Measurement of transverse Emittance

The emittance characterizes the whole beam quality, assuming linear behavior as described by second order differential equation.

It is defined within the phase space as: $\varepsilon_x = \frac{1}{\pi}\int_A dx\,dx'$

The measurement is based on determination of:

either profile width $\boldsymbol{\sigma_x}$ and angular width $\boldsymbol{\sigma_x'}$ at one location
or $\boldsymbol{\sigma_x}$ at different locations and linear transformations.

Different devices are used at transfer lines:

- Lower energies E_{kin} < 100 MeV/u: slit-grid device, pepper-pot (suited in case of non-linear forces).
- All beams: Quadrupole variation & 'three grid' method using linear transformations (**not** well suited in the presence of non-linear forces)

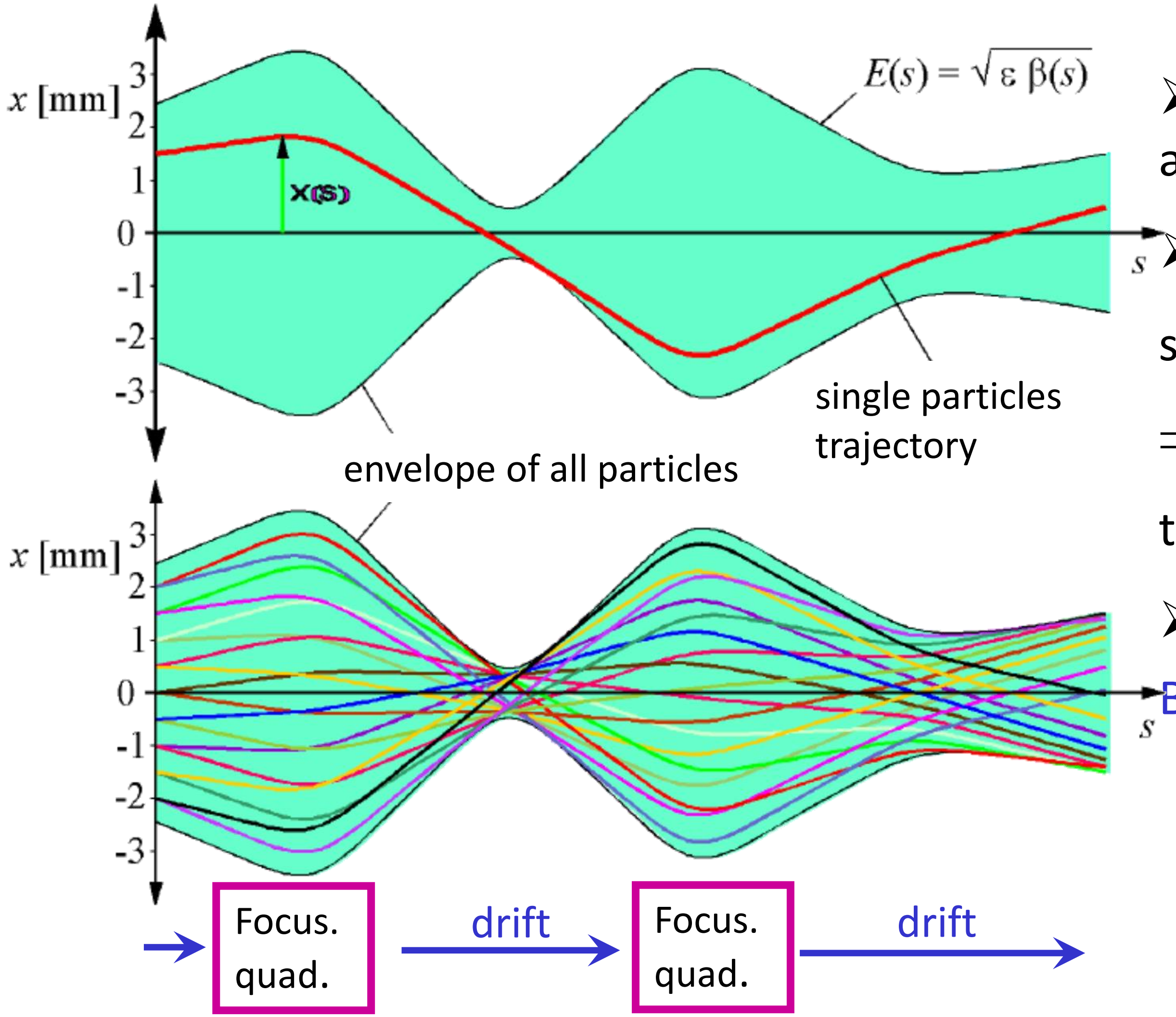
Synchrotron: lattice functions results in stability criterion

$\Rightarrow$ beam width delivers emittance: $\varepsilon_x = \frac{1}{\beta_x(s)}\left[\sigma_x^2 - \left(D(s)\frac{\Delta p}{p}\right)\right]$ and $\varepsilon_y = \frac{\sigma_y^2}{\beta_y(s)}$

Outline:

- Definition and some properties of transverse emittance
- Slit-Grid device: scanning method
- Quadrupole strength variation and position measurement
- Summary

Excurse: Particle Trajectory and Characterization of many Particles



- ➢ Single particle trajectories are forming a beam
- ➢ They have a distribution of start positions and angles

⇒ Characteristic quantity is the **beam envelope**

- ➢ **Goal:**

Behavior of whole ensemble

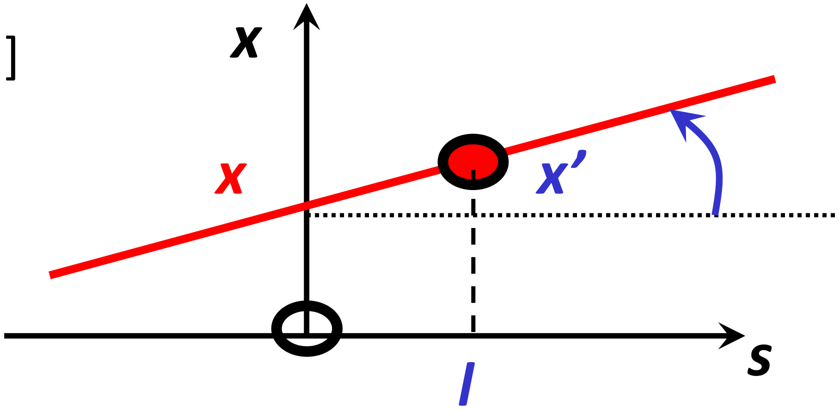
Plot: Wille

Excurse: Definition of Offset and Divergence

Horizontal and vertical coordinates at $s = 0$:

- x **:** Offset from reference orbit in [mm]
- x'**:** Angle of trajectory in unit [mrad]

  $x' = dx / ds$



Assumption: par-axial beams:

- x is small compared to ρ_0
- Small angle with $p_x / p_s << 1$

Longitudinal coordinate:

- Longitudinal orbit difference: $l = - v_0 \cdot (t - t_0)$ in unit [mm]
- Momentum deviation: $\delta = (p - p_0) / p_0$ sometimes in unit [mrad] = [‰]

For **continuous** beam: l has no meaning $\Rightarrow$ set $l \equiv 0$ **!**

Reference particle: No offset $x \equiv y \equiv l \equiv 0$ & no 'angle' $x' \equiv y' \equiv \delta \equiv 0$ for all s

Excurse: Definition of Coordinates and Properties of the Transfer Matrix

The basic vector is 6 dimensional:

$$\vec{x}(s) = \begin{pmatrix} x \\ x' \\ y \\ y' \\ l \\ \delta \end{pmatrix} = \begin{pmatrix} \text{hori. spatial deviation} \\ \text{horizontal divergence} \\ \text{vert. spatial deviation} \\ \text{vertical divergence} \\ \text{longitudinal deviation} \\ \text{momentum deviation} \end{pmatrix} = \begin{pmatrix} [\text{mm}] \\ [\text{mrad}] \\ [\text{mm}] \\ [\text{mrad}] \\ [\text{mm}] \\ [‰] \end{pmatrix}$$

The transformation of a single particle from a location s_0 to s_1 is given by the Transfer Matrix R:

$$\vec{x}(s_1) = \mathrm{R}(s) \cdot \vec{x}(s_0)$$

- Matrix elements: coupling between components ***R_{11}= (x|x), R_{12}= (x|x‘), R_{13}= (x|y)…***

- <u>If</u> all forces are symmetric along the reference orbit than the horizontal and vertical plane are decoupled:
  ⇒ sub-matrix is sufficient

$$\mathrm{R} = \begin{pmatrix} (x|x) & (x|x') & 0 & 0 & 0 & (x|\delta) \\ (x'|x) & (x'|x') & 0 & 0 & 0 & (x'|\delta) \\ 0 & 0 & (y|y) & (y|y') & 0 & 0 \\ 0 & 0 & (y'|y) & (y'|y') & 0 & 0 \\ (l|x) & (l|x') & 0 & 0 & 1 & (l|l) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- Sub-matrix
  here shown for drift ***L***:

$$\boxed{\mathrm{R}_x = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}} \quad \boxed{\mathrm{R}_y = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}} \quad \boxed{\mathrm{R}_s = \begin{pmatrix} 1 & L/\gamma^2 \\ 0 & 1 \end{pmatrix}}$$

Excurse: Some Examples for linear Transformations

The 2-dim sub-space (***x,x'***) can be used in case there is coupling like dispersion $R_{16}=(x\,|\,\delta)=0$

Important examples are:

- Drift with length ***L***: $\mathbf{R}_{\mathrm{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$
- Horizontal **focusing** with quadrupole constant ***k*** end effective length ***l***:

$$\mathbf{R}_{\mathrm{focus}} = \begin{pmatrix} \cos\sqrt{k}\,l & \frac{1}{\sqrt{k}}\sin\sqrt{k}\,l \\ -\sqrt{k}\cdot\sin\sqrt{k}\,l & \cos\sqrt{k}\,l \end{pmatrix} \quad \Rightarrow \quad \mathrm{R}^{\text{thin lens}}_{\text{focus}} = \begin{pmatrix} 1 & 0 \\ -1/\,f & 1 \end{pmatrix}$$

- Horizontal **de-focusing** with quadrupole constant ***k*** end effective length ***l***:

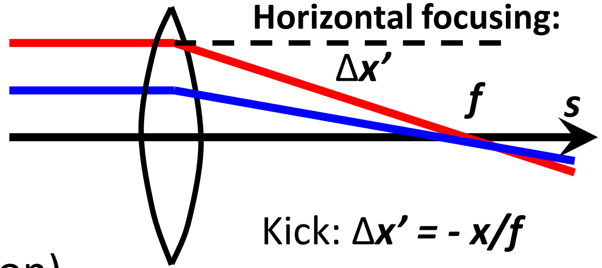
$$\mathbf{R}_{\mathrm{de-focus}} = \begin{pmatrix} \cosh\sqrt{k}\,l & \frac{1}{\sqrt{k}}\sinh\sqrt{k}\,l \\ \sqrt{k}\cdot\sinh\sqrt{k}\,l & \cosh\sqrt{k}\,l \end{pmatrix} \quad \Rightarrow \quad \mathrm{R}^{\text{thin lens}}_{\text{de-focus}} = \begin{pmatrix} 1 & 0 \\ 1/\,f & 1 \end{pmatrix}$$

