LONGITUDINAL BEAM DYNAMICS

Elias Métral (CERN BE Department)

This course started with the one of Frank Tecker (CERN-BE) in 2010 (I took over from him in 2011), who inherited it from Roberto Corsini (CERN-BE), who gave this course in the previous years, based on the transparencies written by Louis Rinolfi (CERN-BE) who held the course at JUAS from 1994 to 2002 (see CERN/PS 2000-008 (LP)):
http://cdsweb.cern.ch/record/446961/files/ps-2000-008.pdf

Material from Joel LeDuff's Course at the CERN Accelerator School held at Jyvaskyla, Finland the 7-18 September 1992 (CERN 94-01) has been used as well:
http://cdsweb.cern.ch/record/235242/files/p253.pdf
http://cdsweb.cern.ch/record/235242/files/p289.pdf

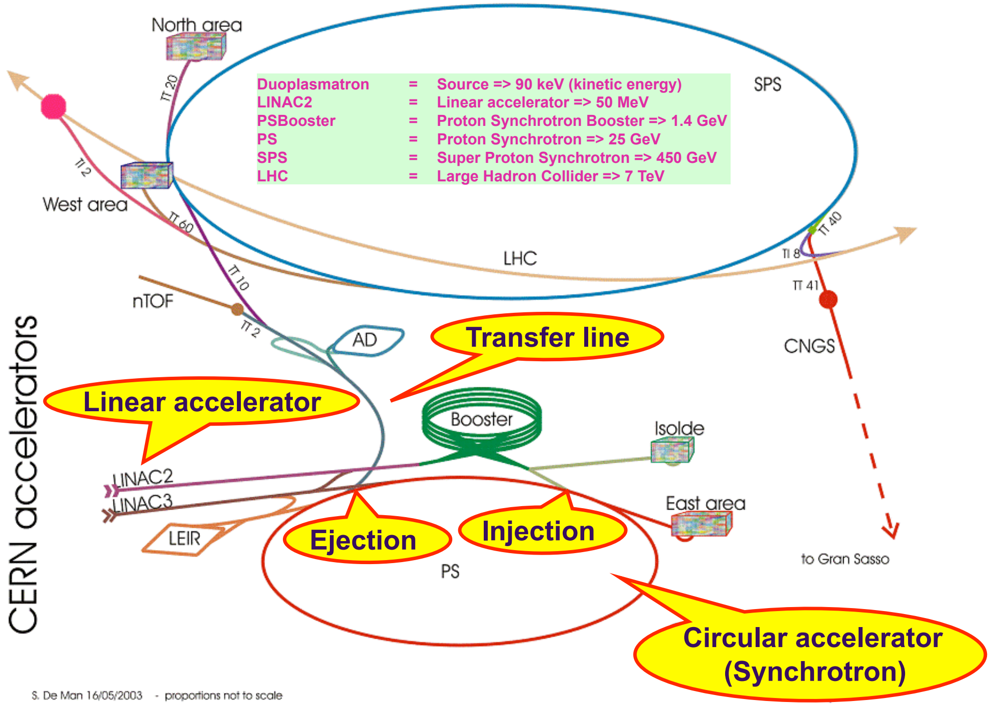
I attended the course given by Louis Rinolfi in 1996 and was his assistant in 2000 and 2001 (and the assistant of Michel Martini for his course on transverse beam dynamics)

This course and related exercises / exams (as well as other courses) can be found in my web page: http://emetral.web.cern.ch/emetral/

Assistant: Benoit Salvant (CERN BE Department)

PURPOSE OF THIS COURSE

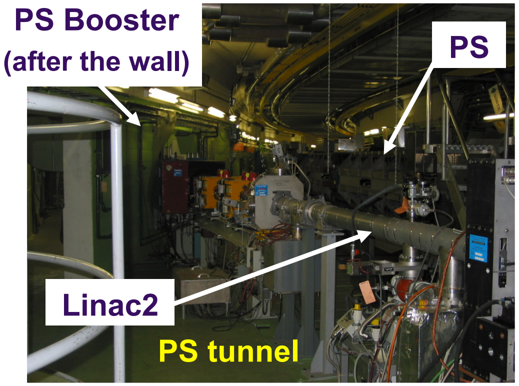
Discuss the oscillations of the particles in the longitudinal plane of synchrotrons, called SYNCHROTRON OSCILLATIONS (similarly to the betatron oscillations in the transverse planes), and derive the basic equations

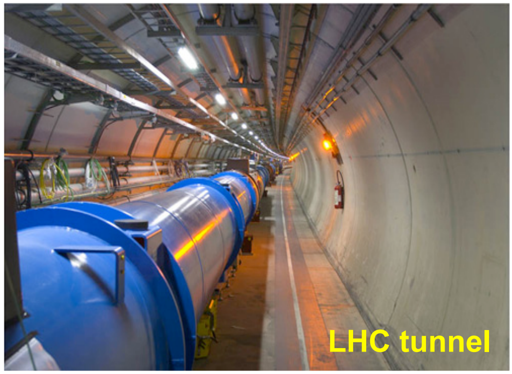


Duoplasmatron = Source => 90 keV (kinetic energy)
LINAC2 = Linear accelerator => 50 MeV
PSBooster = Proton Synchrotron Booster => 1.4 GeV
PS = Proton Synchrotron => 25 GeV
SPS = Super Proton Synchrotron => 450 GeV
LHC = Large Hadron Collider => 7 TeV

S. De Man 16/05/2003 - proportions not to scale

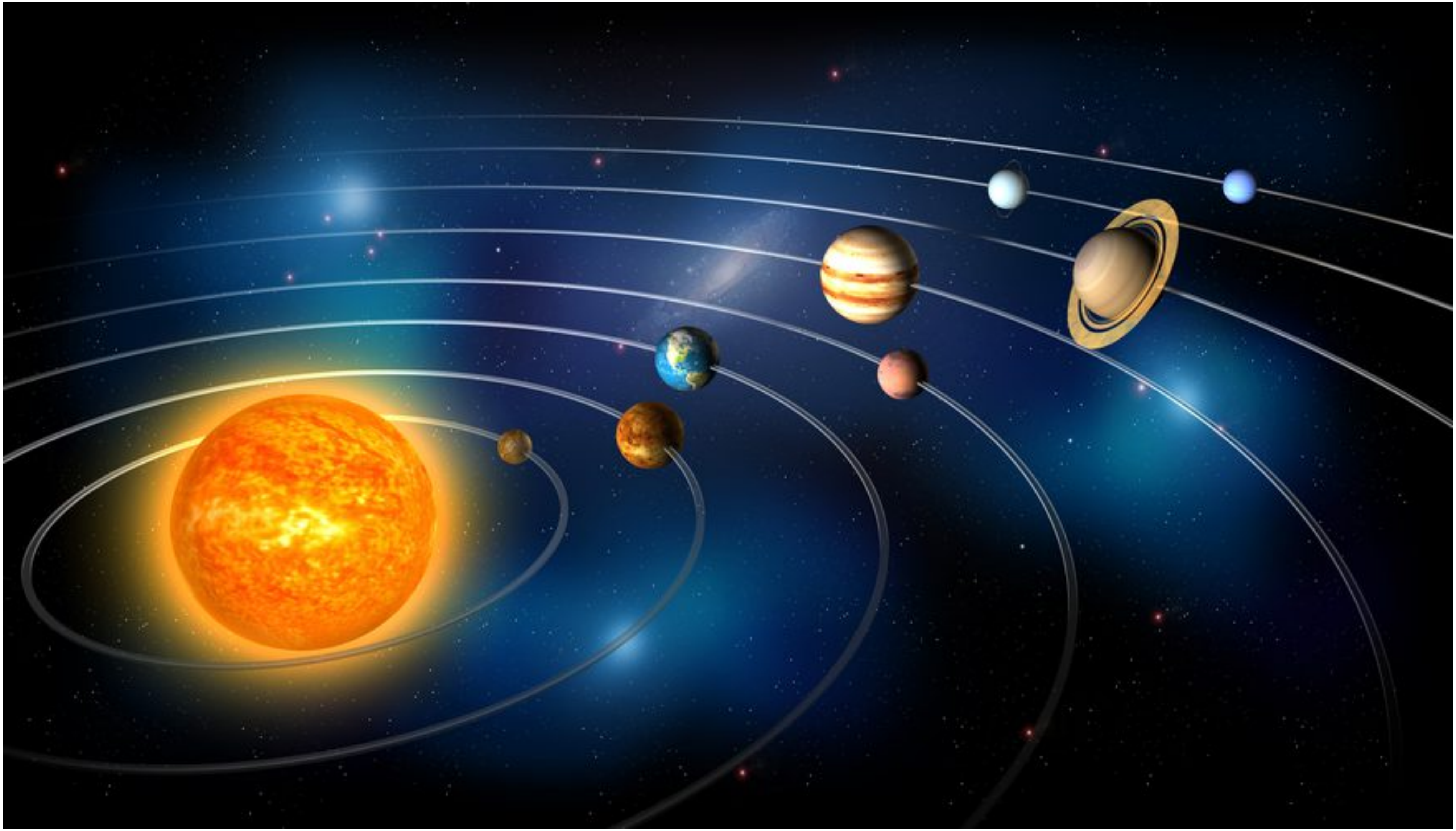
Example of the LHC p beam in the injector chain





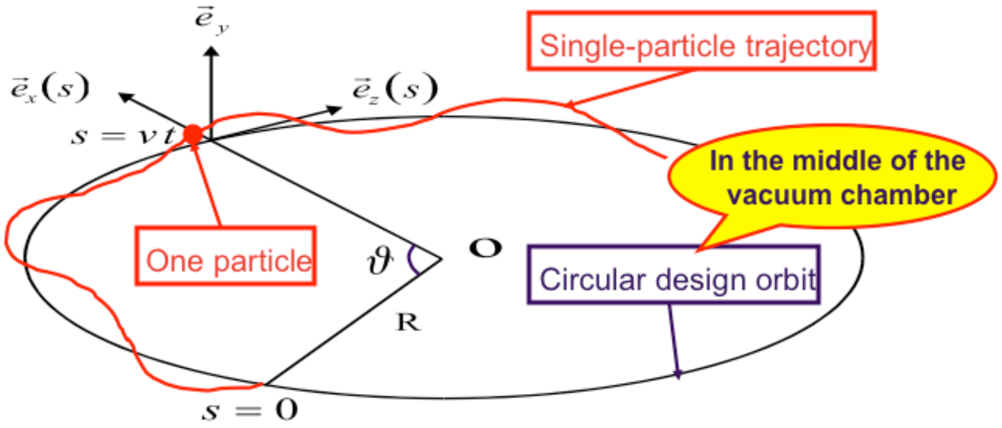
$1\ \text{eV} = 1.6 \times 10^{-19}\ \text{J}$ (energy gain by an e- accelerated by a potential difference of 1 Volt)

PURPOSE OF THIS COURSE



PURPOSE OF THIS COURSE

IN REAL SPACE



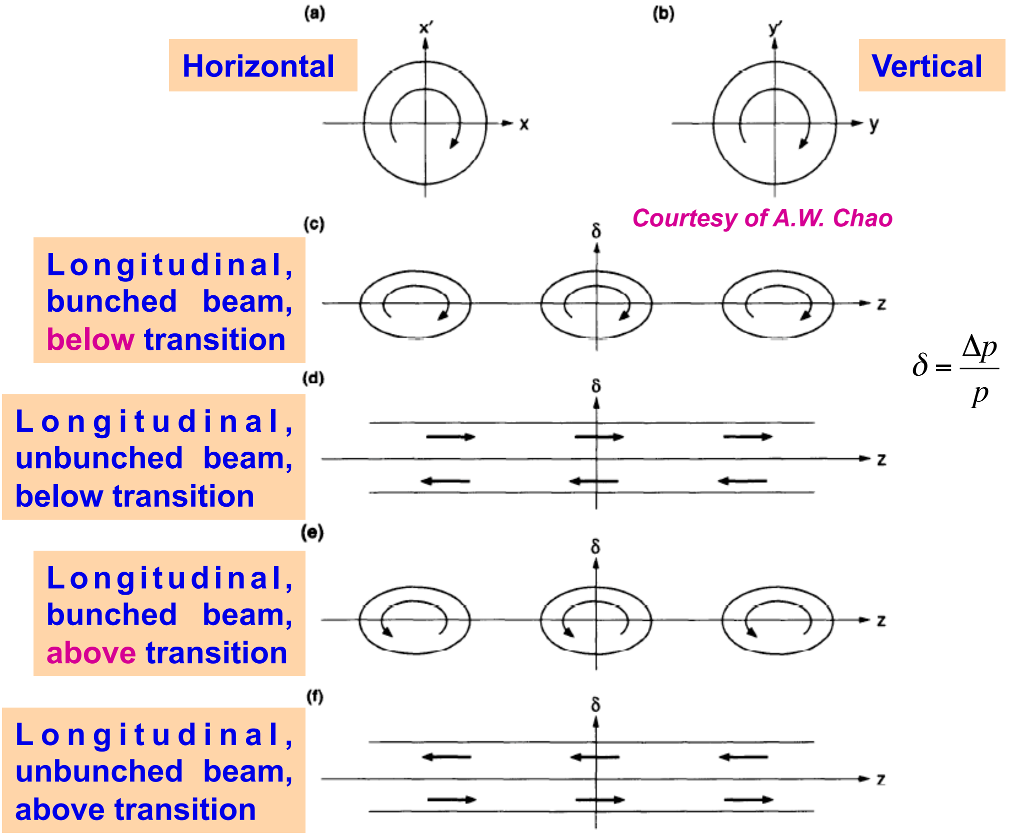
Single-particle trajectory

In the middle of the vacuum chamber

One particle

Circular design orbit

IN PHASE SPACE



Horizontal

Vertical

Courtesy of A.W. Chao

Longitudinal, bunched beam, below transition

Longitudinal, unbunched beam, below transition

Longitudinal, bunched beam, above transition

Longitudinal, unbunched beam, above transition

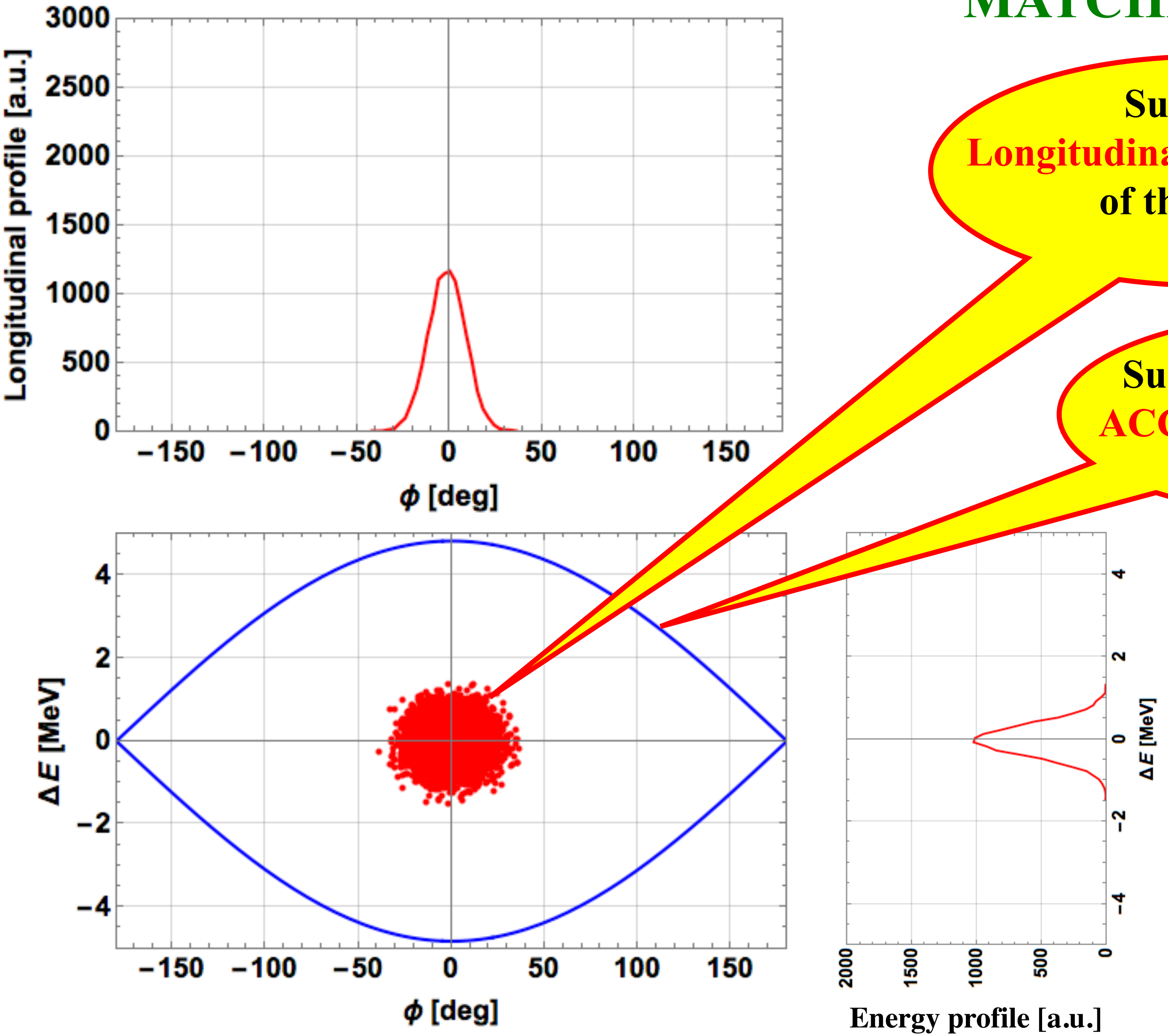
$$\delta = \frac{\Delta p}{p}$$

PURPOSE OF THIS COURSE

Some movies (in phase space) to have a better idea of what we will work on during this course and what you will be able to understand and do after this course…

“MATCHED” AND “MISMATCHED” BUNCH

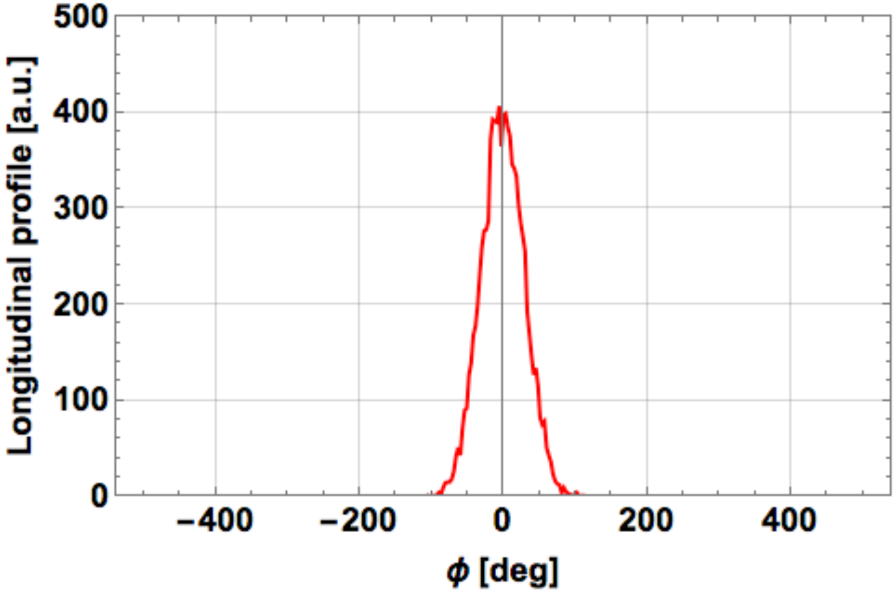
MATCHED BUNCH



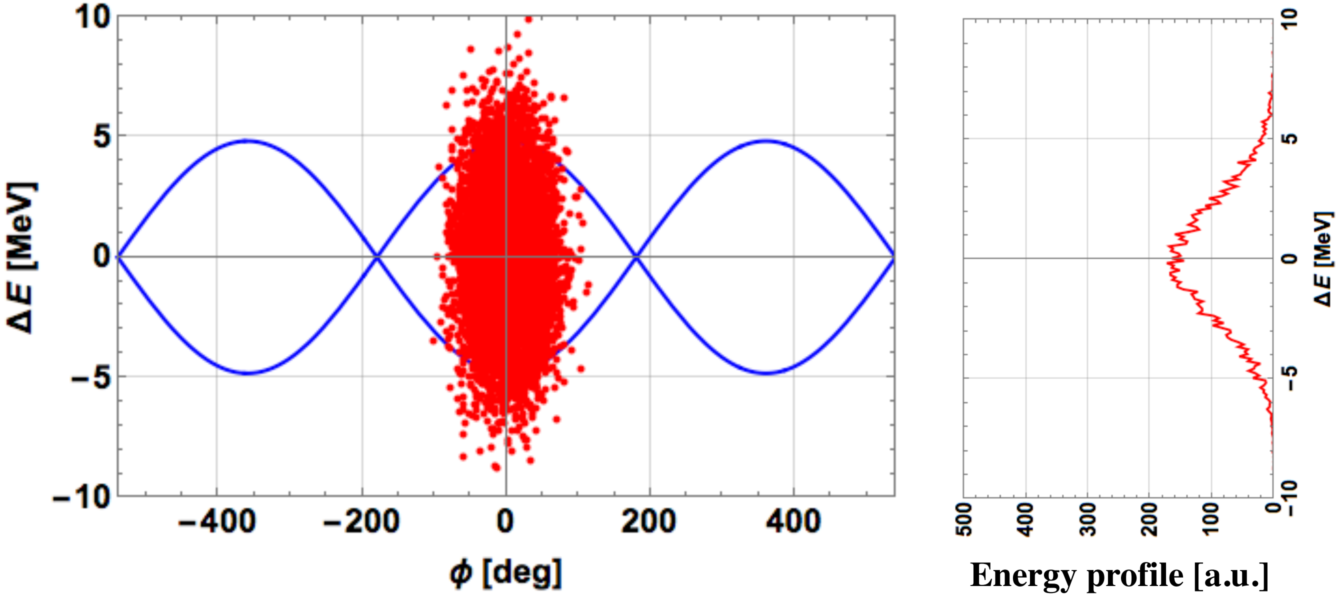
Surface = Longitudinal EMITTANCE of the bunch $= \varepsilon_L$

Surface = Longitudinal ACCEPTANCE of the RF bucket

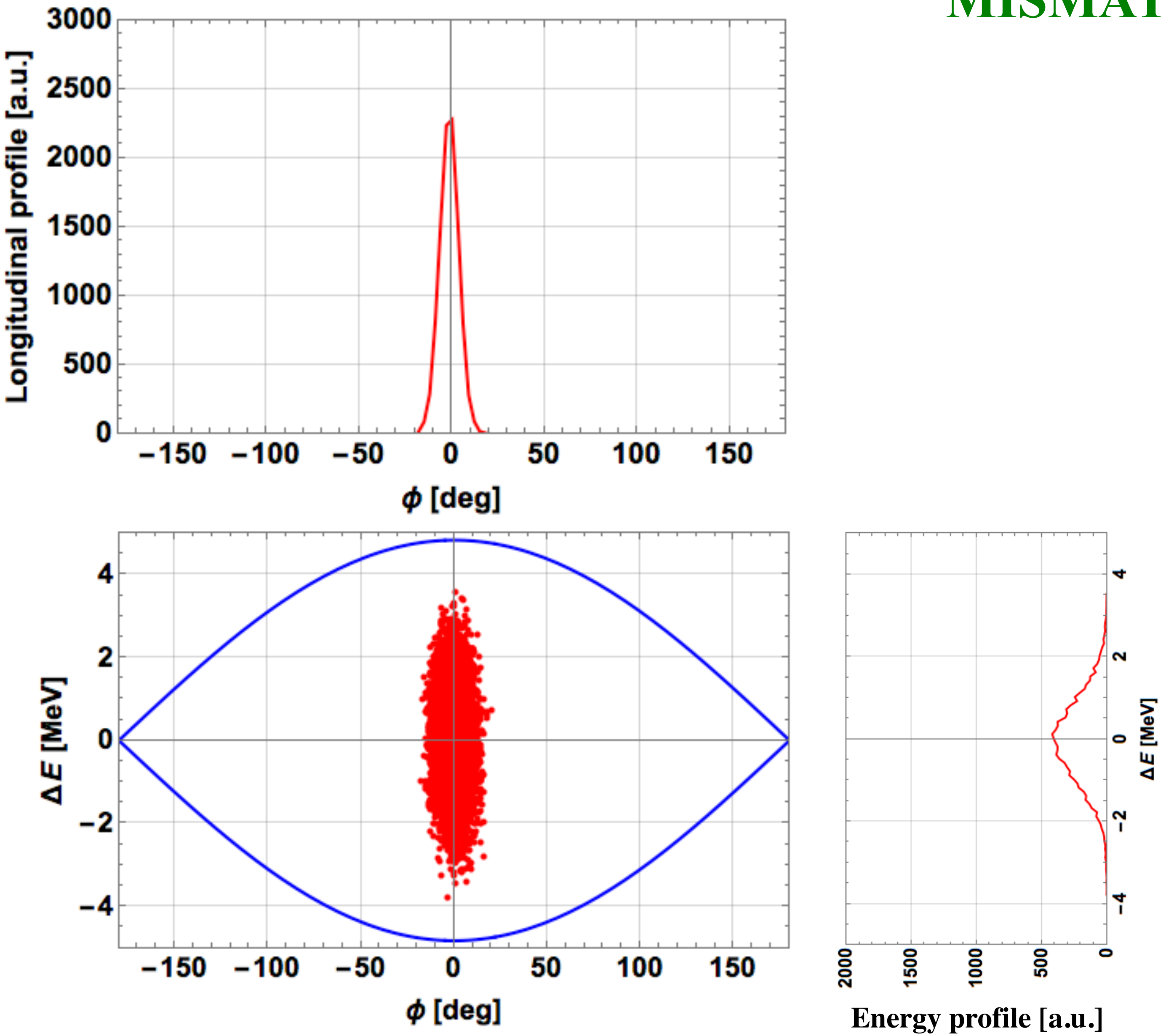
MISMATCHED BUNCH



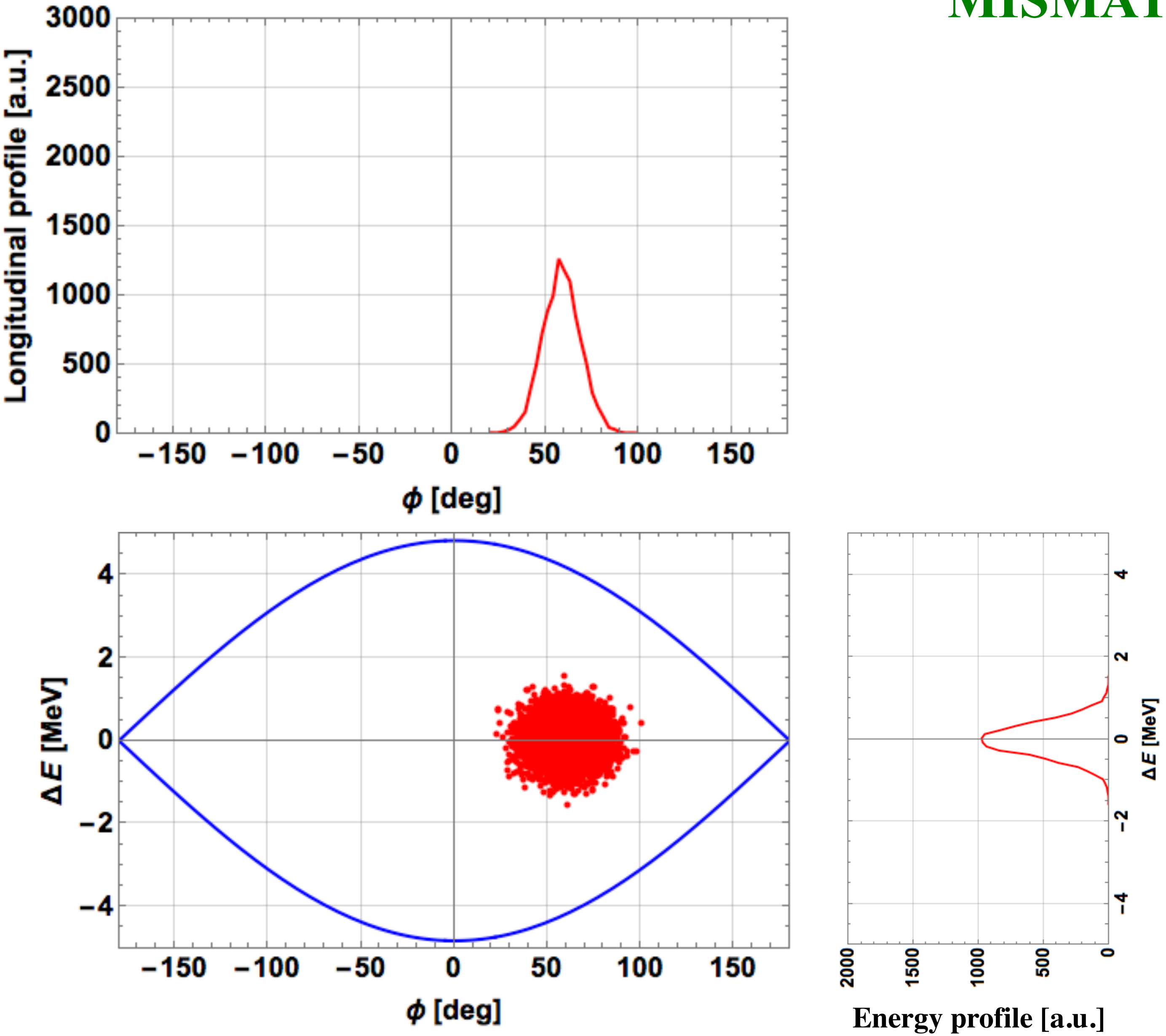
Leads to an increase of the longitudinal emittance and/or particle losses



