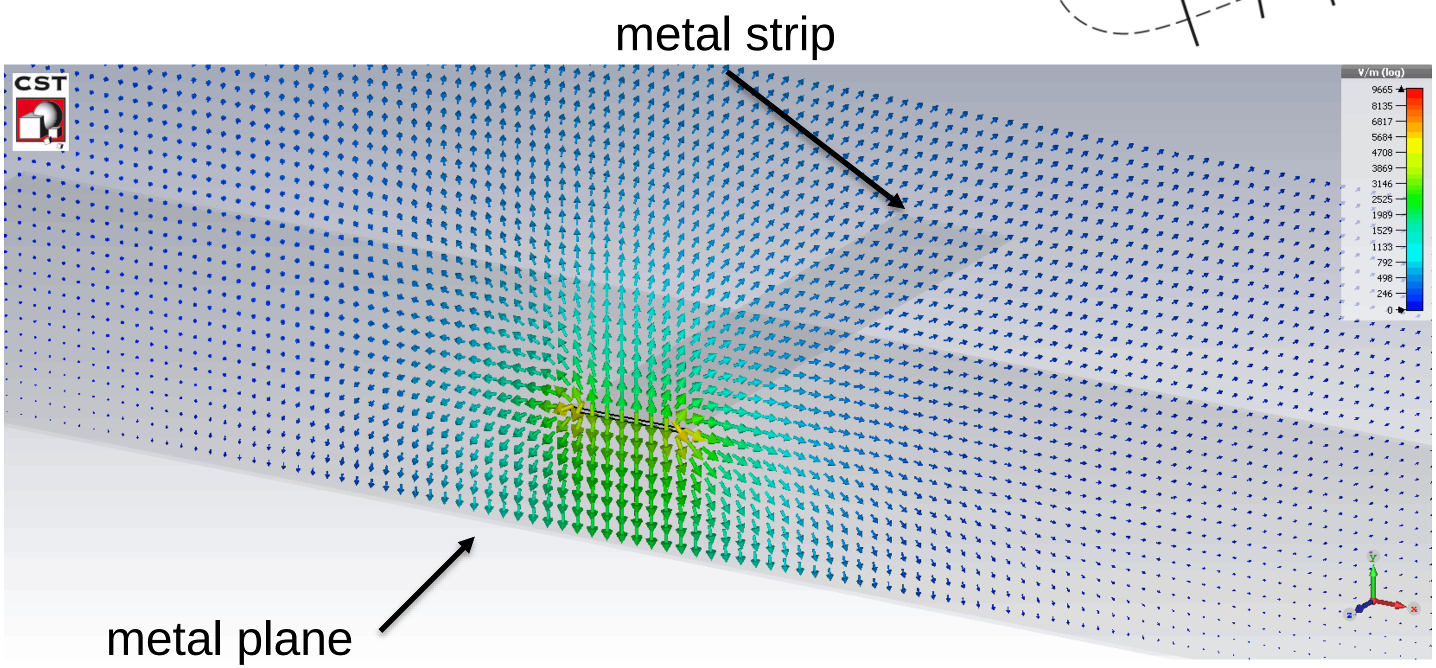
Common TEM waveguides







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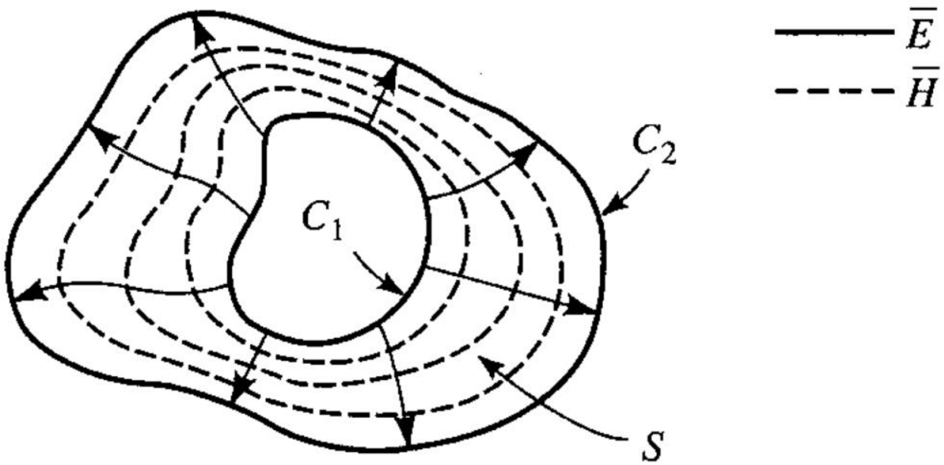
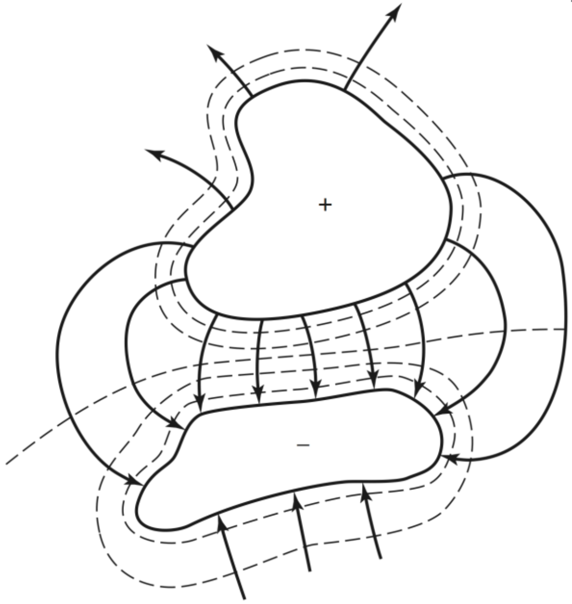
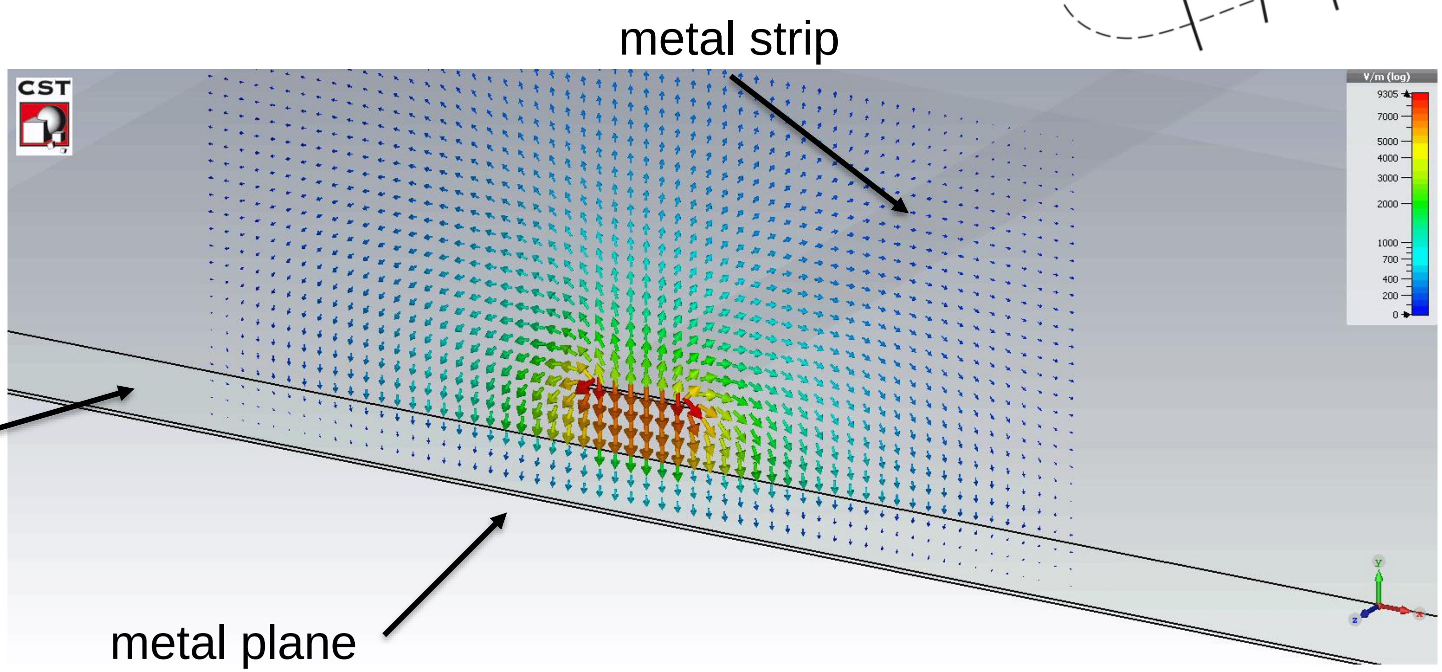
Common TEM waveguides: stripline with dielectrics







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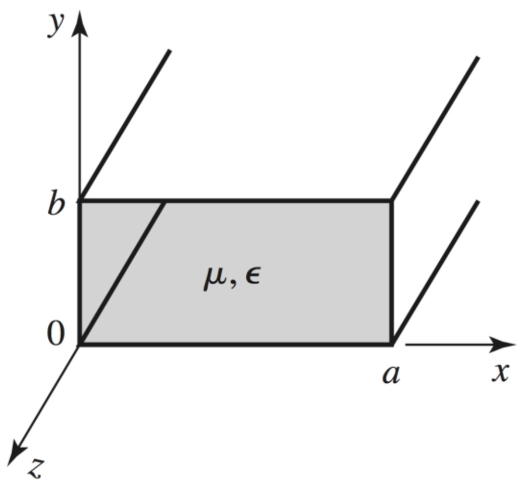
1. Write the Helmotz equations for potentials

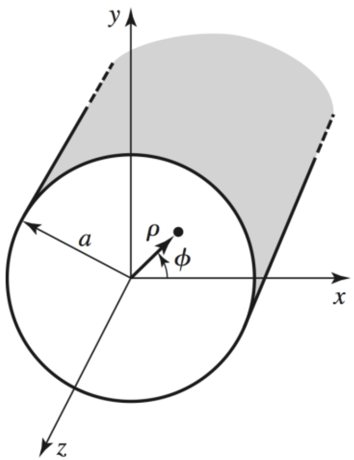
TM waves $\nabla_t^2 A_z + k_t^2 A_z = 0$

TE waves $\nabla_t^2 F_z + k_t^2 F_z = 0$

$$k_t^2 = k^2 - \beta^2 = \omega^2 \mu\epsilon - \beta^2$$

$$\epsilon = \epsilon_r \epsilon_0 \left(1 - j \tan\delta\right)$$





Cartesian coordinates

$$\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Cylindrical coordinates

$$\nabla_t^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}$$

2.

$$A_z(x, y) = X(x)Y(y)$$

$$A_z(\rho, \phi) = R(\rho)\Phi(\phi)$$

Separation of variables

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General solution for fields in cylindrical waveguide

3. Eigenvalue problem: Eigenvalues + Eigen-function

TM $\nabla_t^2 A_z + k_t^2 A_z = 0$ $\qquad k_t \qquad A_z,\ F_z$

TE $\nabla_t^2 F_z + k_t^2 F_z = 0$

4. Compute the fields and apply the boundary conditions

$$\vec{e} = \vec{e}_t + \hat{z}\ e_z$$

$$\vec{h} = \vec{h}_t + \hat{z}\ h_z$$

$\longrightarrow$

$$\vec{e}_{m,n} \qquad \vec{h}_{m,n}$$

$$\beta_{m,n} = \sqrt{\omega^2 \mu \epsilon - k_t^2\ _{(m,n)}}$$

Mode (m,n)

5.

It can be complex

$$\vec{E} = \sum_{m,n} a_{m,n}\ \vec{e}_{m,n}\ e^{-j\beta_{m,n} z} \qquad \vec{H} = \sum_{m,n} b_{m,n}\ \vec{h}_{m,n}\ e^{-j\beta_{m,n} z}$$

It depends on the sources

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Rectangular waveguides

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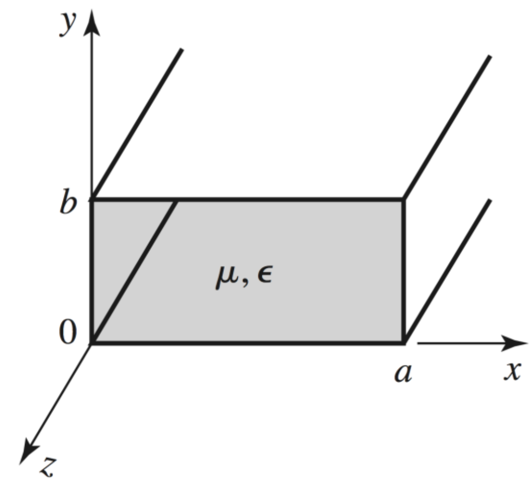
Rectangular waveguides: TE mode

Example

$$F_z = X(x)Y(y)$$

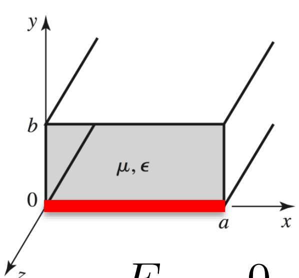
Write the Helmotz equation



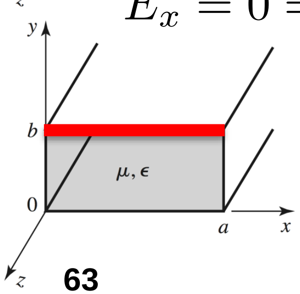
$$X(x) =$$

$$Y(y) =$$

$$e_x = -\frac{1}{\epsilon}\frac{\partial F_z}{\partial y} = -\frac{1}{\epsilon}XY' :$$



$$E_x = 0 \Longrightarrow e_x = 0$$



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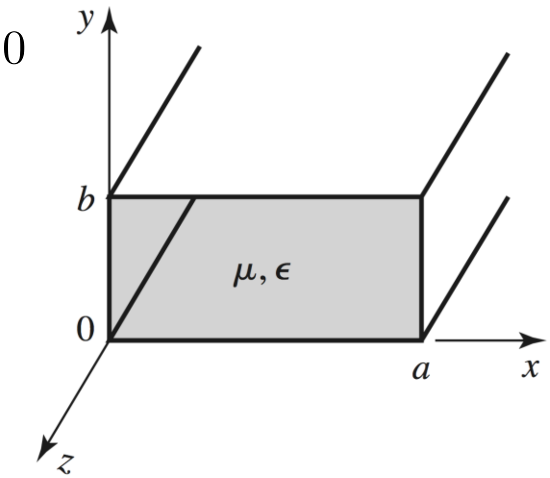
Rectangular waveguides: TE mode

Example

$$F_z = X(x)Y(y) \qquad \nabla_t^2 F_z + k_t^2 F_z = YX'' + XY'' + k_t^2 XY = 0$$



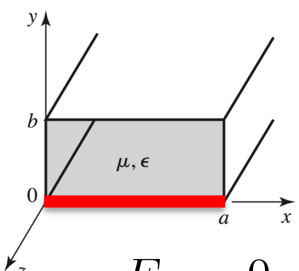
$$\frac{X''}{X} + \frac{Y''}{Y} + k_t^2 = 0 \qquad -k_x^2 - k_y^2 + k_t^2 = 0$$

constraint condition

$$\frac{X''}{X} = -k_x^2 \longrightarrow X(x) = C_1 \cos(k_x x) + D_1 \sin(k_x x)$$