Ideal quad.: field gradient $g = B_{pole}/a$, B_{pole} field at poles, ***a*** aperture

→ quadrupole constant $k = |\,g\,|\,/\,(B\rho)_0$

Thin lens approximation: $l \rightarrow 0 \;\Rightarrow\; kl \rightarrow \mathrm{const} \;\Rightarrow\; kl \equiv 1/f$

⇒ simple transfer matrix (math. proof by 1st order Taylor expansion)



Excurse: Definition of Beam Matrix

The basic vector is 6 dimensional:

$$\vec{x}(s) = \begin{pmatrix} x \\ x' \\ y \\ y' \\ l \\ \delta \end{pmatrix} = \begin{pmatrix} \text{hori. spatial deviation} \\ \text{horizontal divergence} \\ \text{vert. spatial deviation} \\ \text{vertical divergence} \\ \text{longitudinal deviation} \\ \text{momentum deviation} \end{pmatrix} = \begin{pmatrix} [\text{mm}] \\ [\text{mrad}] \\ [\text{mm}] \\ [\text{mrad}] \\ [\text{mm}] \\ [‰] \end{pmatrix}$$

The transformation of a single particle from a location s_0 to s_1 is given by the Transfer Matrix R:

$$\vec{x}(s_1) = \mathrm{R}(\mathrm{s}) \cdot \vec{x}(s_0)$$

The transformation of a the envelope from a location s_0 to s_1 is given by the Beam Matrix σ:

$$\sigma(s_1) = \mathrm{R}(\mathrm{s}) \cdot \sigma(s_0) \cdot \mathrm{R}^{\mathrm{T}}(\mathrm{s})$$

6-dim Beam Matrix with *decoupled* hor. & vert. plane:

$$\boldsymbol{\sigma} = \begin{pmatrix} \boxed{\sigma_{11}} & \sigma_{12} & 0 & 0 & \sigma_{15} & \sigma_{16} \\ \sigma_{12} & \sigma_{22} & 0 & 0 & \sigma_{25} & \sigma_{26} \\ 0 & 0 & \boxed{\sigma_{33}} & \sigma_{34} & 0 & 0 \\ 0 & 0 & \sigma_{34} & \sigma_{44} & 0 & 0 \\ \sigma_{15} & \sigma_{25} & 0 & 0 & \boxed{\sigma_{55}} & \sigma_{56} \\ \sigma_{16} & \sigma_{26} & 0 & 0 & \sigma_{56} & \sigma_{66} \end{pmatrix}$$

horizontal
vertical
longitudinal
hor.-long. coupling
→ 13 values

Beam width for the three coordinates:

$$x_{rms} = \sqrt{\sigma_{11}}$$

$$y_{rms} = \sqrt{\sigma_{33}}$$

$$l_{rms} = \sqrt{\sigma_{55}}$$

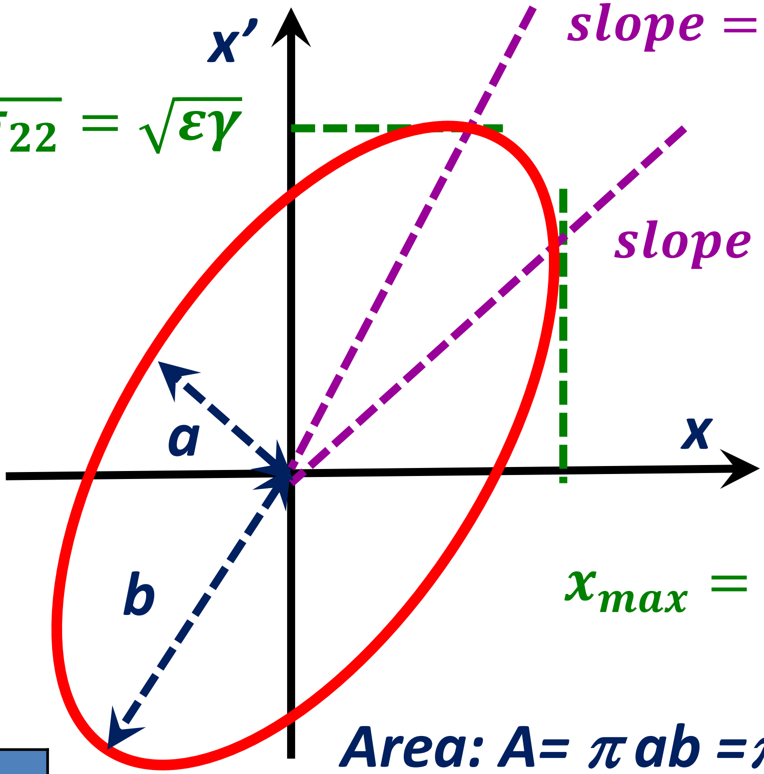
Horizontal beam matrix:

$$\sigma_{11} = \langle x^2 \rangle$$

$$\sigma_{12} = \langle x\, x' \rangle$$

$$\sigma_{22} = \langle x'^2 \rangle$$

Excurse: Geometrical Parameters of Phase Space Ellipsoid



$$slope = \frac{\sigma_{22}}{\sigma_{12}} = -\frac{\gamma}{\alpha}$$

$$x'_{max} = \sqrt{\sigma_{22}} = \sqrt{\varepsilon\gamma}$$

$$slope = \frac{\sigma_{12}}{\sigma_{11}} = -\frac{\alpha}{\beta}$$

$$x_{max} = \sqrt{\sigma_{11}} = \sqrt{\varepsilon\beta}$$

The phase space distribution is

described by an ellipse

with the characteristics parameters

Area: A= π ab =$\pi\,\varepsilon$

Corresponding parameters:

Beam Matrix	Twiss Parameter	Statistics
σ_{11}	$\varepsilon\beta$	$<x^2>$
σ_{22}	$\varepsilon\gamma$	$<x'^2>$
σ_{12}	$-\varepsilon\alpha$	$<xx'>$

The determinate is preserved:

$$\varepsilon = \sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \Leftrightarrow 1 = \beta\gamma - \alpha^2$$

Proof: $\sigma(s_1) = R \cdot \sigma(s_0) \cdot R^T$

$$\Rightarrow \det(\sigma(s_1)) = \det(\mathrm{R}) \cdot \det(\sigma(s_0)) \cdot \det(\mathrm{R}^\mathrm{T})$$

$$= \det(\sigma(s_0))$$

Excurse: The Emittance for Gaussian and non-Gaussian Beams

The beam distribution can be non-Gaussian, e.g. at:

- beams behind ion source
- space charged dominated beams at LINAC & synchrotron
- cooled beams in storage rings

General description of emittance

using terms of 2-dim distribution:

$$\varepsilon_{rms} = \sqrt{\underbrace{\langle x^2\rangle\langle x'^2\rangle}_{\textbf{Variances}} - \underbrace{\langle xx'\rangle^2}_{\textbf{Covariance i.e. correlation}}}$$

It describes the value for 1 standard derivation

$$\langle x\rangle \equiv \mu = \frac{\int\int x\cdot\rho(x,x')\,dxdx'}{\int\int \rho(x,x')\,dxdx'} \qquad \langle x'\rangle \equiv \mu' = \frac{\int\int x'\cdot\rho(x,x')\,dxdx'}{\int\int \rho(x,x')\,dxdx'}$$

$$\langle x^n\rangle = \frac{\int\int (x-\mu)^n\cdot\rho(x,x')\,dxdx'}{\int\int \rho(x,x')\,dxdx'} \qquad \langle x'^n\rangle = \frac{\int\int (x'-\mu')^n\cdot\rho(x,x')\,dxdx'}{\int\int \rho(x,x')\,dxdx'}$$

$$\text{covariance}\;:\; \langle xx'\rangle = \frac{\int\int (x-\mu)(x'-\mu')\cdot\rho(x,x')\,dxdx'}{\int\int \rho(x,x')\,dxdx'}$$

For discrete distribution:

$$\langle x\rangle = \frac{\sum\limits_{i,j}\rho(i,j)\cdot x_i x'_j}{\sum\limits_{i,j}\rho(i,j)}$$

and correspondingly for all other moments

Definition of transverse Emittance

The emittance characterizes the whole beam quality: $\varepsilon_x = \frac{1}{\pi}\int_A dx\,dx'$

Ansatz:

Beam matrix at one location: $\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} = \varepsilon \cdot \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$ with $\vec{x} = \begin{pmatrix} x \\ x' \end{pmatrix}$

It describes a 2-dim probability distr.

The value of emittance is:

$$\varepsilon_x = \sqrt{\det \boldsymbol{\sigma}} = \sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2}$$

For the profile and angular measurement:

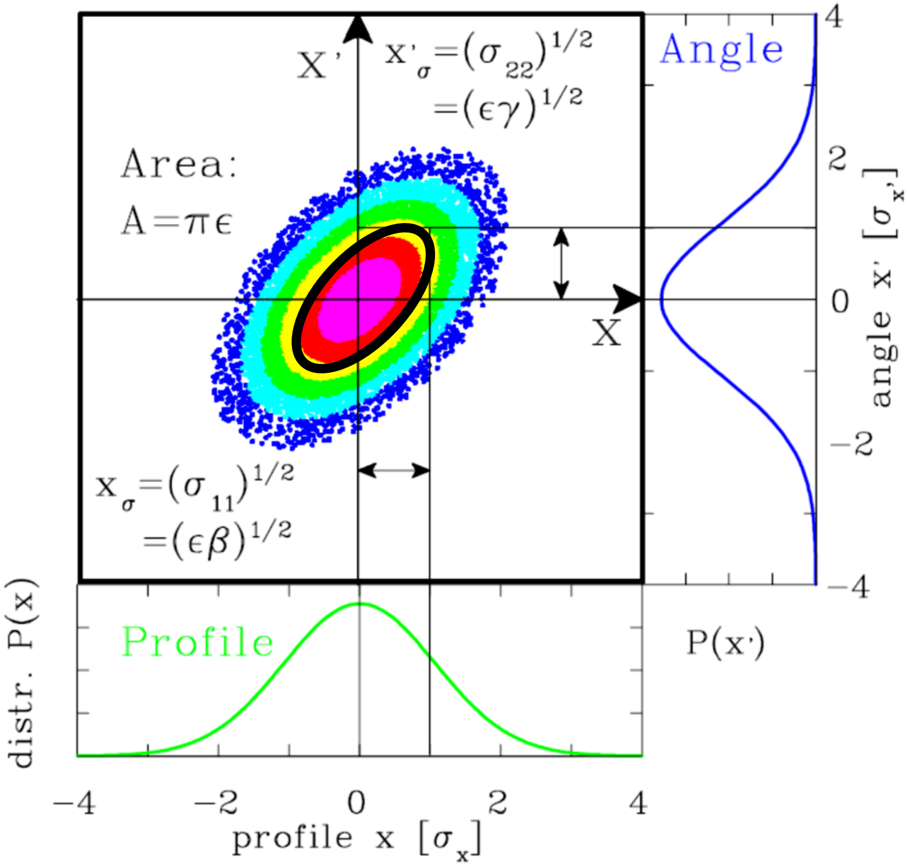
$$x_\sigma = \sqrt{\sigma_{11}} = \sqrt{\varepsilon\beta} \quad \text{and}$$

$$x'_\sigma = \sqrt{\sigma_{22}} = \sqrt{\varepsilon\gamma}$$

Geometrical interpretation:

All points $\boldsymbol{x}$ fulfilling $\boldsymbol{x}^t \cdot \boldsymbol{\sigma}^{-1} \cdot \boldsymbol{x} = 1$ are located on a **ellipse**

$$\sigma_{22}x^2 - 2\sigma_{12}xx' + \sigma_{11}x'^2 = \det \boldsymbol{\sigma} = \varepsilon_x^2$$



The Emittance for Gaussian Beams

The density function for a 2-dim Gaussian distribution is:

$$\rho(x,x') = \frac{1}{2\pi\epsilon}\exp\left[-\frac{1}{2}\;\vec{x}^T\;\sigma^{-1}\;\vec{x}\;\right]$$

$$= \frac{1}{2\pi\epsilon}\exp\left[\frac{-1}{2\det\sigma}\left(\sigma_{22}x^2 - 2\sigma_{12}xx' + \sigma_{11}x'^2\right)\right]$$

It describes an ellipse with the characteristics

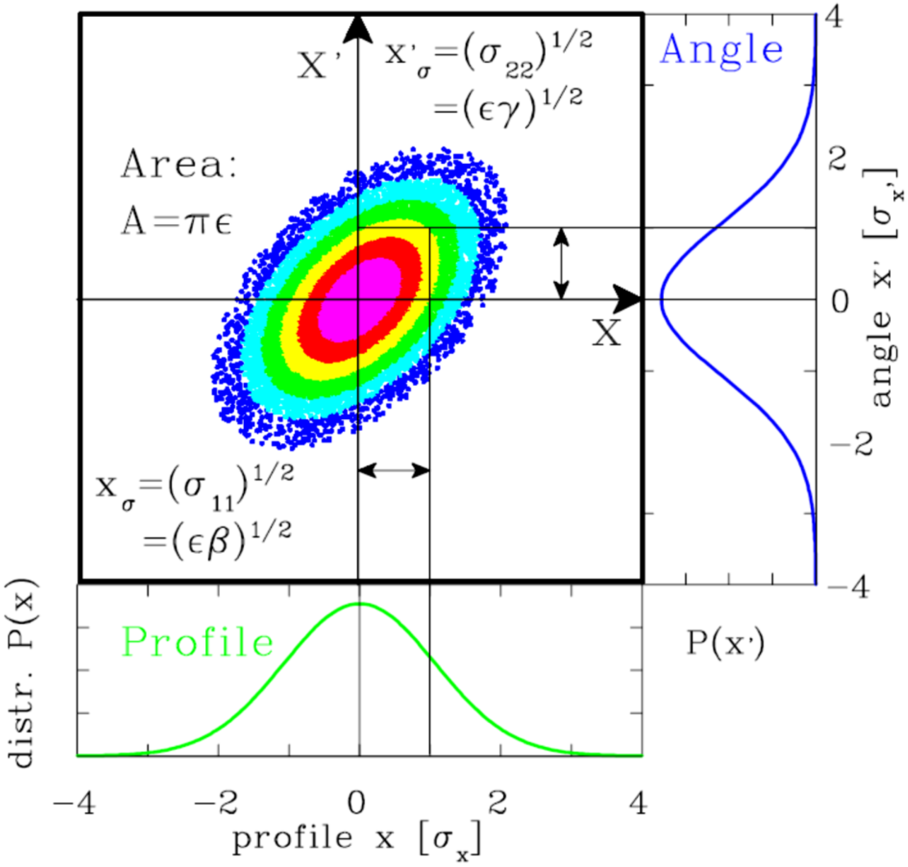
profile and angle Gaussian distribution of width

$$x_\sigma \equiv \sqrt{\left\langle x^2 \right\rangle} = \sqrt{\sigma_{11}} \quad \text{and}$$

$$x'_\sigma \equiv \sqrt{\left\langle x'^2 \right\rangle} = \sqrt{\sigma_{22}}$$

and the correlation or covariance

$$\mathrm{cov} \equiv \sqrt{\left\langle xx' \right\rangle} = \sqrt{\sigma_{12}}$$



For $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ it is $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

assuming $\det(\mathbf{A}) = ad\text{-}bc \neq 0 \Leftrightarrow$ matrix invertible

The Emittance for Gaussian and non-Gaussian Beams

The beam distribution can be non-Gaussian, e.g. at:

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- space charged dominated beams at LINAC & synchrotron
- cooled beams in storage rings

General description of emittance

using terms of 2-dim distribution:

$$\varepsilon_{rms} = \sqrt{\underbrace{\langle x^2\rangle\langle x'^2\rangle}_{\textbf{Variances}} - \underbrace{\langle xx'\rangle^2}_{\textbf{Covariance i.e. correlation}}}$$

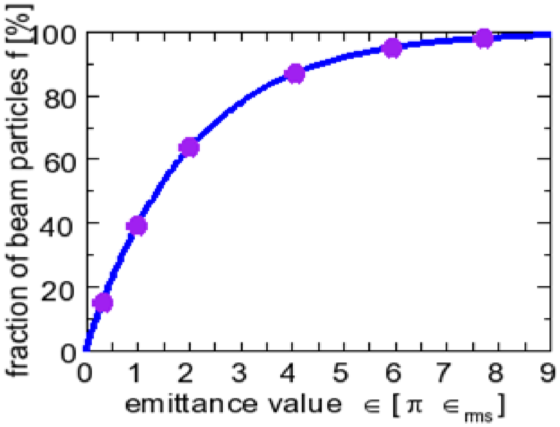
It describes the value for 1 standard derivation

For Gaussian beams only: ε_{rms} ↔ interpreted as area containing a fraction ***f*** of ions:

$$\varepsilon(f) = -2\pi\varepsilon_{rms}\cdot\ln(1-f)$$

Care:

No common definition of emittance concerning the fraction ***f***



Emittance ε(f)	Fraction f
$1\cdot\varepsilon_{rms}$	15 %
$\pi\cdot\varepsilon_{rms}$	39 %
$2\pi\cdot\varepsilon_{rms}$	63 %
$4\pi\cdot\varepsilon_{rms}$	86 %
$8\pi\cdot\varepsilon_{rms}$	98 %

Outline:

- Definition and some properties of transverse **emittance**
- **Slit-Grid device: scanning method**

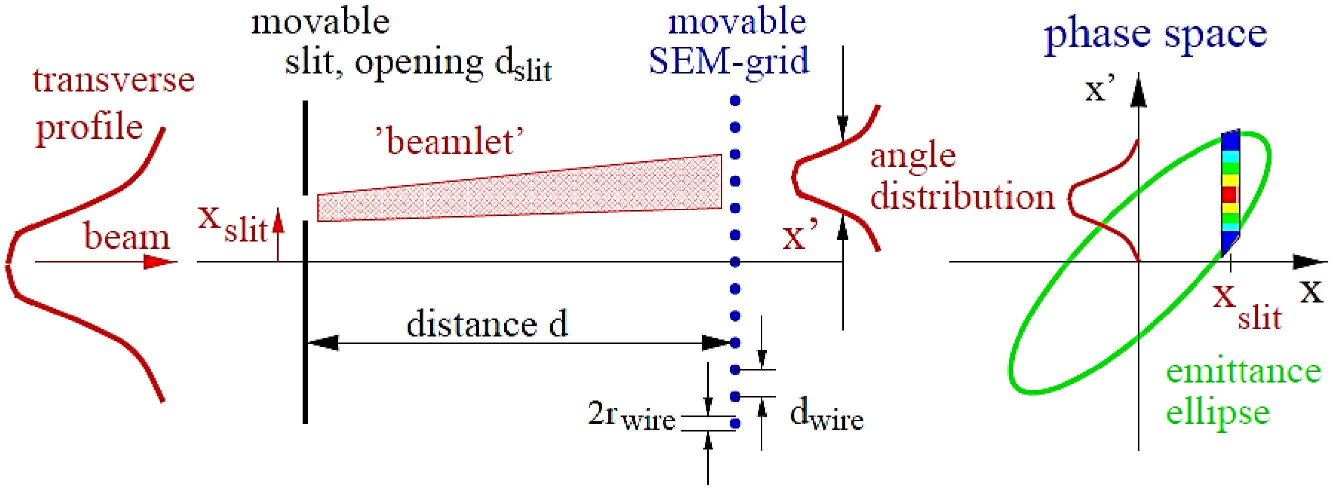
  scanning slit → beam position & grid → angular distribution
- **Quadrupole strength variation and position measurement**
- **Summary**

The Slit-Grid Measurement Device

Slit-Grid: Direct determination of position and angle distribution.

Used for protons with $E_{kin} < 100$ MeV/u $\Rightarrow$ range $R < 1$ cm.

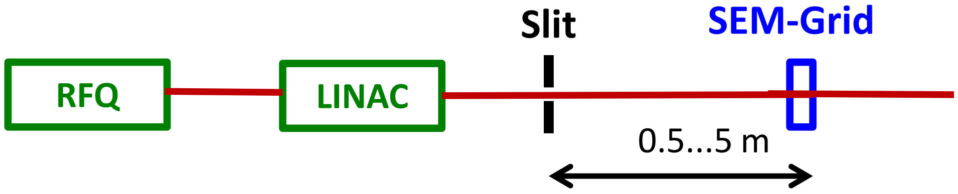
Hardware **Analysis**



Slit: position ***P(x)*** with typical width: 0.1 to 0.5 mm

Distance: typ. 0.5 to 5 m (depending on beam energy 0. 1 ... 100 MeV)

SEM-Grid: angle distribution ***P(x')***



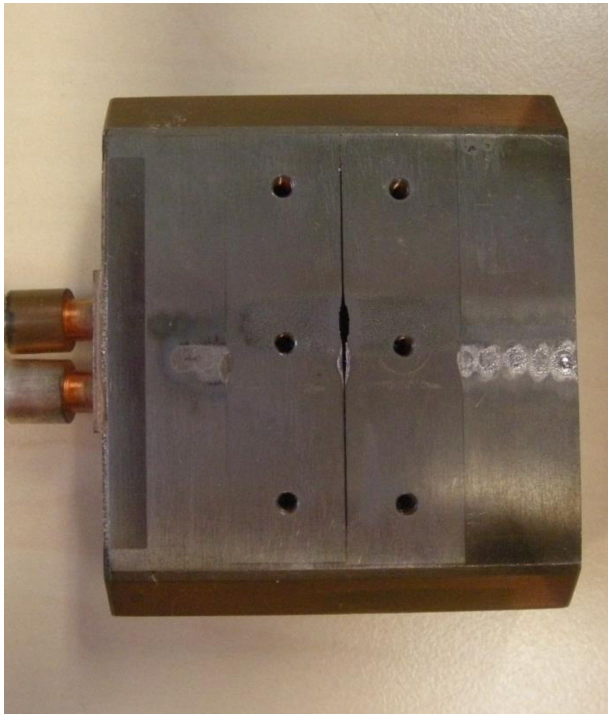
Slit & SEM-Grid

Slit with e.g. 0.1 mm thickness

→ Transmission only from ***Δx***.