MISMATCHED BUNCH

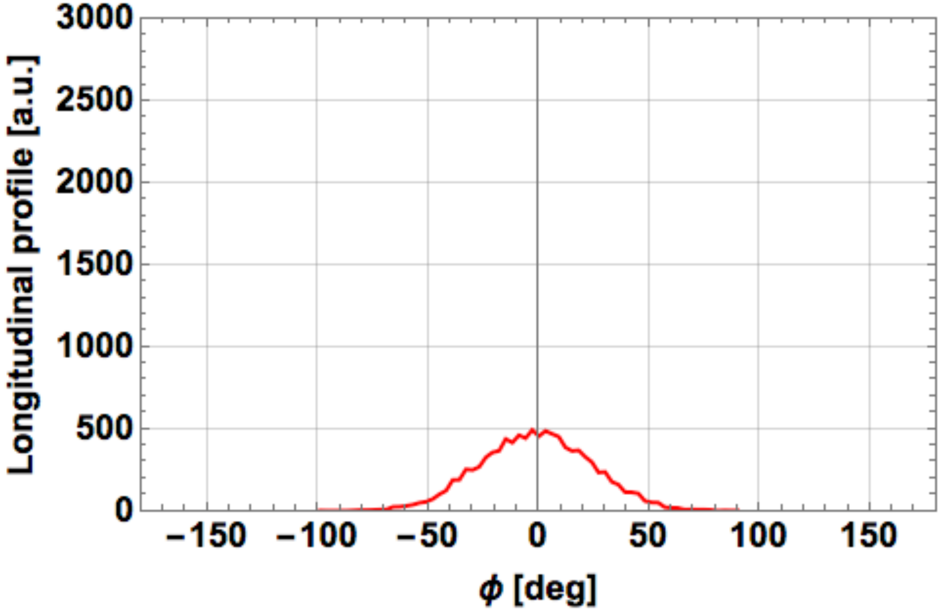


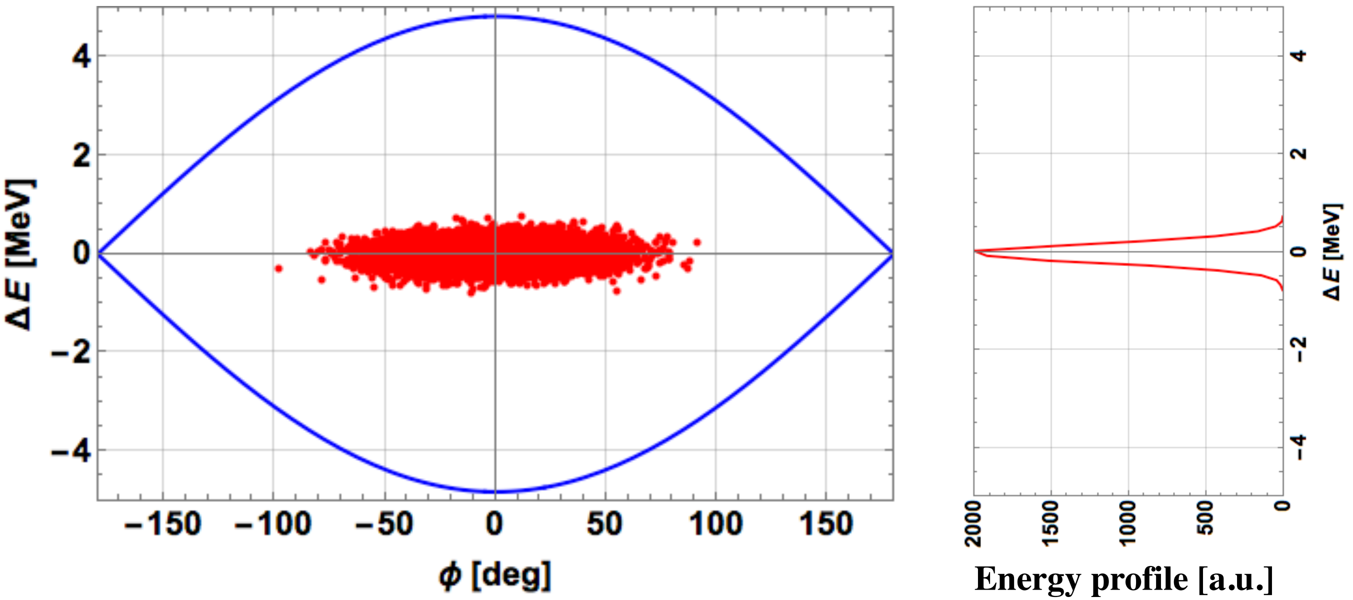
MISMATCHED BUNCH



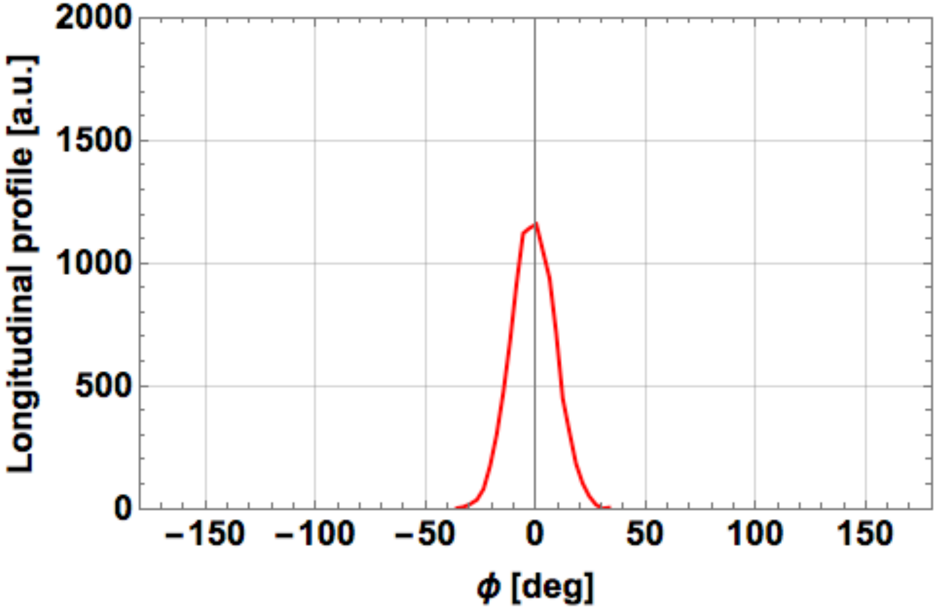
SOME “RF GYMNASTICS”

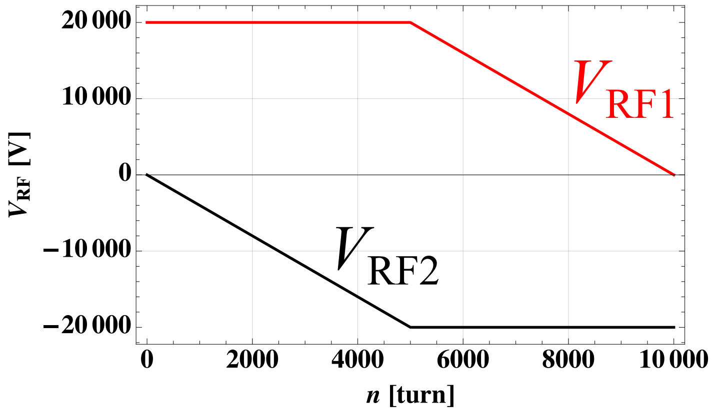
BUNCH ROTATION (to shorten bunches before extraction)

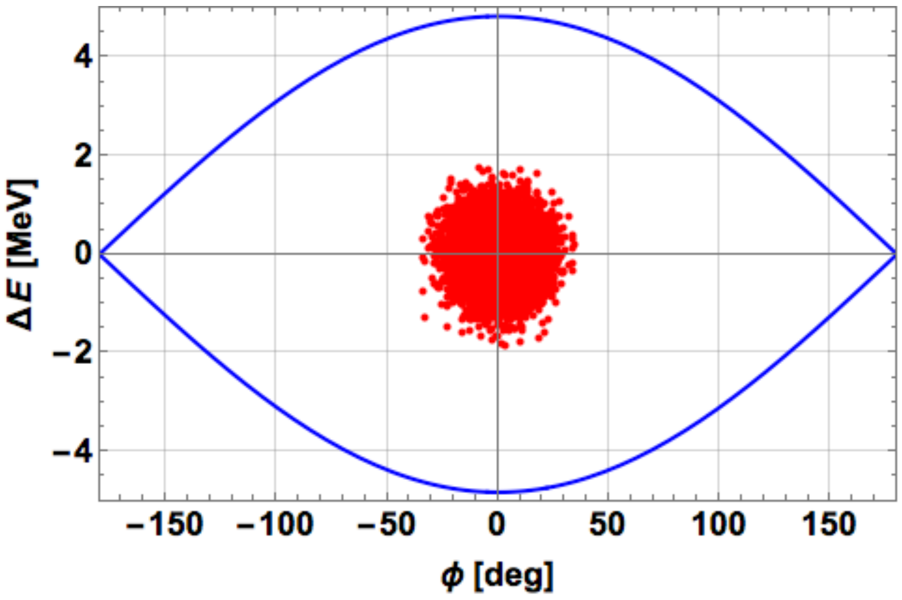


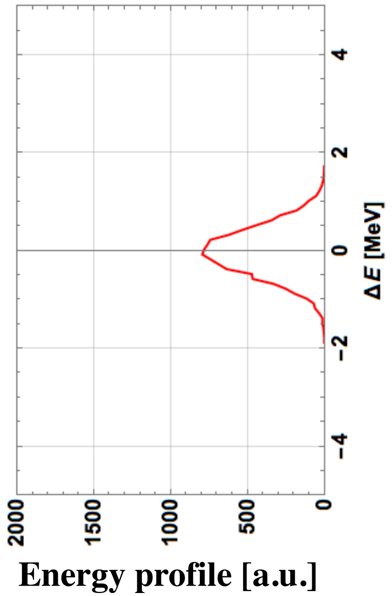


BUNCH (DOUBLE) SPLITTING

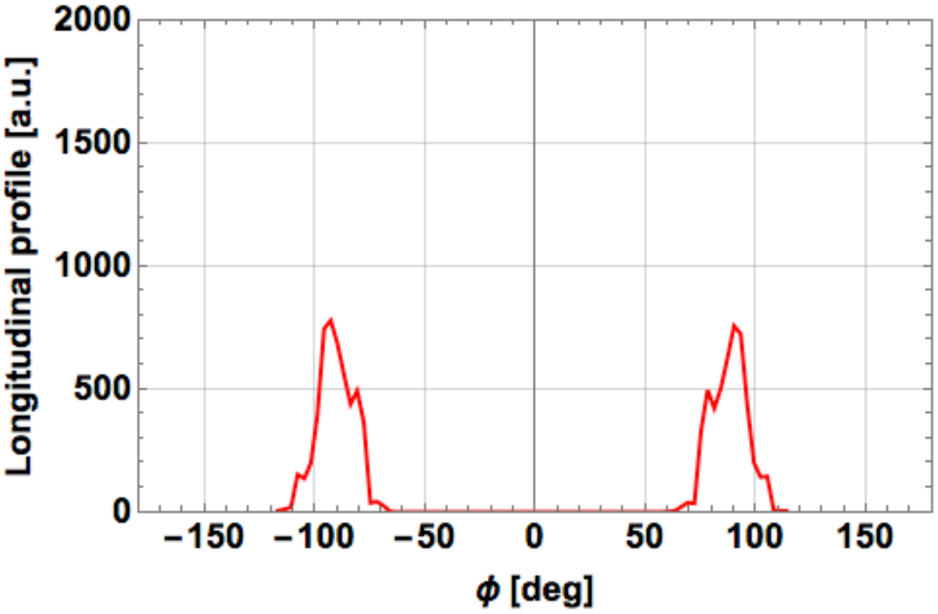


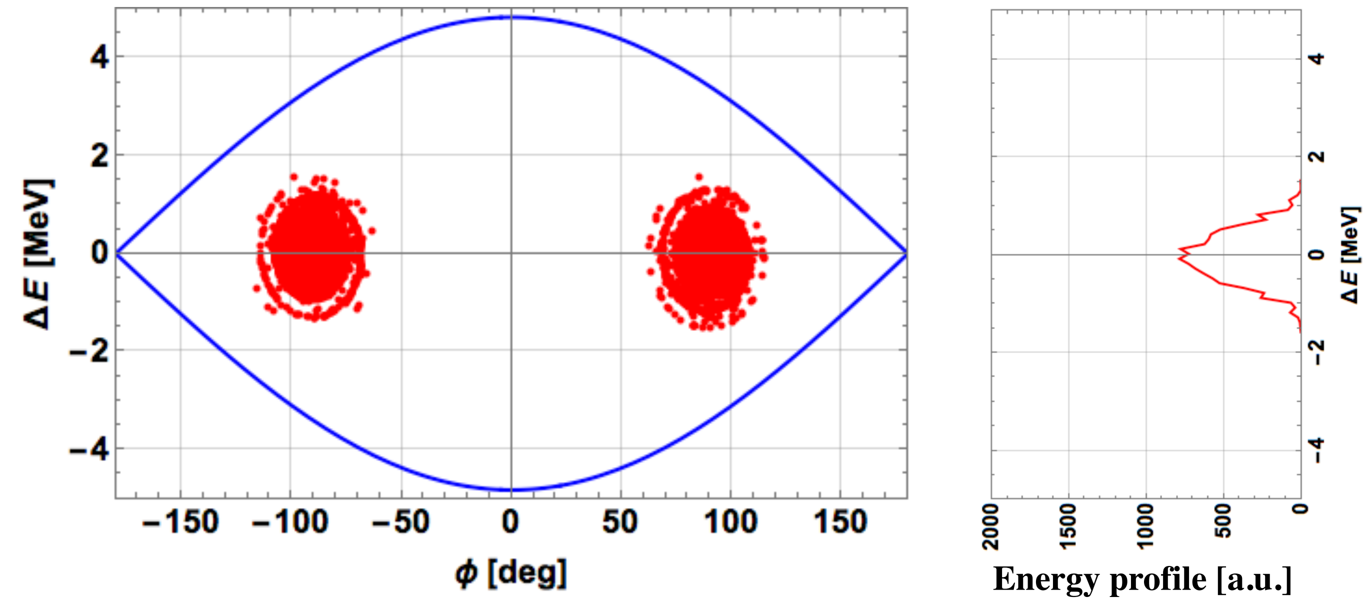




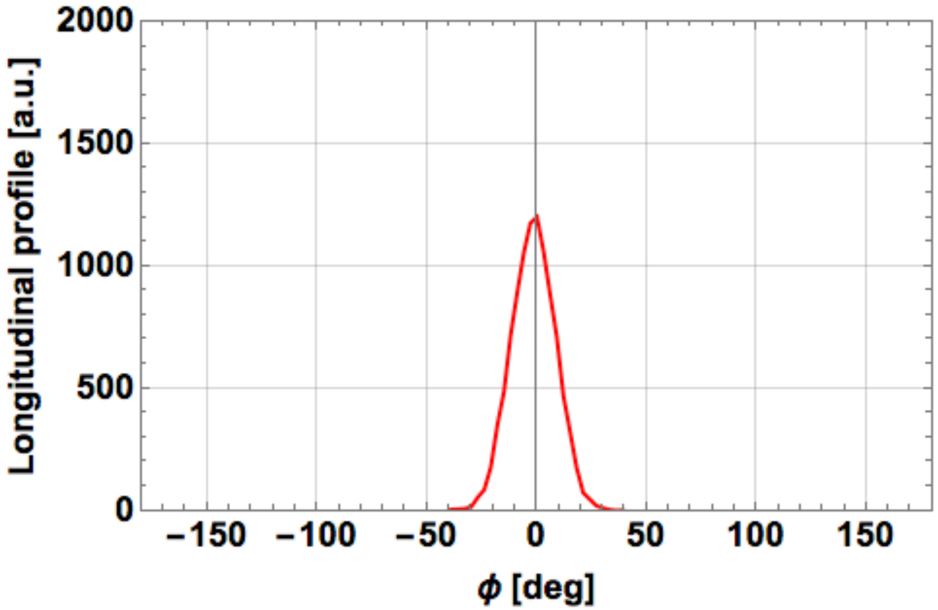


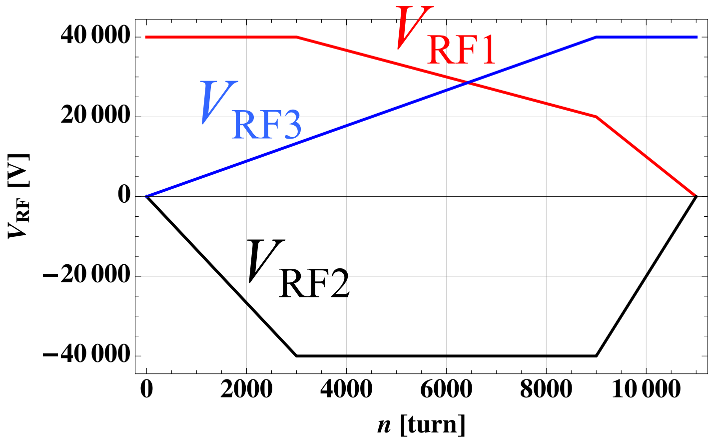
BUNCH MERGING
(reverse process)

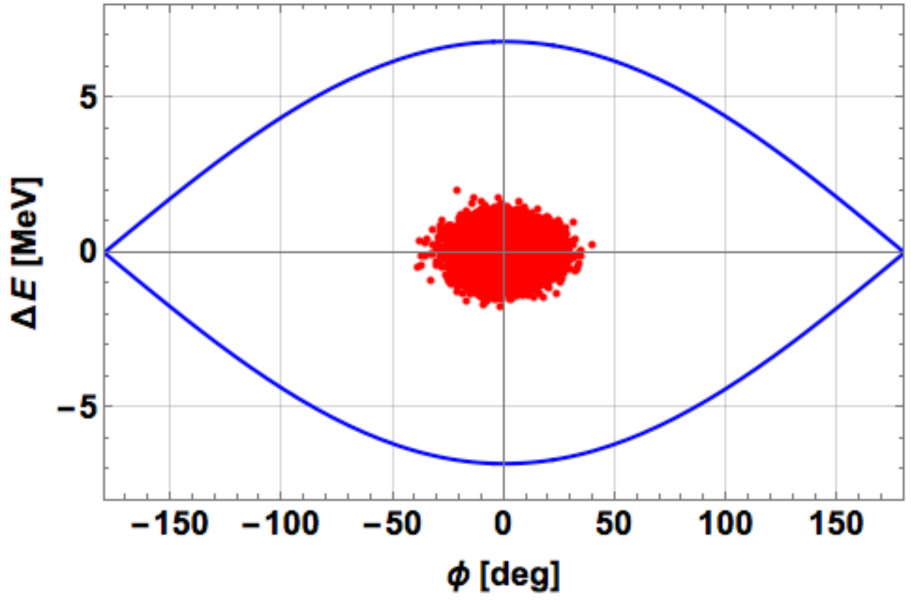


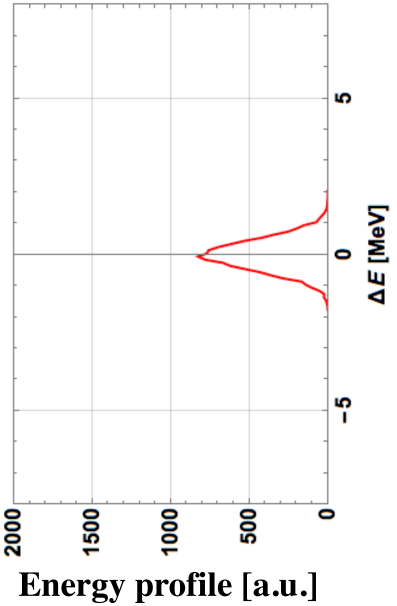


BUNCH TRIPLE SPLITTING









JUAS - TIMETABLE 2020 - WEEK 4

Schedule 2020	Monday Feb 3	Tuesday Feb 4	Wednesday Feb 5	Thursday Feb 6	Friday Feb 7
09:00	**Longitudinal Dynamics** *E. Métral/B. Salvant*	**Longitudinal Dynamics** *E. Métral/B. Salvant*	**Space charge** *M. Migliorati*	**Mini-workshop Accelerator Design** *Ph. Bryant*	**Longitudinal Dynamics** *E. Métral/B. Salvant*
10:00	**Coffee Break**	**Coffee Break**	**Coffee Break**	**Coffee Break**	**Coffee Break**
10:15	**Longitudinal Dynamics** *E. Métral/B. Salvant*	**Longitudinal Dynamics** *E. Métral/B. Salvant*	**Space charge** *M. Migliorati*	**Mini-workshop Accelerator Design** *Ph. Bryant*	**Longitudinal Dynamics** *E. Métral/B. Salvant*
11:15	**Longitudinal Dynamics** *E. Métral/B. Salvant*	**Longitudinal Dynamics** *E. Métral/B. Salvant*	**Space charge** *M. Migliorati*	**Mini-workshop Accelerator Design** *Ph. Bryant*	**Longitudinal Dynamics** *E. Métral/B. Salvant*
12:15	**WORKING LUNCH**	**BREAK**	**BREAK**	**BREAK**	**BREAK**
14:00	**Space charge** *M. Migliorati*	**Space charge** *M. Migliorati*	**Longitudinal Dynamics** *E. Métral/B. Salvant*	**Mini-workshop Accelerator Design** *Ph. Bryant*	**Presentation of Accelerator Design** *Students*
15:00	**Space charge** *M. Migliorati*	**Space charge** *M. Migliorati*	**Longitudinal Dynamics** *E. Métral/B. Salvant*	**Mini-workshop Accelerator Design** *Ph. Bryant*	**Presentation of Accelerator Design** *Students*
16:00	**Coffee Break**	**Coffee Break**	**Coffee Break**	**Coffee Break**	**Coffee Break**
16:15	**Space charge** *M. Migliorati*	**Space charge** *M. Migliorati*	**Longitudinal Dynamics** *E. Métral/B. Salvant*	**Mini-workshop Accelerator Design** *Ph. Bryant*	**Presentation of Accelerator Design** *Students*
17:15			**Novel High Gradient Particle Accelerators** Seminar *R. Assmann*	**Future High-Energy Linear Colliders** Seminar (incl. ESIPAP students) *L. Rinolfi*	
18:15			**AFTER WORK AT ESI**		

+ Examination on TH 13/02/2020 (09:00 to 10:30)