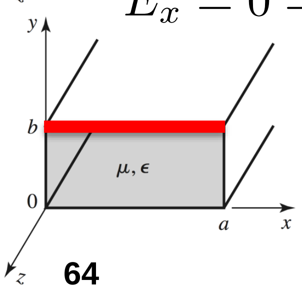
$$\frac{Y''}{Y} = -k_y^2 \longrightarrow Y(y) = C_2 \cos(k_y y) + D_2 \sin(k_y y)$$

$$e_x = -\frac{1}{\epsilon}\frac{\partial F_z}{\partial y} = -\frac{1}{\epsilon} XY' = -\frac{k_y}{\epsilon}\left[C_1 \cos(k_x x) + D_1 \sin(k_x x)\right]\left[-C_2 \sin(k_y y) + D_2 \cos(k_y y)\right]$$



$$e_x(0 \le x \le a, y = 0) = \ldots [-C_2 \cdot 0 + D_2 \cdot 1] = 0 \iff D_2 = 0$$

$$E_x = 0 \Longrightarrow e_x = 0$$



$$e_x(0 \le x \le a, y = b) = \ldots [-C_2 \sin(k_y b)] = 0 \iff k_y b = n\pi$$

$$n = 0, 1, 2, \ldots$$

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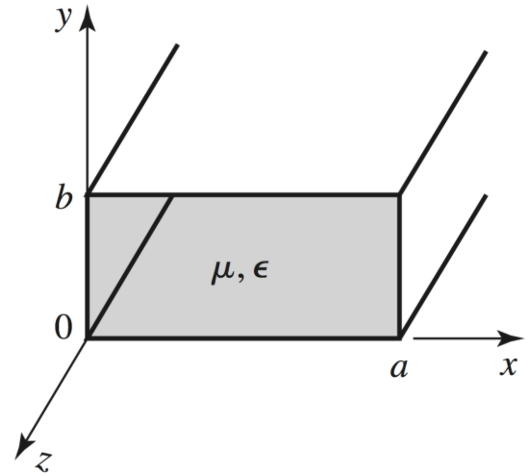
Eigenvalues and cut-off frequencies (TE mode, rect. WG)

$$k_t^2 = k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \omega^2\mu\epsilon - \beta^2$$

constraint condition



$$\vec{H} = \sum_{m,n} b_{m,n}\ \vec{h}_{m,n}\ e^{-j\beta_{m,n}z}$$

$$\beta_{m,n} = \sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$\vec{E} = \sum_{m,n} a_{m,n}\ \vec{e}_{m,n}\ e^{-j\beta_{m,n}z}$$

Cut-off frequencies f_c such that $\beta_{m,n} = 0$

$$(f_c)_{m,n} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$m,\ n = 0, 1, 2, \ldots \qquad m = n \neq 0$$

$f < (f_c)_{m,n}$ **mode m, n is attenuated exponentially (evanescent mode)**

$f > (f_c)_{m,n}$ **mode m, n is propagating with no attenuation**

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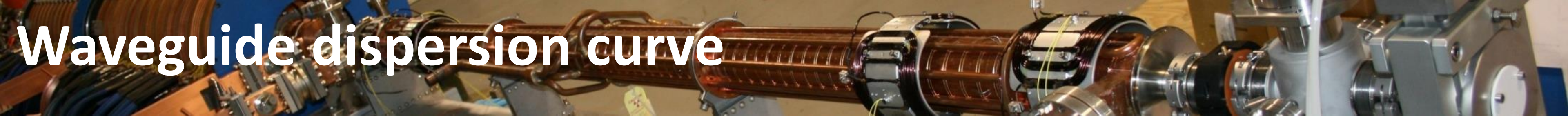


Waveguide dispersion curve

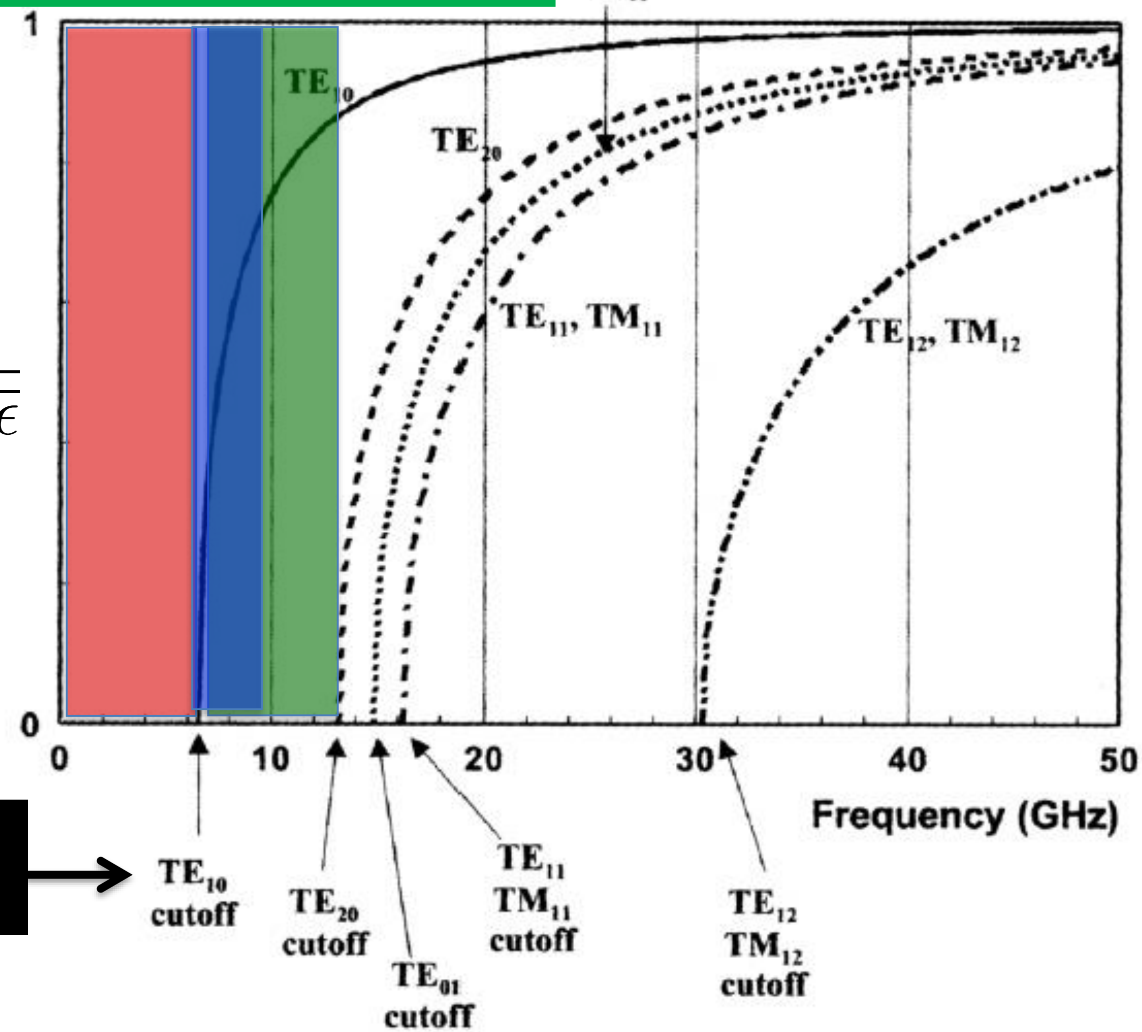
Courtesy of S. Pisa

Unimodal propagation $1.25\,(f_c)_1 < f < 0.95\,(f_c)_2$

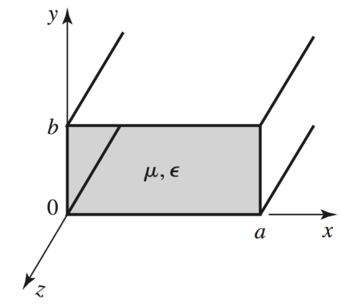
Cut-off

High dispersion

$$\frac{\beta_{m,n}}{\omega\sqrt{\mu\epsilon}}$$



Fundamental mode →



Same curve for TE and TM mode, but n=0 or m=0 is possible only for TE modes.

In any metallic waveguide the fundamental mode is TE.

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Single mode operation of a rectangular waveguide

Exercise

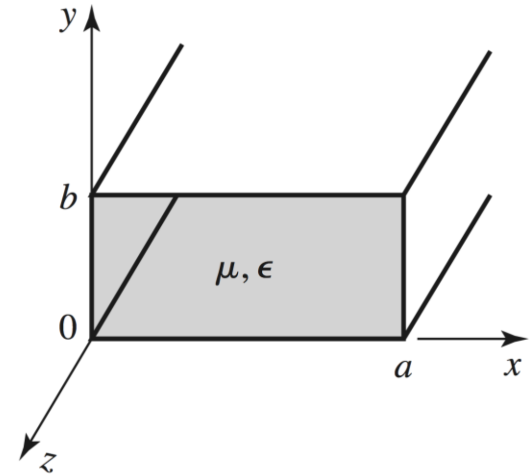
1. **Find the smallest ratio a/b allowing the largest bandwidth of single mode operation**

2. **State the largest bandwidth of single mode operation**

3. **Defining the single mode bandwidth (BW) as**

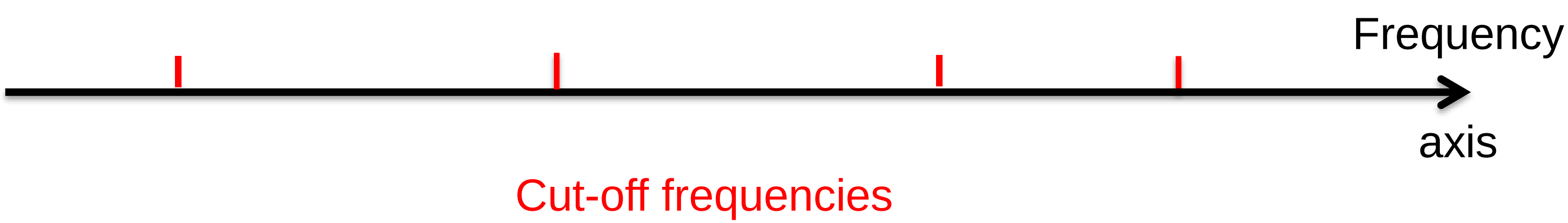
$$1.25\ (f_c)_1 < f < 0.95\ (f_c)_2$$

Find the single mode BW for WR-90 waveguide (a=22.86mm and b=10.16 mm)



Hint: $$(f_c)_{m,n} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \qquad m,\ n = 0, 1, 2, \ldots \quad m = n \neq 0$$

Place the possible cut-off frequencies for different modes on the frequency axis



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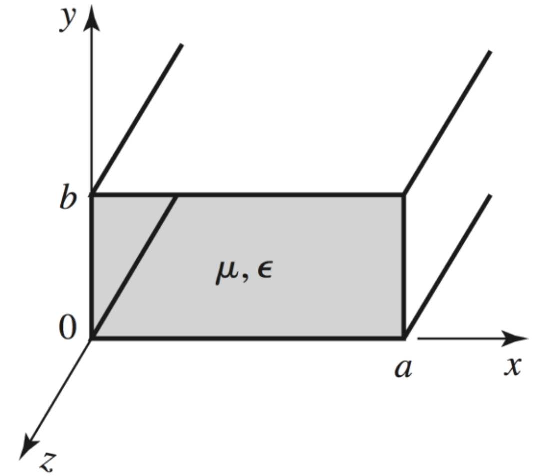
Single mode operation of a rectangular waveguide



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Find the single mode BW for WR-90 waveguide (a=22.86mm and b=10.16 mm)

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Eigenfunctions and mode pattern (TE mode, rect. WG)

$$E_x^{+,(m,n)} = a_{m,n}\frac{k_y}{\epsilon}\cos(k_x x)\sin(k_y y)e^{-j\beta z}$$

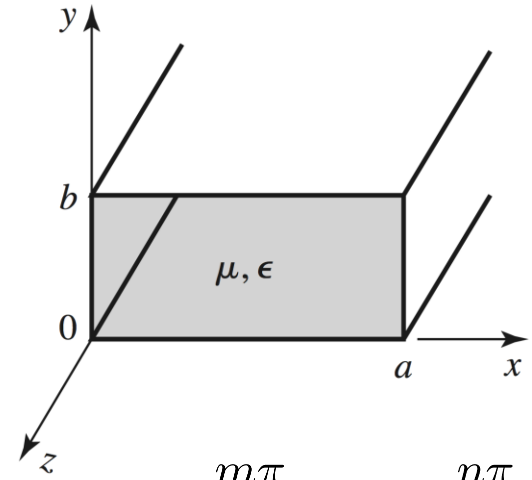
$$E_y^{+,(m,n)} = -a_{m,n}\frac{k_x}{\epsilon}\sin(k_x x)\cos(k_y y)e^{-j\beta z}$$

$$E_z^{+,(m,n)} = 0$$

$$H_x^{+,(m,n)} = a_{m,n}\frac{k_x\beta}{\omega\mu\epsilon}\sin(k_x x)\cos(k_y y)e^{-j\beta z}$$

$$H_y^{+,(m,n)} = a_{m,n}\frac{k_y\beta}{\omega\mu\epsilon}\cos(k_x x)\sin(k_y y)e^{-j\beta z}$$

$$H_z^{+,(m,n)} = -ja_{m,n}\frac{k_t^2}{\omega\mu\epsilon}\cos(k_x x)\cos(k_y y)e^{-j\beta z}$$



$$k_x = \frac{m\pi}{a} \qquad k_y = \frac{n\pi}{b}$$

$$\beta = \sqrt{\omega^2\mu\epsilon - k_x^2 - k_y^2}$$

$$TE_{m,n}^{+z}$$

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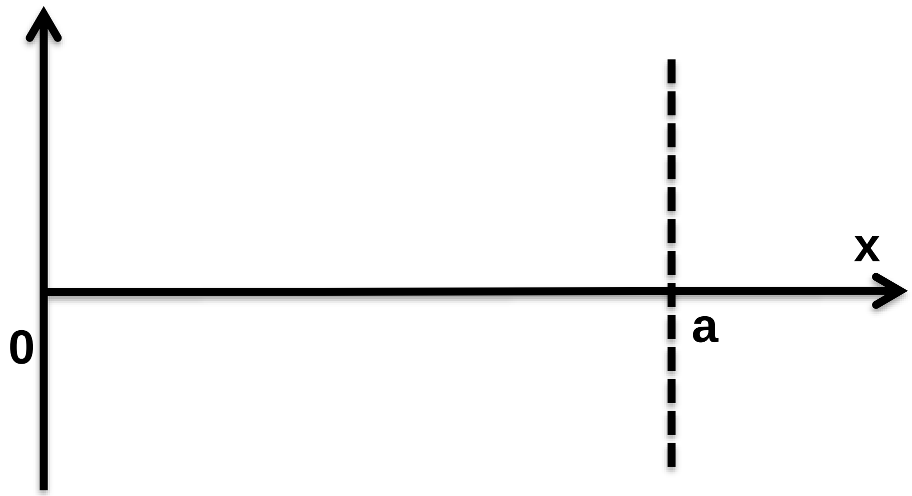
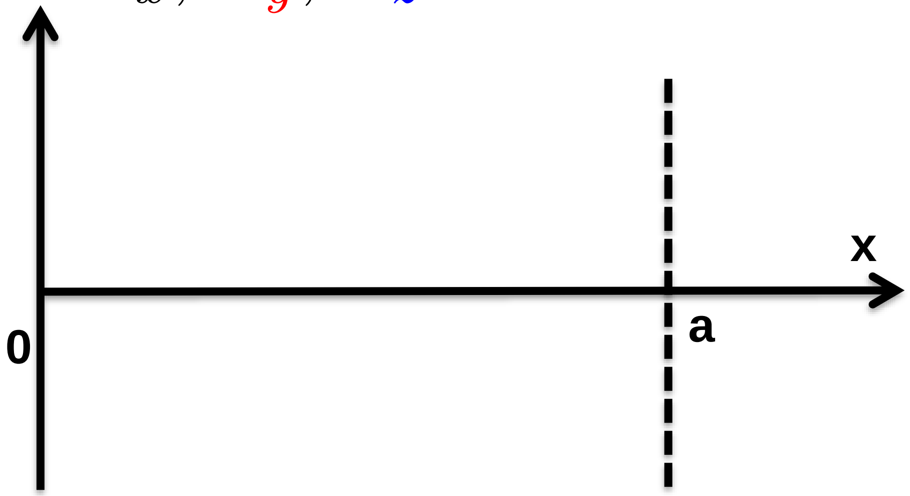
Let's draw …