Example: Slit of width 0.1 mm (defect)

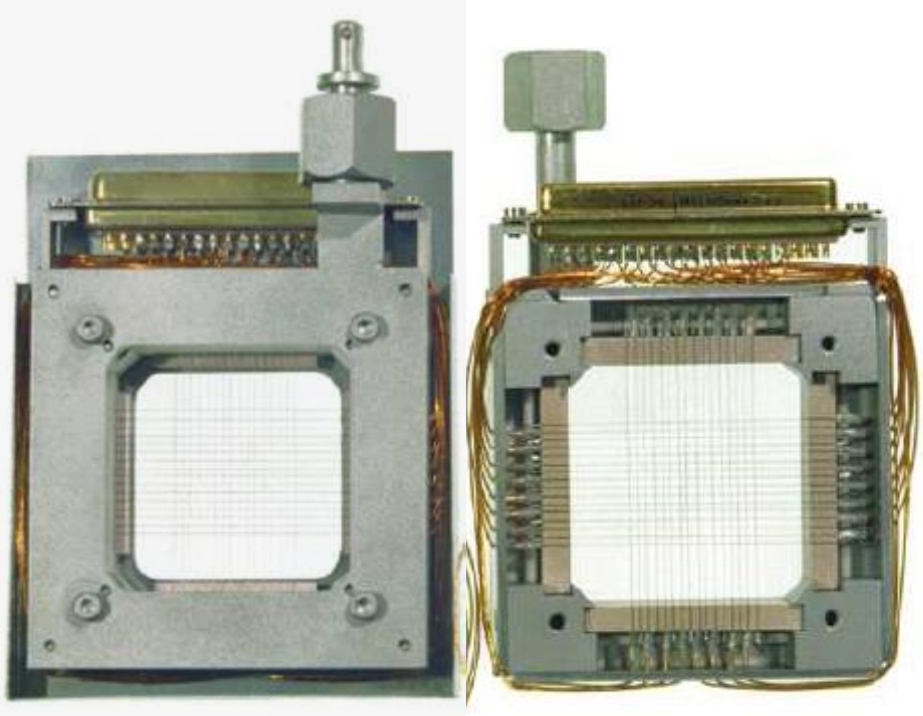
Moved by stepping motor, 0.1 mm resolution



Beam surface interaction: e^- emission

→ measurement of current.

Example: 15 wire spaced by 1.5 mm:



Each wire is equipped with one I/U converter

different ranges settings by $\boldsymbol{R_i}$

→ very large dynamic range up to 10^6.

Display of Measurement Results

The distribution of the ions is depicted as a function of

- Position [mm]
- Angle [mrad]

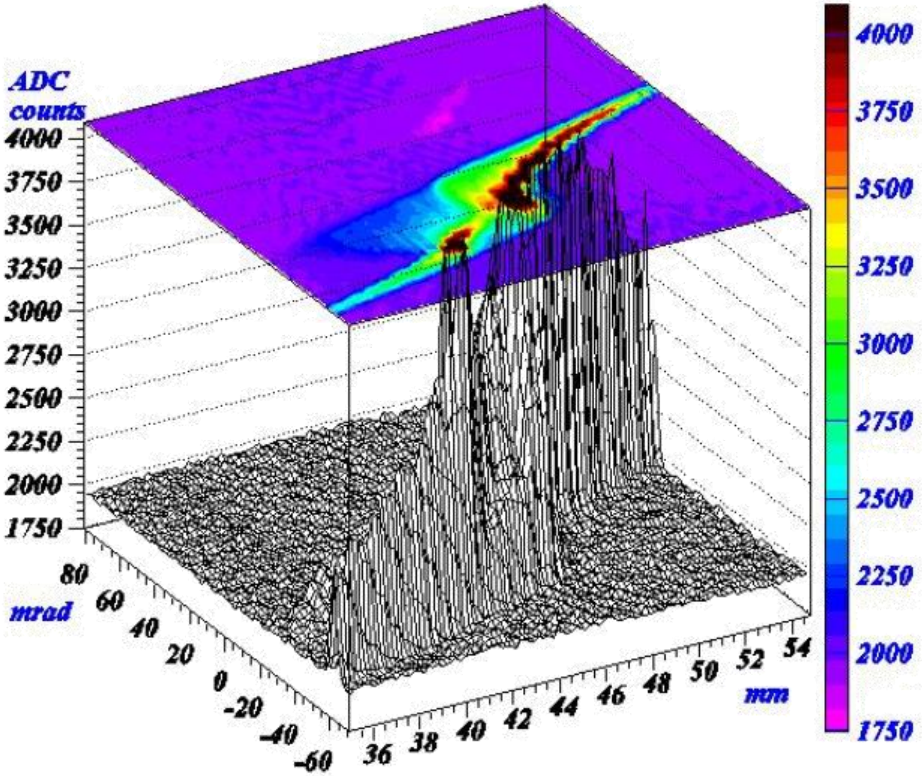
The distribution can be visualized by

- Mountain plot
- Contour plot

Calc. of 2nd moments $<x^2>$, $<x'^2>$ & $<xx'>$

Emittance value ε_{rms} from

$$\varepsilon_{rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$



Problems:

- Finite **binning** results in limited resolution
- **Background** → large influence on $<x^2>$, $<x'^2>$ and $<xx'>$

Or fit of distribution i.e. ellipse to data

⇒ **Effective emittance only**

Beam: Ar^{4+}, 60 KeV, 15 μA

at Spiral2 Phoenix ECR source.

Plot from P. Ausset, DIPAC 2009

The Resolution of a Slit-Grid Device

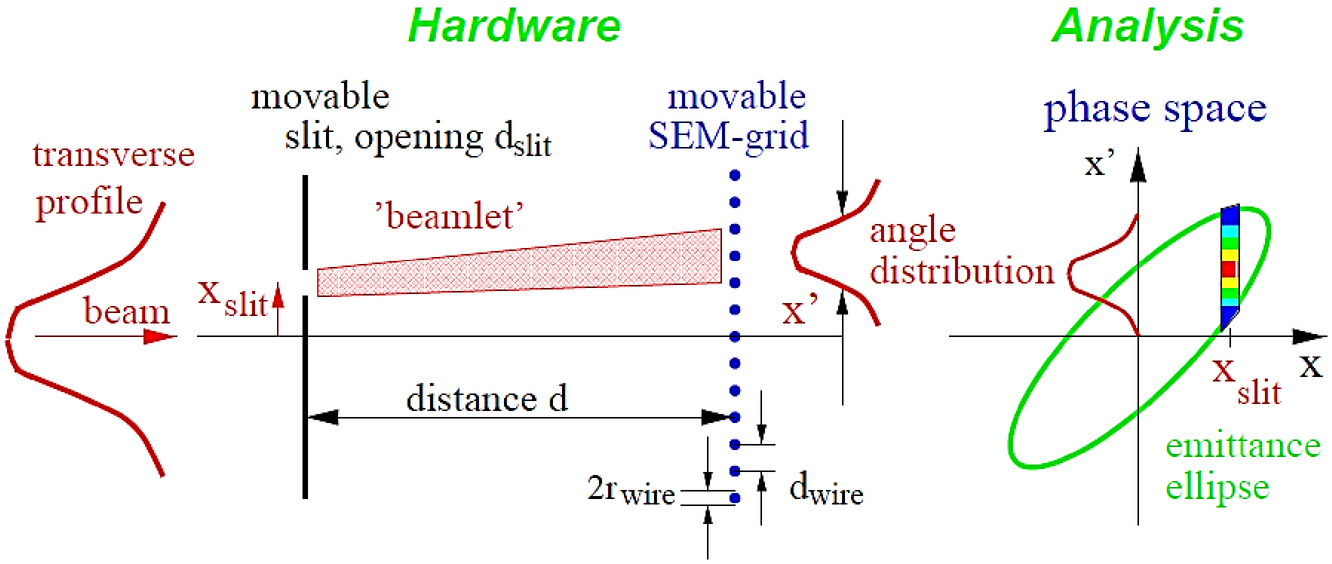
The width of the slit d_{slit} gives the resolution in space $\Delta x = d_{slit}$.

The angle resolution is $\Delta x' = (d_{wire} + 2r_{wire})/d$

$\Rightarrow$ discretization element $\Delta x \cdot \Delta x'$.

By scanning the SEM-grid the angle resolution can be improved.

Problems for small beam sizes or parallel beams.

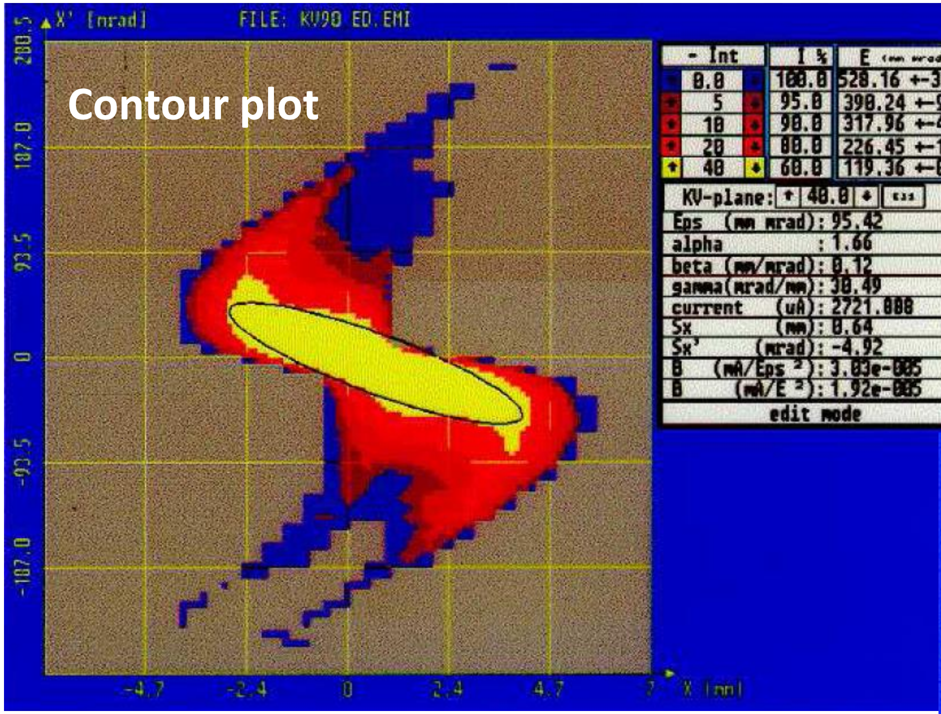


For pulsed LINACs: Only one measurement each pulse → long measuring time required.