LESSON I

Fields & forces

Acceleration by time-varying electric field

Relativistic equations

LESSON II

Particle acceleration => Synchrotrons

Transit time factor

Main RF parameters

Momentum compaction

Transition energy

LESSON III

Equations related to synchrotrons

Synchronous particle

Synchrotron oscillations

Principle of phase stability

LESSON IV

RF acceleration for synchronous particle

RF acceleration for non-synchronous particle

Small amplitude oscillations

Large amplitude oscillations - the RF bucket

Synchrotron frequency and tune

Tracking

Nonadiabatic theory needed "close" to transition

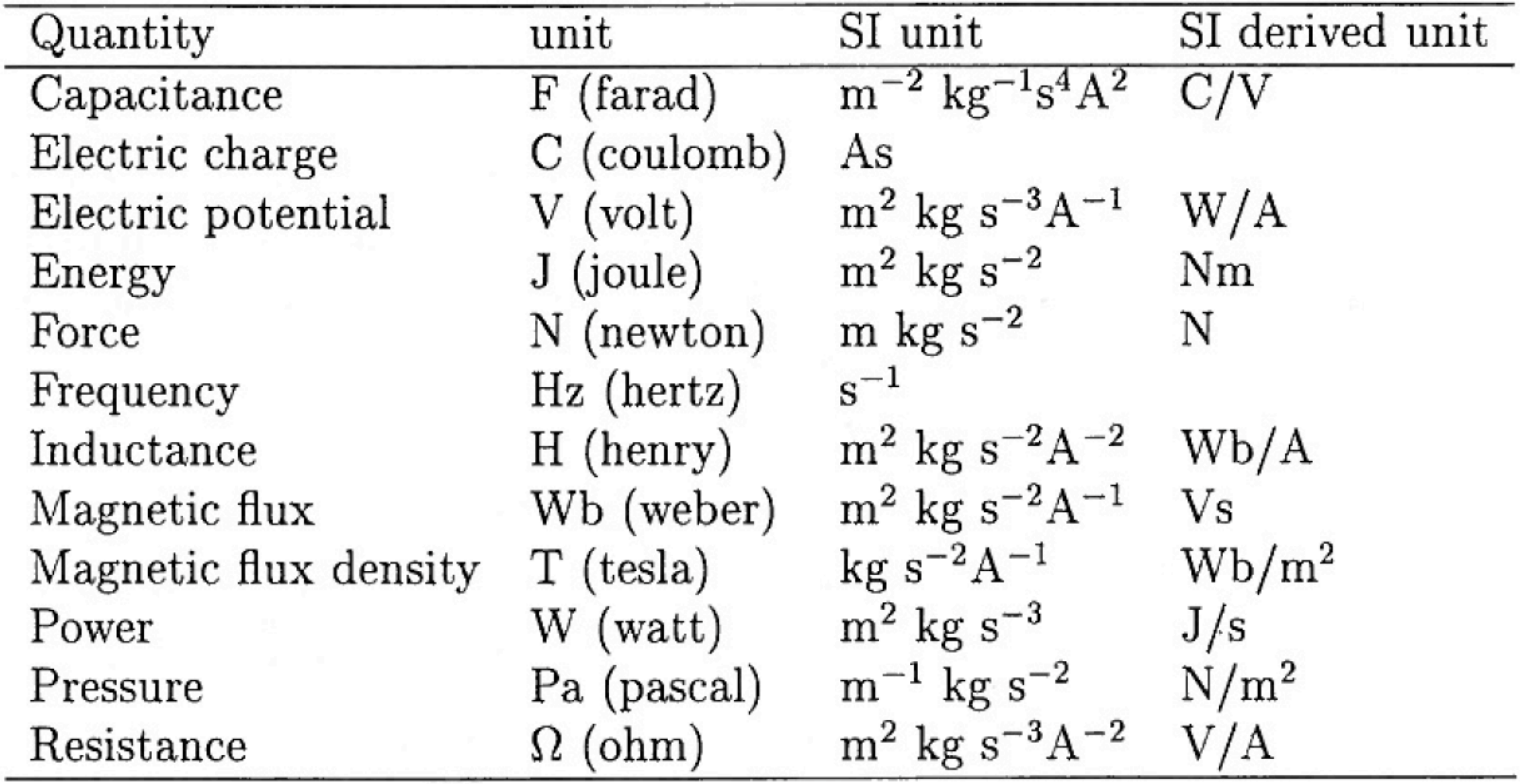
Double RF systems

LESSON V

Measurement of the longitudinal bunch profile and Tomography

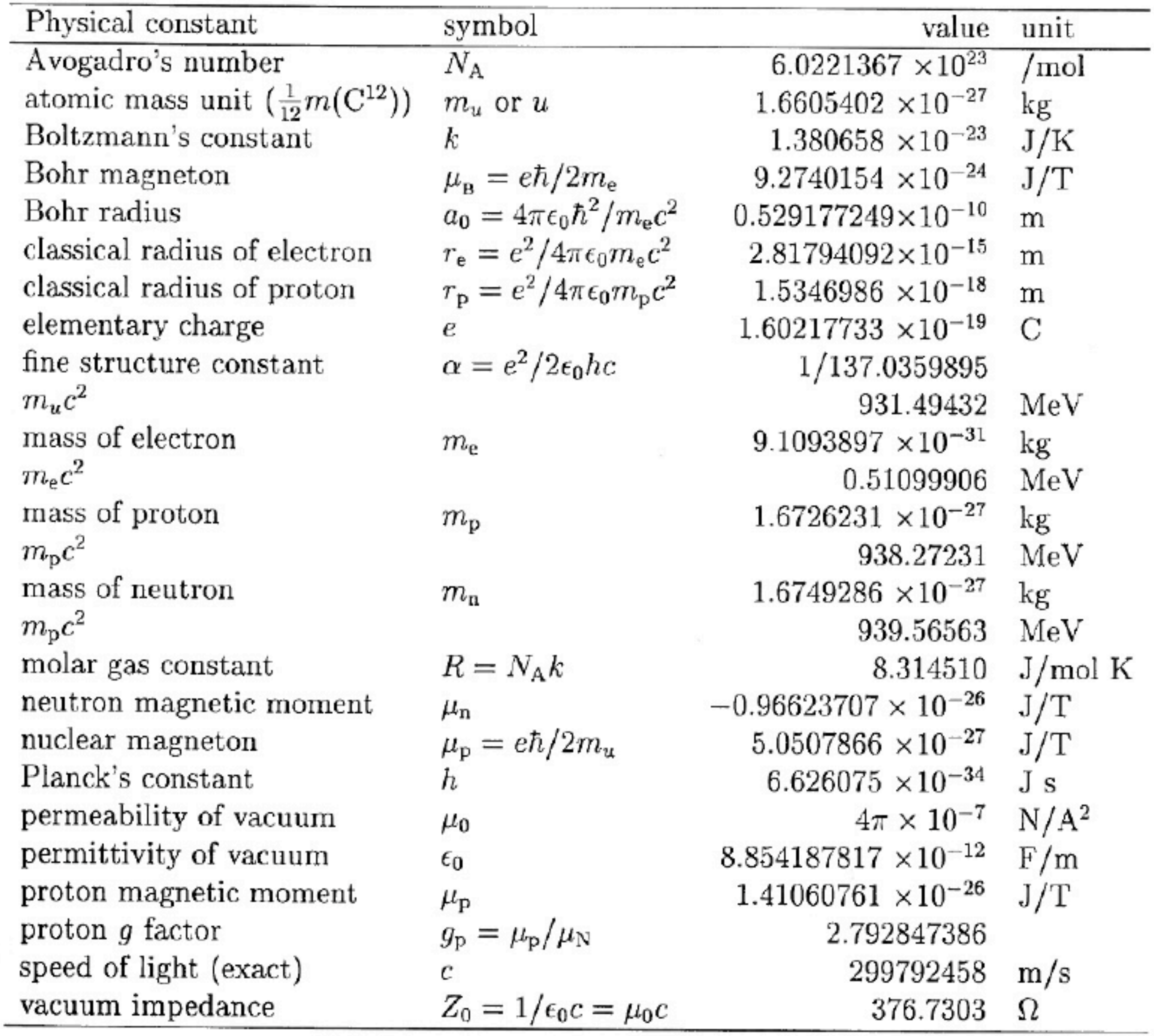
The pyHEADTAIL simulation code (by Benoit Salvant)

Units of physical quantities



Quantity	unit	SI unit	SI derived unit
Capacitance	F (farad)	$m^{-2}\ kg^{-1} s^4 A^2$	C/V
Electric charge	C (coulomb)	As	
Electric potential	V (volt)	$m^2\ kg\ s^{-3} A^{-1}$	W/A
Energy	J (joule)	$m^2\ kg\ s^{-2}$	Nm
Force	N (newton)	$m\ kg\ s^{-2}$	N
Frequency	Hz (hertz)	s^{-1}	
Inductance	H (henry)	$m^2\ kg\ s^{-2} A^{-2}$	Wb/A
Magnetic flux	Wb (weber)	$m^2\ kg\ s^{-2} A^{-1}$	Vs
Magnetic flux density	T (tesla)	$kg\ s^{-2} A^{-1}$	Wb/m^2
Power	W (watt)	$m^2\ kg\ s^{-3}$	J/s
Pressure	Pa (pascal)	$m^{-1}\ kg\ s^{-2}$	N/m^2
Resistance	Ω (ohm)	$m^2\ kg\ s^{-3} A^{-2}$	V/A

Fundamental physical constants



Physical constant	symbol	value	unit
Avogadro's number	N_A	6.0221367×10^{23}	/mol
atomic mass unit ($\frac{1}{12}m(C^{12})$)	m_u or u	$1.6605402 \times 10^{-27}$	kg
Boltzmann's constant	k	1.380658×10^{-23}	J/K
Bohr magneton	$\mu_B = e\hbar/2m_e$	$9.2740154 \times 10^{-24}$	J/T
Bohr radius	$a_0 = 4\pi\epsilon_0\hbar^2/m_e c^2$	$0.529177249 \times 10^{-10}$	m
classical radius of electron	$r_e = e^2/4\pi\epsilon_0 m_e c^2$	$2.81794092 \times 10^{-15}$	m
classical radius of proton	$r_p = e^2/4\pi\epsilon_0 m_p c^2$	$1.5346986 \times 10^{-18}$	m
elementary charge	e	$1.60217733 \times 10^{-19}$	C
fine structure constant	$\alpha = e^2/2\epsilon_0 hc$	$1/137.0359895$	
$m_u c^2$		931.49432	MeV
mass of electron	m_e	$9.1093897 \times 10^{-31}$	kg
$m_e c^2$		0.51099906	MeV
mass of proton	m_p	$1.6726231 \times 10^{-27}$	kg
$m_p c^2$		938.27231	MeV
mass of neutron	m_n	$1.6749286 \times 10^{-27}$	kg
$m_p c^2$		939.56563	MeV
molar gas constant	$R = N_A k$	8.314510	J/mol K
neutron magnetic moment	μ_n	$-0.96623707 \times 10^{-26}$	J/T
nuclear magneton	$\mu_p = e\hbar/2m_u$	$5.0507866 \times 10^{-27}$	J/T
Planck's constant	h	6.626075×10^{-34}	J s
permeability of vacuum	μ_0	$4\pi \times 10^{-7}$	N/A^2
permittivity of vacuum	ϵ_0	$8.854187817 \times 10^{-12}$	F/m
proton magnetic moment	μ_p	$1.41060761 \times 10^{-26}$	J/T
proton g factor	$g_p = \mu_p/\mu_N$	2.792847386	
speed of light (exact)	c	299792458	m/s
vacuum impedance	$Z_0 = 1/\epsilon_0 c = \mu_0 c$	376.7303	Ω

LESSON I

Fields & forces

Acceleration by time-varying electric field

Relativistic equations

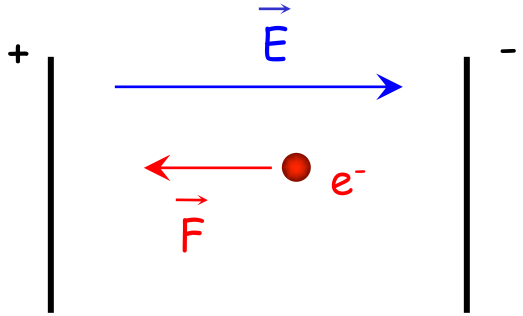
Fields and force

Equation of motion for a particle of charge q

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{dt}} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

$\vec{p} = m\,\vec{v}$	Momentum
$\vec{v}$	Velocity
$\vec{E}$	Electric field
$\vec{B}$	Magnetic field

Constant electric field

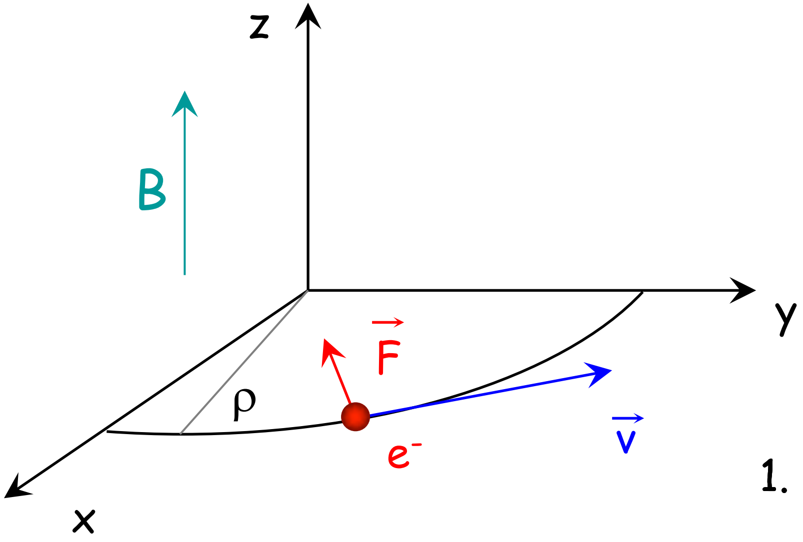


$$\frac{d\vec{p}}{dt} = -e\,\vec{E}$$

1. Direction of the force always parallel to the field
2. Trajectory can be modified, velocity also ⇒ momentum and energy can be modified

This force can be used to accelerate and decelerate particles

Constant magnetic field



$$\frac{d\vec{p}}{dt} = \vec{F} = -e\left(\vec{v} \times \vec{B}\right)$$

1. Direction always perpendicular to the velocity
2. Trajectory can be modified, but not the velocity

$$e\, v\, B = \frac{m\, v^2}{\rho}$$

This force cannot modify the energy

magnetic rigidity: $B\ \rho = \frac{p}{e}$

angular frequency: $\omega = 2\pi f = \frac{e}{m} B$

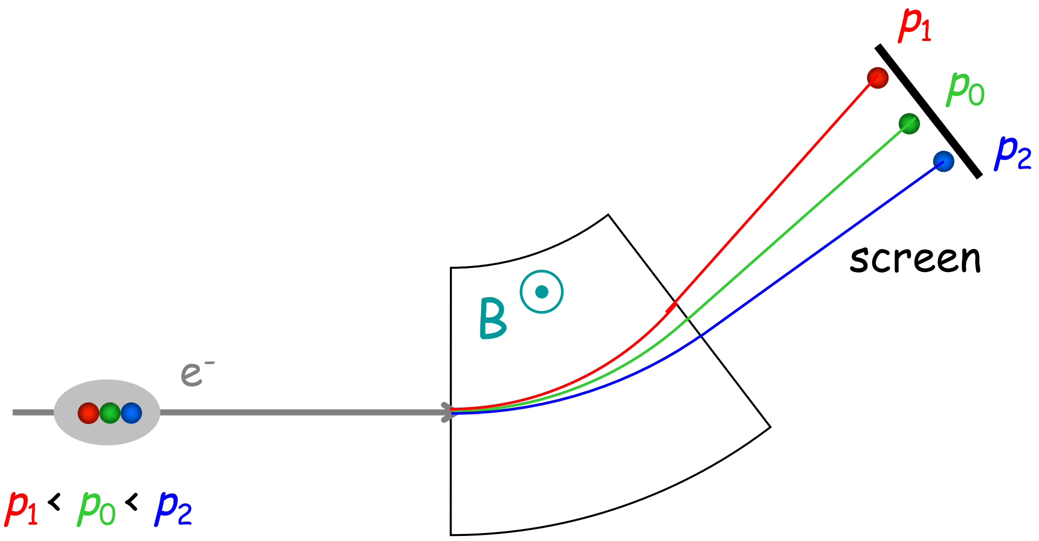
Important relationship:

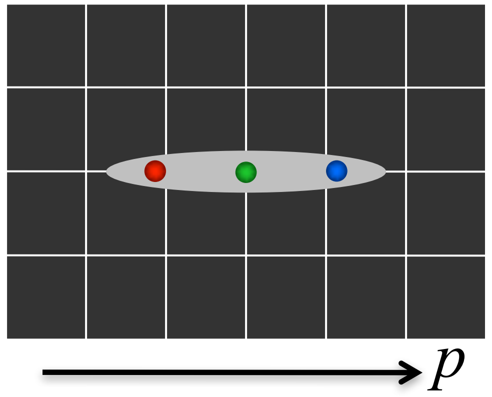
$$B\,\rho = \frac{p}{e} \quad \Rightarrow \quad \rho = \frac{p}{e\,B}$$

Practical units:

$$B\,\rho\,[\mathrm{Tm}] \approx \frac{p\,[\mathrm{GeV/c}]}{0.3}$$

Application: spectrometer





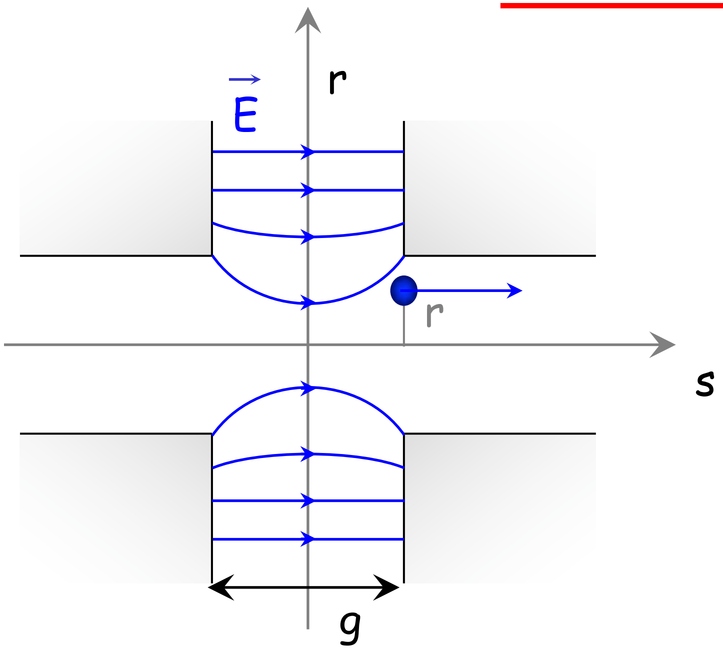
Comparison of magnetic and electric forces

$$|\vec{B}| = 1\ \mathrm{T}$$

$$|\vec{E}| = 10\ \mathrm{MV/m}$$

$$\frac{F_{MAGN}}{F_{ELEC}} = \frac{e\,v\,B}{e\,E} = \beta\,c\,\frac{B}{E} \cong 3 \cdot 10^{8}\,\frac{1}{10^{7}}\,\beta = 30\ \beta$$

Acceleration by time-varying electric field



- Let V_{RF} be the amplitude of the RF voltage across the gap g
- The particle crosses the gap at a distance r
- The energy gain is:

$$\boxed{\Delta E = e \int_{-g/2}^{g/2} \vec{E}(s,r,t)\, d\vec{s}}$$

[MeV]

[n]
(1 for electrons or protons)

[MV/m]

In the cavity gap, the electric field is supposed to be:

$$E(s,r,t) = E_1(s,r) \cdot E_2(t)$$

In general, $E_2(t)$ is a sinusoidal time variation with angular frequency ω_{RF}

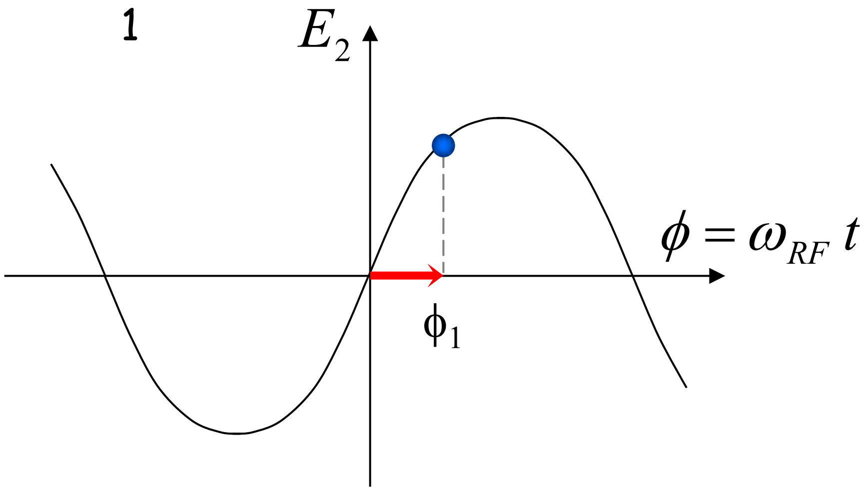
$$E_2(t) = E_\circ \sin\Phi(t) \qquad \text{where} \qquad \Phi(t) = \int_{t_0}^{t} \omega_{RF} dt + \Phi_0$$

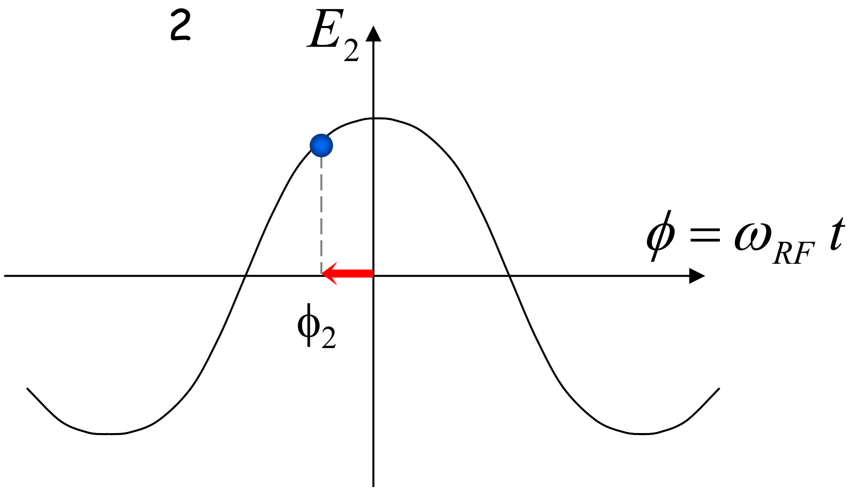
Convention

1. For circular accelerators, the origin of time is taken at the zero crossing of the RF voltage with positive slope

2. For linear accelerators, the origin of time is taken at the positive crest of the RF voltage

Time t= 0 chosen such that:





$$E_2(t) = E_\circ \sin(\omega_{RF}\, t)$$

$$E_2(t) = E_\circ \cos(\omega_{RF}\, t)$$

Relativistic Equations

$$E = m c^2$$

normalized velocity

$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}}$$

energy

$$E = E_{kin} + E_0$$

total

kinetic

rest

$$\frac{\text{total energy}}{\text{rest energy}}$$

$$\gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}}$$

momentum

$$p = mv = \beta \frac{E}{c} = \beta \gamma m_0 c$$

energy	momentum	mass
eV	eV/c	eV/c^2

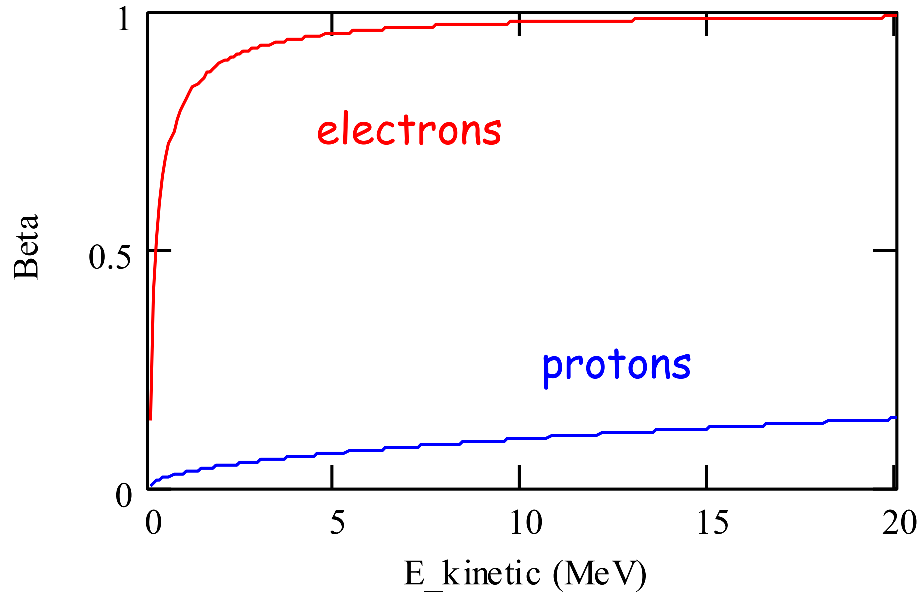
$$p^2 c^2 = E^2 - E_0{}^2$$

$$\gamma = 1 + \frac{E_{kin}}{E_0}$$

$$p\left[\mathrm{GeV/c}\right] \cong 0.3 \ \ B\left[\mathrm{T}\right] \ \ \rho\left[\mathrm{m}\right]$$

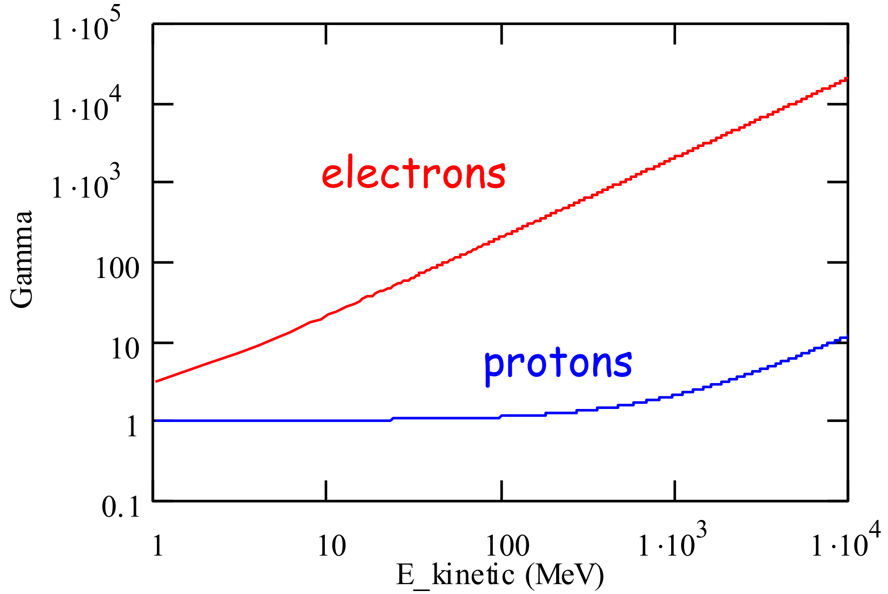
normalized velocity

$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}}$$



$$\frac{\text{total energy}}{\text{rest energy}}$$

$$\gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}}$$



First derivatives

$$\mathrm{d}\beta = \beta^{-1}\,\gamma^{-3}\,\mathrm{d}\gamma$$

$$\mathrm{d}(cp) = E_0\,\gamma^3\,\mathrm{d}\beta$$

$$\mathrm{d}\gamma = \beta\left(1-\beta^2\right)^{-3/2}\,\mathrm{d}\beta$$

Logarithmic derivatives

$$\frac{\mathrm{d}\beta}{\beta} = (\beta\,\gamma)^{-2}\,\frac{\mathrm{d}\gamma}{\gamma}$$

$$\frac{\mathrm{d}p}{p} = \frac{\gamma^2}{\gamma^2-1}\,\frac{\mathrm{d}E}{E} = \frac{\gamma}{\gamma+1}\,\frac{\mathrm{d}E_{kin}}{E_{kin}}$$

$$\frac{\mathrm{d}\gamma}{\gamma} = \left(\gamma^2-1\right)\frac{\mathrm{d}\beta}{\beta}$$