Exercise

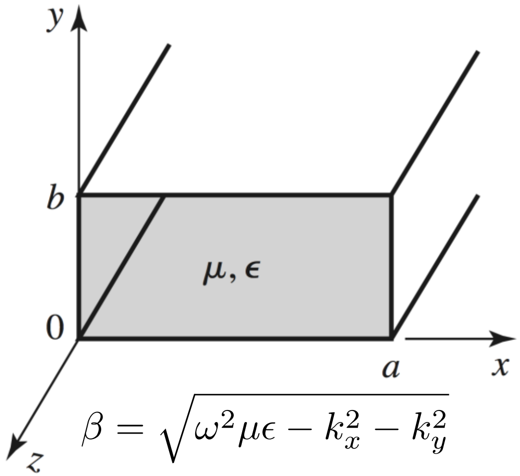
E_x, E_y



H_x, H_y, H_z



$$TE_{1,0}$$



$$\beta = \sqrt{\omega^2\mu\epsilon - k_x^2 - k_y^2}$$

$$k_x = \frac{m\pi}{a} \qquad k_y = \frac{n\pi}{b} \qquad TE^{+z}_{m,n}$$

$$E_x^{+,(m,n)} = a_{m,n}\frac{k_y}{\epsilon}\cos(k_x x)\sin(k_y y)e^{-j\beta z}$$

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$$H_z^{+,(m,n)} = -ja_{m,n}\frac{k_t^2}{\omega\mu\epsilon}\cos(k_x x)\cos(k_y y)e^{-j\beta z}$$

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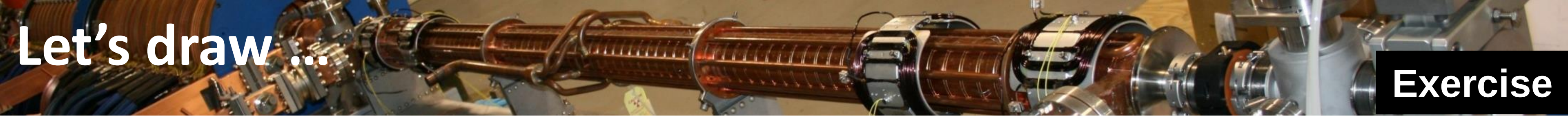
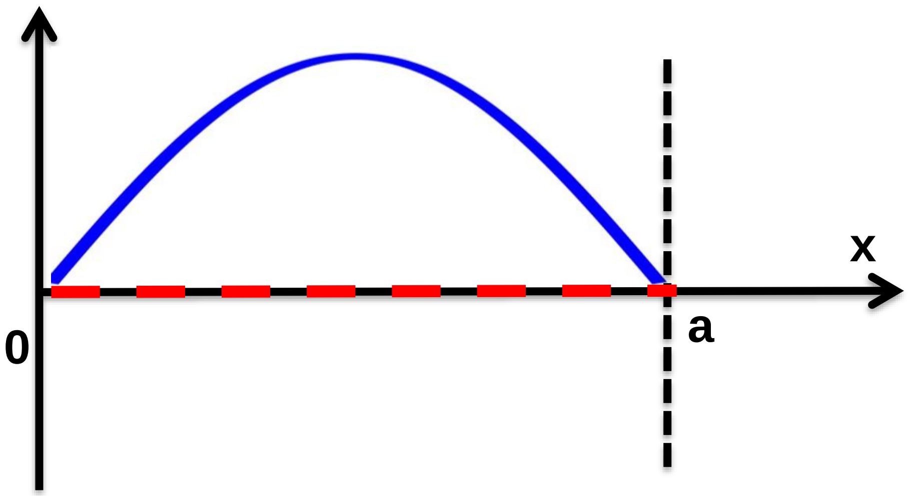


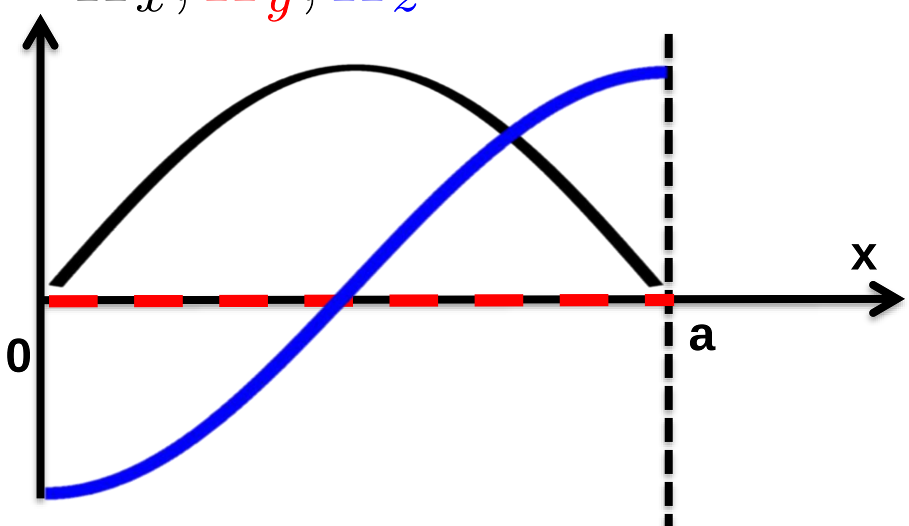
Let's draw …

Exercise

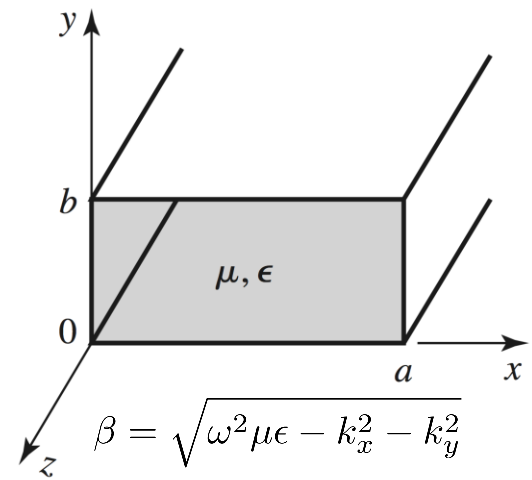
E_x, E_y



H_x, H_y, H_z



$TE_{1,0}$



$$\beta = \sqrt{\omega^2\mu\epsilon - k_x^2 - k_y^2}$$

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$$H_z^{+,(m,n)} = -ja_{m,n}\frac{k_t^2}{\omega\mu\epsilon}\cos(k_x x)\cos(k_y y)e^{-j\beta z}$$

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Eigenfunctions and mode pattern (TE mode, rect. WG)

Exercise

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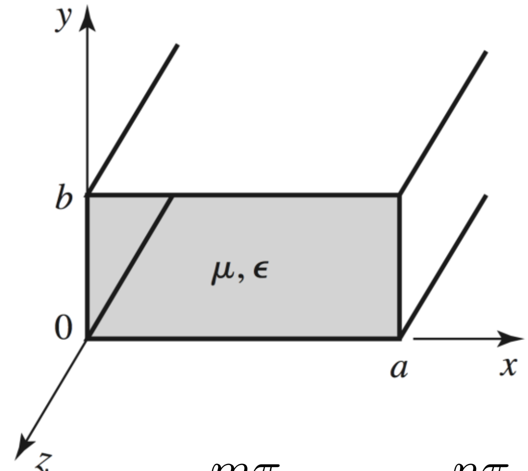
$$H_y^{+,(m,n)} = a_{m,n}\frac{k_y\beta}{\omega\mu\epsilon}\cos(k_x x)\sin(k_y y)e^{-j\beta z}$$

$$H_z^{+,(m,n)} = -ja_{m,n}\frac{k_t^2}{\omega\mu\epsilon}\cos(k_x x)\cos(k_y y)e^{-j\beta z}$$



$$k_x = \frac{m\pi}{a} \qquad k_y = \frac{n\pi}{b}$$

$$\beta = \sqrt{\omega^2\mu\epsilon - k_x^2 - k_y^2}$$

$$TE_{m,n}^{+z}$$

Draw the field pattern in the xz plane for TE10

E field
H field

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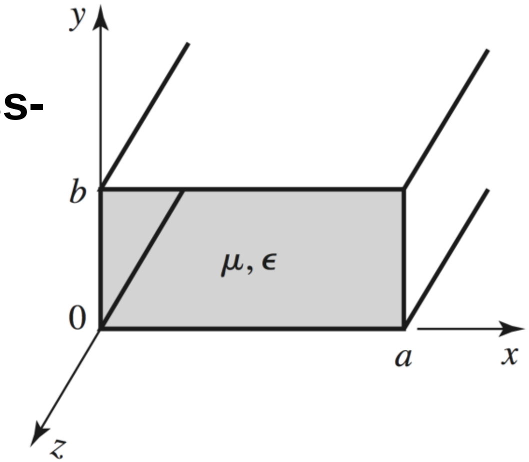
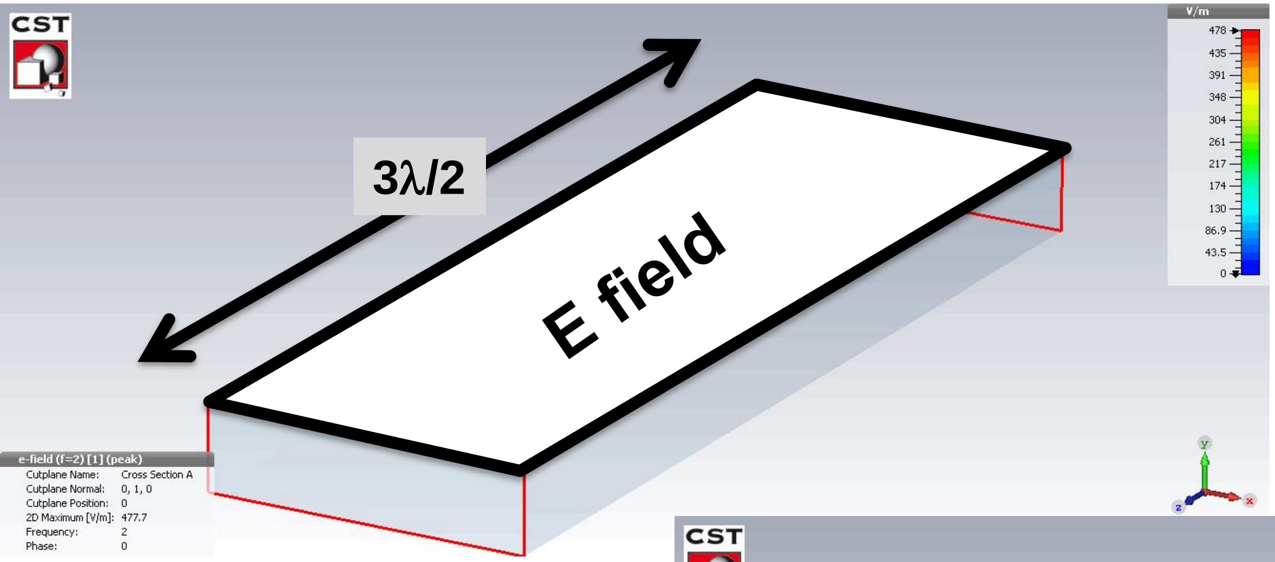
Field pattern (TE10 mode, rect. WG)

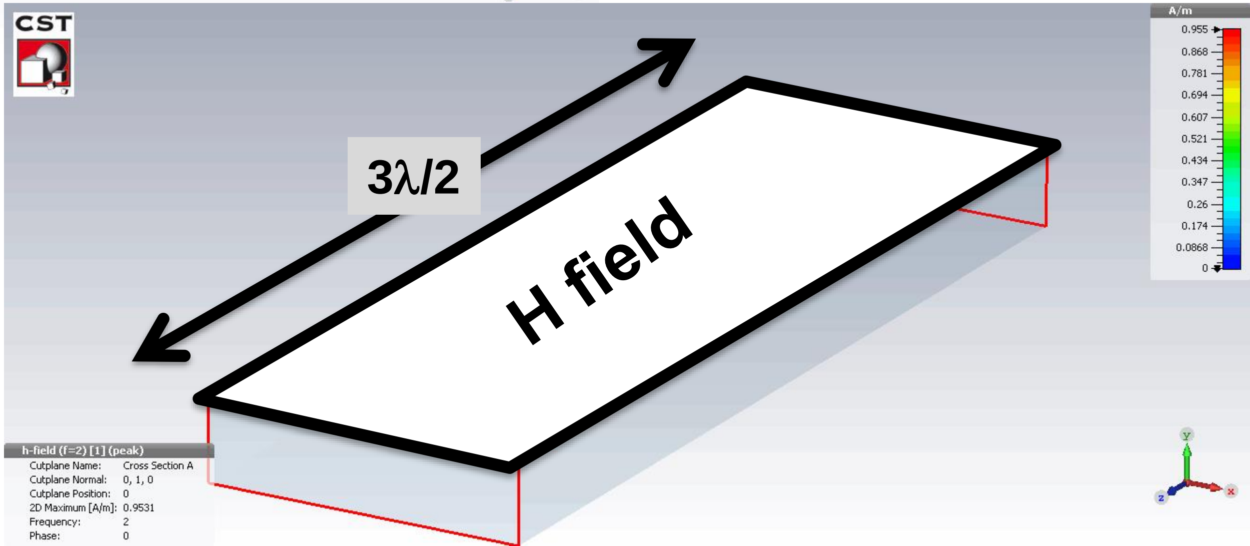
Exercise

$TE^{+z}_{m,n}$

m (n) is the number of half periods (or maxima/minima) along the x (y) axis in the cross-section.





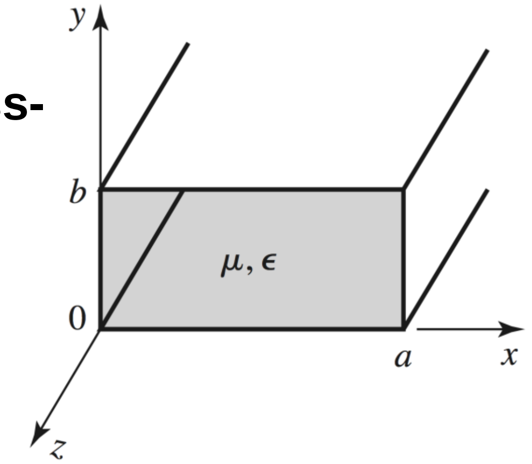


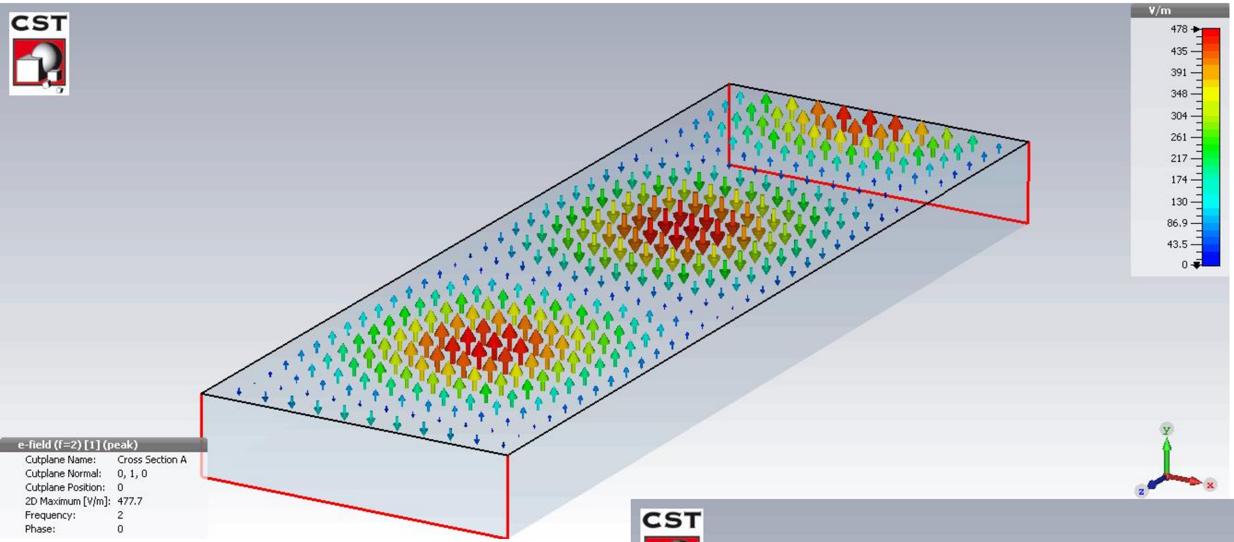
Simulations by L. Ficcadenti

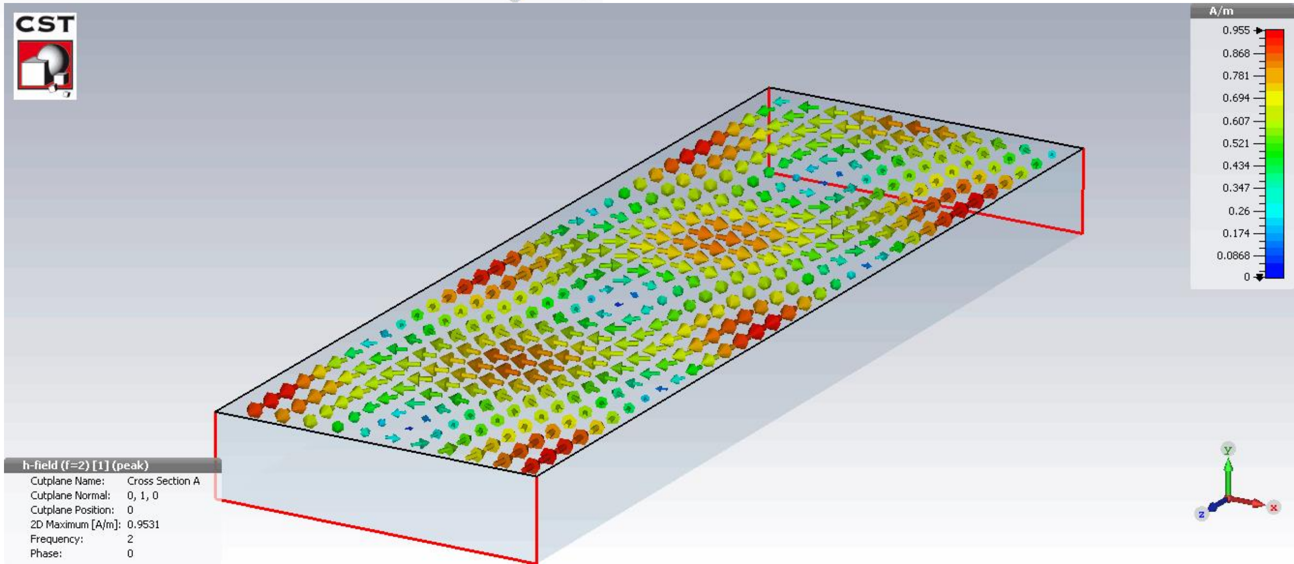
Field pattern (TE10 mode, rect. WG)