Result of an Slit-Grid Emittance Measurement

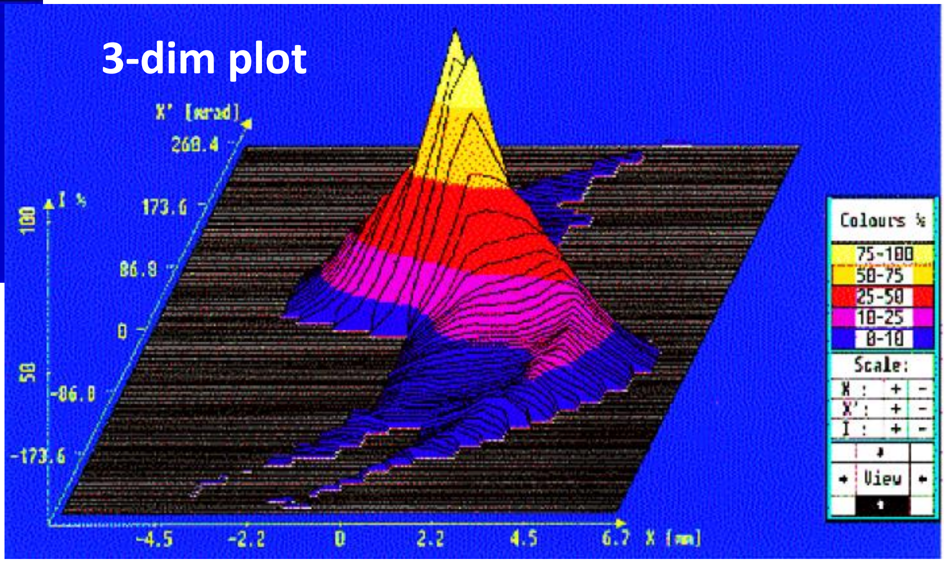
Result for a beam behind ion source: ➢ here aberration in quadrupoles due to large beam size

- ➢ different evaluation and plots possible
- ➢ can monitor any distribution


Contour plot

- Int	I %	E (mm mrad)
0.0	100.0	528.16 +-39.1
5	95.0	398.24 +-9.1
10	90.0	317.96 +-4.1
20	80.0	226.45 +-1.8
40	60.0	119.36 +-0.5

KV-plane: 40.0
Eps (mm mrad): 95.42
alpha : 1.66
beta (mm/mrad): 0.12
gamma(mrad/mm): 30.49
current (uA): 2721.000
Sx (mm): 0.64
Sx' (mrad): -4.92
B (mA/Eps ²): 3.03e-005
B (mA/E ²): 1.92e-005
edit mode


3-dim plot

Low energy ion beam:
→ well suited for emittance showing space-charge effects or aberrations.

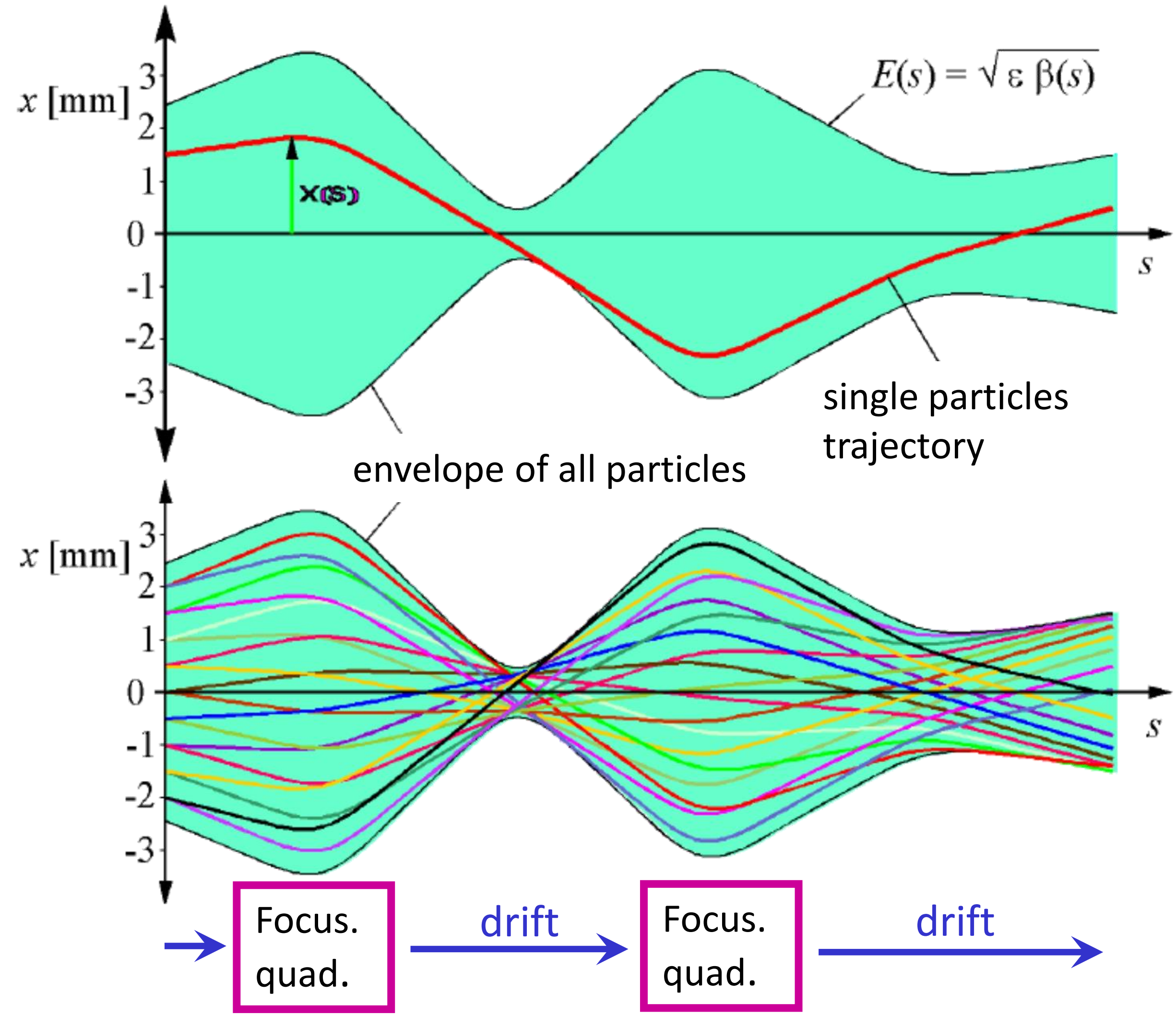
Outline:

- Definition and some properties of transverse emittance
- Slit-Grid device: scanning method

  scanning slit → beam position & grid → angular distribution
- **Quadrupole strength variation and position measurement**

  emittance from several profile measurement and beam optical calculation
- Summary

Excurse: Particle Trajectory and Characterization of many Particles



Plot: Wille

- ➢ Single particle trajectories are forming a beam
- ➢ They have a distribution of start positions and angles

⇒ Characteristic quantity is the **beam envelope**

Transformation of envelope

- ➢ **Goal:**

Behavior of whole ensemble Via ‘Beam Matrix’ σ

$$\boldsymbol{\sigma}(s) = \begin{pmatrix}\sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22}\end{pmatrix}$$

Excurse: Conservation of Emittance

Liouville's Theorem:

The phase space density can not changes with conservative e.g. linear forces.

The beam distribution at one location s_0 is described by the beam matrix $\boldsymbol{\sigma}(s_0)$

This beam matrix is transported from location s_0 to s_1 via the transfer matrix

$$\boxed{\sigma(s_1) = R \cdot \sigma(s_0) \cdot R^T}$$

6-dim beam matrix with *decoupled* horizontal, vertical and longitudinal plane:

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 & 0 & 0 & 0 \\ \sigma_{12} & \sigma_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} & \sigma_{34} & 0 & 0 \\ 0 & 0 & \sigma_{34} & \sigma_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{55} & \sigma_{56} \\ 0 & 0 & 0 & 0 & \sigma_{56} & \sigma_{66} \end{pmatrix}$$

Horizontal beam matrix:

$$\sigma_{11} = \langle x^2 \rangle$$

$$\sigma_{12} = \langle x\, x' \rangle$$

$$\sigma_{22} = \langle x'^2 \rangle$$

Beam width for the three coordinates:

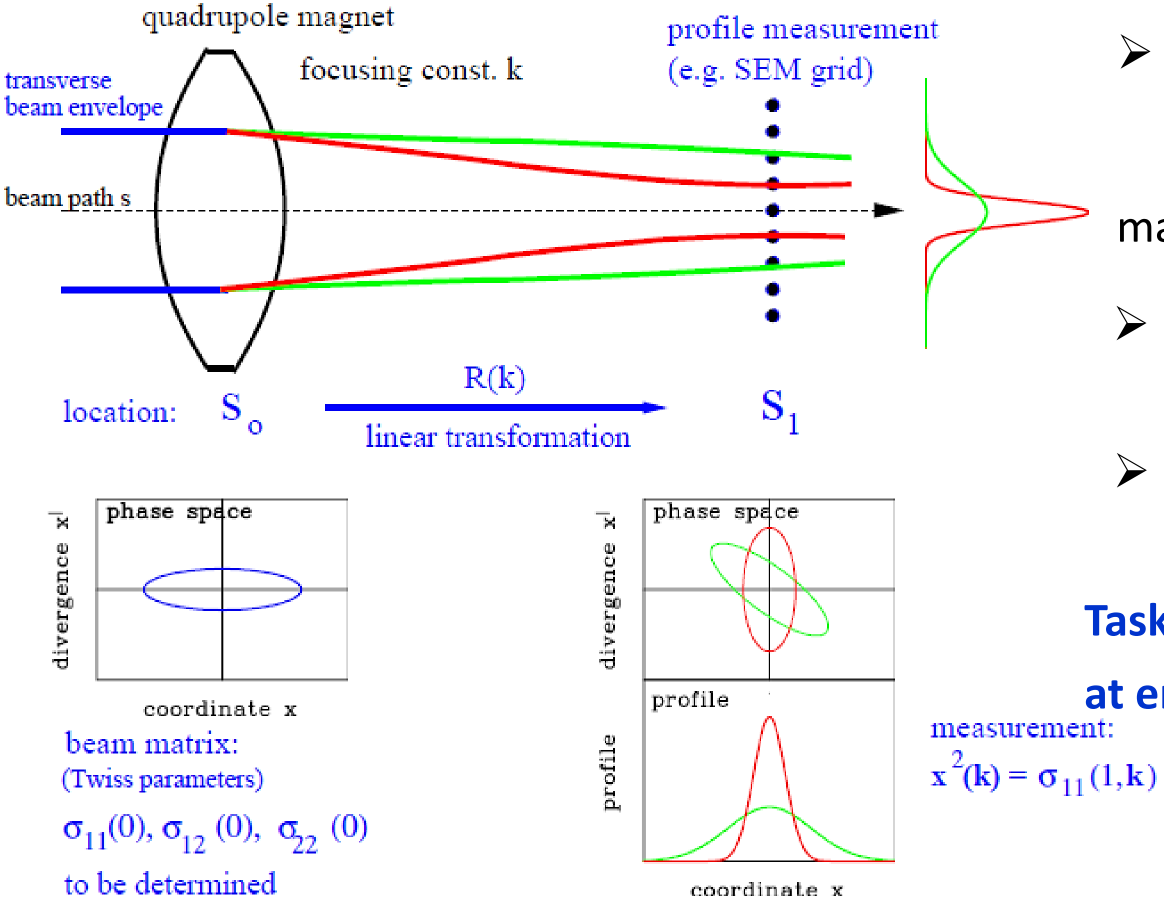
$$x_{rms} = \sqrt{\sigma_{11}} = \sqrt{\langle x^2 \rangle}$$

$$y_{rms} = \sqrt{\sigma_{33}} = \sqrt{\langle y^2 \rangle}$$

$$l_{rms} = \sqrt{\sigma_{55}} = \sqrt{\langle l^2 \rangle}$$

Emittance Measurement by Quadrupole Variation

From a profile determination, the emittance can be calculated via linear transformation, if a well known and constant distribution (e.g. Gaussian) is assumed.