LESSON II

Particle acceleration => Synchrotrons

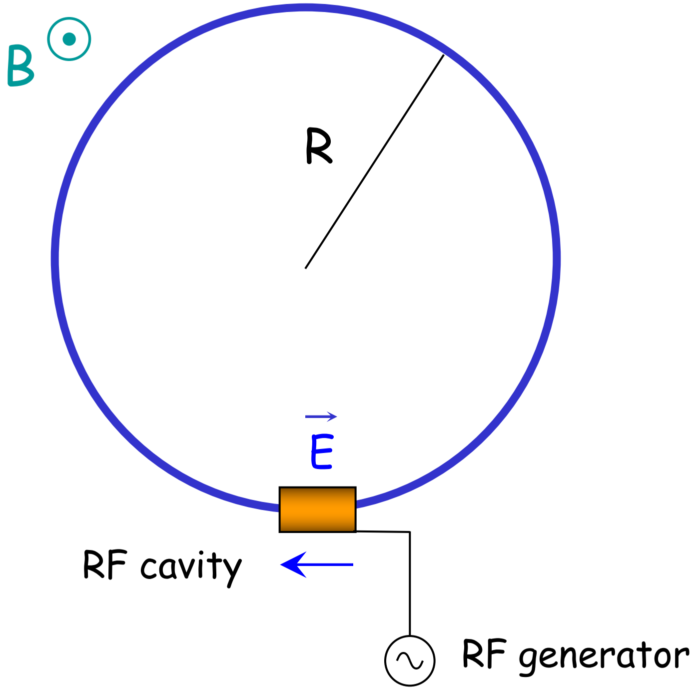
Transit time factor

Main RF parameters

Momentum compaction

Transition energy

Synchrotron



Synchronism condition

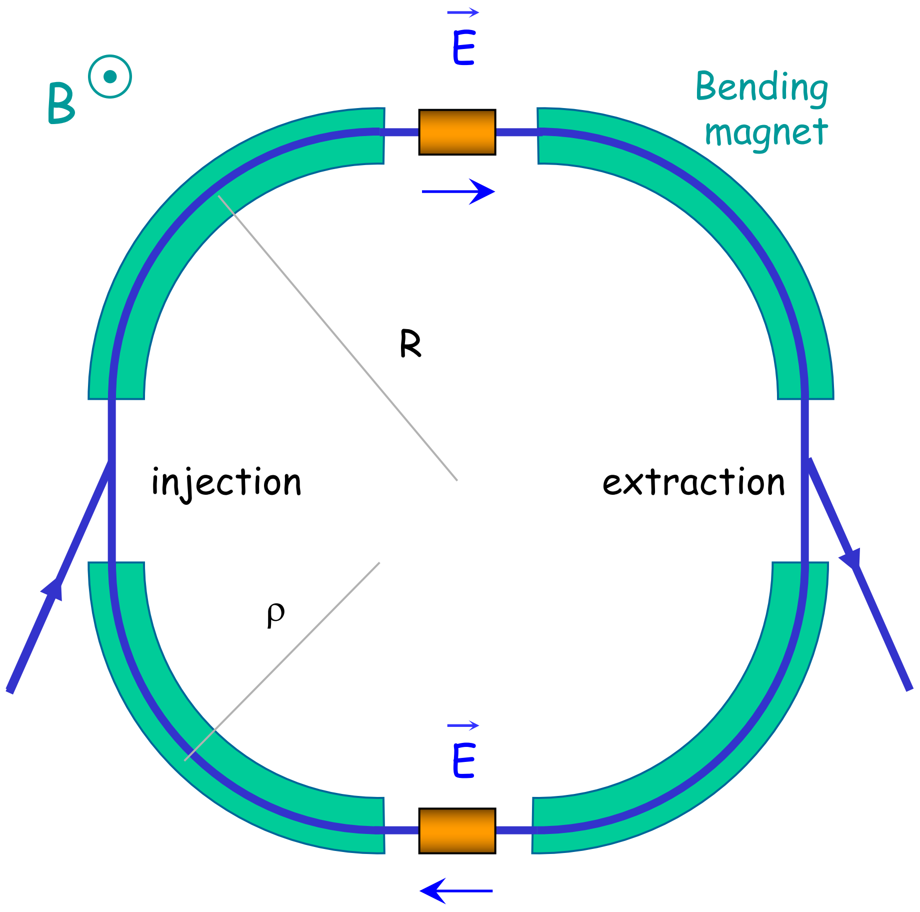
$$T_s = h\, T_{RF}$$

$$\frac{2\pi R}{v_s} = h\, T_{RF}$$

h integer,
harmonic number

1. ω_{RF} and ω increase with energy
2. To keep particles on the closed orbit, B should increase with time

Synchrotron



- In reality, the orbit in a synchrotron is not a circle, straight sections are added for RF cavities, injection and extraction, etc..
- Usually the beam is pre-accelerated in a linac (or a smaller synchrotron) before injection
- The bending radius ρ does not coincide to the machine radius $R = L/2\pi$

LEAR (CERN)
Low Energy Antiproton Ring

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EPA (CERN)
Electron Positron Accumulator

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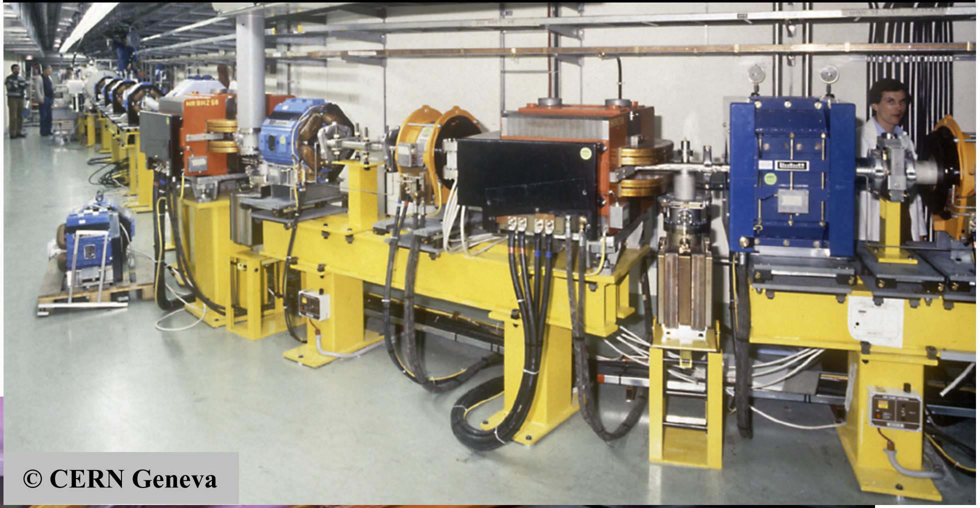

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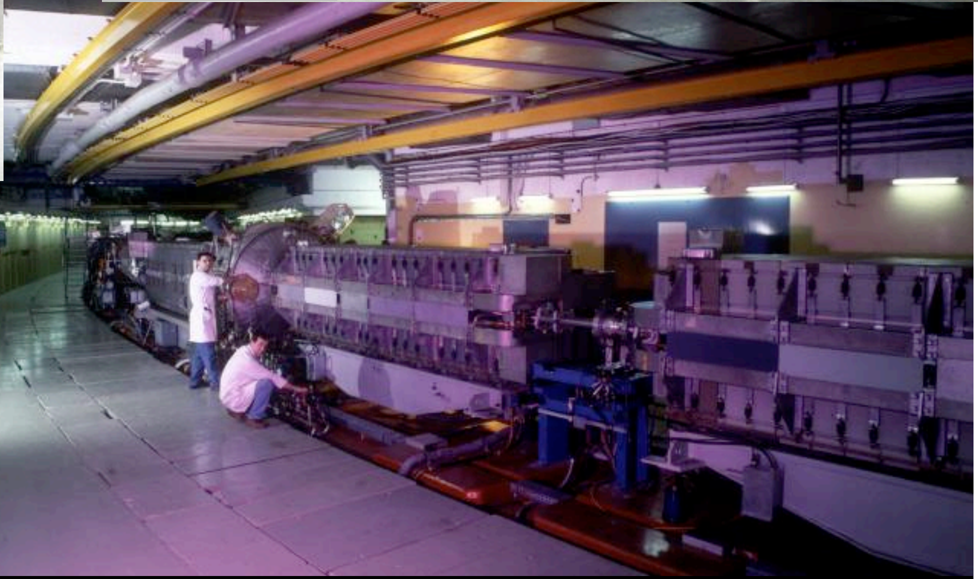
Examples of different proton and electron synchrotrons at CERN



PS (CERN)
Proton Synchrotron

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Parameters for circular accelerators

The basic principles, for the common circular accelerators, are based on the two relations:

1. The Lorentz equation: the orbit radius can be espressed as:

$$R = \frac{\gamma \, v \, m_0}{eB}$$

2. The synchronicity condition: The revolution frequency can be expressed as:

$$f = \frac{e\,B}{2\pi \, \gamma \, m_0}$$

According to the parameter we want to keep constant or let vary, one has different acceleration principles.
They are summarized in the table below:

Machine	**Energy (γ)**	**Velocity**	**Field**	**Orbit**	**Frequency**
Cyclotron	~ 1	var.	const.	~ v	const.
Synchrocyclotron	var.	var.	B(r)	~ p	B(r)/γ(t)
Proton/Ion synchrotron	var.	var.	~ p	R	~ v
Electron synchrotron	var.	const.	~ p	R	const.

Transit time factor

RF acceleration in a gap g

$$E(s,r,t) = E_1(s,r) \cdot E_2(t)$$

Simplified model ➡

$$E_1(s,r) = \frac{V_{RF}}{g} = \text{const.}$$

$$E_2(t) = \sin(\omega_{RF}\, t + \phi_0)$$

At t = 0, s = 0 and v ≠ 0, parallel to the electric field

Energy gain:

$$\Delta E = e \int_{-g/2}^{g/2} E(s,r,t)\, \mathrm{d}s \Rightarrow \Delta E = e\, V_{RF}\, T_a \sin \phi_0$$

where

$$T_a = \frac{\sin \dfrac{\omega_{RF}\, g}{2\, v}}{\dfrac{\omega_{RF}\, g}{2\, v}}$$

T_a is called transit time factor

- $T_a < 1$
- $T_a \rightarrow 1$ if $g \rightarrow 0$

Transit time factor II

In the general case, the transit time factor is given by:

$$T_a = \frac{\int_{-\infty}^{+\infty} E_1(s,r)\,\cos\left(\omega_{RF}\,\frac{s}{v}\right)\mathrm{d}s}{\int_{-\infty}^{+\infty} E_1(s,r)\,\mathrm{d}s}$$

It is the ratio of the peak energy gained by a particle with velocity *v* to the peak energy gained by a particle with infinite velocity.

Main RF parameters

I. Voltage, phase, frequency

In order to accelerate particles, longitudinal fields must be generated in the direction of the desired acceleration

$$E(s,t) = E_1(s) \cdot E_2(t) \qquad E_2(t) = E_0 \sin\left[\int_{t_0}^{t} \omega_{RF} \, \mathrm{d}t + \phi_0\right]$$

$$\omega_{RF} = 2\pi \, f_{RF} \qquad \Delta E = e \, V_{RF} \, T_a \sin \phi_0$$

Such electric fields are generated in RF cavities characterized by the voltage amplitude, the frequency and the phase

II. Harmonic number

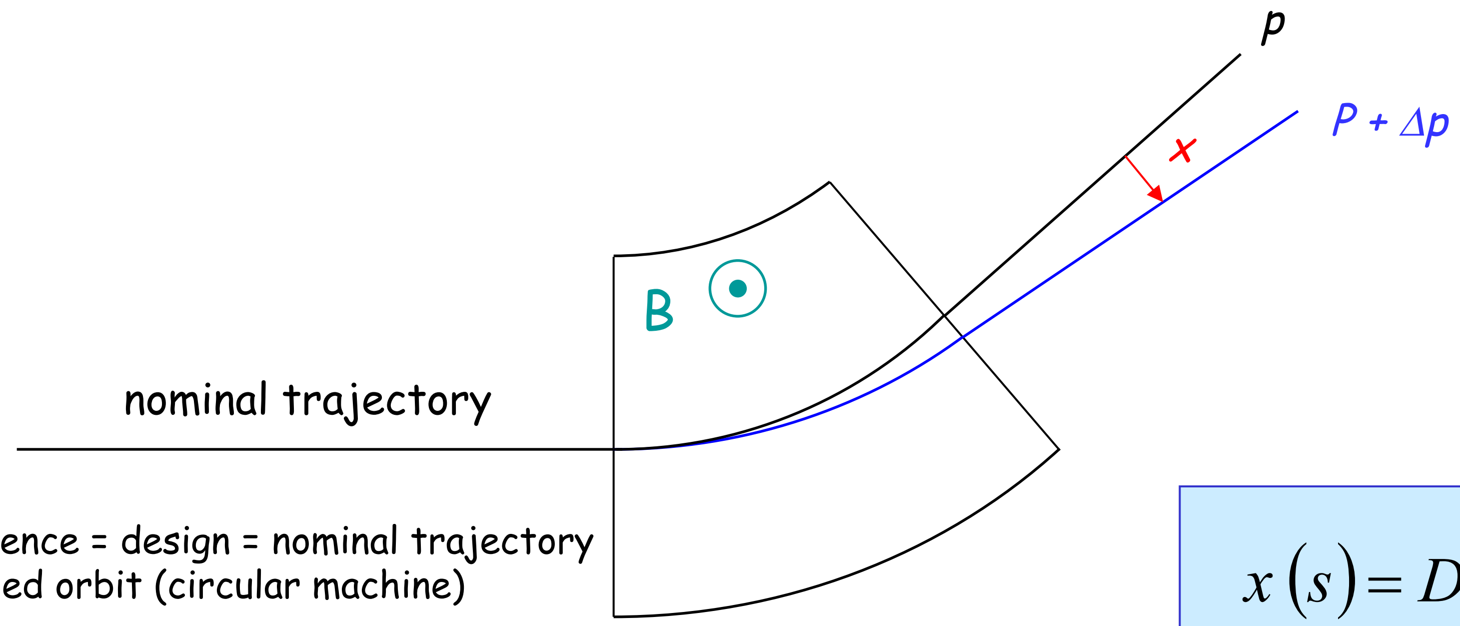
$$T_{rev} = h \, T_{RF} \quad \Rightarrow \quad f_{RF} = h \, f_{rev}$$

f_{rev} = revolution frequency
f_{RF} = frequency of the RF
h = harmonic number

harmonic number in different machines:

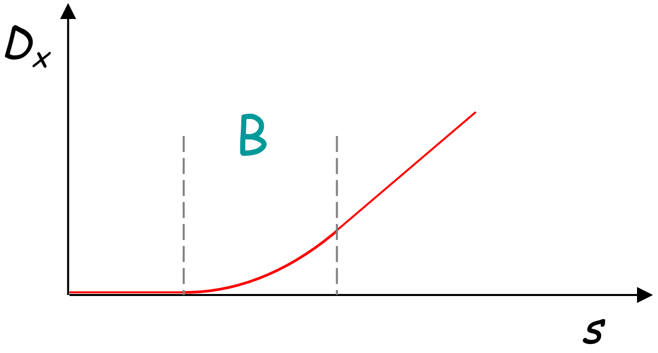
AA	EPA	PS	SPS
1	8	20	4620

Dispersion



reference = design = nominal trajectory
= closed orbit (circular machine)

$$x(s) = D_x(s)\ \frac{\Delta p}{p}$$



Momentum compaction factor in a transport system

In a particle transport system, a nominal trajectory is defined for the nominal momentum *p*.

For a particle with a momentum $p + \Delta p$ the trajectory length can be different from the length *L* of the nominal trajectory.

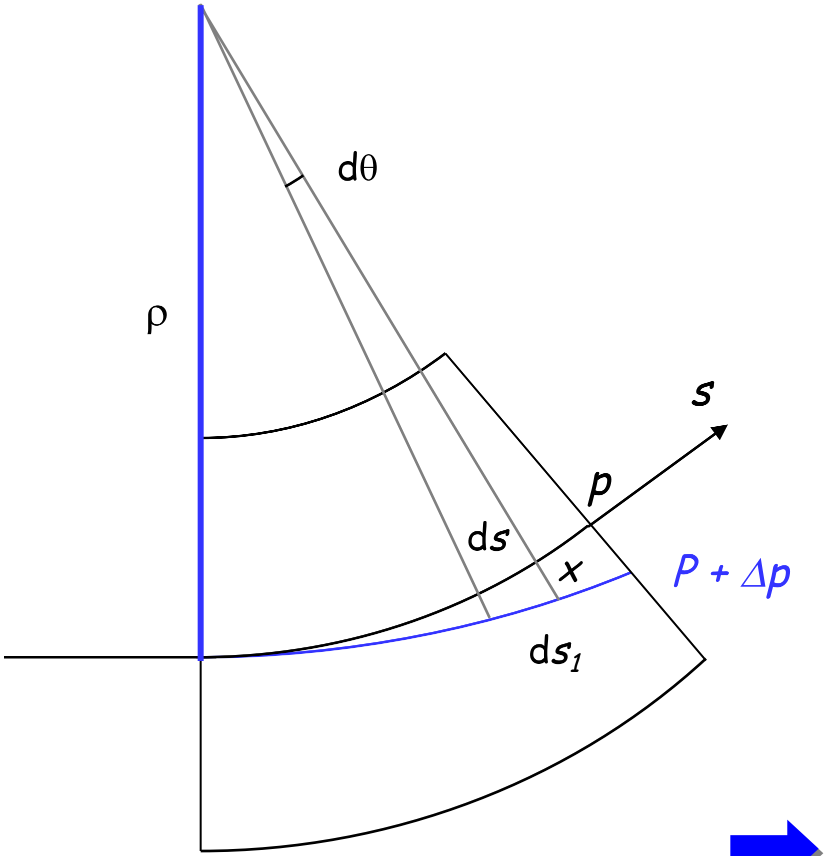
The momentum compaction factor is defined by the ratio:

$$\alpha_p = \frac{dL/L}{dp/p}$$

Therefore, for small momentum deviation, to first order it is:

$$\frac{\Delta L}{L} = \alpha_p \frac{\Delta p}{p}$$

Example: constant magnetic field



$$ds = \rho \, \mathrm{d}\theta$$

$$ds_1 = \left(\rho + x\right) \mathrm{d}\theta$$

$$\frac{ds_1 - ds}{ds} = \frac{\left(\rho + x\right)\mathrm{d}\theta - \rho \, \mathrm{d}\theta}{\rho \, \mathrm{d}\theta} = \frac{x}{\rho} \underset{\uparrow}{=} \frac{D_x}{\rho} \frac{\mathrm{d}p}{p}$$

By definition of dispersion D_x

$$\Rightarrow \boxed{\alpha_p = \frac{1}{L} \int_0^L \frac{D_x(s)}{\rho(s)} \mathrm{d}s}$$

To first order, only the bending magnets contribute to a change of the trajectory length (r = ∞ in the straight sections)

Momentum compaction in a ring

In a circular accelerator, a nominal closed orbit is defined for the nominal momentum p.

For a particle with a momentum deviation Δp produces an orbit length variation ΔC with:

For B = const.

$$\frac{\Delta C}{C} = \alpha_p \frac{\Delta p}{p}$$

$$C = 2\pi R$$

circumference

(average) radius of the closed orbit

The momentum compaction factor is defined by the ratio:

$$\alpha_p = \frac{dC/C}{dp/p} = \frac{dR/R}{dp/p}$$

and

$$\alpha_p = \frac{1}{C}\int_C \frac{D_x(s)}{\rho(s)}\,\mathrm{d}s$$

N.B.: in most circular machines, α_p is positive ⇒ higher momentum means longer circumference

Momentum compaction as a function of energy

$$E = \frac{p\,c}{\beta} \quad \Rightarrow \quad \frac{\mathrm{d}E}{E} = \beta^2 \frac{dp}{p}$$

$$\boxed{\alpha_p = \beta^2 \frac{E}{R} \frac{\mathrm{d}R}{\mathrm{d}E}}$$

Momentum compaction as a function of magnetic field

Definition of average magnetic field

$$< B > = \frac{1}{2\pi R} \int_C B_f \; \mathrm{d}s = \frac{1}{2\pi R} \left(\underbrace{\int_{straights} B_f \; \mathrm{d}s}_{= 0} + \underbrace{\int_{magnets} B_f \; \mathrm{d}s}_{2\pi \rho B_f} \right)$$

$$< B > = \frac{B_f \, \rho}{R} \Rightarrow \frac{\mathrm{d} < B >}{< B >} = \frac{\mathrm{d} B_f}{B_f} + \frac{\mathrm{d}\rho}{\rho} - \frac{\mathrm{d} R}{R}$$

$$B_f \; \rho = \frac{p}{e}$$

$$\Downarrow$$

$$< B > R = \frac{p}{e} \Rightarrow \frac{\mathrm{d} < B >}{< B >} + \frac{\mathrm{d} R}{R} = \frac{\mathrm{d} p}{p}$$

For B_f = const.

$$\boxed{\alpha_p = 1 - \frac{\mathrm{d} < B >}{< B >} \Big/ \frac{\mathrm{d} p}{p}}$$

Transition energy

Proton (ion) circular machine with α_p positive

1. Momentum larger than the nominal ($p + \Delta p$) ⇒ longer orbit ($C+\Delta C$)
2. Momentum larger than the nominal ($p + \Delta p$) ⇒ higher velocity ($v + \Delta v$)

What happens to the revolution frequency $f = v/C$?

- At low energy, v increases faster than C with momentum
- At high energy $v \cong c$ and remains almost constant

➡ There is an energy for which the velocity variation is compensated by the trajectory variation ⇒ transition energy

Below transition:	higher energy ⇒	higher revolution frequency
Above transition:	higher energy ⇒	lower revolution frequency

Transition energy – quantitative approach

We define a parameter η (revolution frequency spread per unit of momentum spread), called slip or slippage factor:

$$\boxed{\eta = \frac{{\mathrm{d}f}/{f}}{{\mathrm{d}p}/{p}} = \frac{{\mathrm{d}\omega}/{\omega}}{{\mathrm{d}p}/{p}}}$$

$$f = \frac{v}{C} \quad \Rightarrow \quad \frac{\mathrm{d}f}{f} = \frac{\mathrm{d}\beta}{\beta} - \frac{\mathrm{d}C}{C}$$

from $p = \frac{m_0\, c\, \beta}{\sqrt{1-\beta^2}}$ ⇒ $\frac{\mathrm{d}\beta}{\beta} = \frac{1}{\gamma^2}\frac{\mathrm{d}p}{p}$

definition of momentum compaction factor: $\frac{\mathrm{d}C}{C} = \alpha_p \frac{\mathrm{d}p}{p}$

$$\boxed{\frac{\mathrm{d}f}{f} = \left(\frac{1}{\gamma^2} - \alpha_p\right)\frac{\mathrm{d}p}{p}}$$

Transition energy – quantitative approach

$$\eta = \frac{1}{\gamma^2} - \alpha_p$$

The transition energy is the energy that corresponds to $\eta = 0$
(α_p is fixed, and γ variable)

$$\gamma_{tr} = \sqrt{\frac{1}{\alpha_p}}$$

The parameter η can also be written as

$$\eta = \frac{1}{\gamma^2} - \frac{1}{\gamma_{tr}^2}$$

- At low energy $\eta > 0$
- At high energy $\eta < 0$

N.B.: for electrons, $\gamma >> \gamma_{tr} \Rightarrow \eta < 0$
for linacs $\alpha_p = 0 \Rightarrow \eta > 0$

LESSON III

Equations related to synchrotrons

Synchronous particle

Synchrotron oscillations

Principle of phase stability

Equations related to synchrotrons

$$\frac{\mathrm{d}p}{p} = \gamma_{tr}{}^{2}\,\frac{\mathrm{d}R}{R} + \frac{\mathrm{d}B}{B}$$

$$\frac{\mathrm{d}p}{p} = \gamma^{2}\,\frac{\mathrm{d}f}{f} + \gamma^{2}\,\frac{\mathrm{d}R}{R}$$

$$\frac{\mathrm{d}B}{B} = \gamma_{tr}{}^{2}\,\frac{\mathrm{d}f}{f} + \left[1 - \left(\frac{\gamma_{tr}}{\gamma}\right)^{2}\right]\frac{\mathrm{d}p}{p}$$

$$\frac{\mathrm{d}B}{B} = \gamma^{2}\,\frac{\mathrm{d}f}{f} + \left(\gamma^{2} - \gamma_{tr}{}^{2}\right)\frac{\mathrm{d}R}{R}$$

$p\,[\mathrm{MeV/c}]$	momentum
$R\,[\mathrm{m}]$	orbit radius
$B\,[\mathrm{T}]$	magnetic field
$f\,[\mathrm{Hz}]$	rev. frequency
γ_{tr}	transition energy

I - Constant radius

$$\boxed{\mathrm{d}R = 0}$$

Beam maintained on the same orbit when energy varies

$$\frac{\mathrm{d}p}{p} = \frac{\mathrm{d}B}{B}$$

$$\frac{\mathrm{d}p}{p} = \gamma^2 \frac{\mathrm{d}f}{f}$$

If *p* increases

➡ *B* increases

f increases

II - Constant energy

$$\boxed{\mathrm{d}p = 0}$$

$$V_{RF} = 0$$

Beam debunches

$$\frac{\mathrm{d}p}{p} = 0 = \gamma_{tr}{}^{2} \frac{\mathrm{d}R}{R} + \frac{\mathrm{d}B}{B}$$

$$\frac{\mathrm{d}p}{p} = 0 = \gamma^{2} \frac{\mathrm{d}f}{f} + \gamma^{2} \frac{\mathrm{d}R}{R}$$

If *B* increases

→ *R* decreases

f increases

III - Magnetic flat-top

$$\boxed{\mathrm{d}B = 0}$$

Beam bunched with constant magnetic field

$$\frac{\mathrm{d}p}{p} = \gamma_{tr}{}^{2} \frac{\mathrm{d}R}{R}$$

$$\frac{\mathrm{d}B}{B} = 0 = \gamma_{tr}{}^{2} \frac{\mathrm{d}f}{f} + \left[1 - \left(\frac{\gamma_{tr}}{\gamma}\right)^{2}\right] \frac{\mathrm{d}p}{p}$$

$$\frac{\mathrm{d}B}{B} = 0 = \gamma^{2} \frac{\mathrm{d}f}{f} + \left(\gamma^{2} - \gamma_{tr}^{2}\right) \frac{\mathrm{d}R}{R}$$