Exercise

$TE^{+z}_{m,n}$ **m (n) is the number of half periods (or maxima/minima) along the x (y) axis in the cross-section.**







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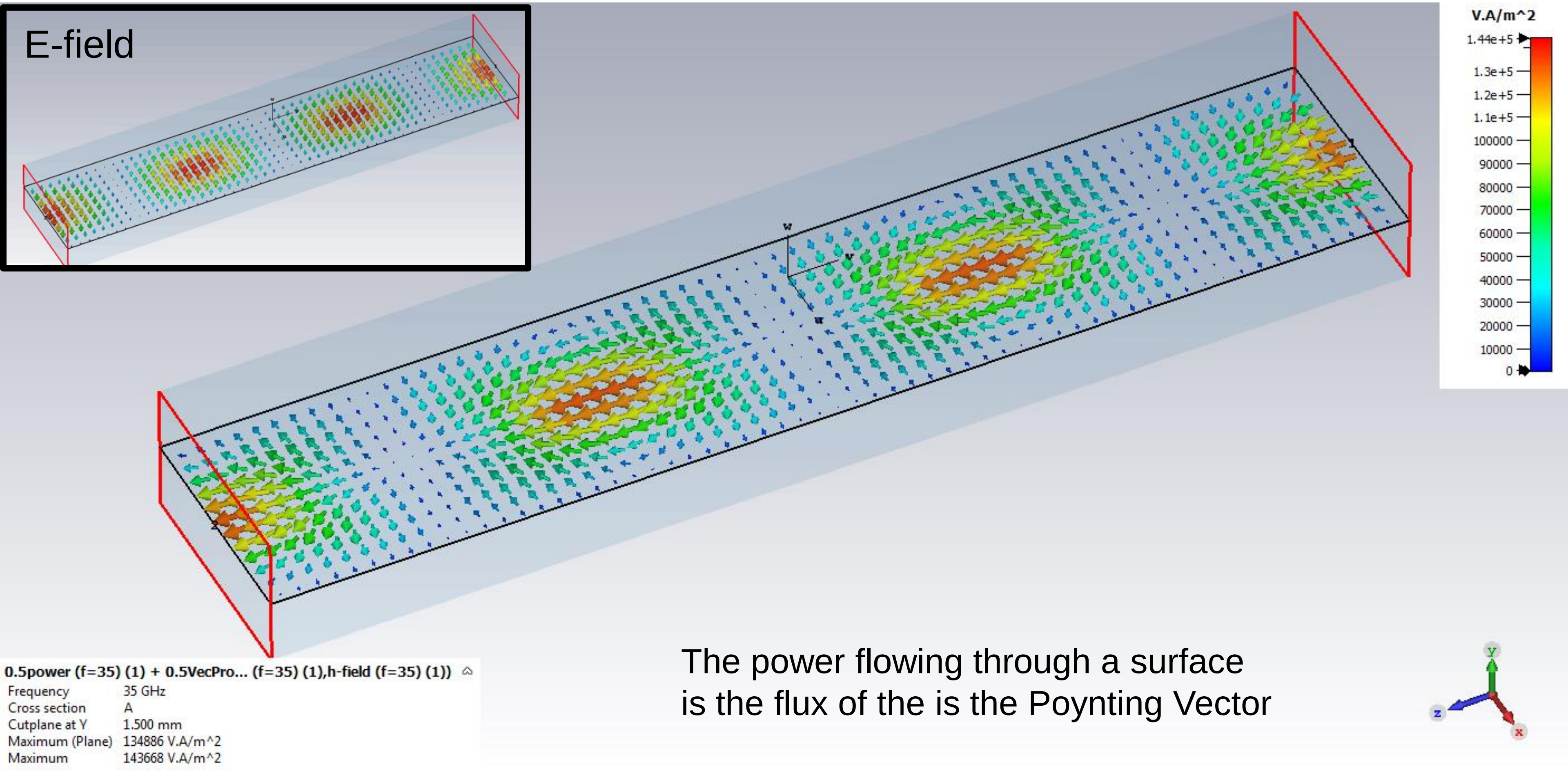
Power Flow pattern (TE10 mode, rect. WG)

Instantaneous power density



The power flowing through a surface
is the flux of the is the Poynting Vector

$$\vec{S}(\vec{r},t) = \vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t) = Re\left\{\frac{\vec{E}(\vec{r},\omega) \times \vec{H}(\vec{r},\omega)^*}{2}\right\} + Re\left\{\frac{\vec{E}(\vec{r},\omega) \times \vec{H}(\vec{r},\omega)\, e^{j2\omega t}}{2}\right\}$$

 Animations by F. Bosco, Sapienza

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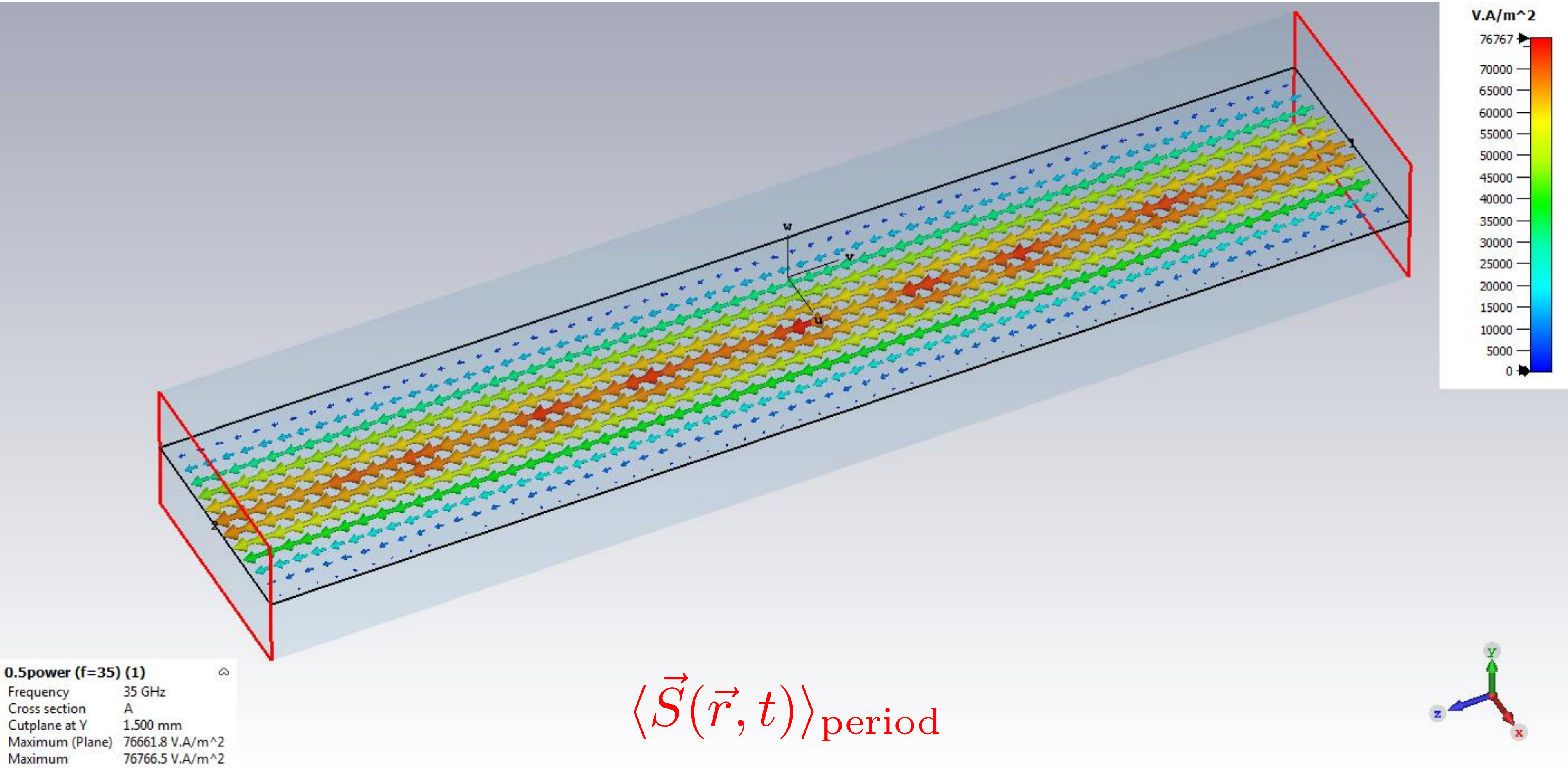
Power Flow pattern (TE10 mode, rect. WG)

Average of the power density over one period



$$\langle \vec{S}(\vec{r},t) \rangle_{\text{period}}$$

$$\vec{S}(\vec{r},t) = \vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t) = Re\left\{ \frac{\vec{E}(\vec{r},\omega) \times \vec{H}(\vec{r},\omega)^*}{2} \right\} + Re\left\{ \frac{\vec{E}(\vec{r},\omega) \times \vec{H}(\vec{r},\omega)\, e^{j2\omega t}}{2} \right\}$$

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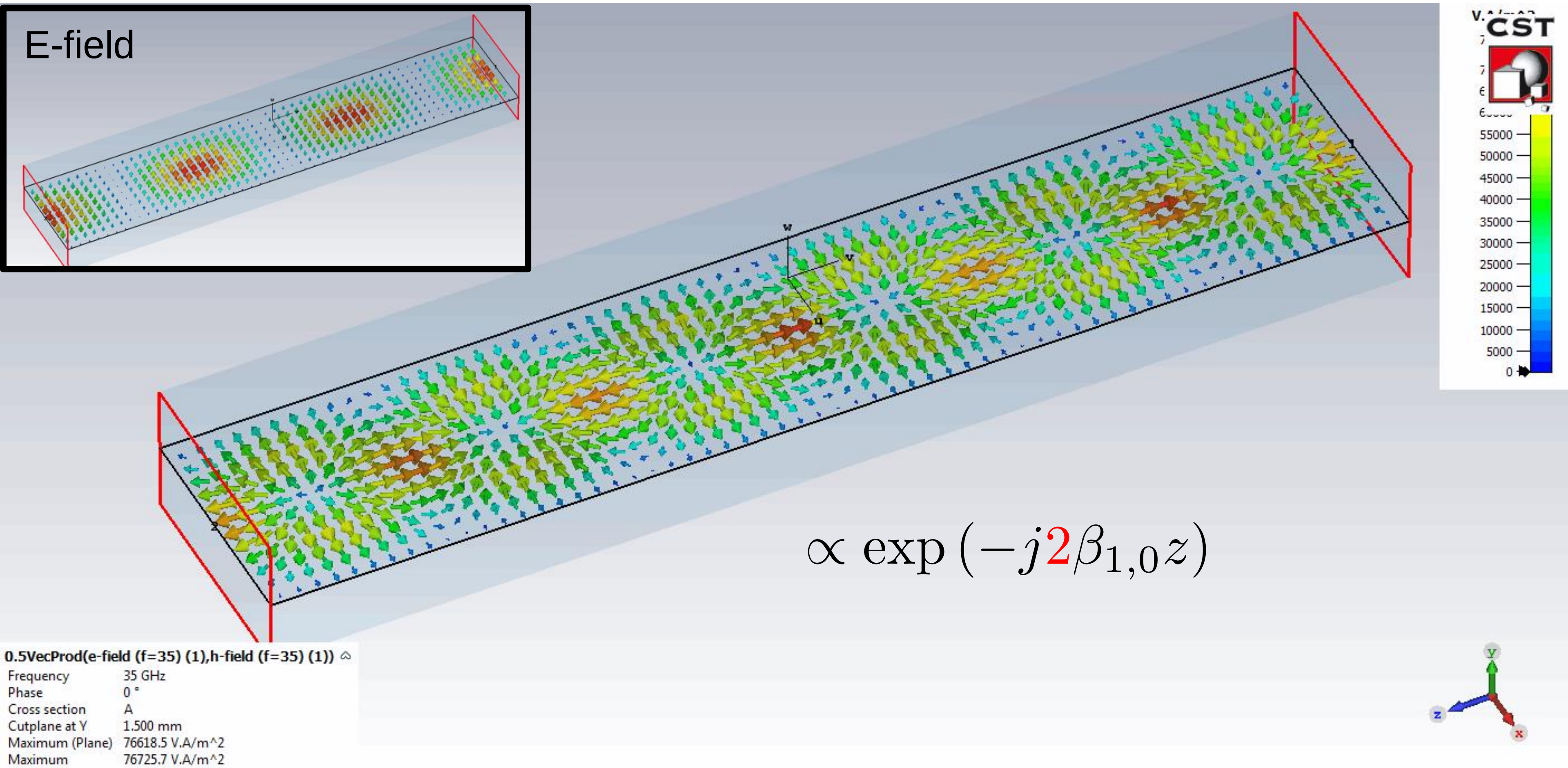
Power Flow pattern (TE10 mode, rect. WG)

Time varying part of the instantaneous power density



$$\propto \exp\left(-j2\beta_{1,0}z\right)$$

$$\vec{S}(\vec{r},t)=\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)=Re\left\{\frac{\vec{E}(\vec{r},\omega)\times\vec{H}(\vec{r},\omega)^*}{2}\right\}+Re\left\{\frac{\vec{E}(\vec{r},\omega)\times\vec{H}(\vec{r},\omega)\,e^{j2\omega t}}{2}\right\}$$