- Measurement of beam width

$$x^2_{max} = \sigma_{11}(1, k)$$

matrix **R(*k*)** describes the focusing.

- With the drift matrix the transfer is

$$\mathbf{R}(k_i) = \mathbf{R}_{drift} \cdot \mathbf{R}_{focus}(k_i)$$

- Transformation of the beam matrix

$$\sigma(1,k_i) = \mathbf{R}(k_i) \cdot \sigma(0) \cdot \mathbf{R}^T(k_i)$$

Task: Calculation of $\sigma(0)$

at entrance s_0 i.e. all three elements

Measurement of transverse Emittance

- The beam width $x_{max}(s_1)$ at s_1 is measured ⇔ matrix element $\sigma_{11}(1, k_i) = x^2_{max}(k_i)$
- Different focusing of quadrupoles k_1 , k_2 ... k_n are used ⇒ $R_{focus}(k_i)$
- After the drift the transfer matrix is $R(k_i) = R_{drift} \cdot R_{focus}(k_i)$
- **Task: Calculation of beam matrix $\sigma(0)$ at entrance s_0 (matrix elements give orientation)**
- **The transformation of the beam matrix is: $\sigma(1,k_i) = R(k_i) \cdot \sigma(0) \cdot R^T(k_i)$**

⇒ **Result: Redundant system of linear equations for matrix elements $\sigma_{ij}(0)$**

$$\sigma_{11}(1, k_1) = R^2_{11}(k_1) \cdot \sigma_{11}(0) + 2\,R_{11}(k_1)\,R_{12}(k_1) \cdot \sigma_{12}(0) + R^2_{12}(k_1) \cdot \sigma_{22}(0) \text{ focusing } k_1$$

...

$$\sigma_{11}(1, k_n) = R^2_{11}(k_n) \cdot \sigma_{11}(0) + 2\,R_{11}(k_n)\,R_{12}(k_n) \cdot \sigma_{12}(0) + R^2_{12}(k_n) \cdot \sigma_{22}(0) \text{ focusing } k_n$$

- To have an error estimation at least three measurements must be done

Assumptions:
- Constant emittance, in particular no space-charge broadening
- Only elliptical shaped beam distribution is considered
- No misalignment, i.e. beam center equals center of the quadrupoles
- If **not** valid: A self-consistent algorithm can be used .

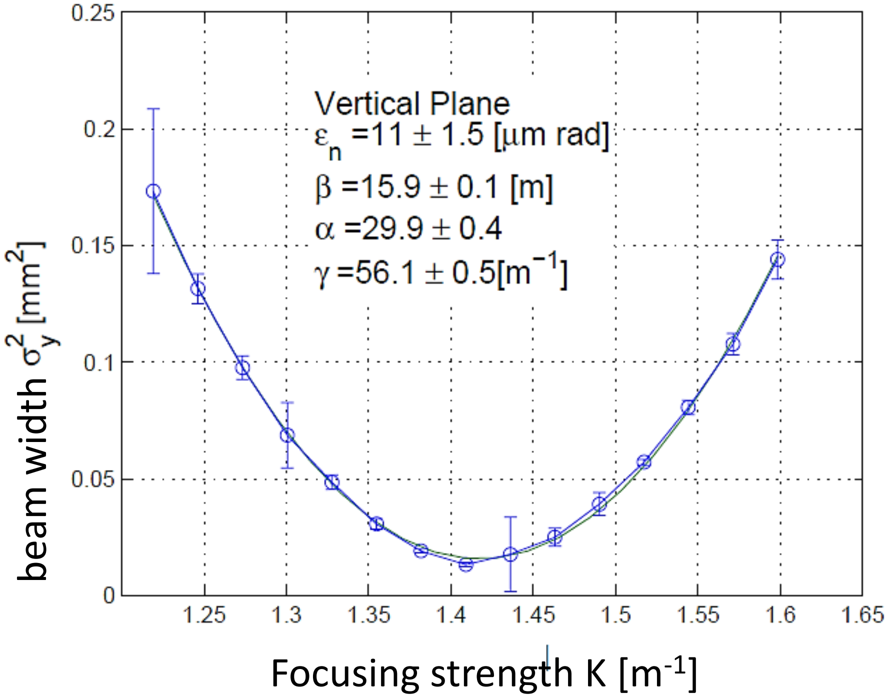
Measurement of transverse Emittance

Using the ‘thin lens approximation’ i.e. the quadrupole has a focal length of ***f***:

$$\mathrm{R}_{focus}(\boldsymbol{K}) = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ K & 1 \end{pmatrix} \Rightarrow \mathrm{R}(L,K) = \mathrm{R}_{drift}(L)\cdot\mathrm{R}_{focus}(K) = \begin{pmatrix} 1+LK & L \\ K & 1 \end{pmatrix}$$

Measurement of the matrix-element $\sigma_{11}(1,K)$ from $\sigma(1,K) = R(K)\cdot\sigma(0)\cdot R^T(K)$

Example: Square of the beam width at ELETTRA 100 MeV e⁻ Linac, YAG:Ce:



G. Penco (ELETTRA) et al., EPAC’08

For completeness: The relevant formulas

$$\begin{aligned} \sigma_{11}(1,K) = {} & L^2\sigma_{11}(0)\cdot K^2 \\ & +2\cdot(\ L\sigma_{11}(0)+L^2\sigma_{12}(0)\ )\cdot K \\ & +L^2\sigma_{22}(0)+\sigma_{11}(0) \\ \equiv {} & a\cdot K^2 - 2ab\cdot K + ab^2 + c \\ = {} & a\cdot(K-b)^2 + c \end{aligned}$$

The three matrix elements at the quadrupole:

$$\sigma_{11}(0) = \frac{a}{L^2}$$

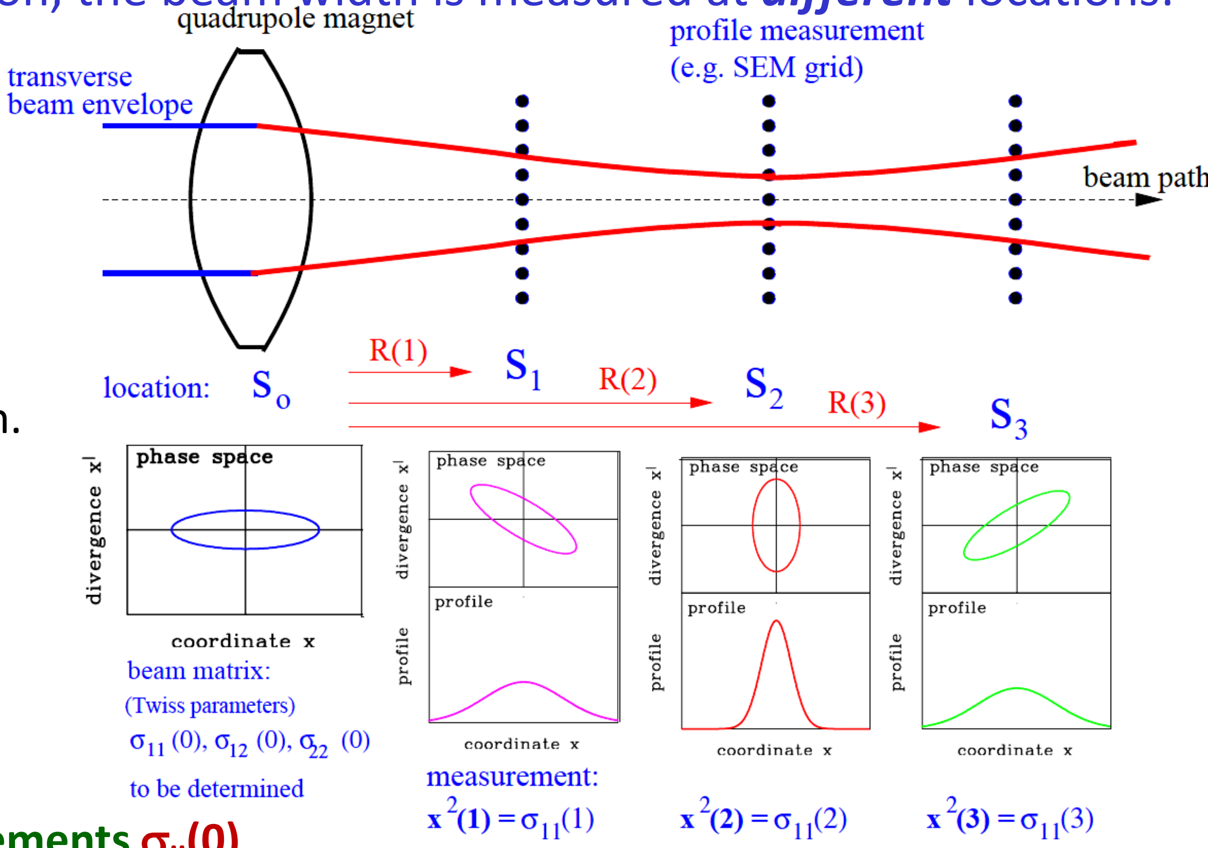
$$\sigma_{12}(0) = -\frac{a}{L^2}\left(\frac{1}{L}+b\right)$$

$$\sigma_{22}(0) = \frac{1}{L^2}\left(ab^2 + c + \frac{2ab}{L} + \frac{a}{L^2}\right)$$

$$\varepsilon_{rms} \equiv \sqrt{\det\sigma(0)} = \sqrt{\sigma_{11}(0)\cdot\sigma_{22}(0) - \sigma_{12}^2(0)} = \sqrt{ac}\,/\,L^2$$

The ‘Three Grid Method’ for Emittance Measurement

Instead of quadrupole variation, the beam width is measured at ***different*** locations:



The procedure is:

- Beam width ***x(i)*** measured at the locations s_i

  ⇒ beam matrix element

  $x^2(i) = \sigma_{11}(i)$.
- The transfer matrix **R(*i*)** is known. (without dipole a 3 × 3 matrix.)
- The transformations are:

  $\sigma(i) = R(i) \cdot \sigma(0) \cdot R^T(i)$

  ⇒ redundant equations:

⇒ **Result: at least equations for elements $\sigma_{ij}(0)$**

$$\sigma_{11}(1) = R_{11}^2(1) \cdot \sigma_{11}(0) + 2\, R_{11}(1)\, R_{12}(1) \cdot \sigma_{12}(0) + R_{12}^2(1) \cdot \sigma_{22}(0) \quad \text{for } \mathbf{R}(1)\text{: } s_0 \rightarrow s_1$$

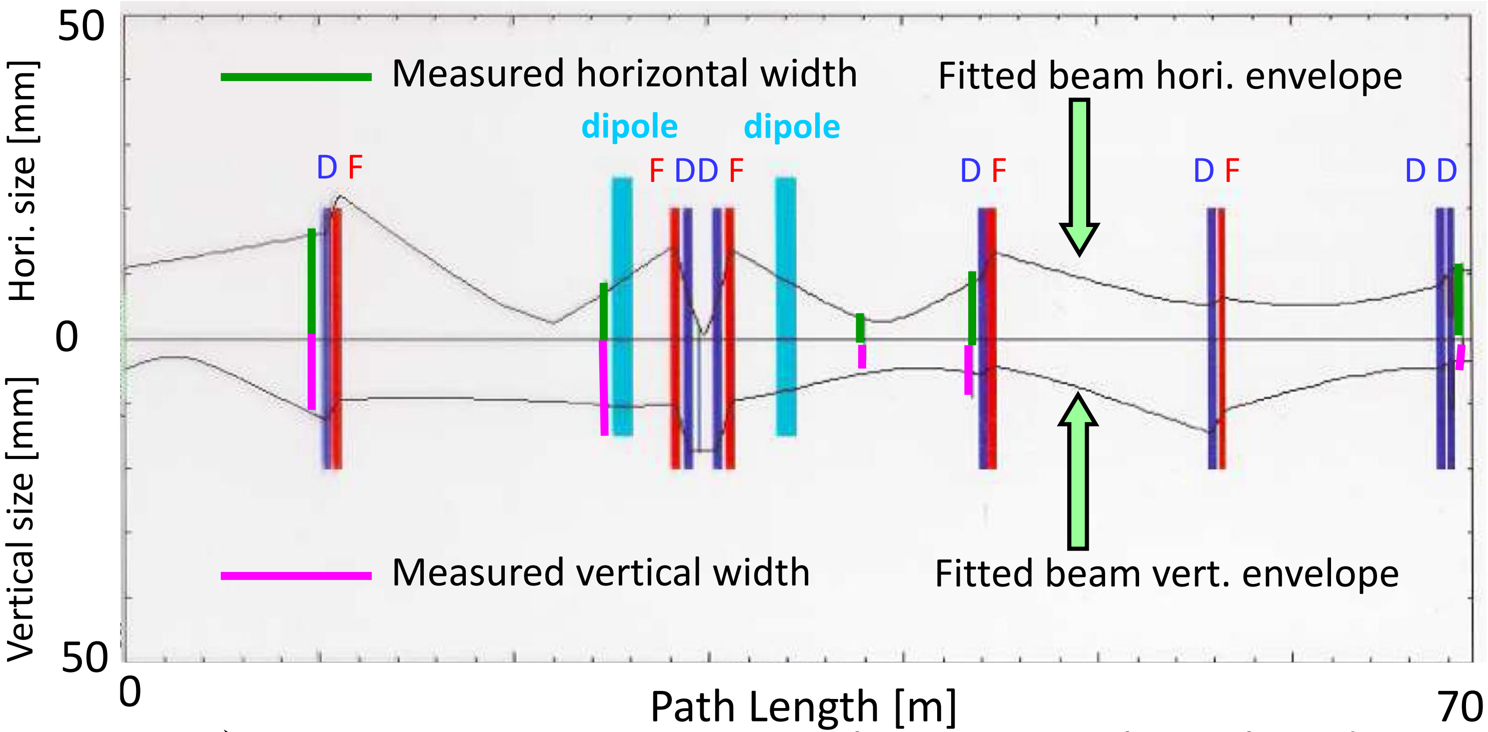
$$\ldots$$

$$\sigma_{11}(n) = R_{11}^2(n) \cdot \sigma_{11}(0) + 2\, R_{11}(n)\, R_{12}(n) \cdot \sigma_{12}(0) + R_{12}^2(n) \cdot \sigma_{22}(0) \quad \text{for } \mathbf{R}(n)\text{: } s_0 \rightarrow s_n$$

Results of a 'Three Grid Method' Measurement

Solution: Solving the linear equations like for quadrupole variation or fitting the profiles with linear optics code (e.g. MADX, TRANSPORT, WinAgile).

Example: The hor. and vert. beam envelope and the beam width at a transfer line:



Assumptions:
- ➢ constant emittance, in particular no space-charge broadening
- ➢100 % transmission i.e. no loss due to vacuum pipe scraping
- ➢ no misalignment, i.e. beam center equals center of the quadrupoles.

Summary for transverse Emittance Measurement

Emittance measurements are very important for comparison to theory.

It includes size (value of ***ε***) and orientation in phase space (***σ_{ij}*** or ***α, β*** and ***γ***)

i.e three independent values $\varepsilon_{rms} = \sqrt{\sigma_{11} \cdot \sigma_{22} - \sigma_{12}} = \sqrt{< x^2 > \cdot < x'^2 > - < xx' >^2}$

Low energy beams → direct measurement of x- and x'-distribution

- ***Slit-grid:*** movable slit → ***x***-profile, grid → ***x'***-profile
- Variances exists: slit-slit, slit-kick, pepperpot .... method

All beams → profile measurement + linear transformation:

- ***Quadrupole variation:*** one location, different setting of a quadrupole
- ***'Three grid method':*** different locations
- ***Assumptions:*** ➢ well aligned beam, no steering
  - ➢ no emittance blow-up due to space charge.

Important remark: For a synchrotron with a *stable beam storage*,

width measurement is sufficient using $x_{rms} = \sqrt{\varepsilon_{rms} \cdot \beta}$

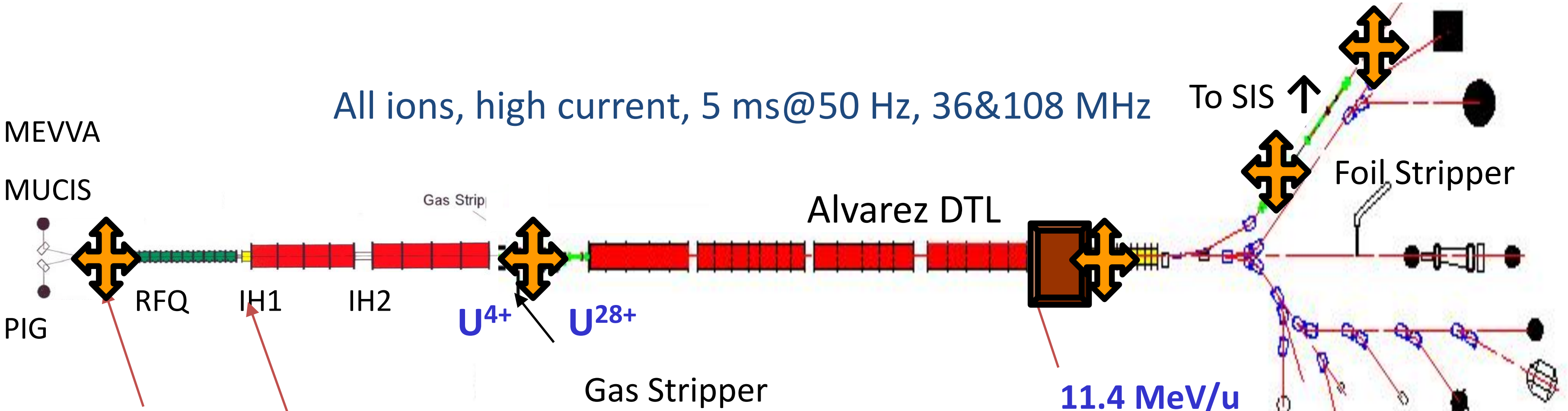
Appendix GSI Ion LINAC: Emittance Measurement Devices

✥ **Slit Grid Emittance**: Standard device, total 9 device

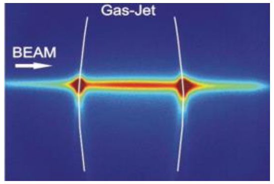
■ **Pepper-pot Emittance**: Special device, total 1 device

Transfer to Synchrotron

All ions, high current, 5 ms@50 Hz, 36&108 MHz



Gas Stripper



2.2 keV/u
β = 0.0022

120 keV/u
β = 0.016

1.4 MeV/u ⇔ β = 0.054

11.4 MeV/u
β = 0.16

Constructed in the 70th, Upgrade 1999,
further upgrades in preparation

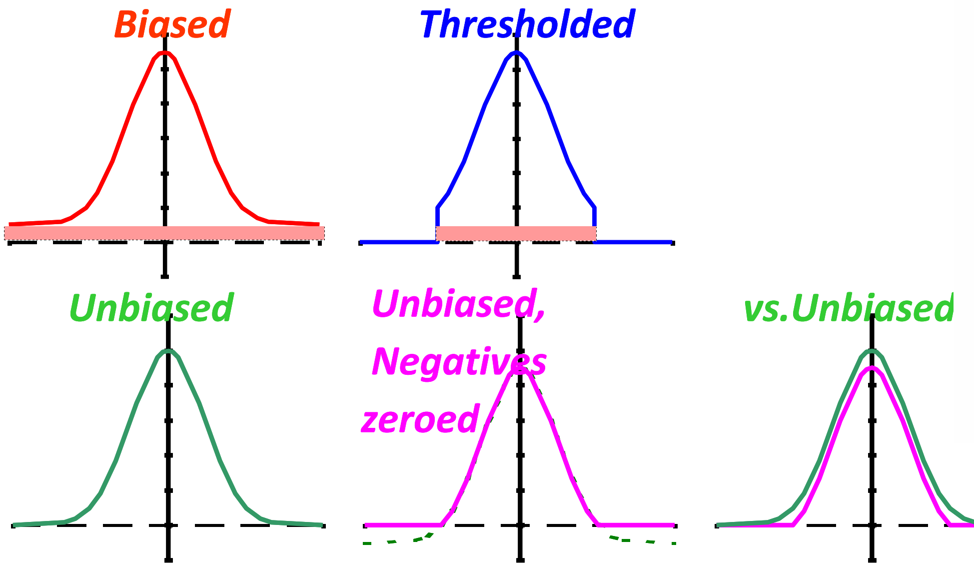
→ Conclusion

Backup slides

The Noise Influence for Emittance Determination

A real measurement of beamlets contains:

- **Noise i.e. fluctuation of the output**
- **Bias i.e. electrical offset from amplifier**