If p increases

⇒ R increases

f increase $\gamma < \gamma_{tr}$

decreases $\gamma > \gamma_{tr}$

IV - Constant frequency

$$\mathrm{d}f = 0$$

Beam driven by an external oscillator

$$\frac{\mathrm{d}p}{p} = \gamma^2 \frac{\mathrm{d}R}{R}$$

$$\frac{\mathrm{d}B}{B} = \left[1 - \left(\frac{\gamma_{tr}}{\gamma}\right)^2\right] \frac{\mathrm{d}p}{p}$$

$$\frac{\mathrm{d}B}{B} = \left(\gamma^2 - \gamma_{tr}^2\right) \frac{\mathrm{d}R}{R}$$

If p increases

➡ R increases

B decreases $\gamma < \gamma_{tr}$

increase $\gamma > \gamma_{tr}$

Four conditions - resume

Beam	**Parameter**	**Variations**
Debunched	$\Delta p = 0$	$B \Uparrow , R \Downarrow , f \Uparrow$
Fixed orbit	$\Delta R = 0$	$B \Uparrow , p \Uparrow , f \Uparrow$
Magnetic flat-top	$\Delta B = 0$	$p \Uparrow , R \Uparrow , f \Uparrow \ (\eta > 0)$ $f \Downarrow \ (\eta < 0)$
External oscillator	$\Delta f = 0$	$B \Uparrow , p \Downarrow , R \Downarrow \ (\eta > 0)$ $p \Uparrow , R \Uparrow \ (\eta < 0)$

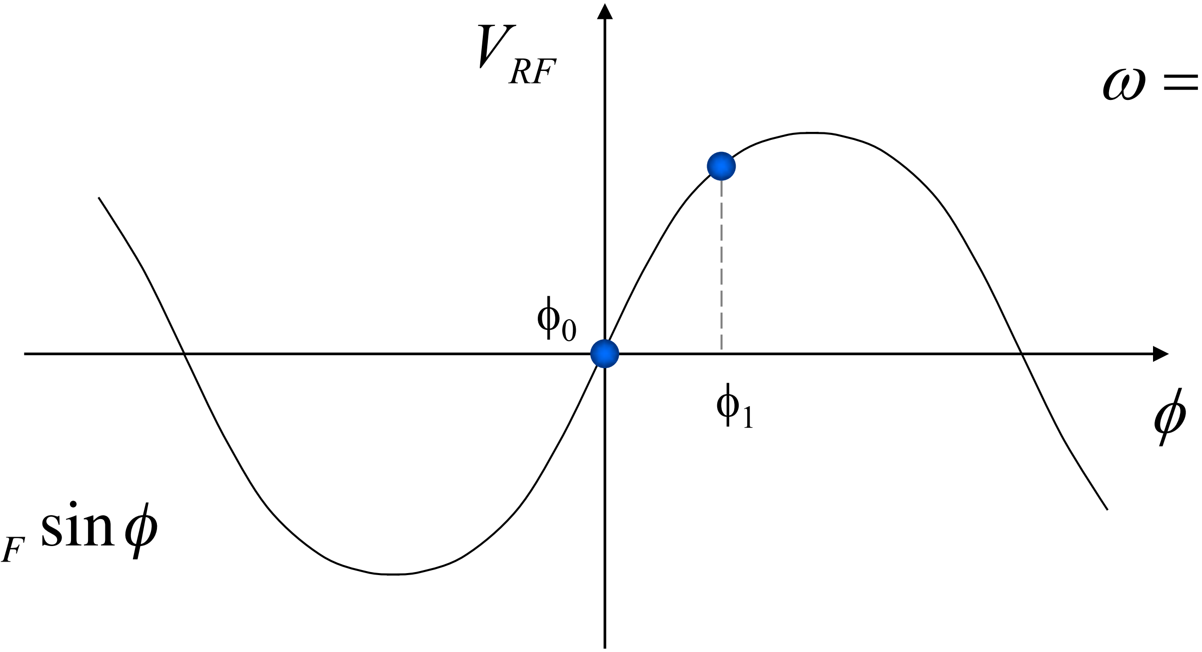
p	momentum
R	orbit radius
B	magnetic field
f	frequency

Synchronous particle

Simple case (no accel.): B = const. $\gamma < \gamma_{tr}$

Synchronous particle: particle that sees always the same phase (at each turn) in the RF cavity

$$\omega = \frac{e\,B}{\gamma\, m_0} = \frac{\omega_{RF}}{h}$$



$$\Delta E = e\,\hat{V}_{RF}\,\sin\phi$$

In order to keep the resonant condition, the particle must keep a constant energy

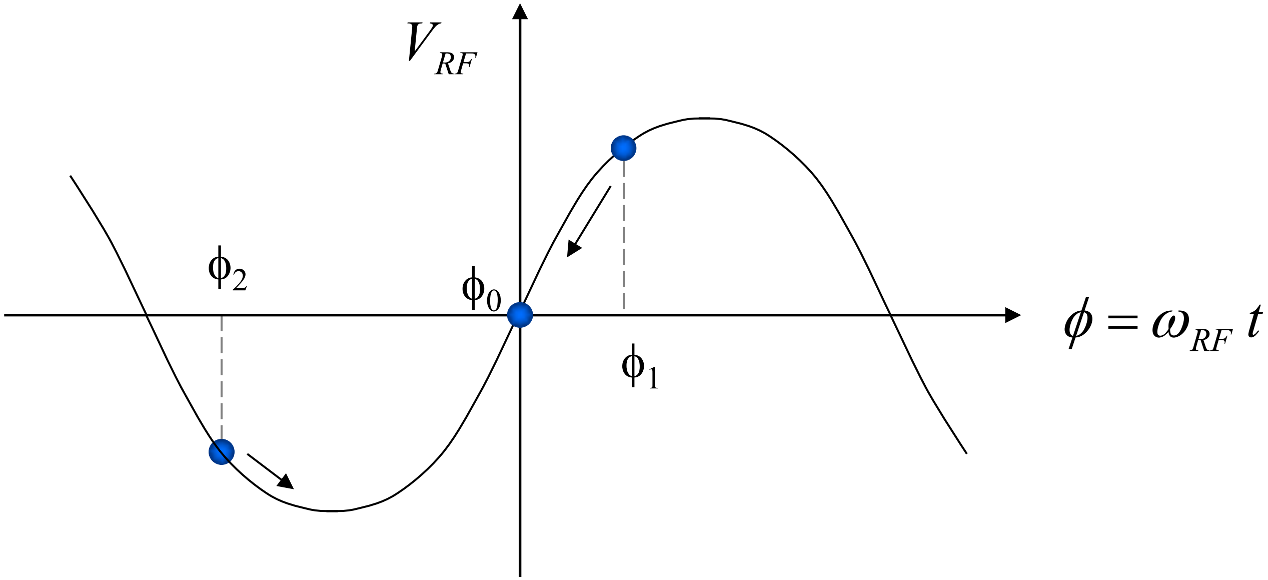
The phase of the synchronous particle must therefore be $\phi_0 = 0$ (circular machines convention)

Let's see what happens for a particle with the same energy and a different phase (e.g., ϕ_1)

Synchrotron oscillations

ϕ_1

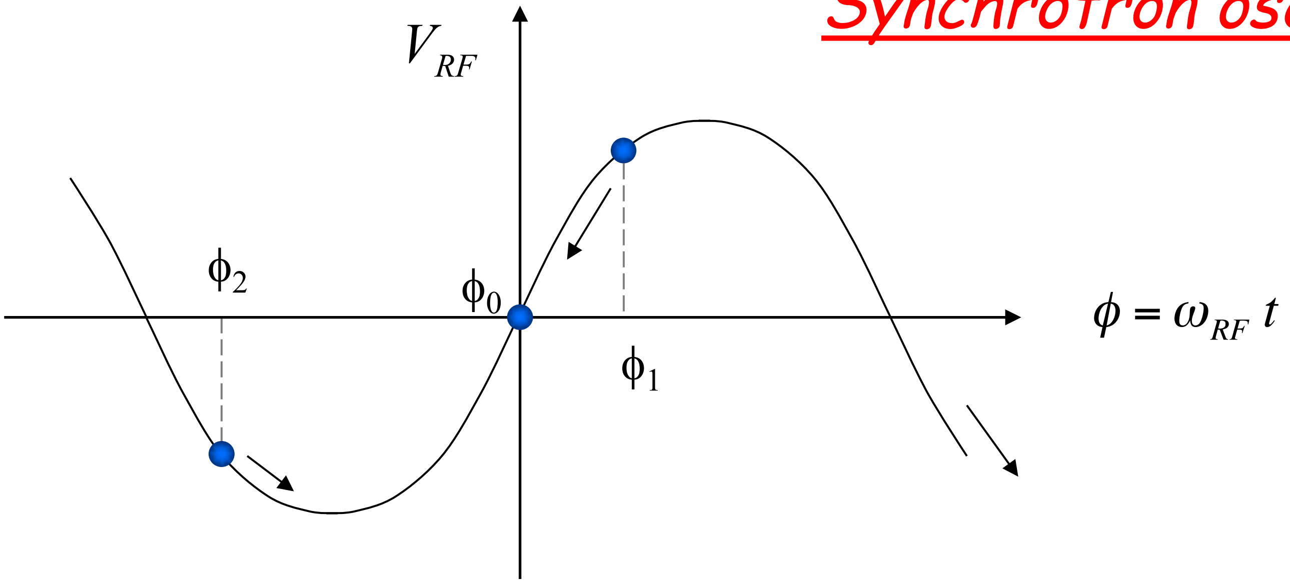
- The particle is accelerated
- Below transition, an increase in energy means an increase in revolution frequency
- The particle arrives earlier – tends toward ϕ_0



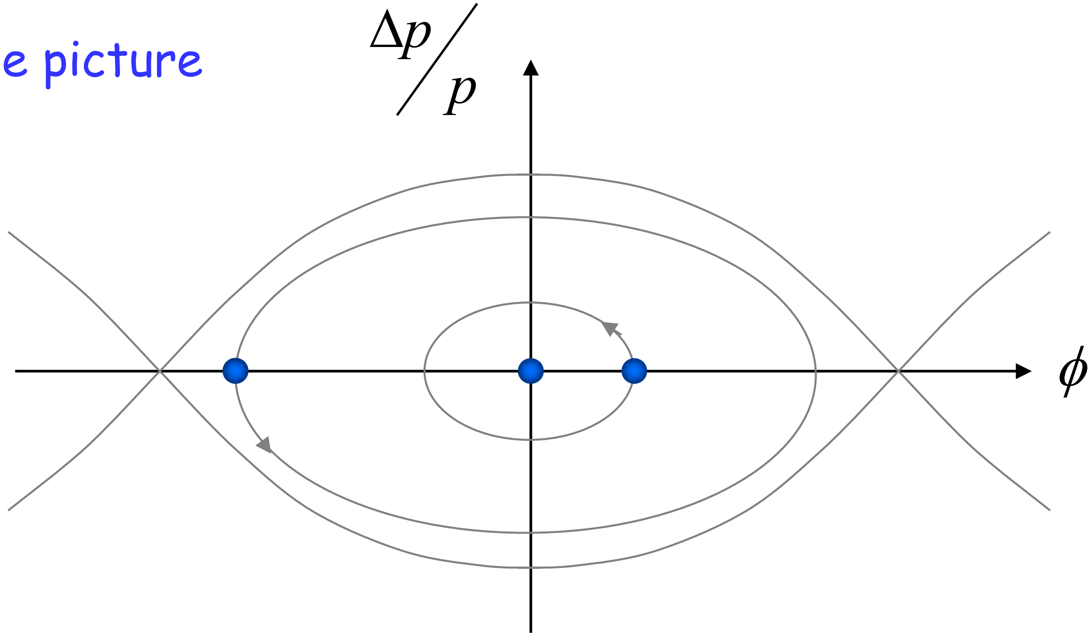
ϕ_2

- The particle is decelerated
- decrease in energy - decrease in revolution frequency
- The particle arrives later – tends toward ϕ_0

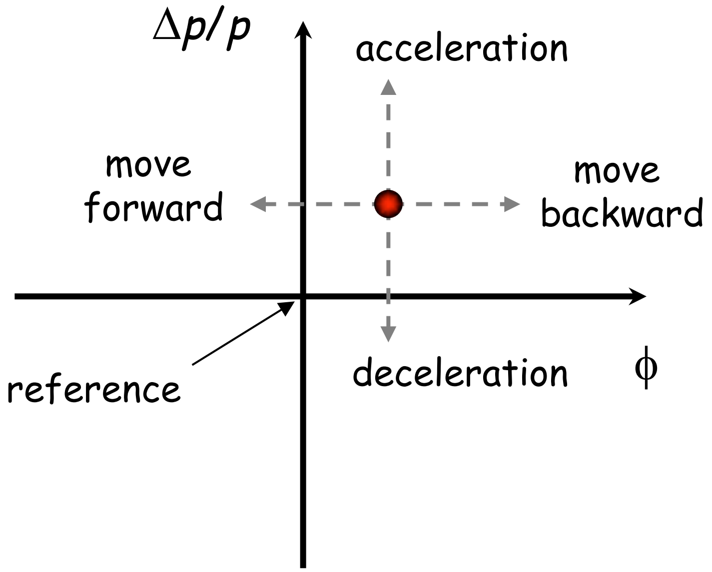
Synchrotron oscillations

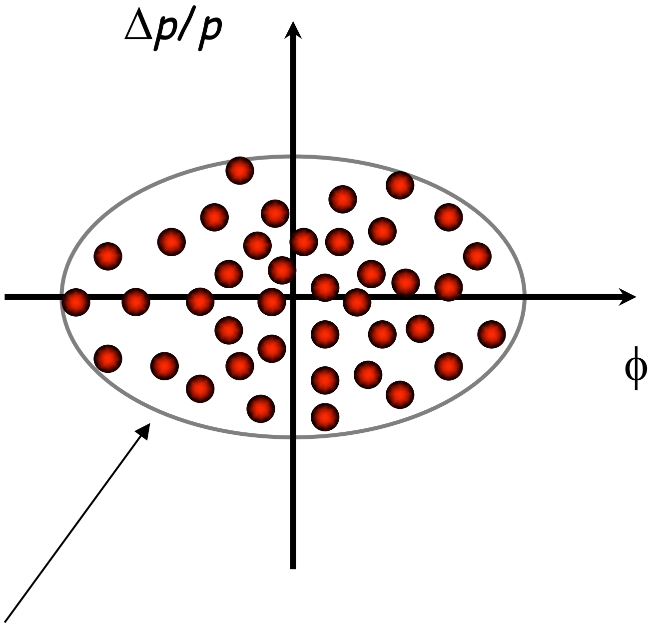


Phase space picture



Longitudinal phase space





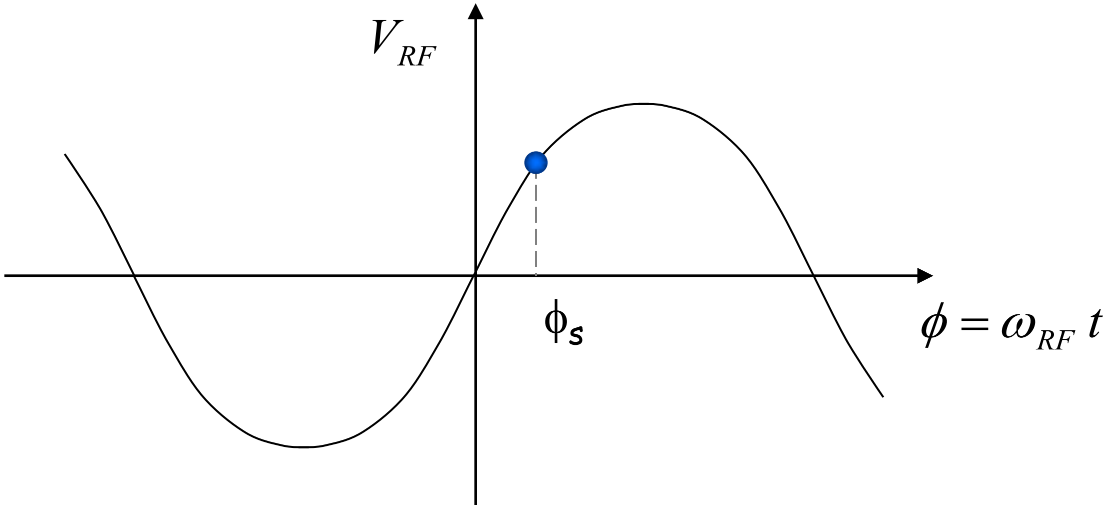
The particle trajectory in the phase space (ϕ, $\Delta p/p$,) describes its longitudinal motion

Emittance: phase space area including all the particles

NB: if the emittance contour correspond to a possible orbit in phase space, its shape does not change with time (matched beam)

Synchronous particle

Case with acceleration B increasing $\gamma < \gamma_{tr}$



$$\Delta E = e \hat{V}_{RF} \sin\phi$$

The phase of the synchronous particle is now $\phi_s > 0$ (circular machines convention)

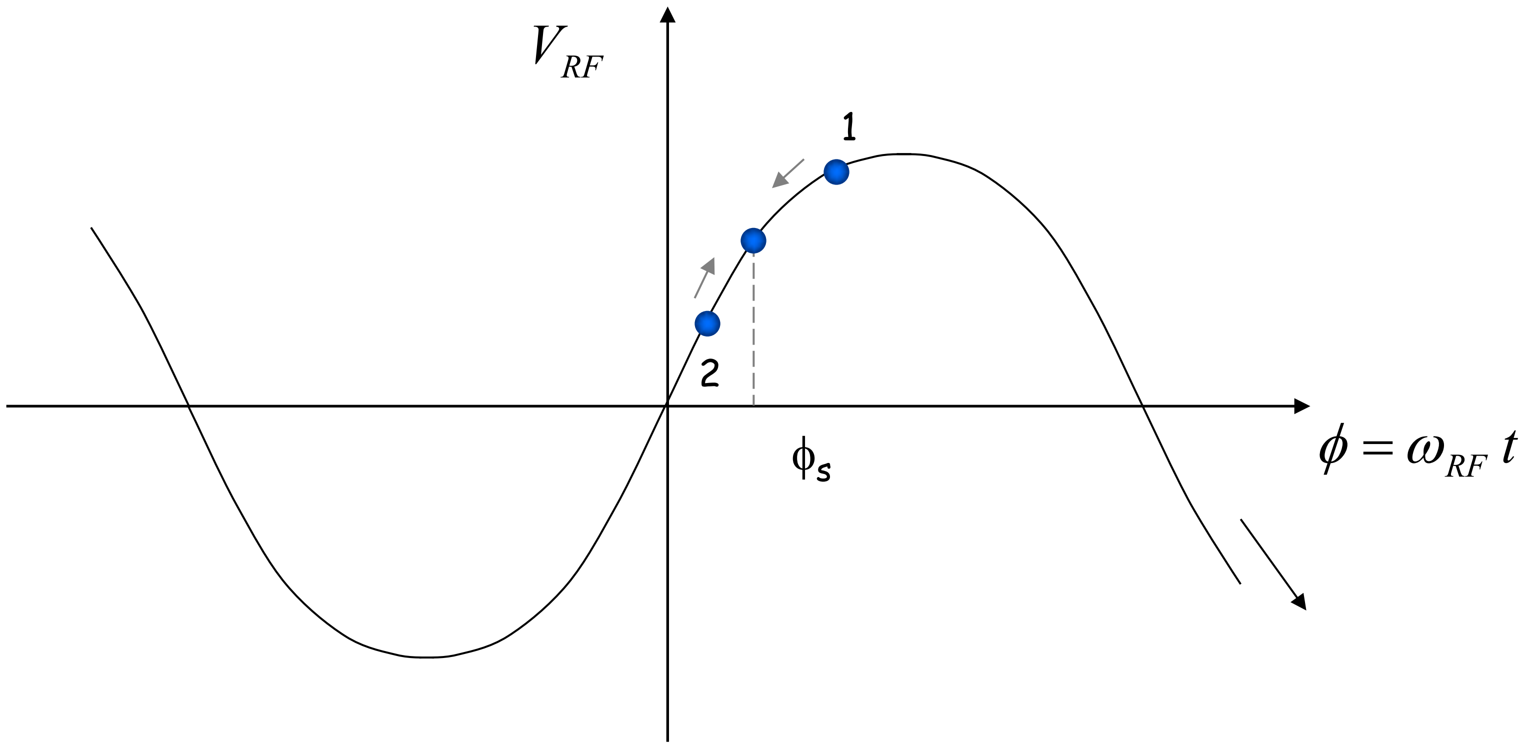
The synchronous particle accelerates, and the magnetic field is increased accordingly to keep the constant radius R

$$R = \frac{\gamma\, v\, m_0}{eB}$$

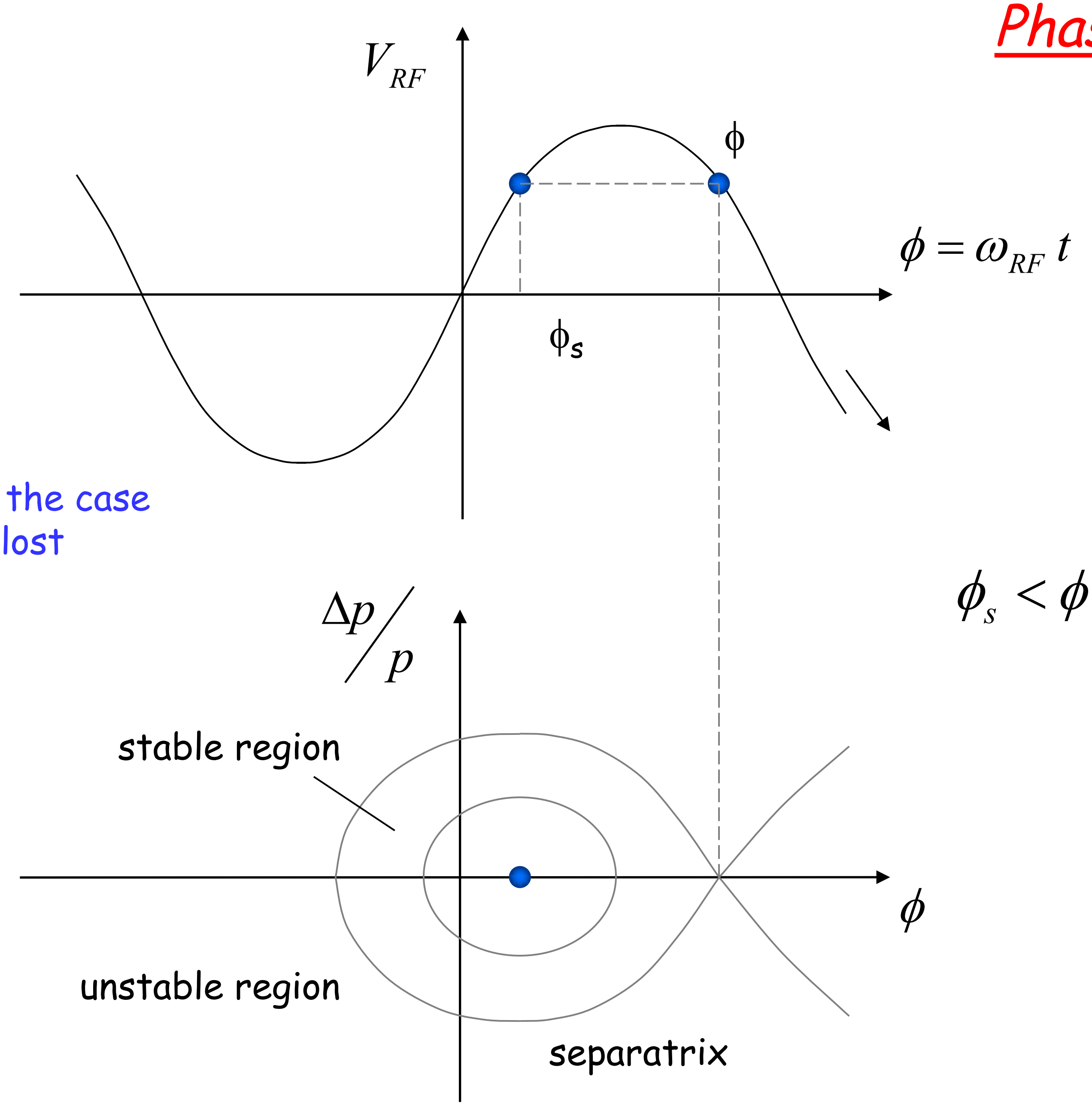
The RF frequency is increased as well in order to keep the resonant condition

$$\omega = \frac{e\,B}{\gamma\, m_0} = \frac{\omega_{RF}}{h}$$

Phase stability



Phase stability



The symmetry of the case with B = const. is lost

$$\phi_s < \phi < \pi - \phi_s$$

LESSON IV

RF acceleration for synchronous particle

RF acceleration for non-synchronous particle

Small amplitude oscillations

Large amplitude oscillations - the RF bucket

Synchrotron frequency and tune

Tracking

Nonadiabatic theory needed "close" to transition

Double RF systems

RF acceleration for synchronous particle - energy gain

Let's assume a synchronous particle with a given $\phi_s > 0$

We want to calculate its rate of acceleration, and the related rate of increase of B, f.

$$p = e\,B\,\rho$$

Want to keep ρ = const

$$\Rightarrow \quad \frac{\mathrm{d}p}{\mathrm{d}t} = e\,\rho\,\frac{\mathrm{d}\,B}{\mathrm{d}t} = e\,\rho\,\dot{B}$$

Over one turn: $$\left(\Delta\,p\right)_{turn} = e\,\rho\,\dot{B}\,T_{rev} = e\,\rho\,\dot{B}\,\frac{2\pi\,R}{\beta c}$$

We know that (relativistic equations) : $$\Delta\,p = \frac{\Delta E}{\beta c}$$

$$\Rightarrow \quad \boxed{\left(\Delta E\right)_{turn} = e\,\rho\,\dot{B}\;2\pi\,R}$$

RF acceleration for synchronous particle - phase

$$(\Delta E)_{turn} = e\,\rho\,\dot{B}\;2\pi\,R$$

On the other hand,
for the synchronous particle:

$$(\Delta E)_{turn} = e\hat{V}_{RF}\sin\phi_s$$

$$e\,\rho\,\dot{B}\;2\pi\,R = e\hat{V}_{RF}\sin\phi_s$$

Therefore:

1. Knowing ϕ_s, one can calculate the increase rate of the magnetic field needed for a given RF voltage:

$$\Rightarrow \boxed{\dot{B} = \frac{\hat{V}_{RF}}{2\pi\,\rho\,R}\sin\phi_s}$$

2. Knowing the magnetic field variation and the RF voltage, one can calculate the value of the synchronous phase:

$$\sin\phi_s = 2\pi\,\rho\,R\;\frac{\dot{B}}{\hat{V}_{RF}} \Rightarrow \boxed{\phi_s = \arcsin\left(2\pi\,\rho\,R\;\frac{\dot{B}}{\hat{V}_{RF}}\right)}$$