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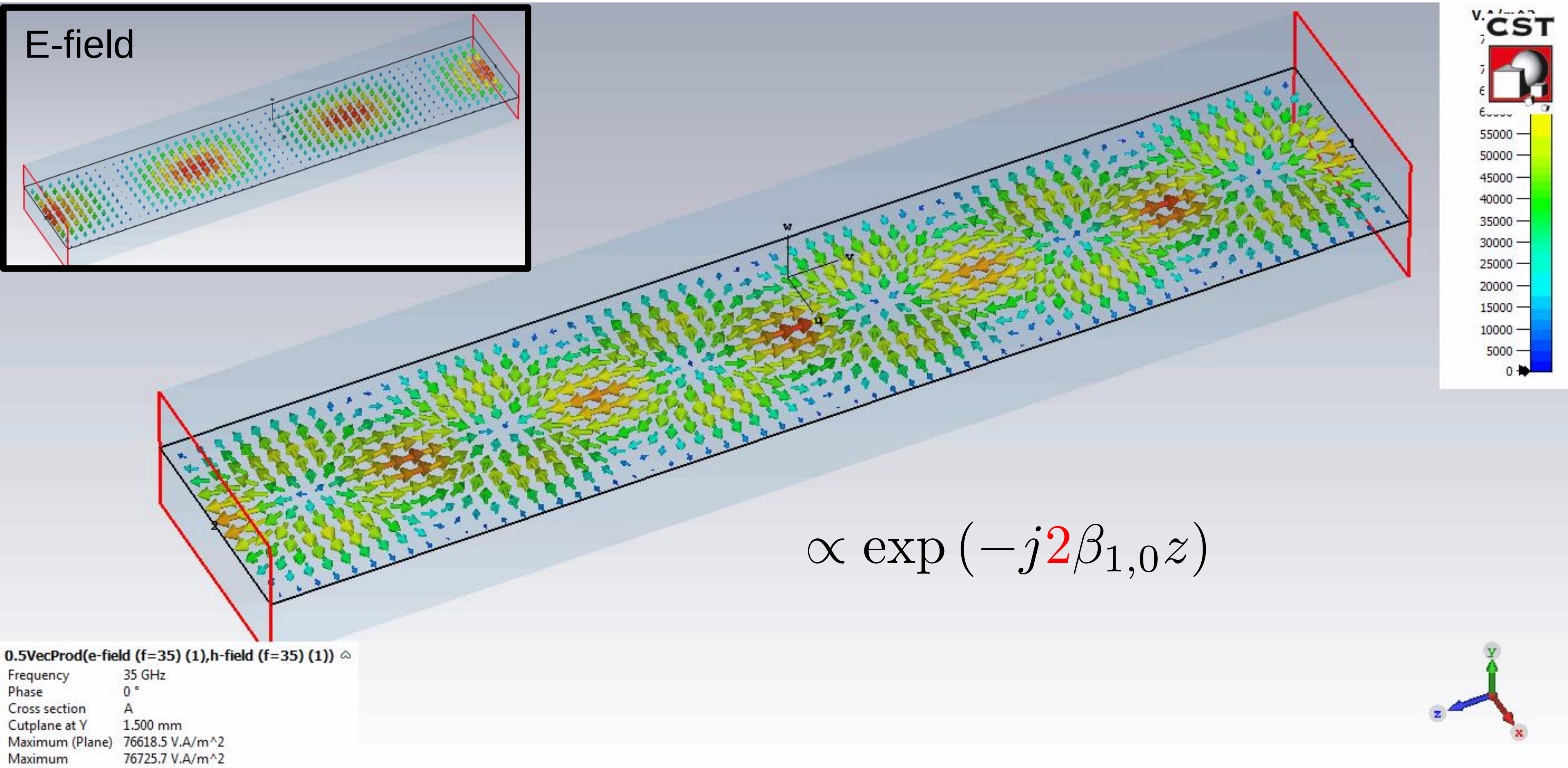
Time varying part of the instantaneous power density



$$\propto \exp\left(-j2\beta_{1,0}z\right)$$

$$\vec{S}(\vec{r},t)=\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)=Re\left\{\frac{\vec{E}(\vec{r},\omega)\times\vec{H}(\vec{r},\omega)^*}{2}\right\}+Re\left\{\frac{\vec{E}(\vec{r},\omega)\times\vec{H}(\vec{r},\omega)e^{j2\omega t}}{2}\right\}$$

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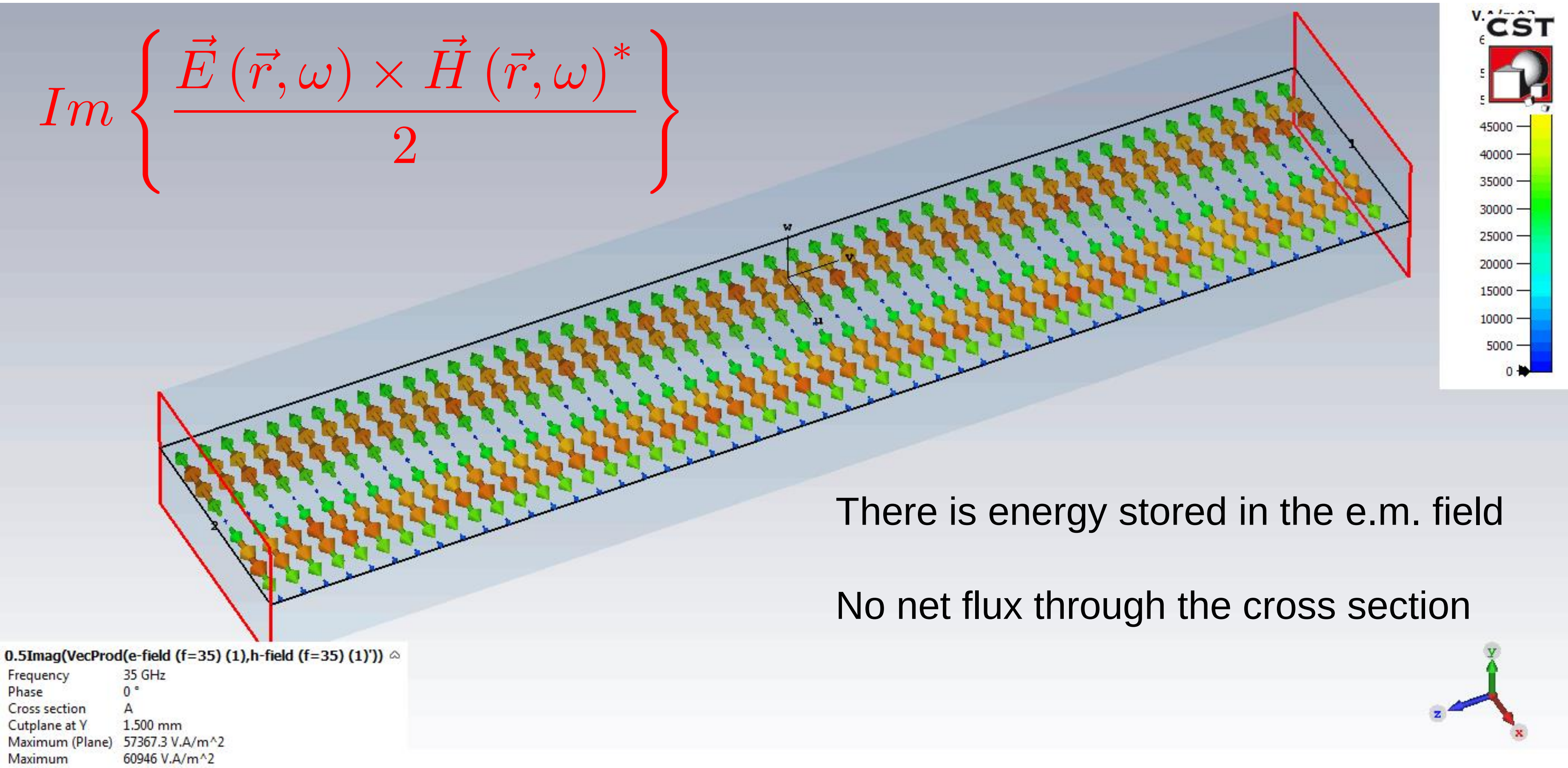
Power Flow pattern (TE10 mode, rect. WG)

Reactive power density

$$Im\left\{\frac{\vec{E}\left(\vec{r},\omega\right)\times\vec{H}\left(\vec{r},\omega\right)^{*}}{2}\right\}$$



There is energy stored in the e.m. field

No net flux through the cross section

$$\vec{S}\left(\vec{r},t\right)=\vec{E}\left(\vec{r},t\right)\times\vec{H}\left(\vec{r},t\right)=Re\left\{\frac{\vec{E}\left(\vec{r},\omega\right)\times\vec{H}\left(\vec{r},\omega\right)^{*}}{2}\right\}+Re\left\{\frac{\vec{E}\left(\vec{r},\omega\right)\times\vec{H}\left(\vec{r},\omega\right)e^{j2\omega t}}{2}\right\}$$

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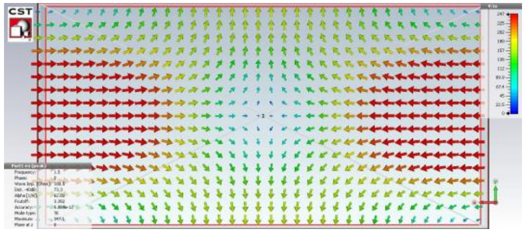
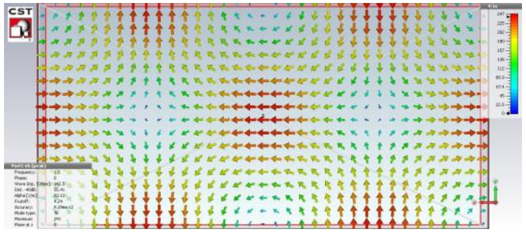
Field pattern at the cross section

$TE^{+z}_{m,n}$ m (n) is the number of half periods (or maxima/minima) along the x (y) axis in the cross-section.

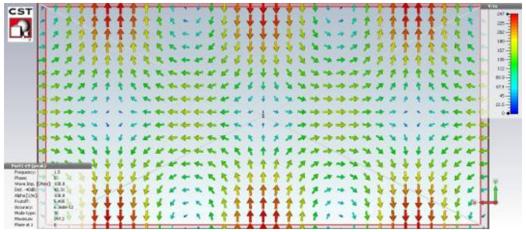
TE??



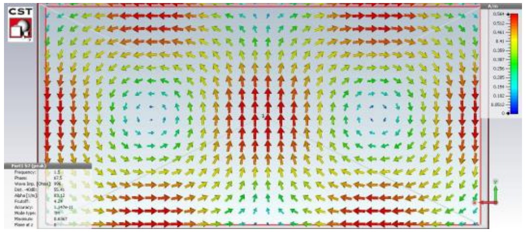
TE??

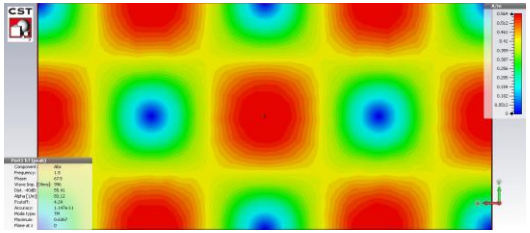


TE??

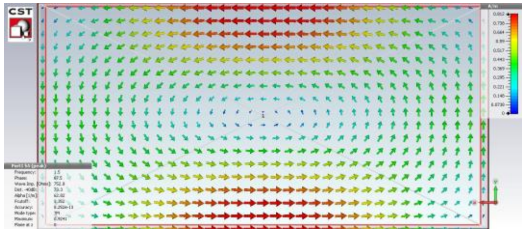


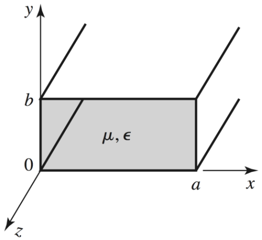
TM??



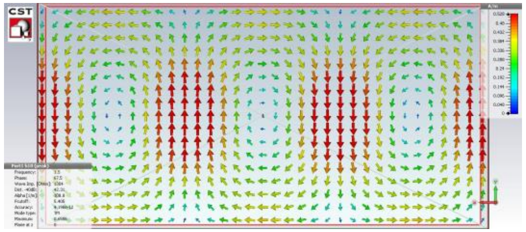


TM??





TM??



Simulations by L. Ficcadenti

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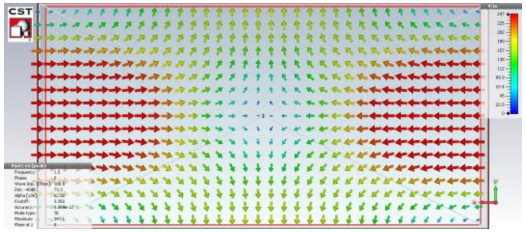
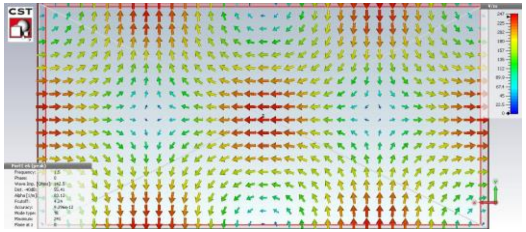
Field pattern at the cross section

$TE^{+z}_{m,n}$ **m (n) is the number of half periods (or maxima/minima) along the x (y) axis in the cross-section.**