RF acceleration for synchronous particle - frequency

$$\omega_{RF} = h\,\omega_s = h\frac{e}{m} < B >$$

$$\left(v = \frac{e}{m} B\rho \right)$$

$$\omega_{RF} = h\frac{e}{m}\frac{\rho}{R} B$$

From relativistic equations:

$$\omega_{RF} = \frac{hc}{R}\sqrt{\frac{B^2}{B^2 + \left(E_0 / ec\rho\right)^2}}$$

Let

$$B_0 \equiv \frac{E_0}{ec\rho}$$

$$\Rightarrow \quad \boxed{f_{RF} = \frac{hc}{2\pi R}\left(\frac{B}{B_0}\right)\frac{1}{\sqrt{1+\left(B/B_0\right)^2}}}$$

Example: PS

At the CERN Proton Synchrotron machine, one has:

$$R = 100 \text{ m}$$

$$\dot{B} = 2.4 \text{ T/s}$$

100 dipoles with l_{eff} = 4.398 m. The harmonic number is 20

Calculate:

1. The energy gain per turn
2. The minimum RF voltage needed
3. The RF frequency when B = 1.23 T (at extraction)

RF acceleration for non synchronous particle

Parameter definition (subscript “s” stands for synchronous particle):

$f = f_s + \Delta f$ revolution frequency

$\phi = \phi_s + \Delta \phi$ RF phase

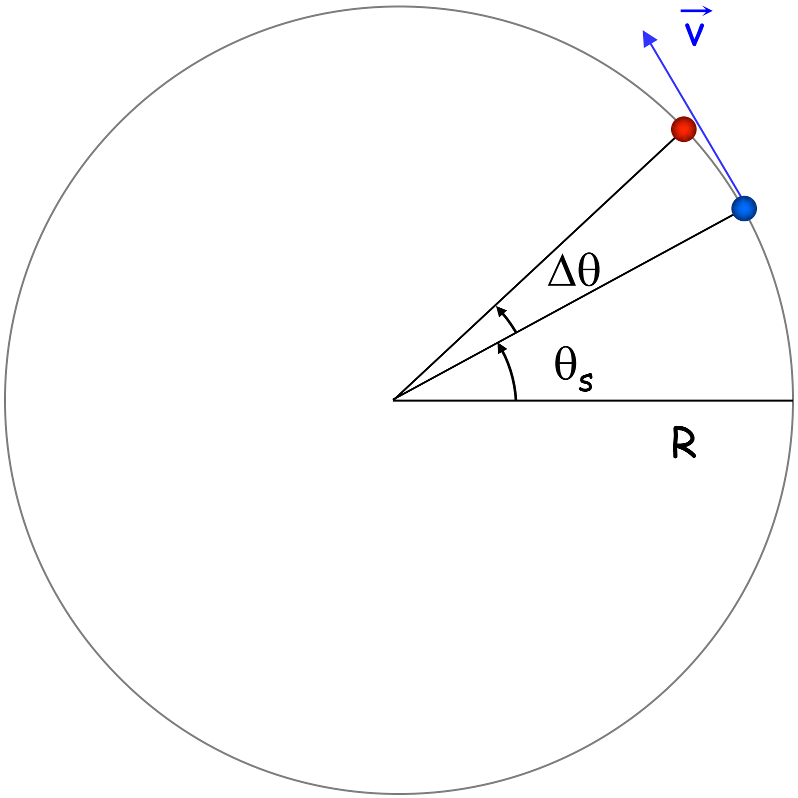
$p = p_s + \Delta p$ Momentum

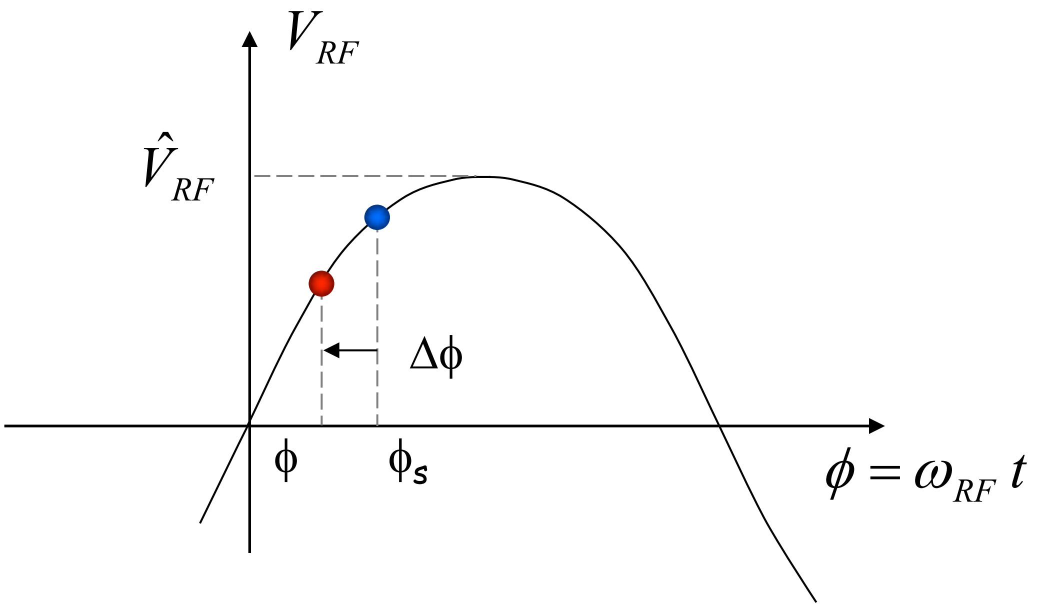
$E = E_s + \Delta E$ Energy

$\theta = \theta_s + \Delta \theta$ Azimuth angle

$$\mathrm{d}s = R\,\mathrm{d}\theta$$

$$\theta(t) = \int_{t_0}^{t} \omega(\tau)\,\mathrm{d}\tau$$





$$\Delta\theta > 0 \quad \Rightarrow \quad \Delta\phi < 0$$

Since $f_{RF} = h\, f_{rev}$

$$\boxed{\Delta\phi = -h\,\Delta\theta}$$

Over one turn θ varies by $2\,\pi$

ϕ varies by $2\,\pi\,h$

Parameters versus $\dot{\phi}$

1. Angular frequency

$$\theta(t) = \int_{t_0}^{t} \omega(\tau)\,\mathrm{d}\tau$$

$$\begin{aligned}\Delta\omega &= \frac{\mathrm{d}}{\mathrm{d}t}(\Delta\theta)\\ &= -\frac{1}{h}\frac{\mathrm{d}}{\mathrm{d}t}(\Delta\phi)\\ &= -\frac{1}{h}\frac{\mathrm{d}}{\mathrm{d}t}(\phi-\phi_s)\\ &= -\frac{1}{h}\frac{\mathrm{d}\phi}{\mathrm{d}t}\end{aligned}$$

$\frac{\mathrm{d}\phi_s}{\mathrm{d}t} = 0$ by definition

➡ $$\boxed{\Delta\omega = -\frac{1}{h}\frac{\mathrm{d}\phi}{\mathrm{d}t}}$$

Parameters versus $\dot{\phi}$

2. Momentum

$$\eta = \frac{\mathrm{d}\omega / \omega}{\mathrm{d}p / p} = \frac{\Delta\omega / \omega}{\Delta p / p}$$

$$\Delta p = \frac{p_s}{\omega_s}\frac{\Delta\omega}{\eta} = \frac{p_s}{\omega_s \eta}\left(-\frac{1}{h}\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)$$

➡

$$\boxed{\Delta p = \frac{-p_s}{\omega_s \eta h}\frac{\mathrm{d}\phi}{\mathrm{d}t}}$$

3. Energy

$$\frac{\mathrm{d}E}{\mathrm{d}p} = v$$

$$\frac{\Delta E}{\Delta p} = v = \omega R$$

➡

$$\boxed{\Delta E = -\frac{R\,p_s}{\eta\,h}\frac{\mathrm{d}\phi}{\mathrm{d}t}}$$

Derivation of equations of motion

Energy gain after the RF cavity

$$(\Delta E)_{turn} = e\hat{V}_{RF} \sin\phi$$

$$(\Delta p)_{turn} = \frac{e}{\omega R} \hat{V}_{RF} \sin\phi$$

Average increase per time unit

$$\frac{(\Delta p)_{turn}}{T_{rev}} = \frac{e}{2\pi R} \hat{V}_{RF} \sin\phi \qquad 2\pi R \dot{p} = e \hat{V}_{RF} \sin\phi$$

valid for any particle !

$$\boxed{2\pi \left(R \dot{p} - R_s \dot{p}_s \right) = e \hat{V}_{RF} \left(\sin\phi - \sin\phi_s \right)}$$

Derivation of equations of motion

$$R\,\dot{p} - R_s\,\dot{p}_s = \left(R_s + \Delta R\right)\left(\dot{p}_s + \Delta\dot{p}\right) - R_s\,\dot{p}_s$$

$$\approx R_s\,\Delta\dot{p} + \dot{p}_s\,\Delta R$$

$$\approx R_s\,\Delta\dot{p} + \dot{p}_s\left(\frac{dR}{dp}\right)_s\Delta p$$

$$= R_s\,\Delta\dot{p} + \frac{dp_s}{dt}\,\frac{dR_s}{dp_s}\Delta p$$

$$= R_s\,\Delta\dot{p} + \dot{R}_s\,\Delta p$$

$$= \frac{d}{dt}\left(R_s\,\Delta_p\right)$$

$$= \frac{d}{dt}\left(\frac{\Delta E}{\omega_s}\right)$$

Derivation of equations of motion

$$2\pi \frac{d}{dt}\left(\frac{\Delta E}{\omega_s}\right) = e\,\hat{V}_{RF}\left(\sin\phi - \sin\phi_s\right)$$

An approximated version of the above is

$$\boxed{\frac{\mathrm{d}(\Delta p)}{\mathrm{d}t} = \frac{e\,\hat{V}_{RF}}{2\pi\,R_s}\left(\sin\phi - \sin\phi_s\right)}$$

Which, together with the previously found equation

$$\boxed{\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{\omega_s\,\eta\,h}{p_s}\,\Delta p}$$

Describes the motion of the non-synchronous particle in the longitudinal phase space ($\Delta p, \phi$)

Equations of motion I

$$\begin{cases} \dfrac{\mathrm{d}(\Delta p)}{\mathrm{d}t} = A\left(\sin\phi - \sin\phi_s\right) \\ \dfrac{\mathrm{d}\phi}{\mathrm{d}t} = B\,\Delta p \end{cases}$$

with $A = \dfrac{e\,\hat{V}_{RF}}{2\pi\,R_s}$

$B = -\dfrac{\eta\,h}{p_s}\dfrac{\beta_s c}{R_s}$

B is not the magnetic field (induction) here!

Equations of motion II

1. First approximation – combining the two equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{B}\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)-A\left(\sin\phi-\sin\phi_s\right)=0$$

We assume that A and B change very slowly compared to the variable $\Delta\phi = \phi - \phi_s$

$$\Rightarrow \quad \boxed{\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2}+\frac{\Omega_{sync}^2}{\cos\phi_s}\left(\sin\phi-\sin\phi_s\right)=0}$$

with $\dfrac{\Omega_{sync}^2}{\cos\phi_s}=-A\,B$ We can also define: $\Omega_0^2=\dfrac{\Omega_{sync}^2}{\cos\phi_s}=\dfrac{e\,\hat{V}_{RF}\,\eta\,h\,c^2}{2\pi\,R_s^2\,E_s}$

Small amplitude oscillations

2. Second approximation

$$\sin\phi = \sin(\phi_s + \Delta\phi)$$
$$= \sin\phi_s \cos\Delta\phi + \cos\phi_s \sin\Delta\phi$$

$\Delta\phi$ small $\Rightarrow$ $\sin\phi \cong \sin\phi_s + \cos\phi_s \, \Delta\phi$

$\dfrac{\mathrm{d}\phi_s}{\mathrm{d}t} = 0$ $\Rightarrow$ $\dfrac{\mathrm{d}^2\phi}{\mathrm{d}t^2} = \dfrac{\mathrm{d}^2}{\mathrm{d}t^2}(\phi_s + \Delta\phi) = \dfrac{\mathrm{d}^2\Delta\phi}{\mathrm{d}t^2}$

by definition

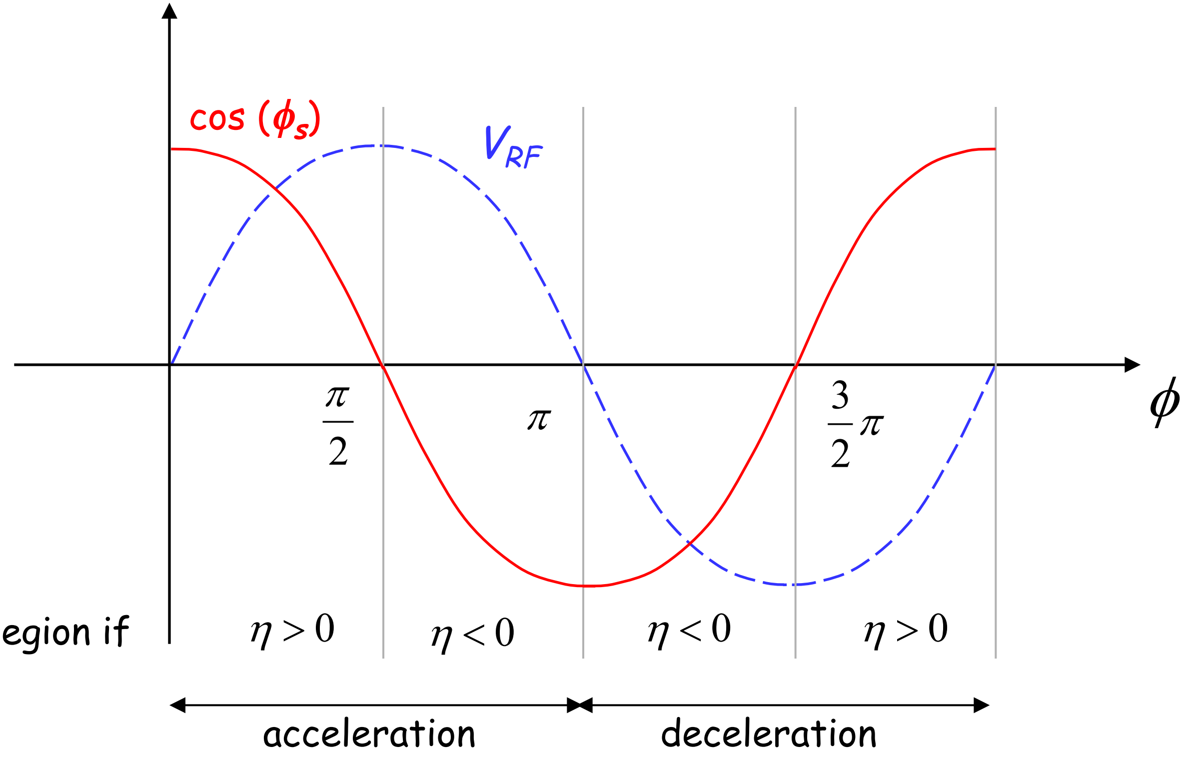
$$\frac{\mathrm{d}^2\Delta\phi}{\mathrm{d}t^2} + \Omega_{sync}^2 \Delta\phi = 0$$

Harmonic oscillator !

Stability condition for ϕ_s

Stability is obtained when the angular frequency of the oscillator, Ω^2_{sync} is real positive:

$$\boxed{\Omega^2_{sync} = \frac{e\,\hat{V}_{RF}\,\eta\,h\,c^2}{2\pi\,R_s^2\,E_s}\cos\phi_s} \quad \Rightarrow \quad \Omega^2_{sync} > 0 \quad \Leftrightarrow \quad \boxed{\eta\cos\phi_s > 0}$$



Small amplitude oscillations - orbits

For $\eta \cos \phi_s > 0$ the motion around the synchronous particle is a stable oscillation:

$$\boxed{\begin{cases} \Delta\phi = \Delta\phi_{\max} \sin\left(\Omega_{sync} t + \phi_0\right) \\ \Delta p = \Delta p_{\max} \cos\left(\Omega_{sync} t + \phi_0\right) \end{cases}}$$

with $\Delta p_{\max} = \dfrac{\Omega_{sync}}{B} \Delta\phi_{\max}$

Synchrotron (angular) frequency and synchrotron tune (for small amplitudes)

$$\boxed{\Omega_{sync} = \omega_s \sqrt{\frac{e\,\hat{V}_{RF}\,h}{2\pi\,\beta^2\,E_s}\,\eta\,\cos\phi_s}}$$

$$\Omega_{sync} = 2\,\pi\,f_{sync}$$

$$\omega_s = 2\,\pi\,f_s$$

Number of synchrotron oscillations per turn:

$$Q_{sync} = \frac{\Omega_{sync}}{\omega_s} = \sqrt{\frac{e\,\hat{V}_{RF}\,h}{2\pi\,\beta^2\,E_s}\,\eta\,\cos\phi_s}$$

"synchrotron tune"

Large amplitude oscillations

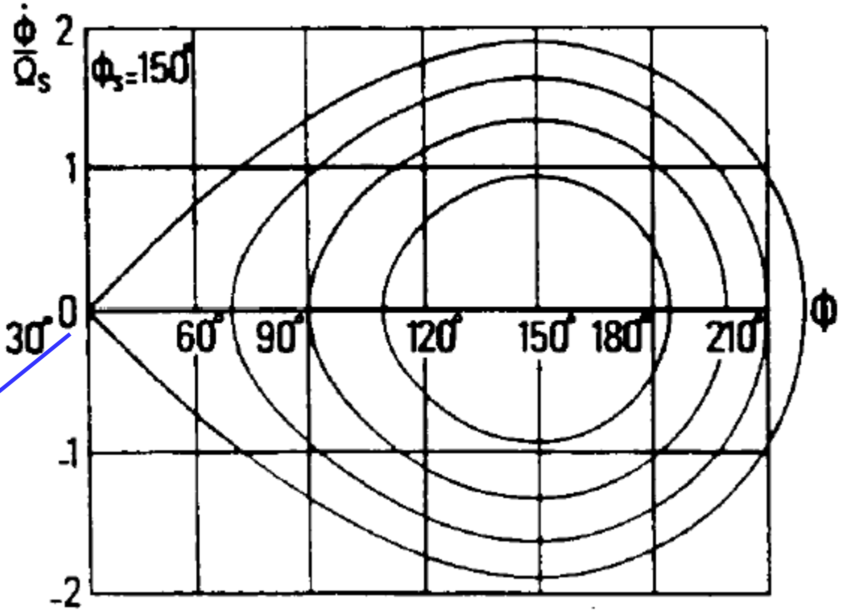
$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\,\phi\text{ - }\sin\,\phi_s\right) = 0$$

Ω_{sync} **will now be noted** Ω_s

Multiplying by $\dot{\phi}$
and integrating

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\,\phi_s\right) = cte$$

Constant of motion



Synchronous phase 150°

here $\dot{\phi} = 0$

$$\phi = \pi - \phi_s$$

Equation of the separatrix

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\;\;\sin\,\phi_s\right) = -\frac{\Omega_s^2}{\cos\phi_s}\left[\cos\left(\pi\text{ - }\phi_s\right) + \left(\pi - \phi_s\right)\sin\,\phi_s\right]$$

Phase space separatrix and particle trajectories

- Equation of the bucket separatrix

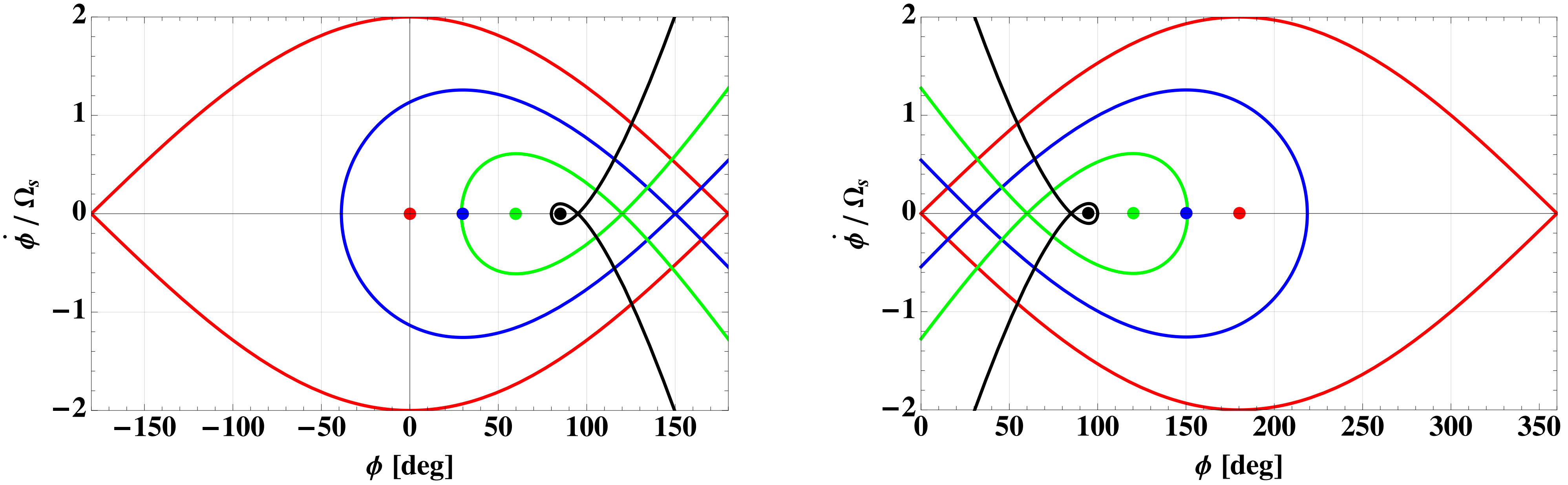
$$\frac{\dot{\phi}}{\Omega_s} = \pm \sqrt{\frac{2}{\cos \phi_s} \left[\cos \phi + \phi \sin \phi_s - \cos \left(\pi - \phi_s \right) - \left(\pi - \phi_s \right) \sin \phi_s \right]}$$

- Equation of a particle trajectory

$$\frac{\dot{\phi}}{\Omega_s} = \pm \sqrt{\frac{2}{\cos \phi_s} \left[\cos \phi + \phi \sin \phi_s \right] + Cte}$$

Phase space separatrix and particle trajectories

- **(Bucket) separatrices**: Below transition
- Above transition



$\phi_s = 0°$ $\phi_s = 30°$

$\phi_s = 60°$ $\phi_s = 85°$

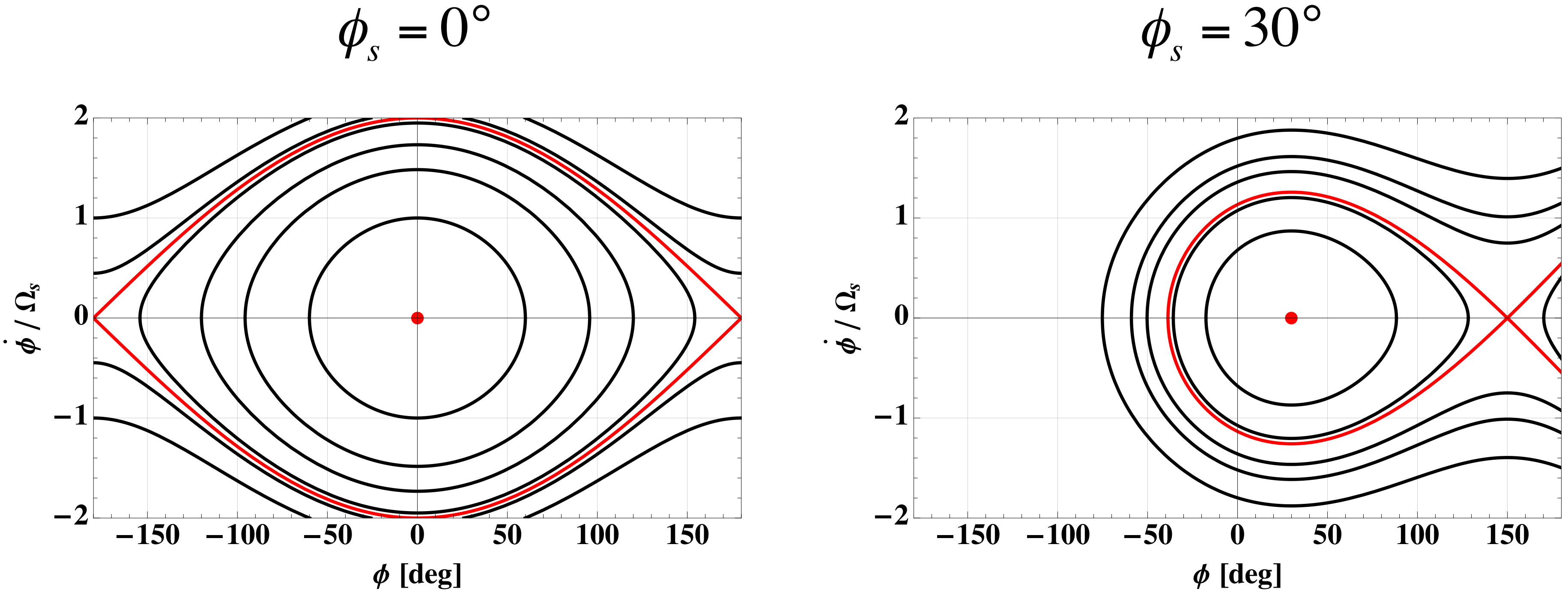
$$\phi_s \Rightarrow \pi - \phi_s$$

$\phi_s = 180°$ $\phi_s = 150°$

$\phi_s = 120°$ $\phi_s = 95°$

Phase space separatrix and particle trajectories

- **Particle trajectories**: Below transition



- Change of variables if one wants to use (Φ, ΔE) or (Δt, ΔE) instead of (Φ, $d\Phi/dt$)

$$\begin{aligned}\Delta\phi &= \phi - \phi_s \\ &= \omega_{RF}\ \Delta t \\ &= h\,\omega_s\,\Delta t\end{aligned}$$

$$\Delta p = \frac{\Delta E}{\beta_s\ c}$$

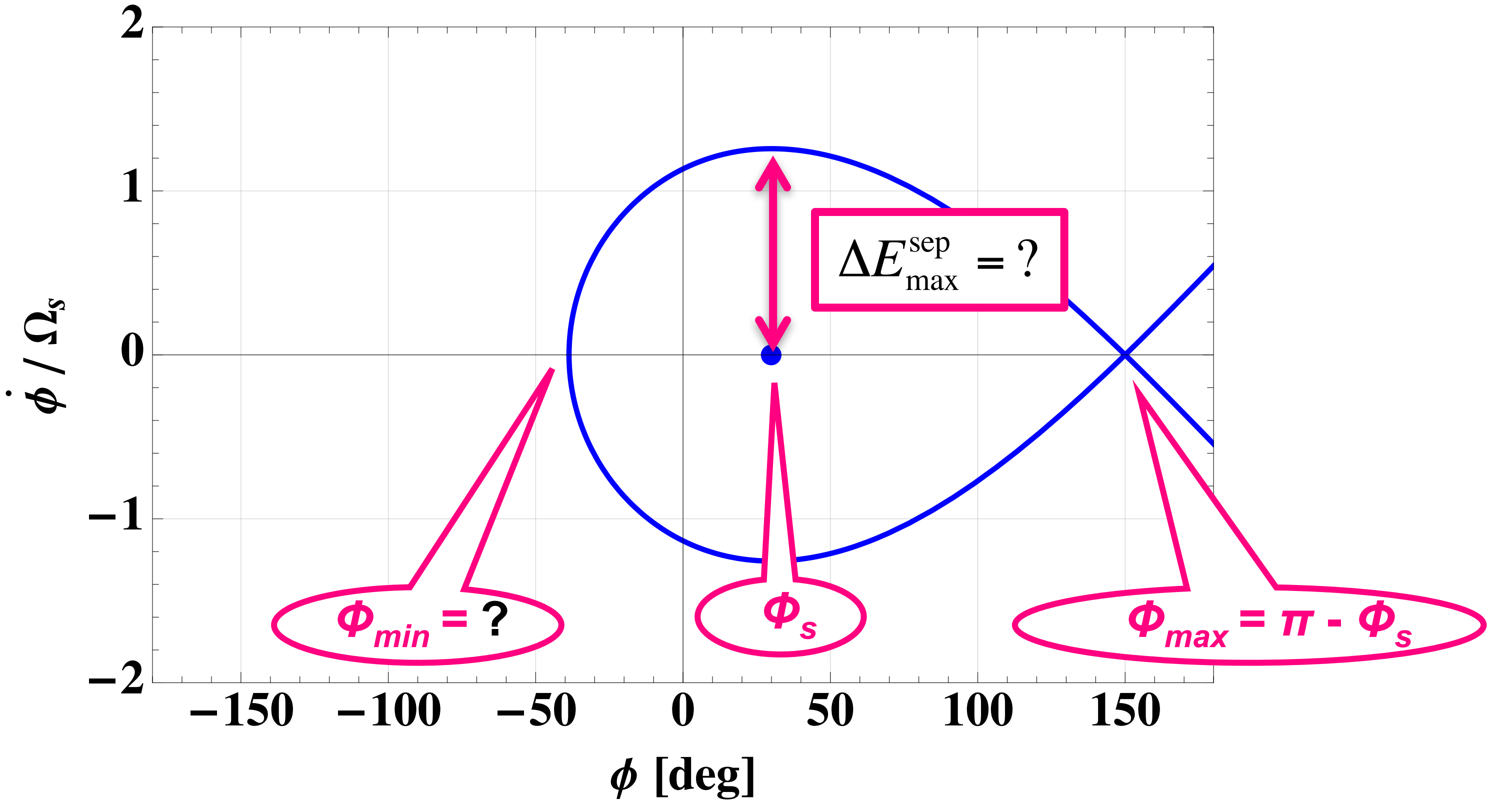
$$\dot{\phi} = -\frac{\eta\,h\,c}{\beta_s\ E_s\ R_s}\,\Delta E$$

=> System of 2 equations to be solved

$$\frac{d}{dt}(\Delta E) = \frac{e\,\hat{V}_{RF}\ \omega_s}{2\pi}\left[\,\sin\left(\,\phi_s + h\,\omega_s\,\Delta t\,\right) - \sin\phi_s\,\right]$$

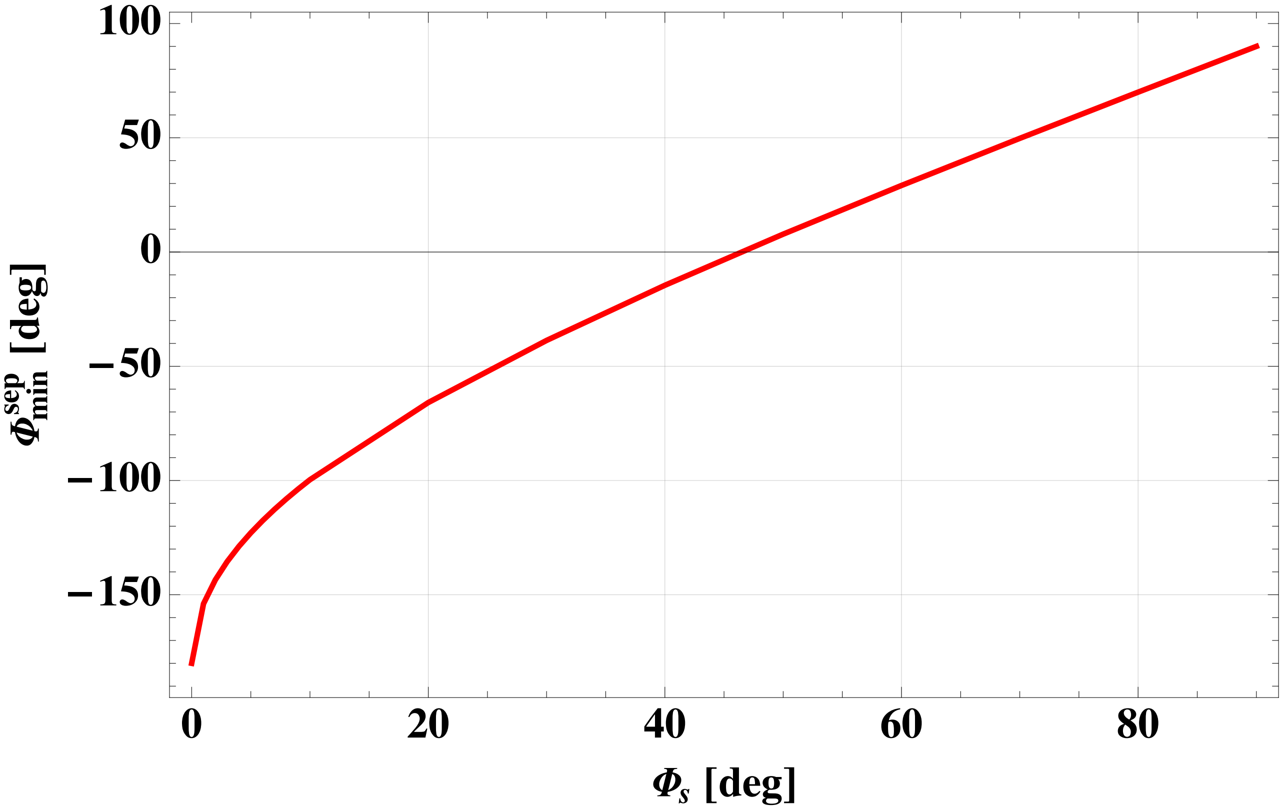
$$\frac{d}{dt}(\Delta t) = -\frac{\eta}{\beta_s^2\ E_s}\,\Delta E$$

- 2 questions


$\dot{\phi}\,/\,\Omega_s$
2
1
0
−1
−2
−150
−100
−50
0
50
100
150
ϕ [deg]
$\Delta E_{\max}^{\text{sep}} = ?$
$\Phi_{min} = ?$
Φ_s
$\Phi_{max} = \pi - \Phi_s$

- $\Phi_{\min}$ is obtained from the equation of the separatrix when $\dot{\phi} = 0$

=> $\boxed{\cos\phi + \phi\sin\phi_s - \cos\left(\pi - \phi_s\right) - \left(\pi - \phi_s\right)\sin\phi_s = 0}$

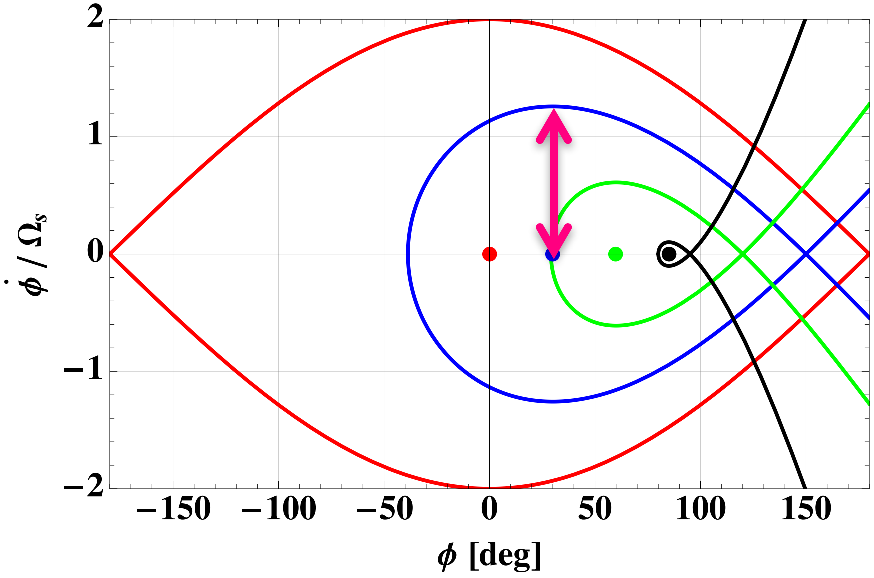


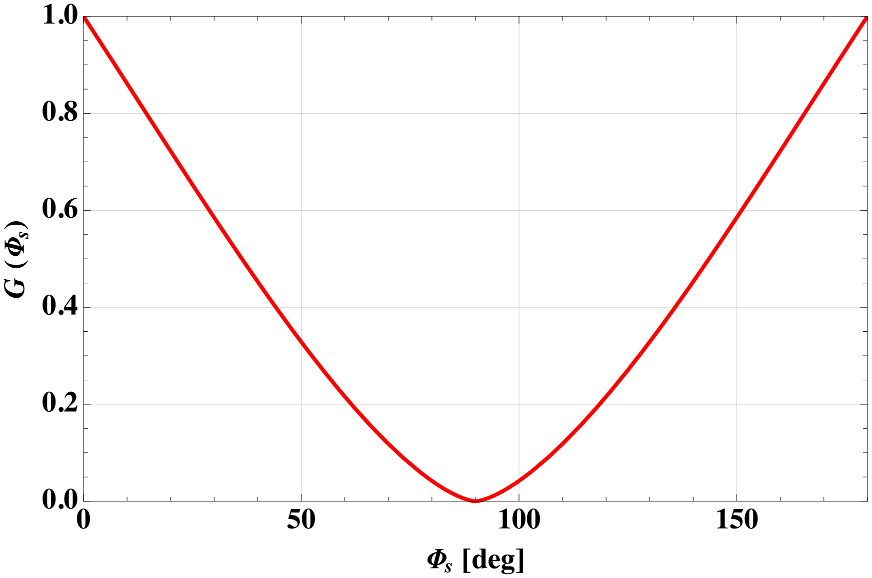
- $\Delta E_{\max}^{\text{sep}}$ is obtained from the equation of the separatrix when $\phi = \phi_s$

$$\boxed{\Delta E_{\max}^{\text{sep}}\left(\phi_s\right) = \sqrt{\frac{2\,\beta_s^2\,E_s\,e\,\hat{V}_{RF}}{\pi\,h\left|\eta\right|}\,G\left(\phi_s\right)}}$$

with

$$\boxed{G\left(\phi_s\right) = \frac{\sqrt{\left|2\cos\phi_s - \left(\pi - 2\phi_s\right)\sin\phi_s\right|}}{\sqrt{2}}}$$





$\boxed{\phi_s = 0°}$ $\boxed{\phi_s = 30°}$ $\boxed{\phi_s = 60°}$ $\boxed{\phi_s = 85°}$

- nTOF bunch in the CERN PS (near transition)

Average machine radius: R [m]	100
Bending dipole radius: ρ [m]	70
$\dot{B}$ [T/s]	2.2
$\hat{V}_{RF}$ [kV]	200
h	8
α_p	0.027
Longitudinal (total) emittance: ε_L [eVs]	2
Number of protons/bunch: N_b [1E10 p/b]	800
Norm. rms. transverse emittance: $\varepsilon^*_{x,y}$ [µm]	5
Trans. average betatron function: $\beta_{x,y}$ [m]	16
Beam pipe [cm × cm]	3.5 × 7
Trans. tunes: $Q_{x,y}$	6.25

20 kV at injection

=> $\gamma_t \approx 6.1$

Tracking

- The motion of the particles can be tracked turn by turn using the recurrence relation (between turn *n* and turn *n*+1)

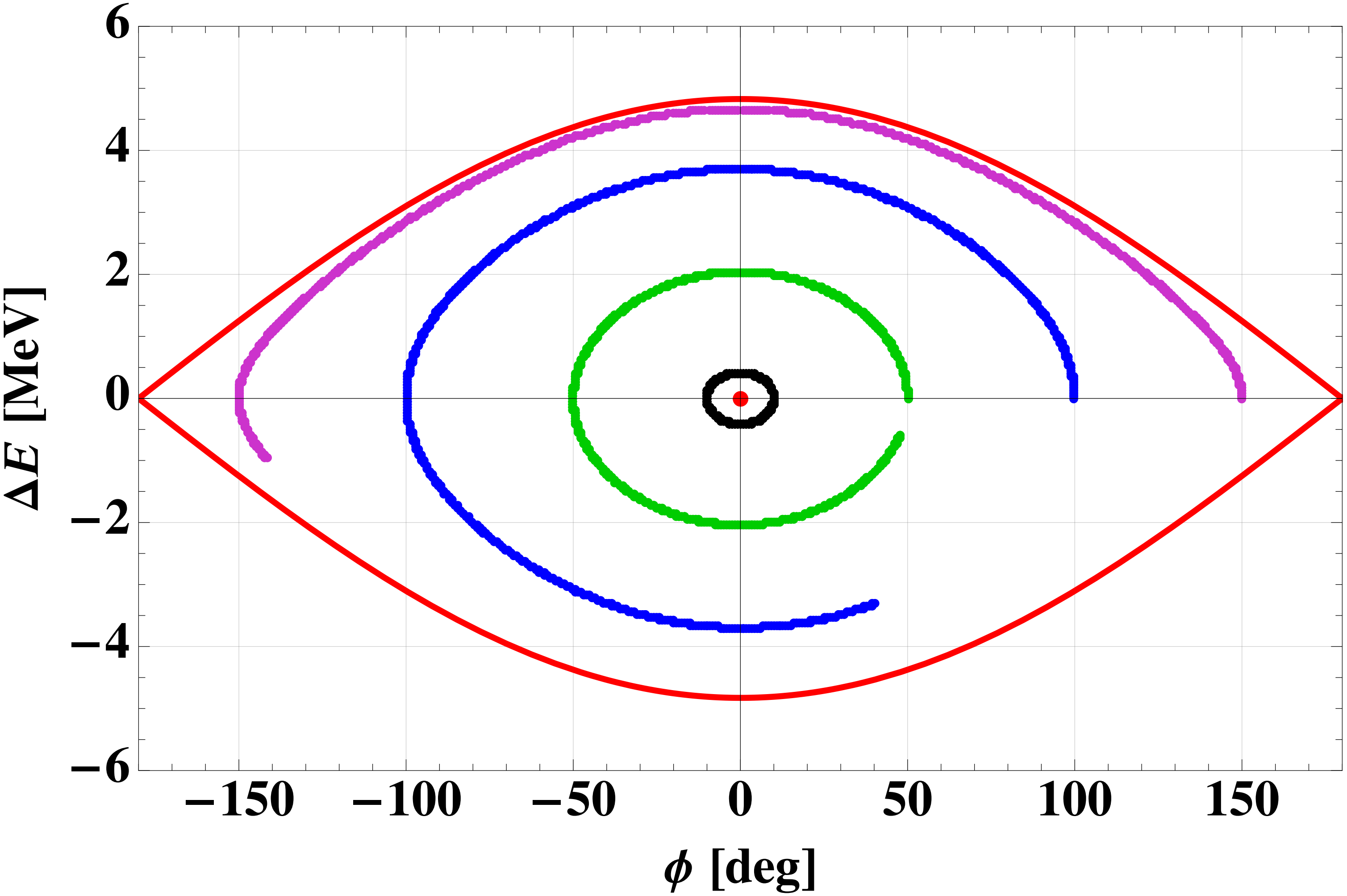
$$\Delta E_{n+1} = \Delta E_n + e\,\hat{V}_{RF}\left[\,\sin\phi_n - \sin\phi_s\,\right]$$

$$\phi_{n+1} = \phi_n - \frac{2\,\pi\,h\,\eta}{\beta_s^2\,E_s}\,\Delta E_{n+1}$$

Tracking applied to the nTOF bunch at PS injection

$$\phi_s = 0 \text{ deg}$$

$$n_{\max} = 758 = 1 / Q_s$$



Tracking applied to the nTOF bunch at PS injection

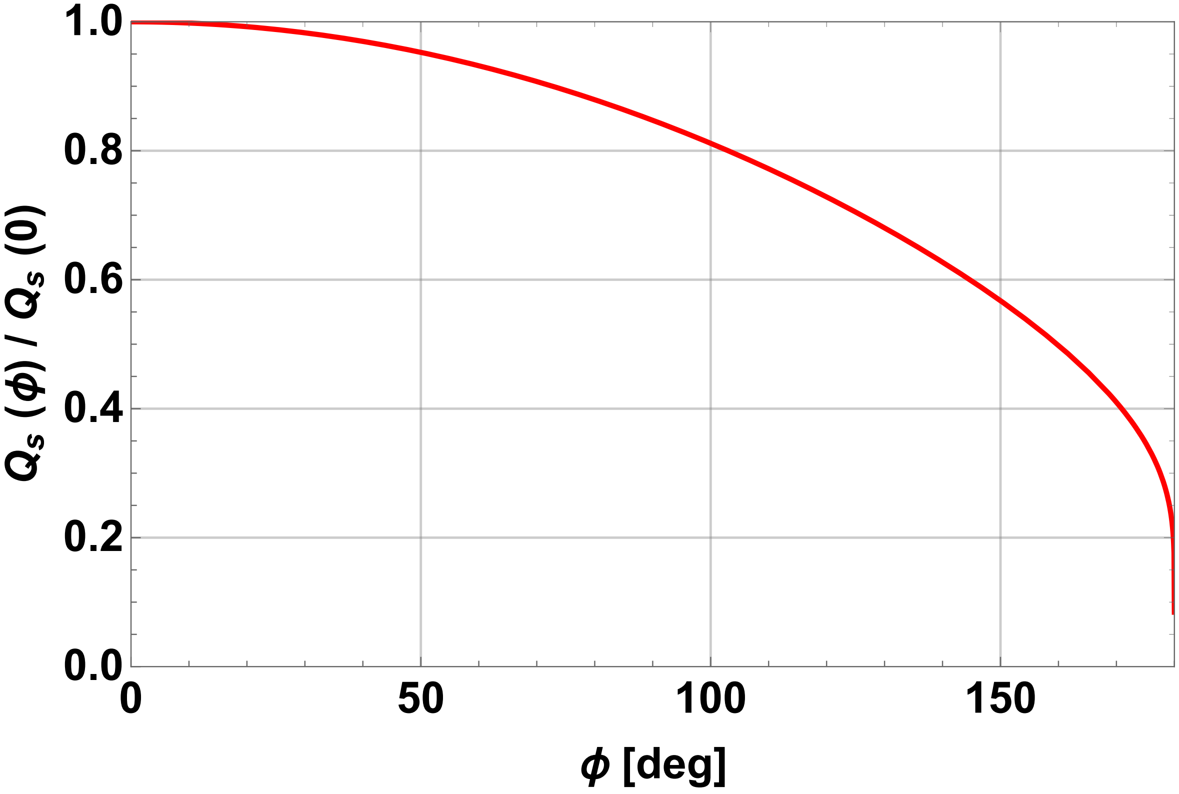
$$\phi_s = 0 \text{ deg}$$

One can show (but the detailed computation is beyond the scope of this course) that

$$\frac{Q_s(\phi)}{Q_s(0)} = \frac{\pi}{2\,K_{cei1}\left[\sin^2(\phi/2)\right]}$$

$$K_{cei1}(x) = \int_0^{\pi/2} \frac{dy}{\sqrt{1 - x\sin^2 y}}$$

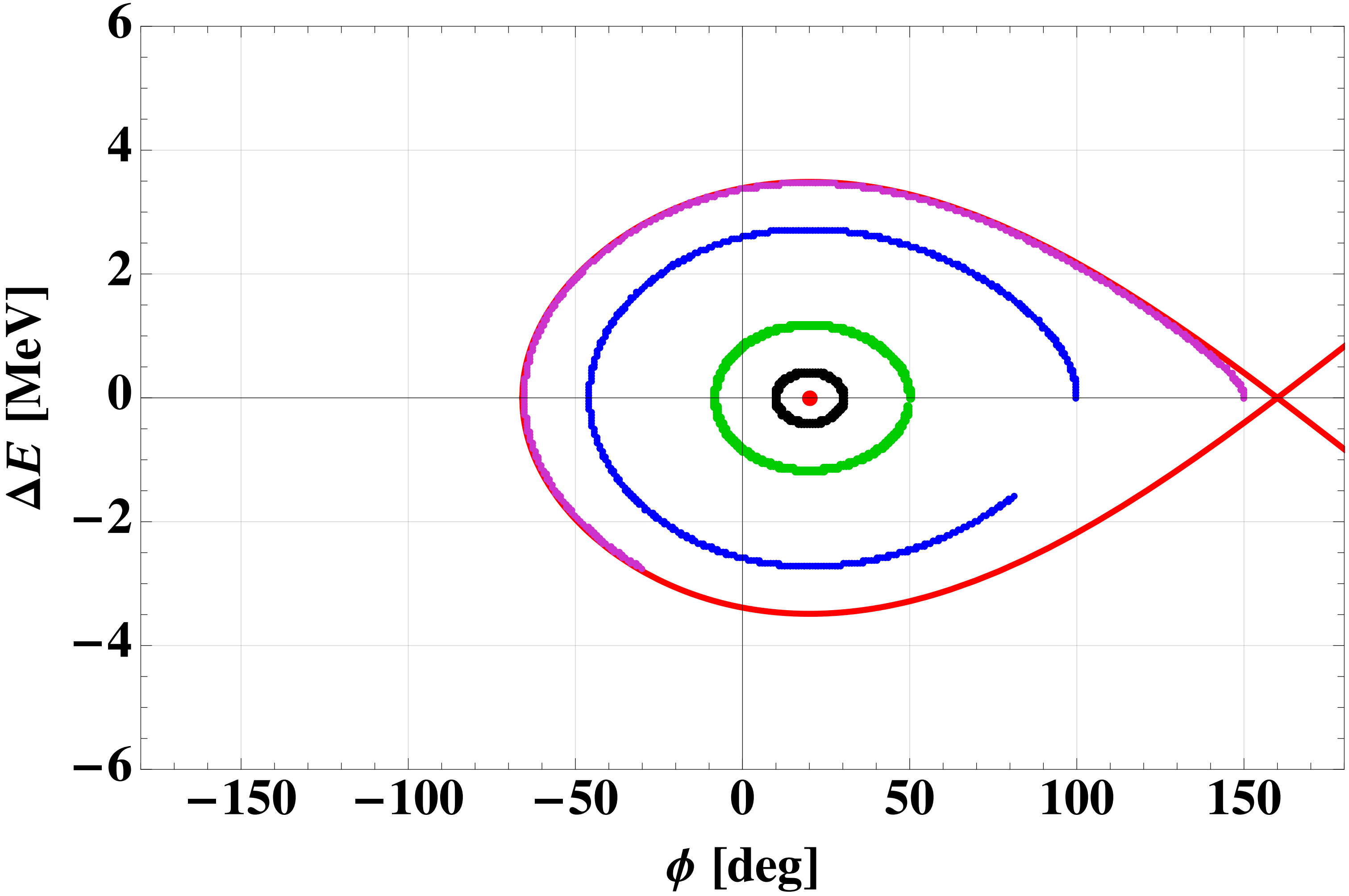
Complete elliptical integral of the first kind



Tracking applied to the nTOF bunch at PS injection

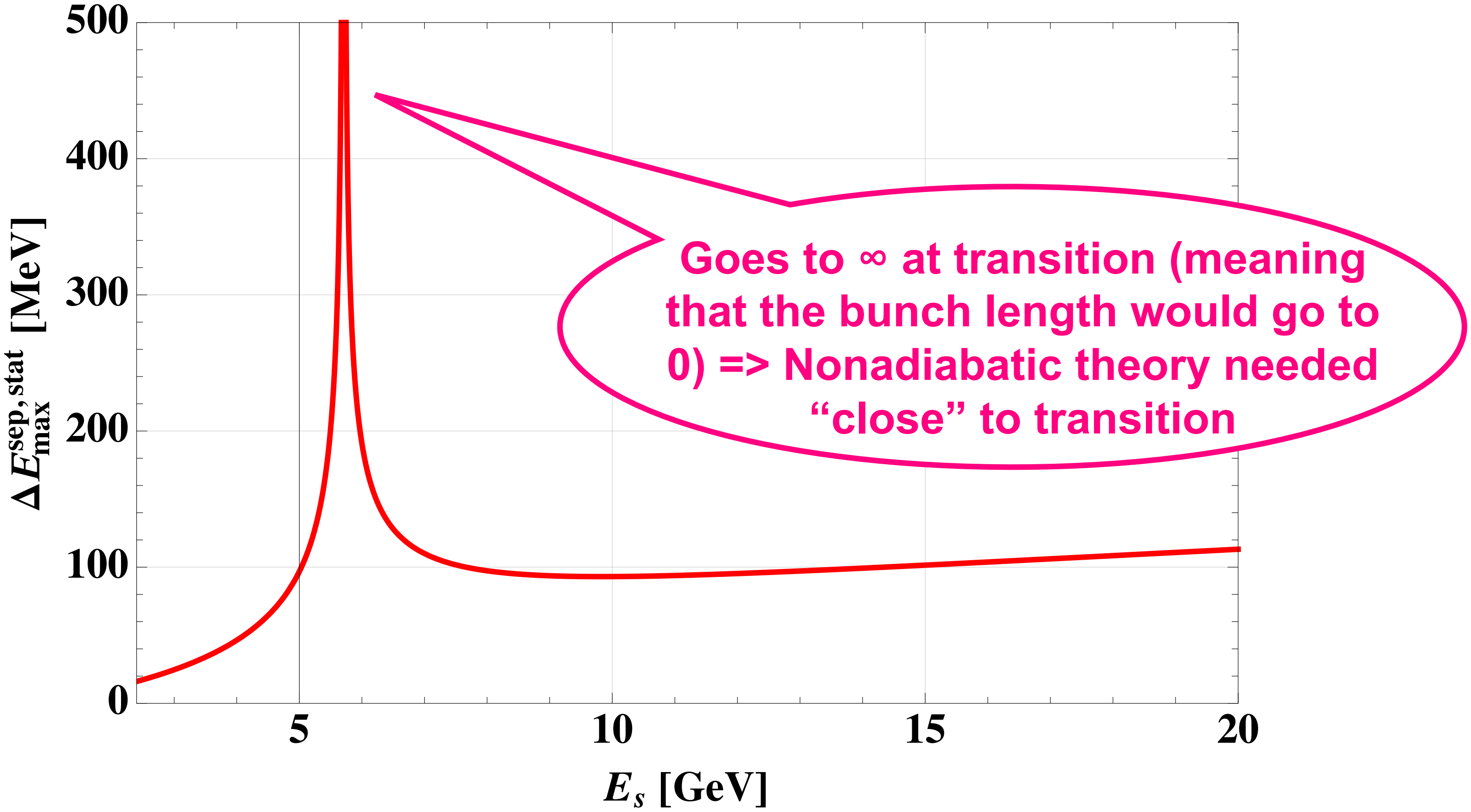
$$\phi_s = 20 \text{ deg}$$

$$n_{\max} = 782 = 1 / Q_s$$



Bucket height near transition (with "adiabatic" theory)