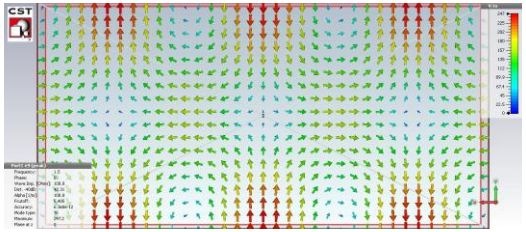
TE11



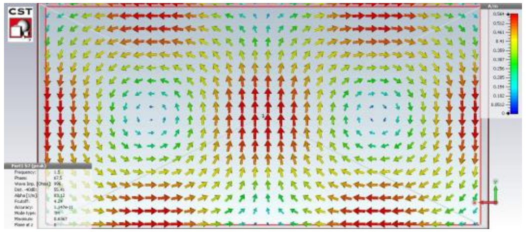
TE21

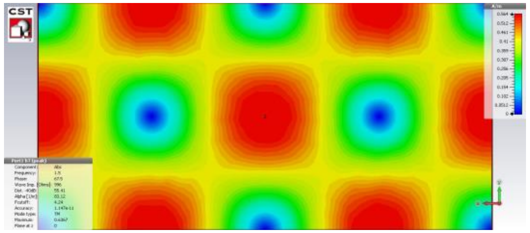


TE31

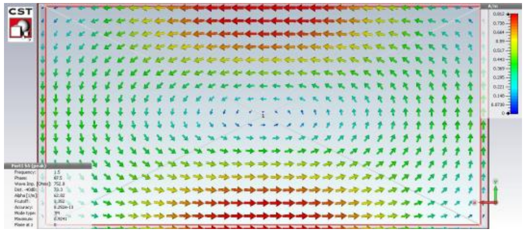


TM21

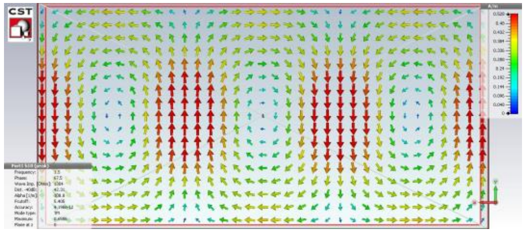


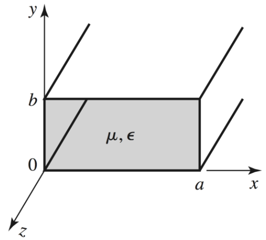


TM11



TM31



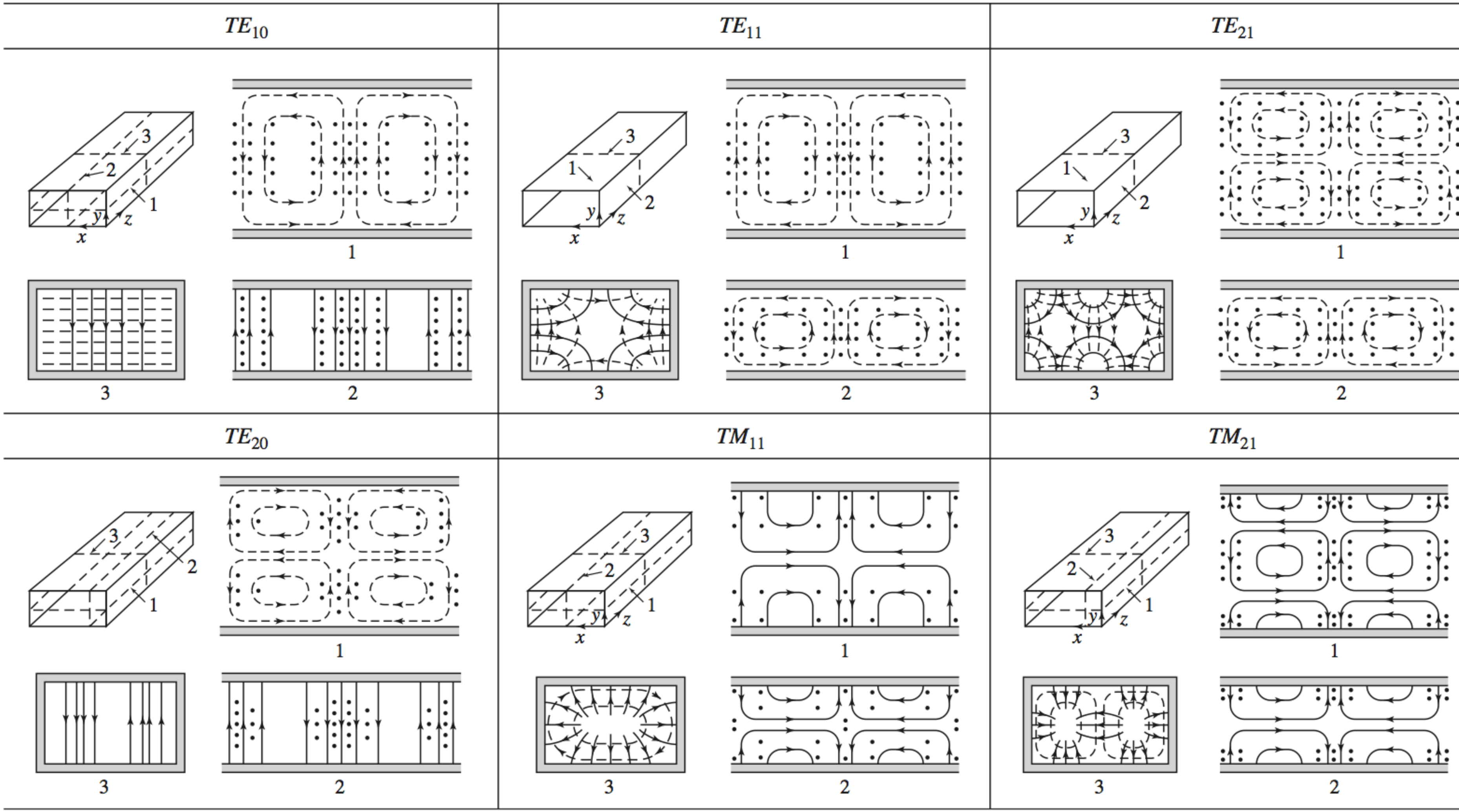


Simulations by L. Ficcadenti

Andrea.Mostacci@uniroma1.it

Field pattern (TE mode, rect. WG)

$TE_{m,n}^{+z}$ m (n) is the number of half periods (or maxima/minima) along the x (y) axis in the cross-section.



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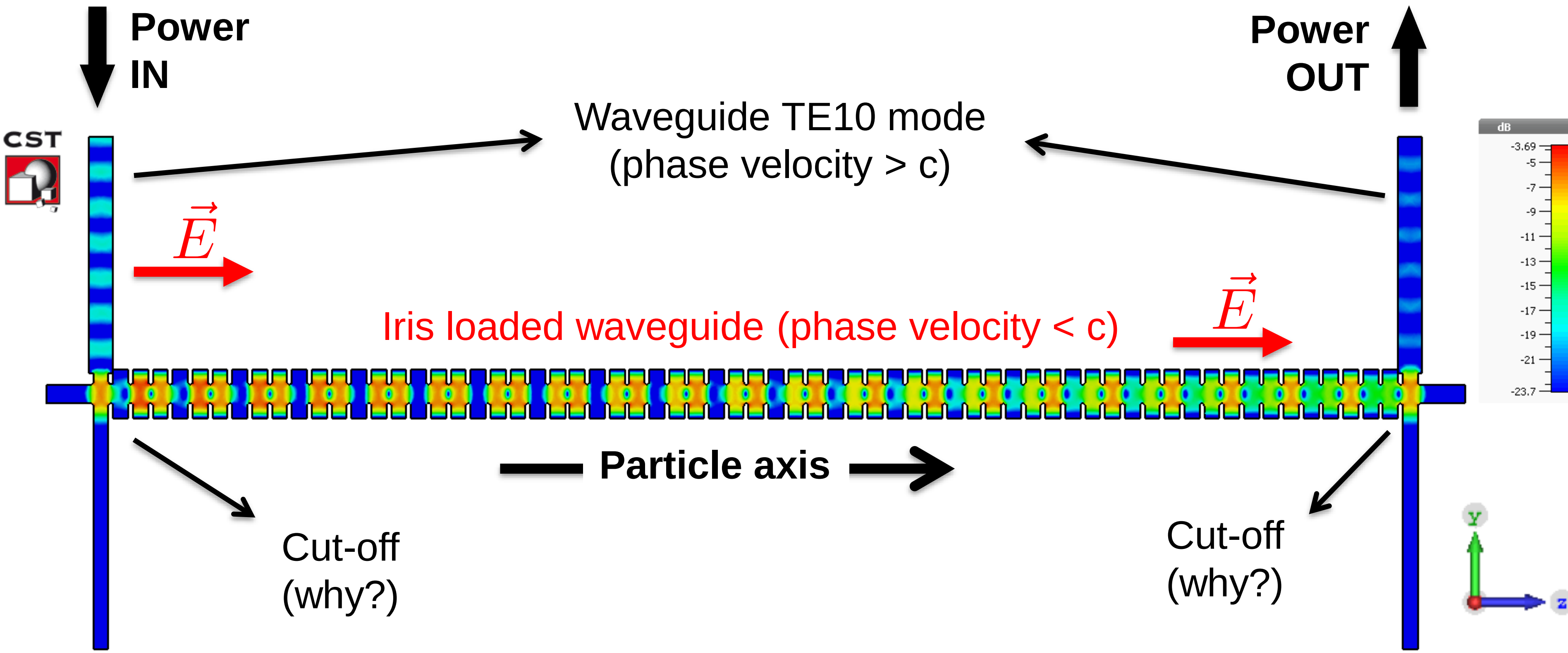
Full EM simulation of a RF accelerating structure

Exercise

X-band (12GHz) accelerating structure for high brightness LINAC

E-field along particle axis, i.e. z-axis (log-scale)



Vacuum port

Vacuum port

Simulation assuming perfect conductor

With phasors, a time animation is identical to phase rotation.

Courtesy of J. Arnesano, Sapienza

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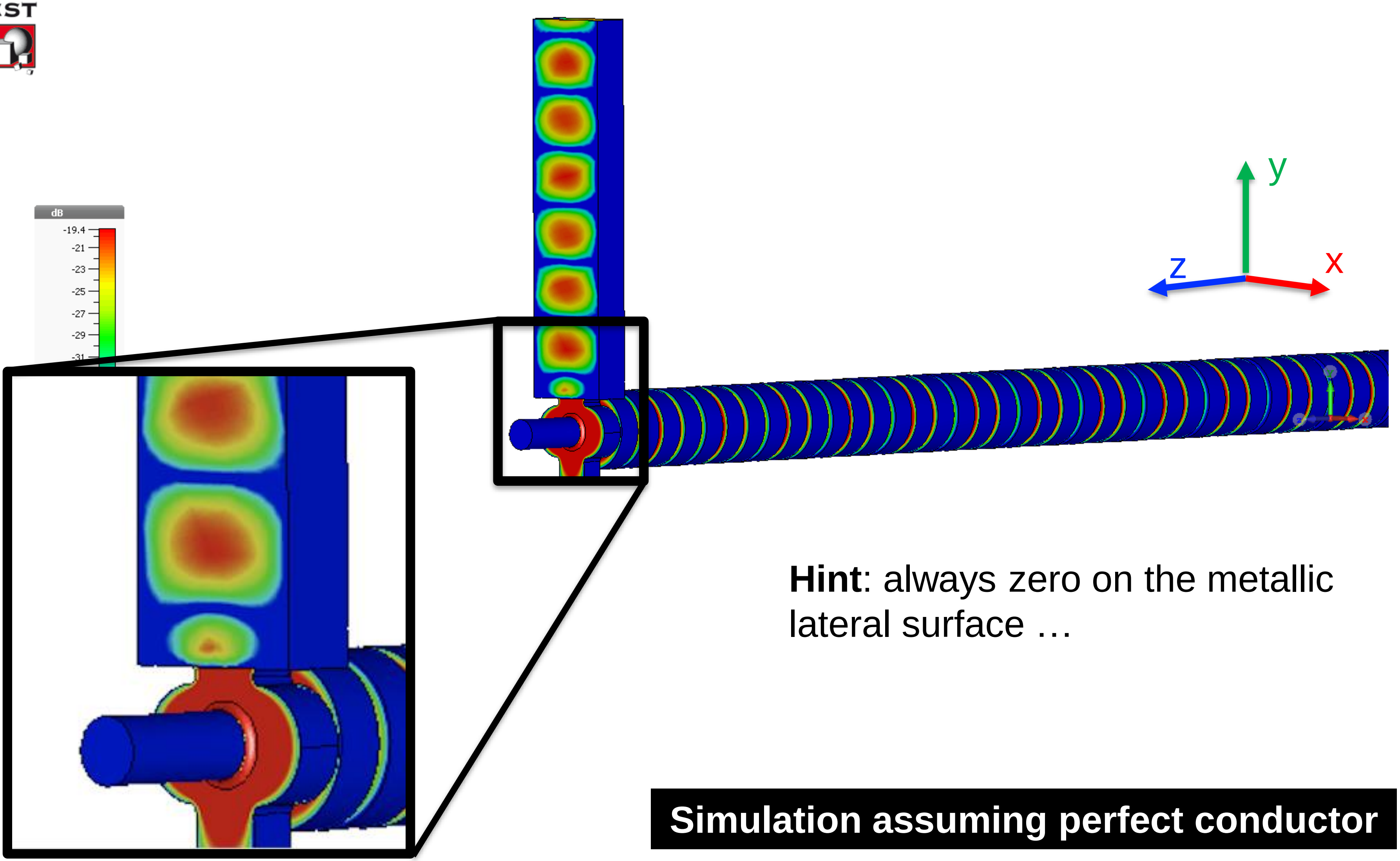
Full EM simulation of a RF accelerating structure

Which field is this one? E or H field?



Hint: always zero on the metallic lateral surface …

Simulation assuming perfect conductor

Courtesy of J. Arnesano, Sapienza

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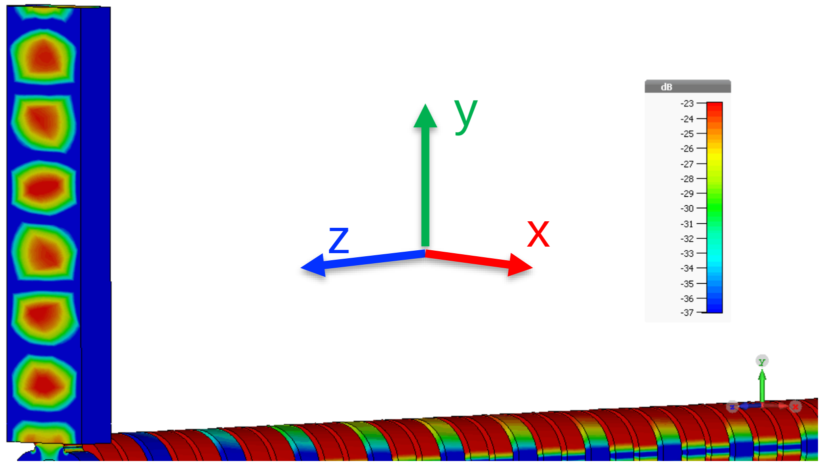
Full EM simulation of a RF accelerating structure

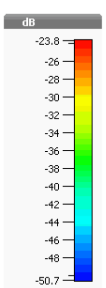
Exercise

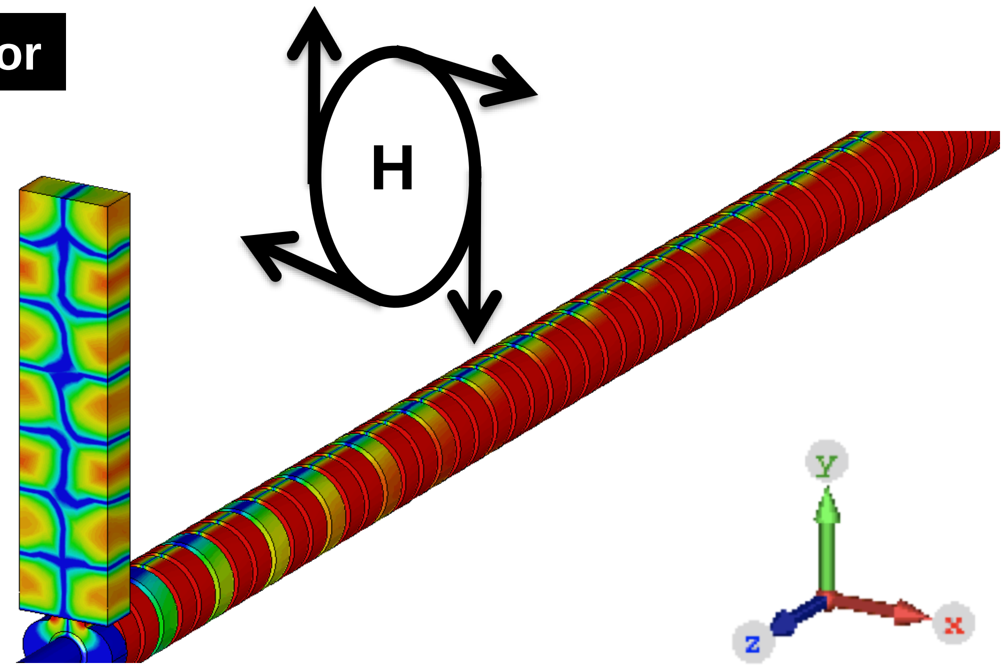


Which field? E or H?

Which component? Along x, y or z?

Simulation assuming perfect conductor





Courtesy of J. Arnesano, Sapienza

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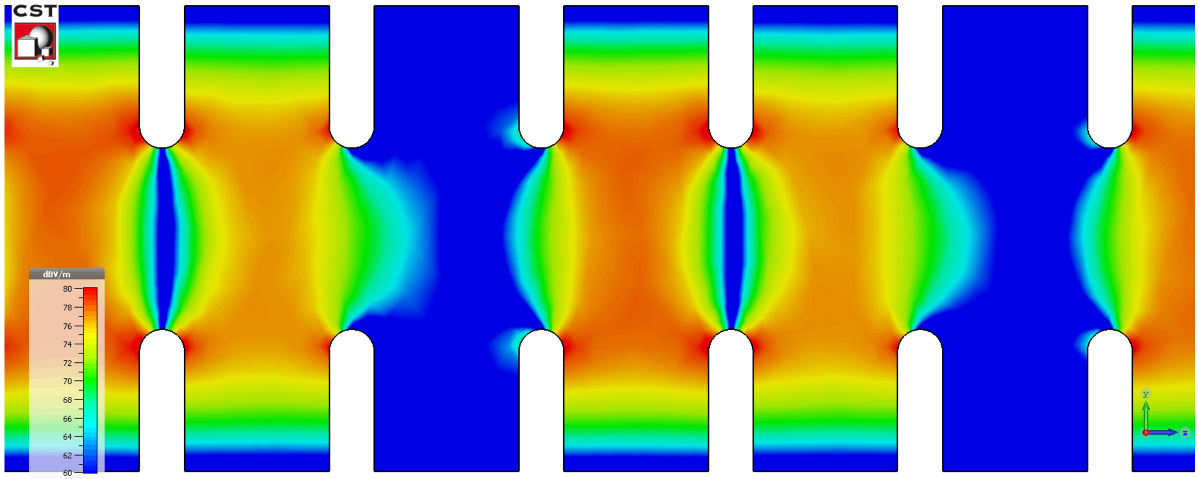
Full EM simulation of a RF accelerating structure

Exercise

Simulation assuming perfect conductor

Accelerating E-field

Particle axis



Which is the periodicity of the accelerating field?