- Case of a stationary bucket in the PS with the nTOF bunch from injection (~ 2.4 GeV total energy) till top energy (~ 20 GeV total energy) assuming a constant RF voltage (200 kV)



Goes to ∞ at transition (meaning that the bunch length would go to 0) => Nonadiabatic theory needed "close" to transition

Nonadiabatic theory needed "close" to transition

- Reminder: the (general, nonlinear) equations, which have to be solved, using the variables ($\Delta\Phi$, ΔE), are

$$\frac{d\,\Delta\phi}{d\,t} = -\frac{h\,\eta\,\omega_s}{\beta_s^2\,E_s}\,\Delta E$$

$$\frac{d\,\Delta E}{d\,t} = \frac{e\,\hat{V}_{RF}\,\omega_s}{2\,\pi}\left[\sin\left(\phi_s + \Delta\phi\right) - \sin\phi_s\right]$$

- Assuming here only small amplitude particles

$$\frac{d\,\Delta E}{d\,t} = \frac{e\,\hat{V}_{RF}\,\omega_s}{2\,\pi}\left[\sin\left(\phi_s + \Delta\phi\right) - \sin\phi_s\right] \approx \frac{e\,\hat{V}_{RF}\,\omega_s}{2\,\pi}\cos\phi_s\,\Delta\phi$$

Nonadiabatic theory needed "close" to transition

=> $$\frac{d}{dt}\left(\frac{\beta_s^2\,E_s}{h\,\eta\,\omega_s}\,\frac{d\,\Delta\phi}{dt}\right)-\frac{e\,\hat{V}_{RF}\,\omega_s}{2\pi}\cos\phi_s\,\Delta\phi=0$$

where in general β_s , E_s , η and ω_s depend on time

◆ Until now we assumed that these parameters were slowly moving => Adiabatic theory

◆ However, close to transition the particle will not be able to catch up with the rapid modification of the bucket shape and a nonadiabatic theory is needed

Nonadiabatic theory needed "close" to transition

- Neglecting the slow time variations of all the parameters except $\dfrac{\eta}{E_s}$,

  one has to solve

$$\frac{d}{dt}\left(\frac{E_s}{\eta}\,\frac{d\,\Delta\phi}{dt}\right)-\frac{h\;e\,\hat{V}_{RF}\;\omega_s^2\cos\phi_s}{2\,\pi\,\beta_s^2}\,\Delta\phi=0$$

- Assuming then that $\gamma=\gamma_t+\dot{\gamma}\;t$, with *t* = 0 at transition,

$$-\eta=\frac{1}{\gamma_t^2}-\frac{1}{\gamma^2}\approx\frac{2\,\dot{\gamma}\,t}{\gamma_t^3}$$

$$E_s=\gamma\,E_0\approx\gamma_t\,E_0$$

$$\boxed{\frac{\eta}{E_s}\approx-\frac{2\,\dot{\gamma}\,t}{\gamma_t^4\;E_0}}$$

Nonadiabatic theory needed "close" to transition

- The (small amplitude) equation which needs to be solved close to transition is

$$\frac{d}{d\,t}\left(\frac{T_c^3}{\left|t\right|}\,\frac{d\,\Delta\phi}{d\,t}\right)+\Delta\phi=0$$

with T_c a nonadiabatic time defined by (with E_0 in eV)

$$T_c=\left(\frac{\beta_s^2\,E_0\,\gamma_t^4}{4\,\pi\,f_s^2\,\dot{\gamma}\,h\,\hat{V}_{RF}\left|\cos\phi_s\right|}\right)^{1/3}$$

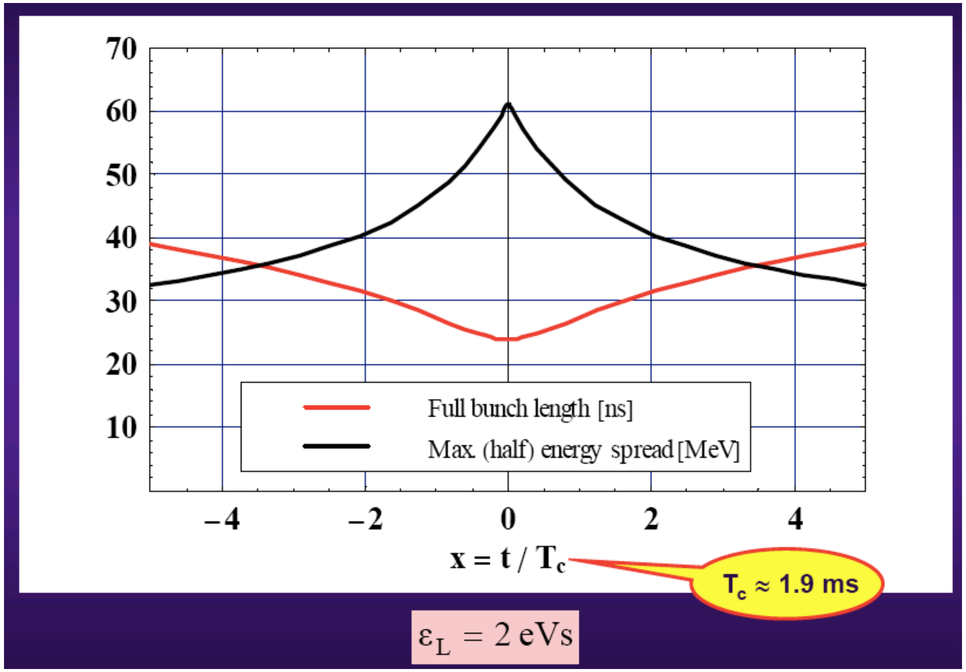
~ 1.9 ms for the nTOF bunch in the CERN PS

Nonadiabatic theory needed "close" to transition

- This equation can be solved but the detailed computation is beyond the scope of this course => See for instance (for those interested)
  - K.Y. Ng, "Physics of Intensity Dependent Beam Instabilities", World Scientific (2006), p. 691
  - E. Métral, USPAS 2009 course, Albuquerque, USA: http://emetral.web.cern.ch/emetral/USPAS09course/EnvelopeEquations.pdf

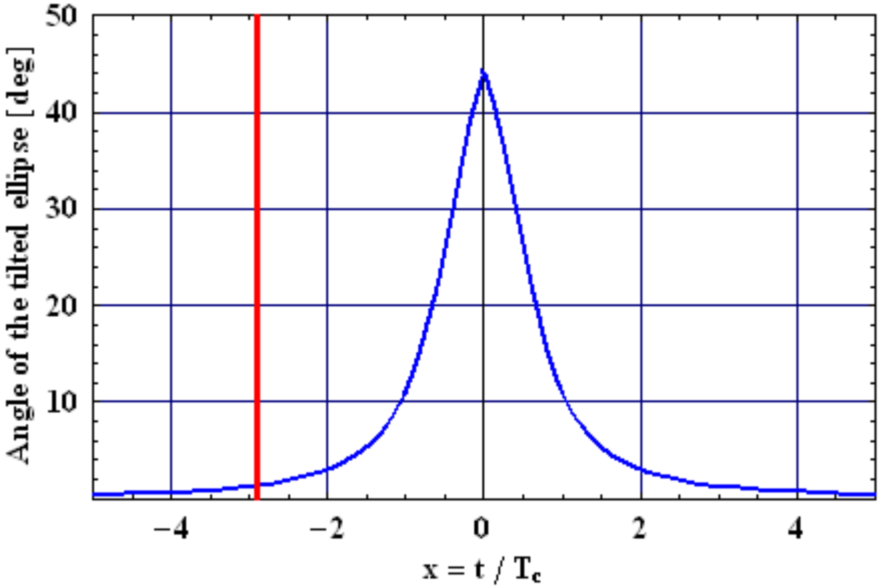
Nonadiabatic theory needed "close" to transition

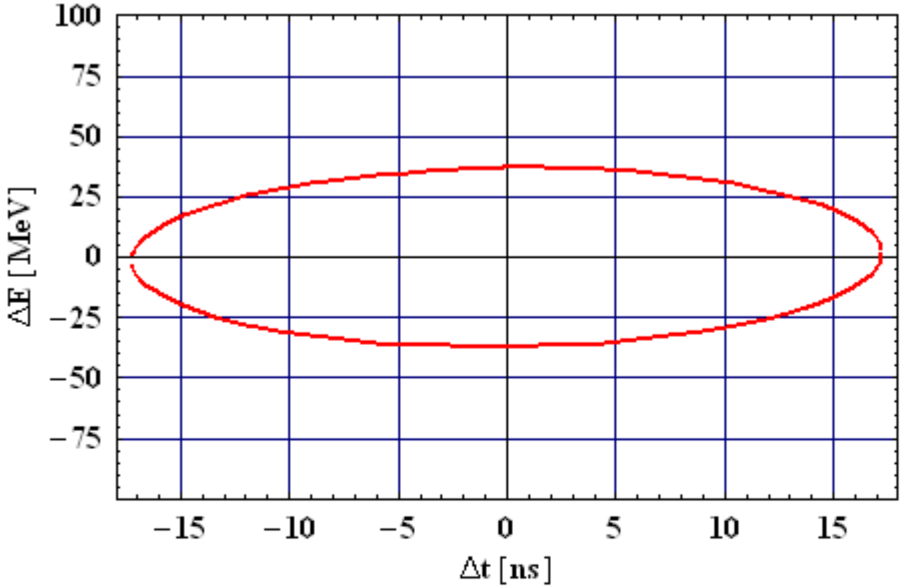
- ◆ Numerical (analytical) result for the case of the nTOF bunch in the CERN PS



$T_c \approx 1.9$ ms

$\varepsilon_L = 2$ eVs





Double RF systems

- Show that the motion of the particles can be tracked turn by turn using the recurrence relation (between turn *n* and turn *n*+1)

RF voltage of the 2nd harmonic

Harmonic number of the 2nd harmonic

$$\Delta E_{n+1} = \Delta E_n + e\,\hat{V}_{RF}\left[\sin\phi_n - \sin\phi_s + \frac{V_{RF2}}{\hat{V}_{RF}}\left\{\sin\left[\phi_{s2} + \frac{h_2}{h}\left(\phi_n - \phi_s\right)\right] - \sin\phi_{s2}\right\}\right]$$

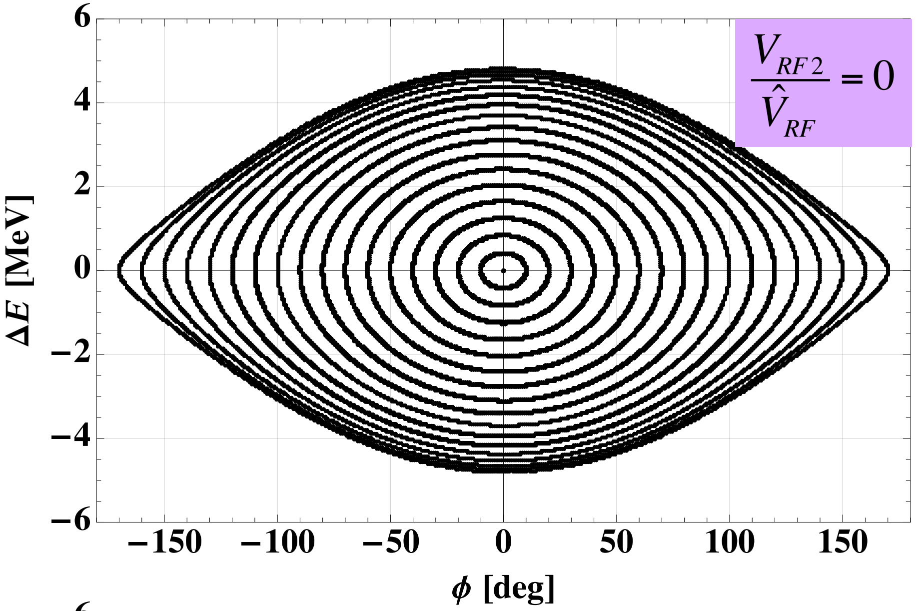
$$\phi_{n+1} = \phi_n - \frac{2\,\pi\,h\,\eta}{\beta_s^2\,E_s}\,\Delta E_{n+1}$$

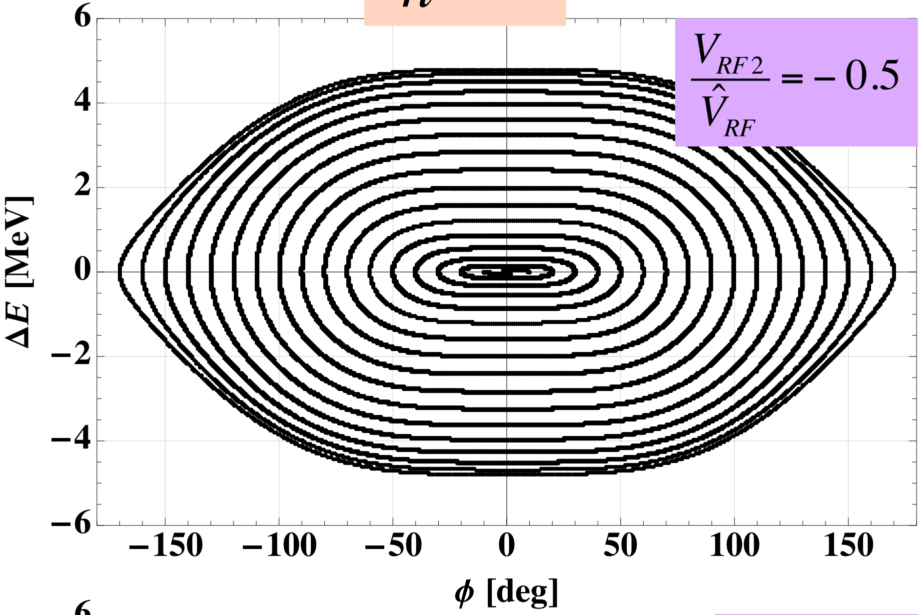
Synchronous phase of the 2nd harmonic

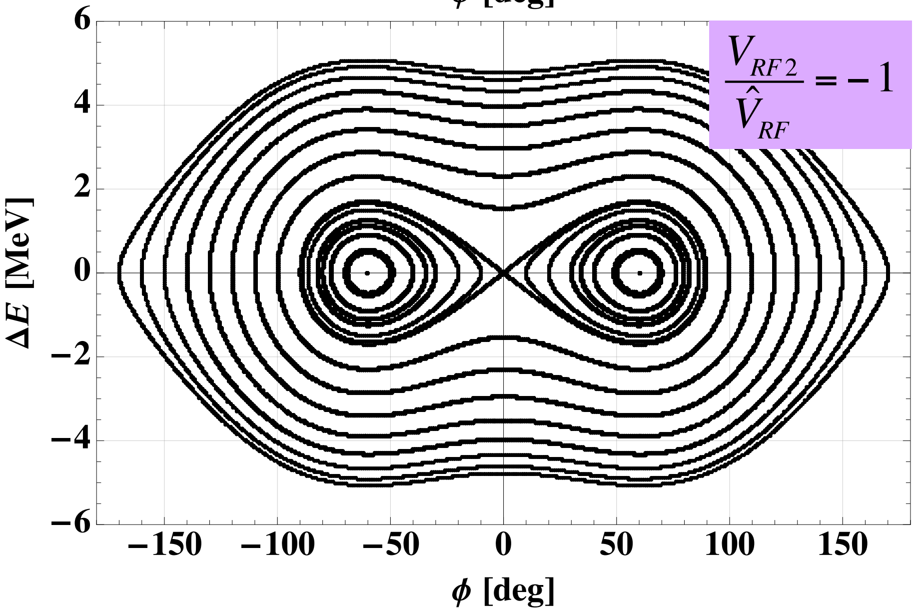
Double RF systems

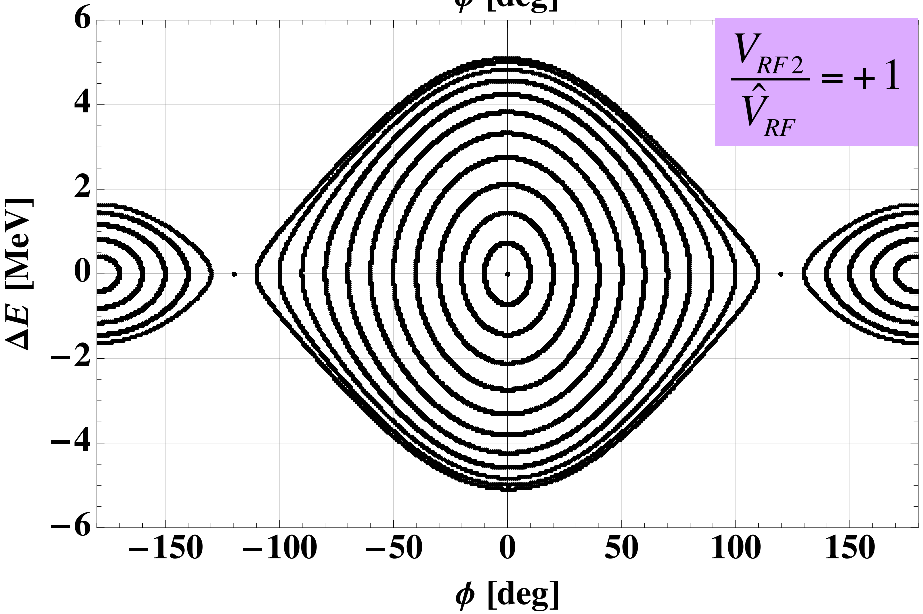
$$\frac{h_2}{h} = 2$$

$$\phi_s = \phi_{s2} = 0$$









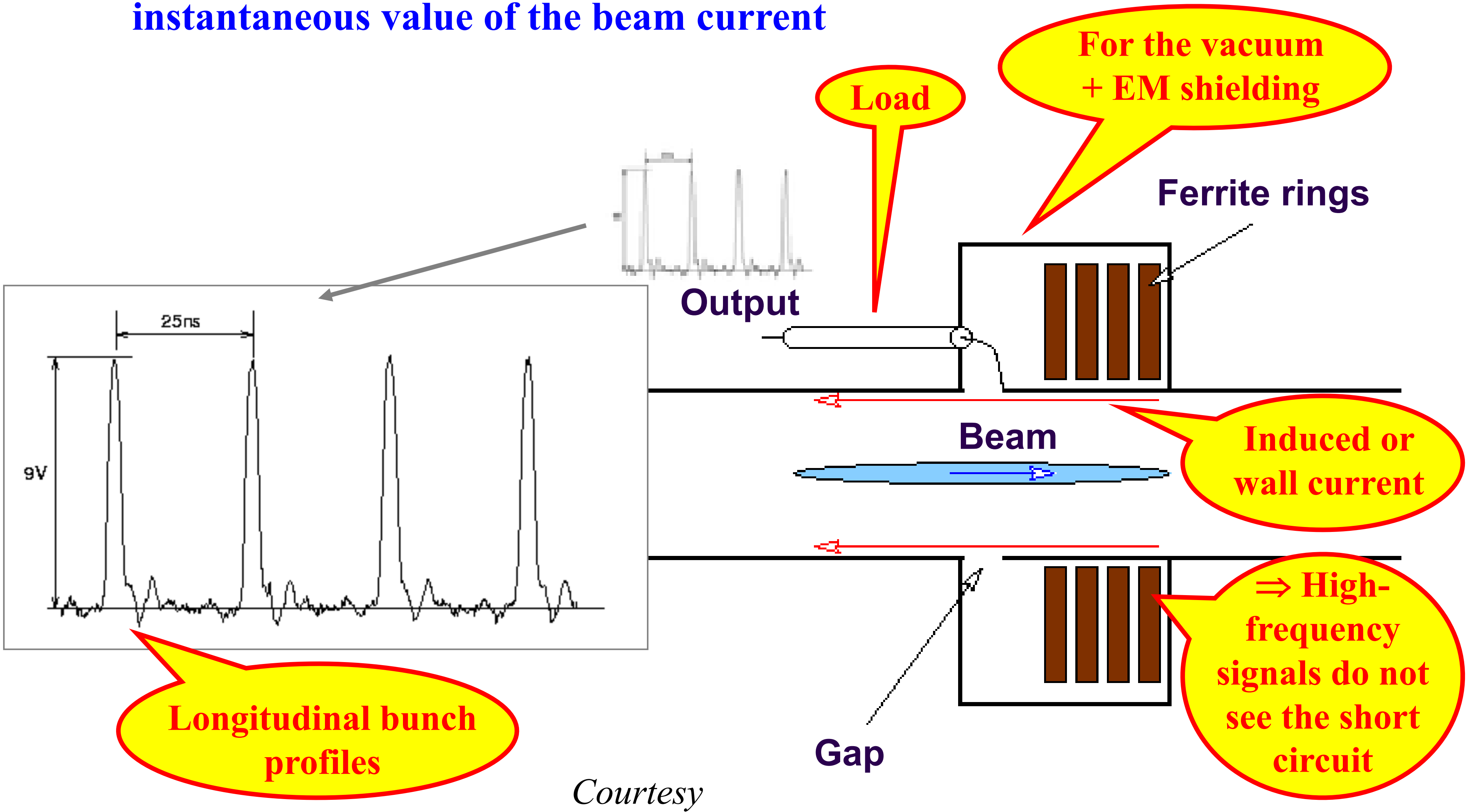
LESSON V

Measurement of the longitudinal bunch profile and Tomography

The pyHEADTAIL simulation code (by Benoit Salvant)

Measurement of the longitudinal bunch profile

=> **WALL CURRENT MONITOR = Device used to measure the instantaneous value of the beam current**



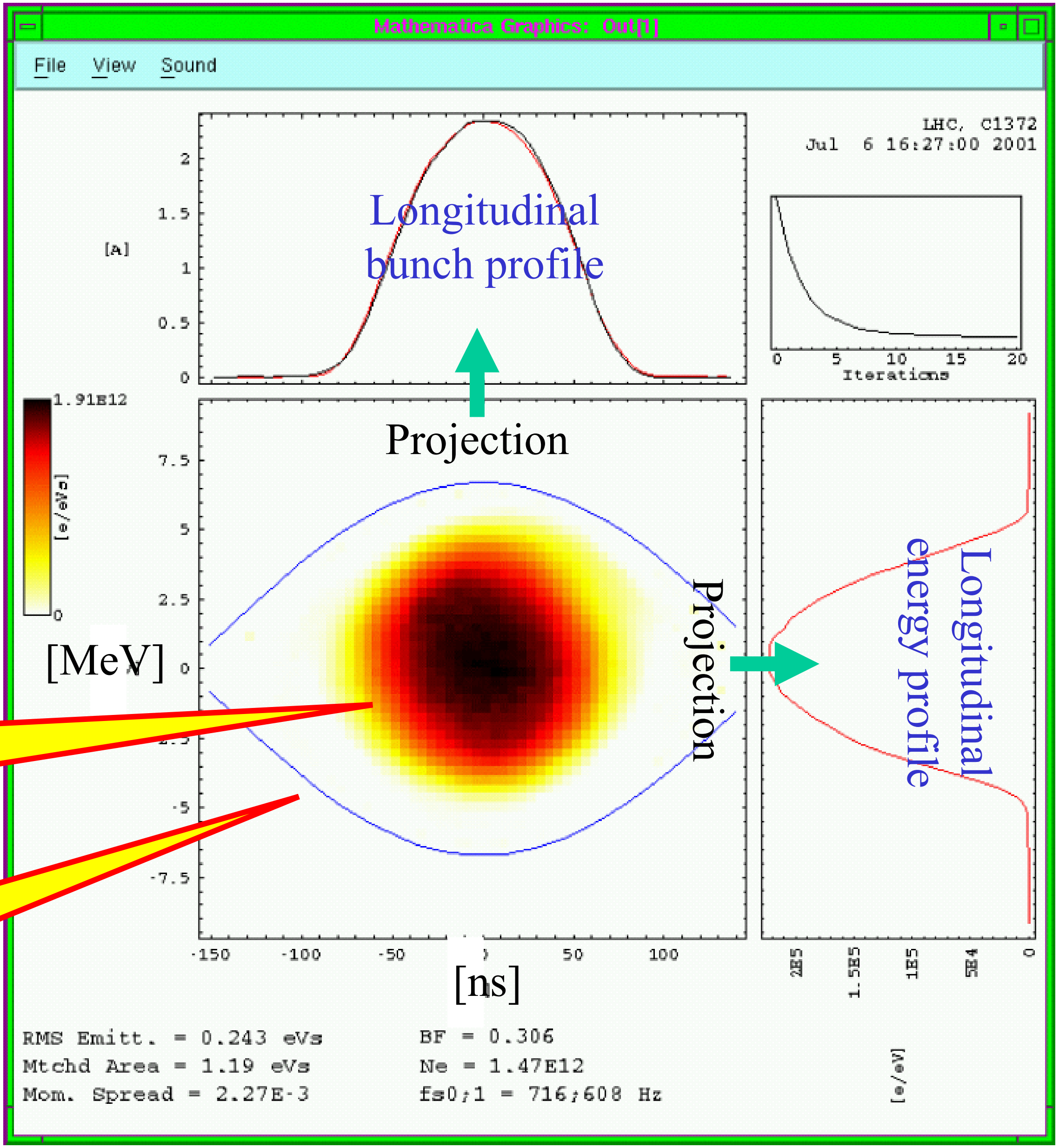
Courtesy J. Belleman

A Wall Current Monitor

Tomography

TOMOSCOPE (developed by S. Hancock, CERN/BE/RF)

The aim of **TOMOGRAPHY** is to estimate an unknown distribution (here the 2D longitudinal distribution) using only the information in the bunch profiles



```
RMS Emitt. = 0.243 eVs        BF = 0.306
Mtchd Area = 1.19 eVs         Ne = 1.47E12
Mom. Spread = 2.27E-3         fs0;1 = 716;608 Hz
```

Surface = Longitudinal EMITTANCE of the bunch = ε_L [eV.s]

Surface = Longitudinal ACCEPTANCE of the bucket

The pyHEADTAIL simulation code

See Tutorial by Benoit Salvant