3 cell periodicity **$2\pi/3$ phase advance**

Courtesy of J. Arnesano, Sapienza
Andrea.Mostacci@uniroma1.it

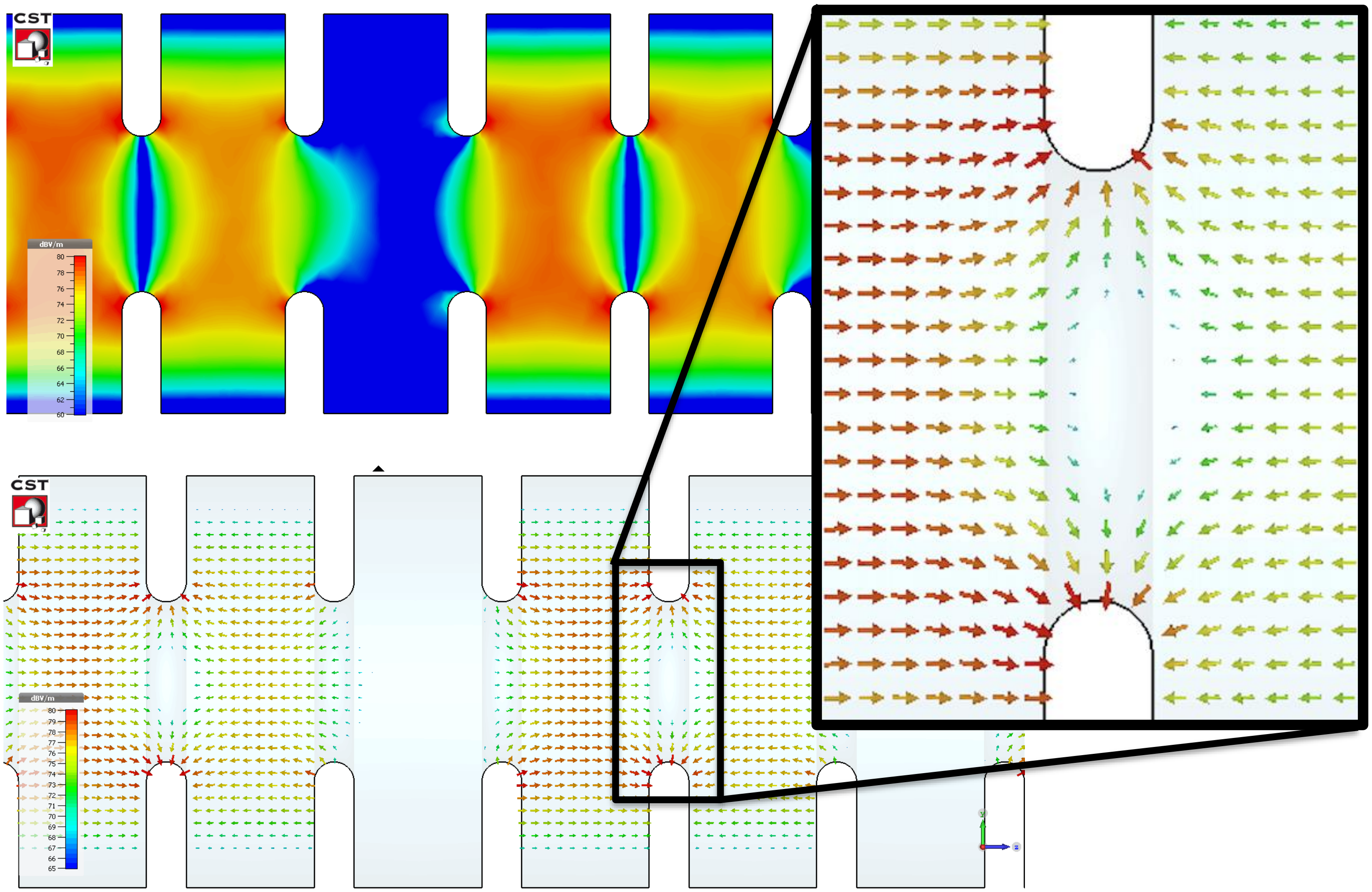
Full EM simulation of a RF accelerating structure

Exercise

Simulation assuming perfect conductor

Accelerating E-field

Particle axis



3 cell periodicity **2π/3 phase advance**

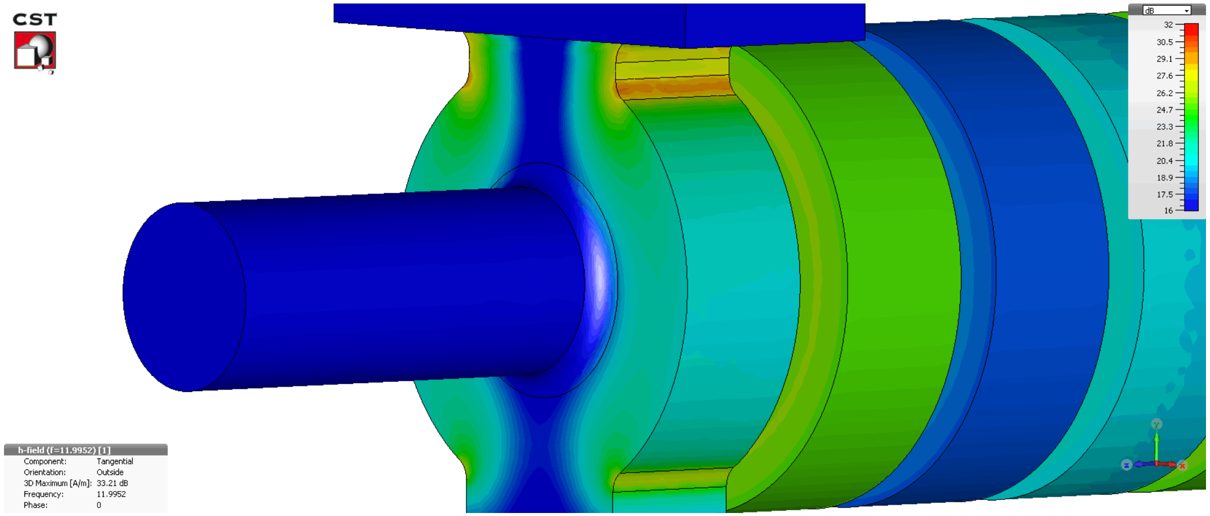
 Courtesy of J. Arnesano, Sapienza Andrea.Mostacci@uniroma1.it

Full EM simulation of a RF accelerating structure

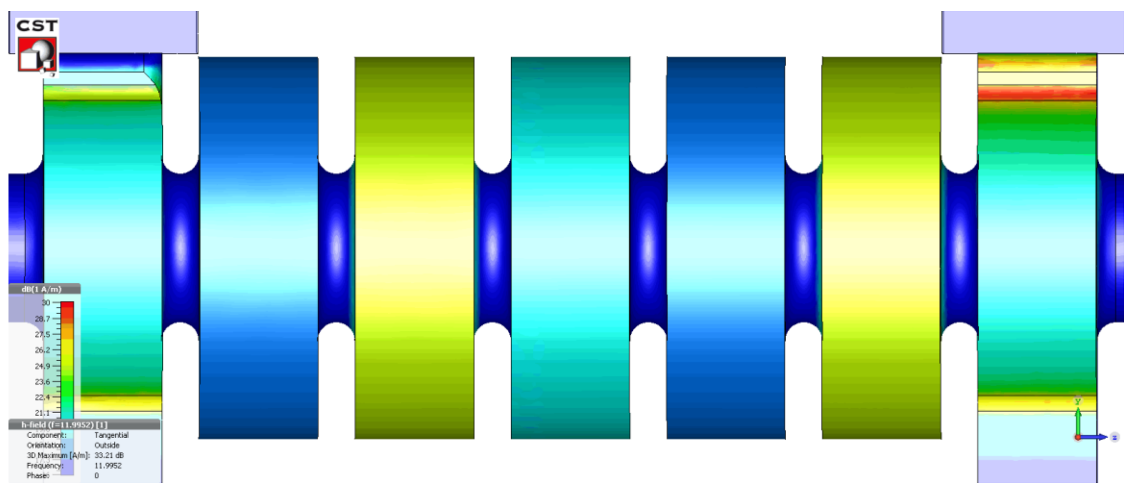
Exercise

Temperature breakdown: seek for maximum power loss

$$P_t = \frac{R_s}{2}\int_S |\hat{n} \times \vec{H}|^2 dS$$



Simulation with perfect conductor



Courtesy of J. Arnesano, Sapienza

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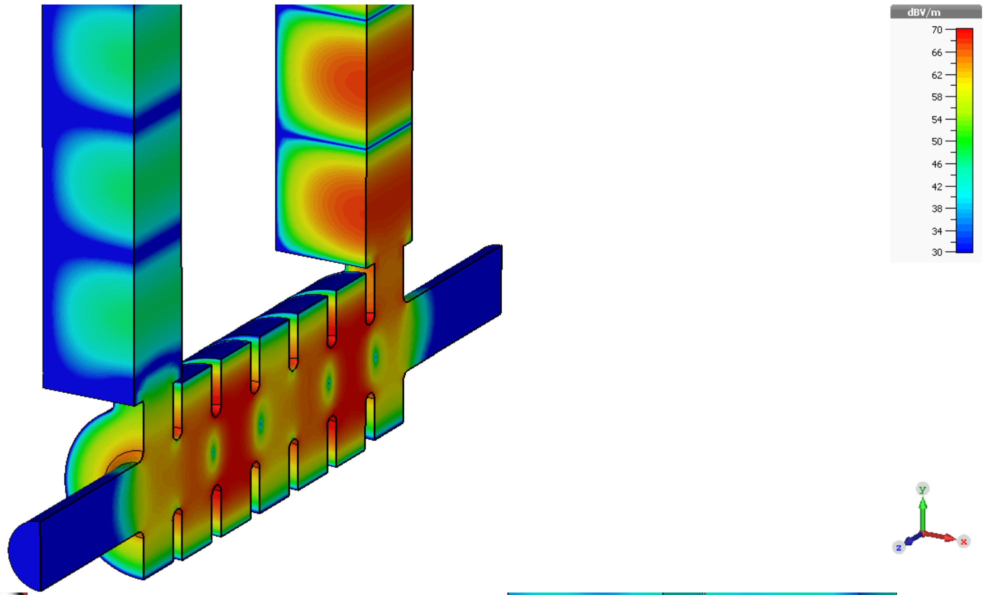
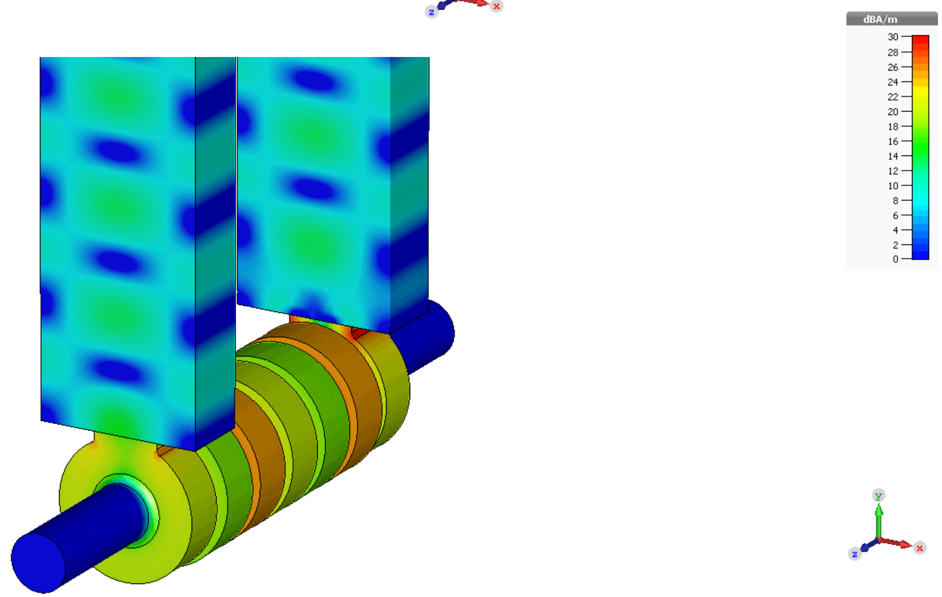
Full EM simulation of a RF accelerating structure

Exercise

Comparison between a tuned coupler and a detuned coupler



Input coupler detuned



Input coupler tuned

Simulation with perfect conductor

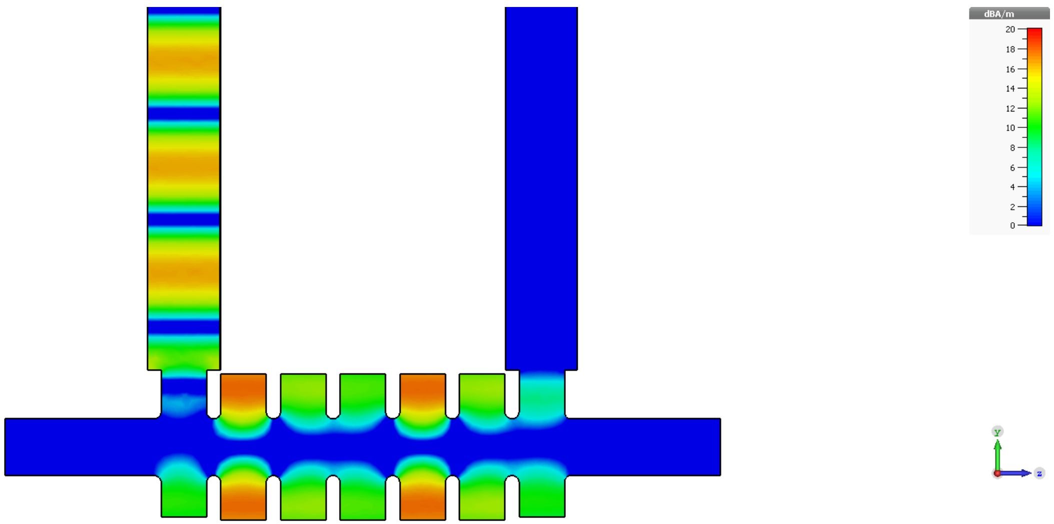
Courtesy of J. Arnesano, Sapienza

Andrea.Mostacci@uniroma1.it

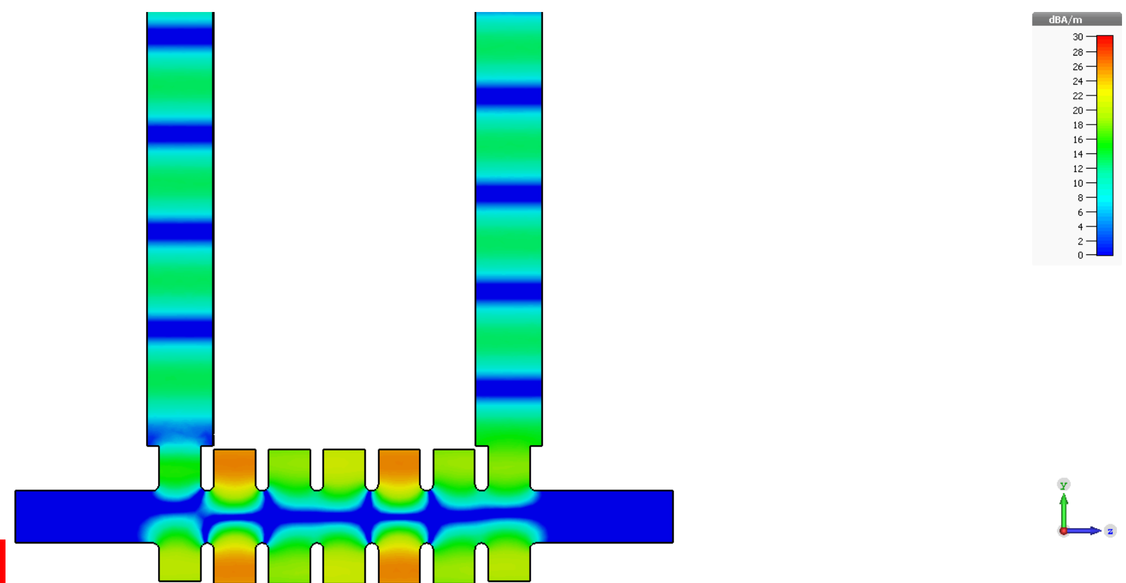
Full EM simulation of a RF accelerating structure

Exercise

Comparison between a tuned coupler and a detuned coupler



Input and output coupler detuned



Input and output coupler tuned

Simulation with perfect conductor

Courtesy of J. Arnesano, Sapienza

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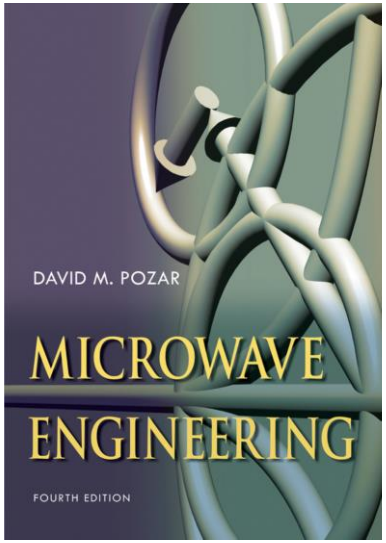
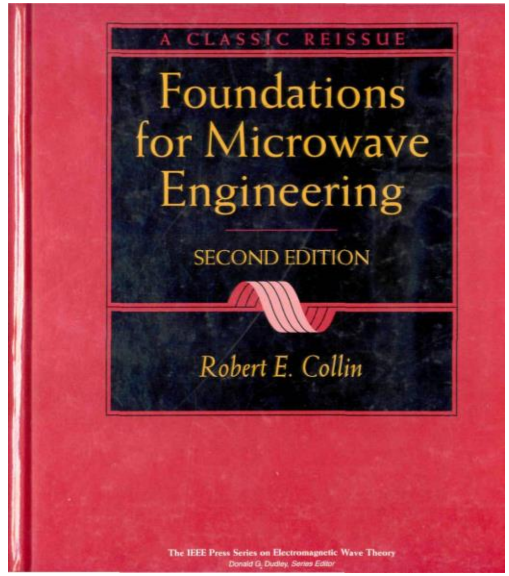
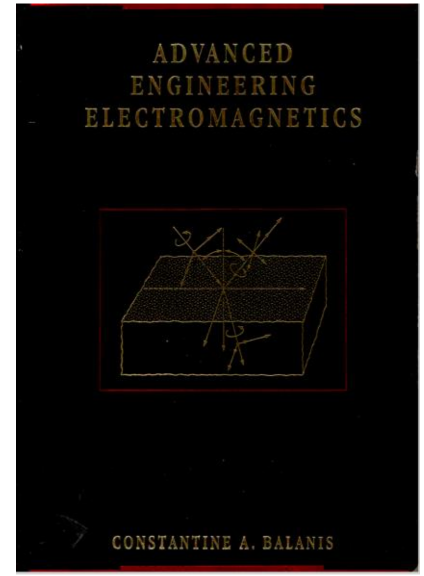
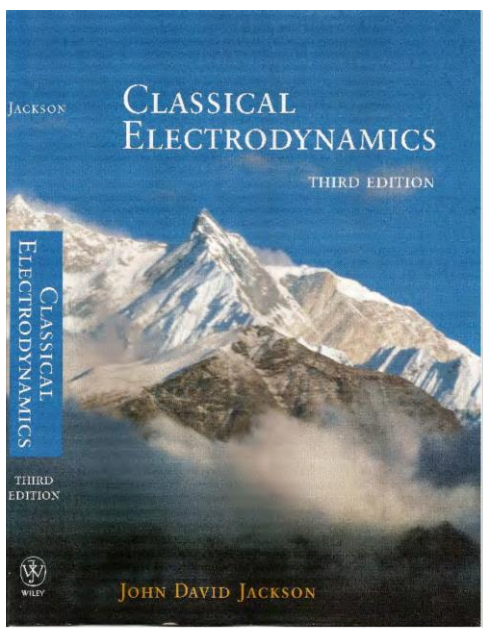
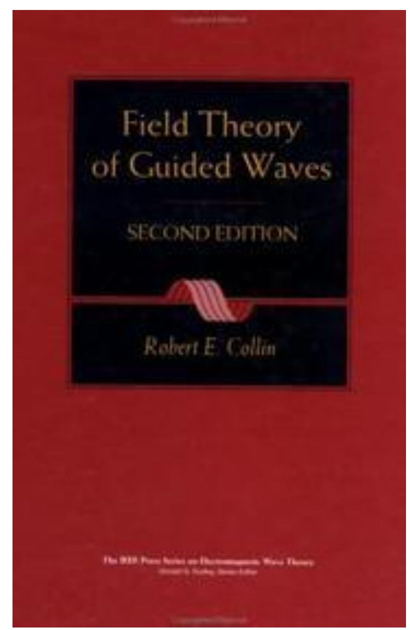
Famous books (personal cut …)











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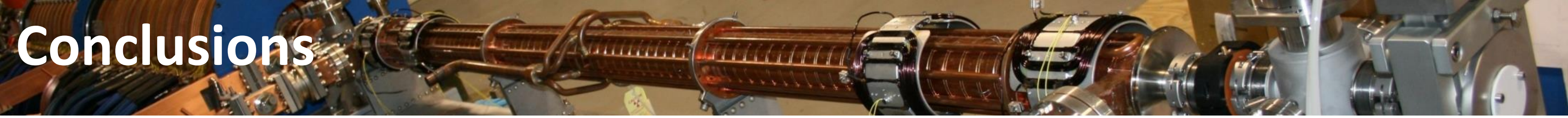
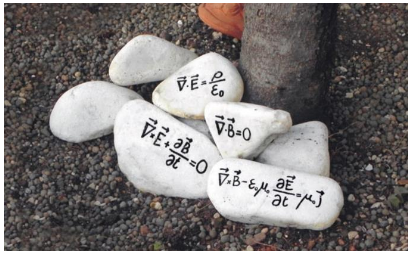
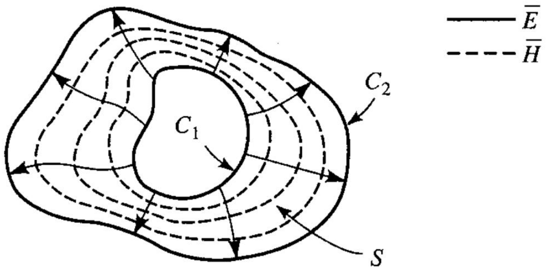
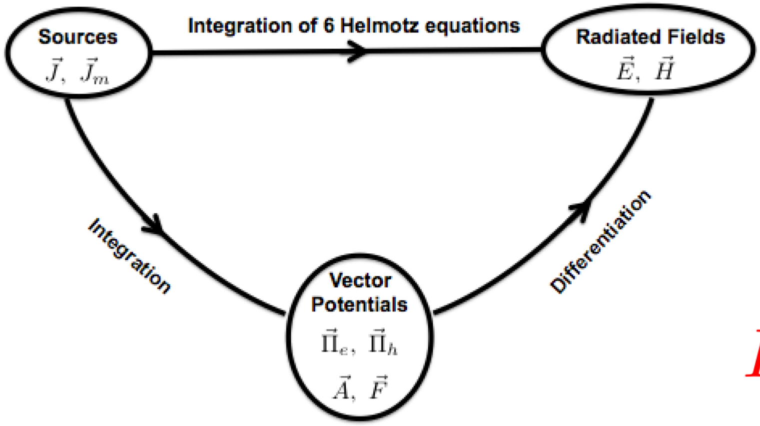
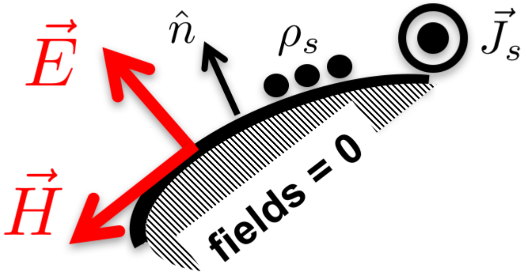
Conclusions

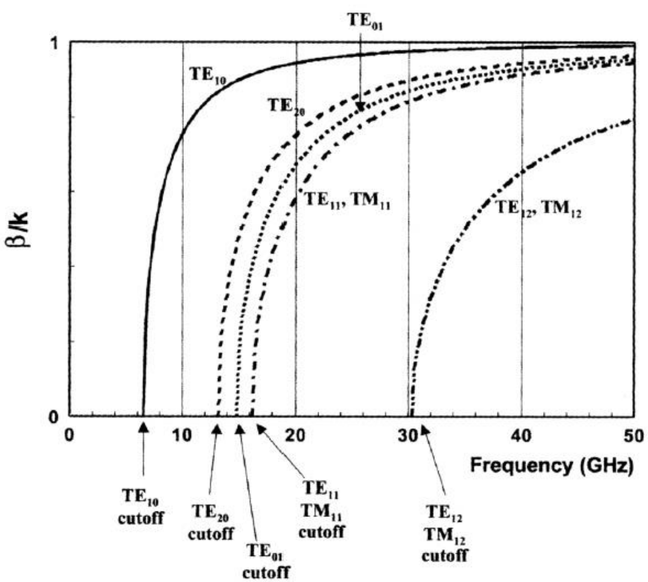


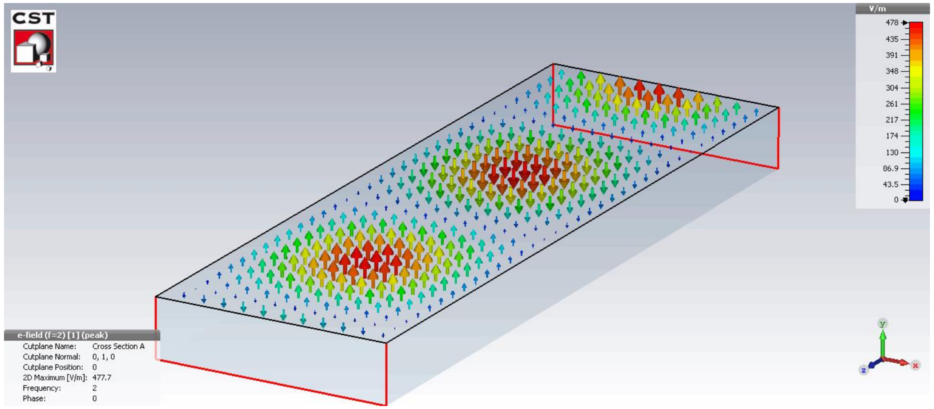


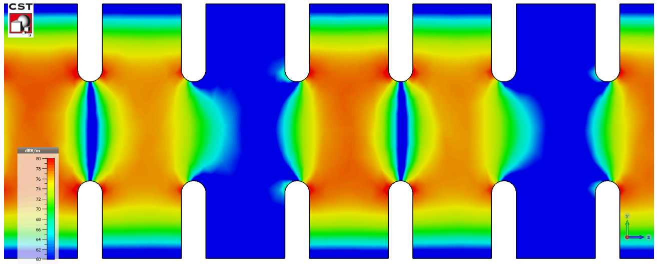












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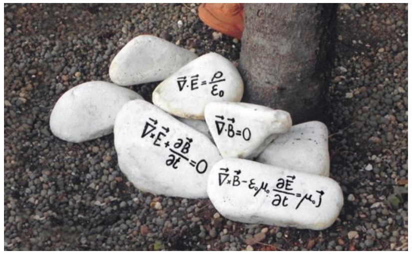
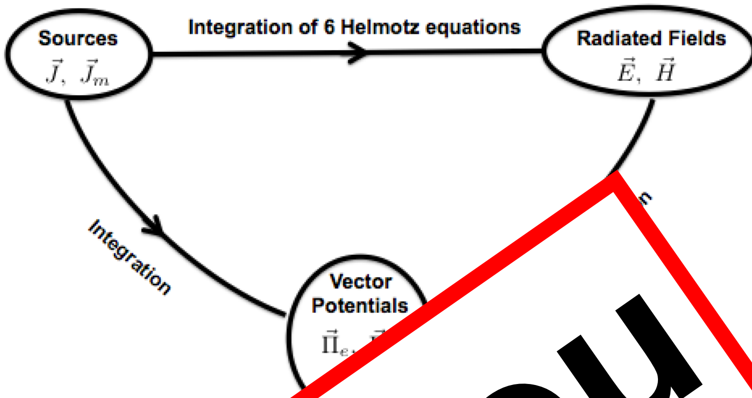
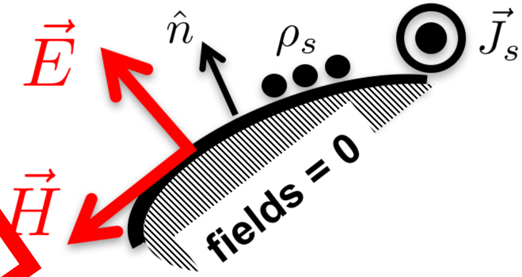
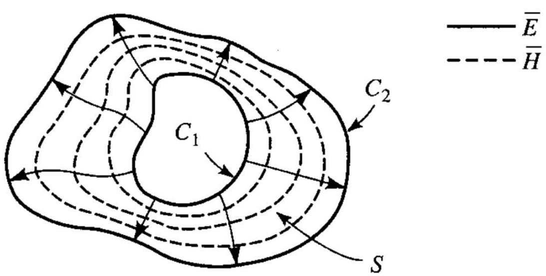
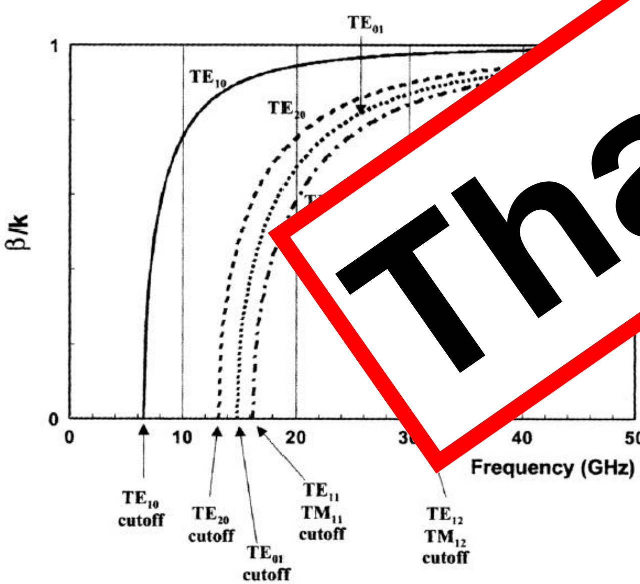
Conclusions

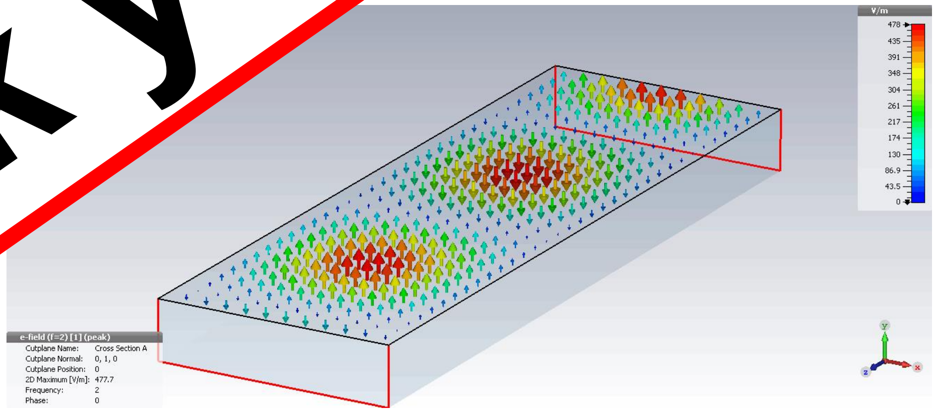


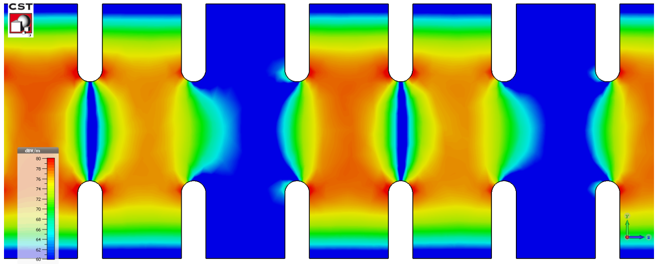












Thank you

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