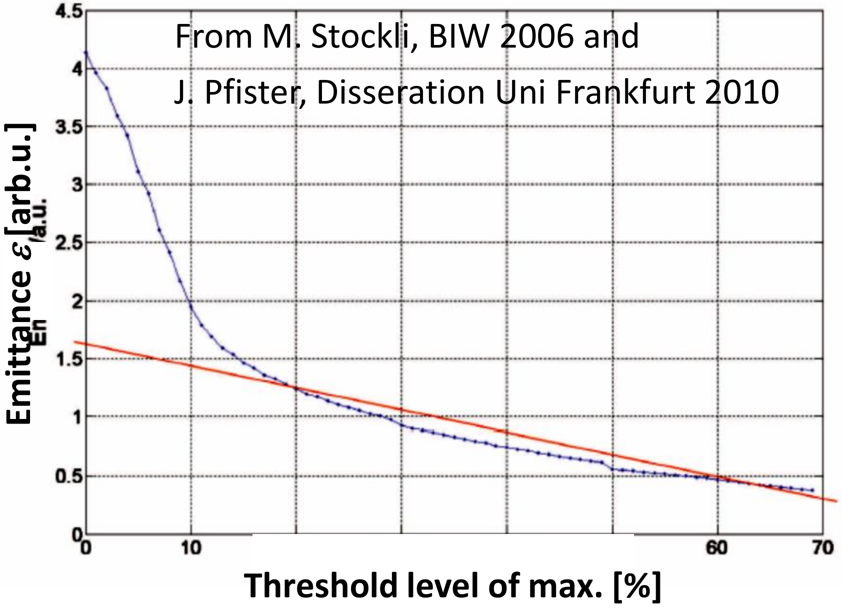


→ Strong influence of noise reduction to numerical values of ***<x>, <x'^{2}>*** and ***<xx'>*** and on $\boldsymbol{\varepsilon_{rms}}$

⇒Algorithm & cut-level must be given for evaluation

General: Typical error ***Δε/ε > 10 %***

Example: Dependence of $\boldsymbol{\varepsilon_{rms}}$ on threshold



From M. Stockli, BIW 2006 and

J. Pfister, Disseration Uni Frankfurt 2010

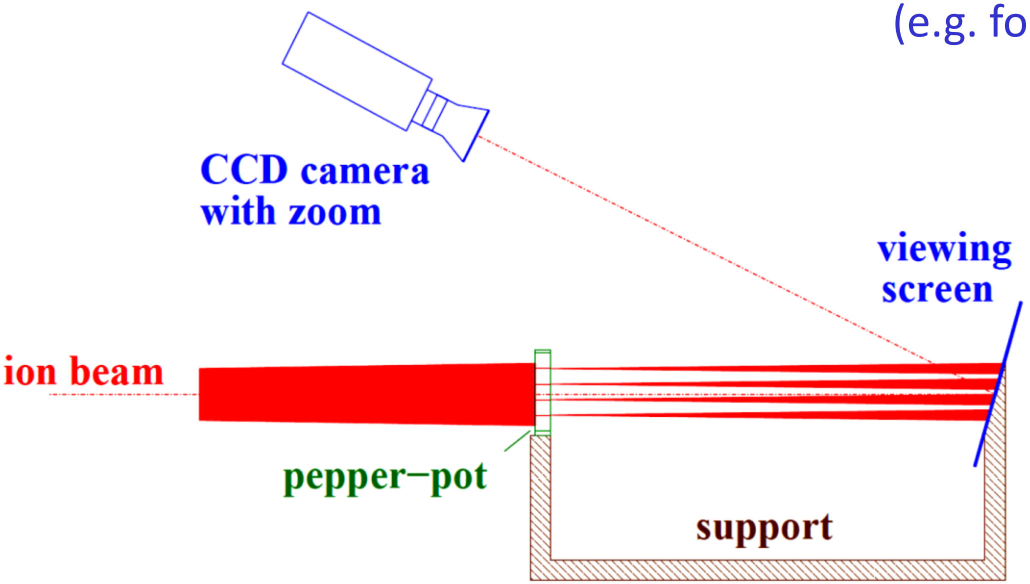
$$\langle x'^2\rangle = \frac{\int x'^2\cdot\rho(x,x')\,dx\,dx'}{\int\rho(x,x')\,dx\,dx'}$$ for continous values

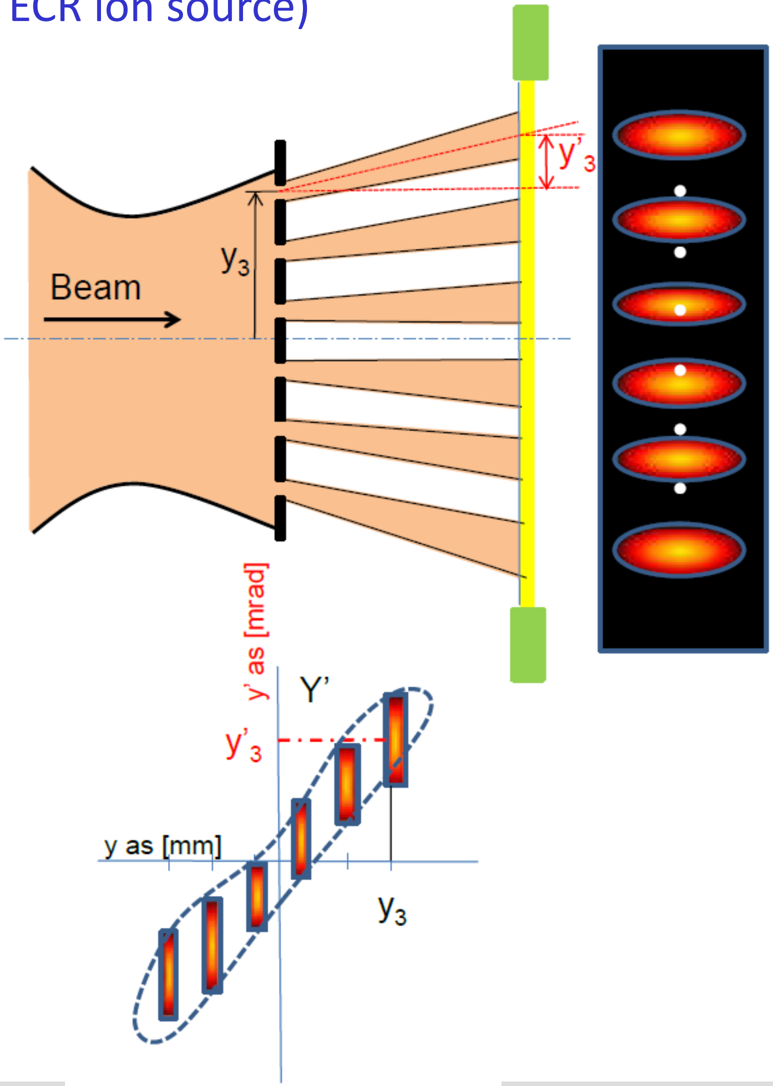
$$= \frac{\sum_{i,j} x'^{\,2}_{ij}\cdot P(x_{ij},x'_{ij})}{\sum_{i,j}P(x_{ij},x'_{ij})}$$ for discrete values

$$\varepsilon_{rms} = \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

The Pepperpot Emittance Device

- For pulsed LINAC: Measurement within one pulse is an advantage
- If horizontal and vertical direction coupled → 2-dim evaluation **required** (e.g. for ECR ion source)





Good **spatial** resolution if many holes are illuminated.

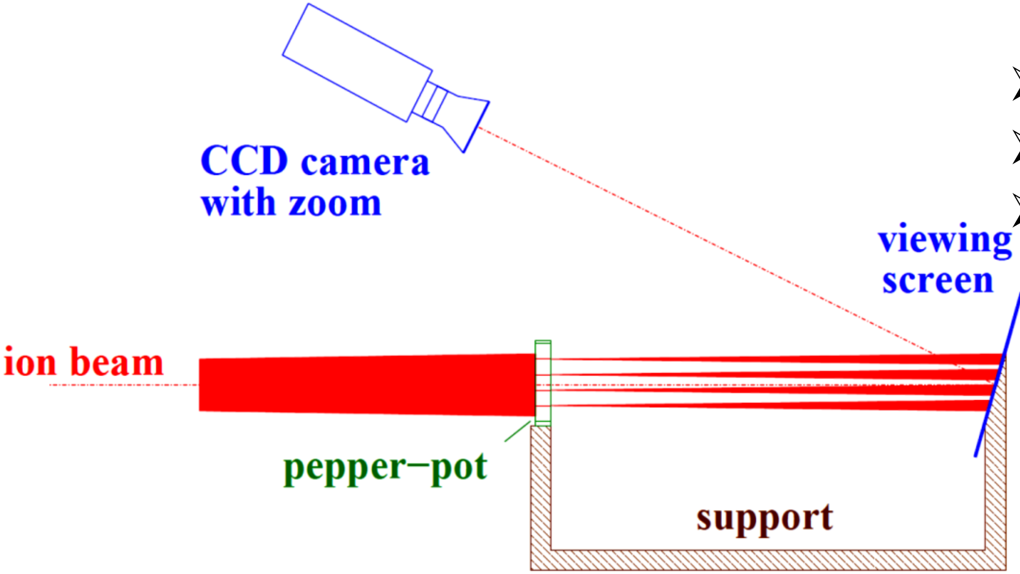
Good **angle** resolution ***only*** if spots do not overlap.

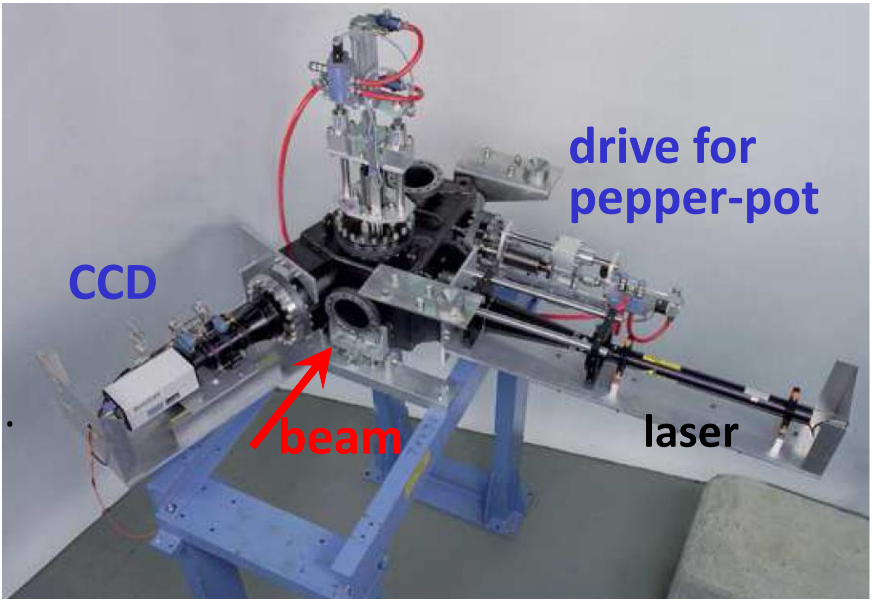
Partly from H.R. Kremers et al., ECRIS 2010

The Pepperpot Emittance Device at GSI UNILAC

Example GSI-LINAC 0.12 to 11 MeV/u:

- ***Pepper-pot***: 15 × 15 holes with Ø 0.1mm on a 50 × 50 mm^2 copper plate
- ***Distance***: pepper-pot-screen: 25 cm
- ***Screen:*** Al_2O_3, Ø 50 mm
- ***Data acquisition***: high resolution CCD





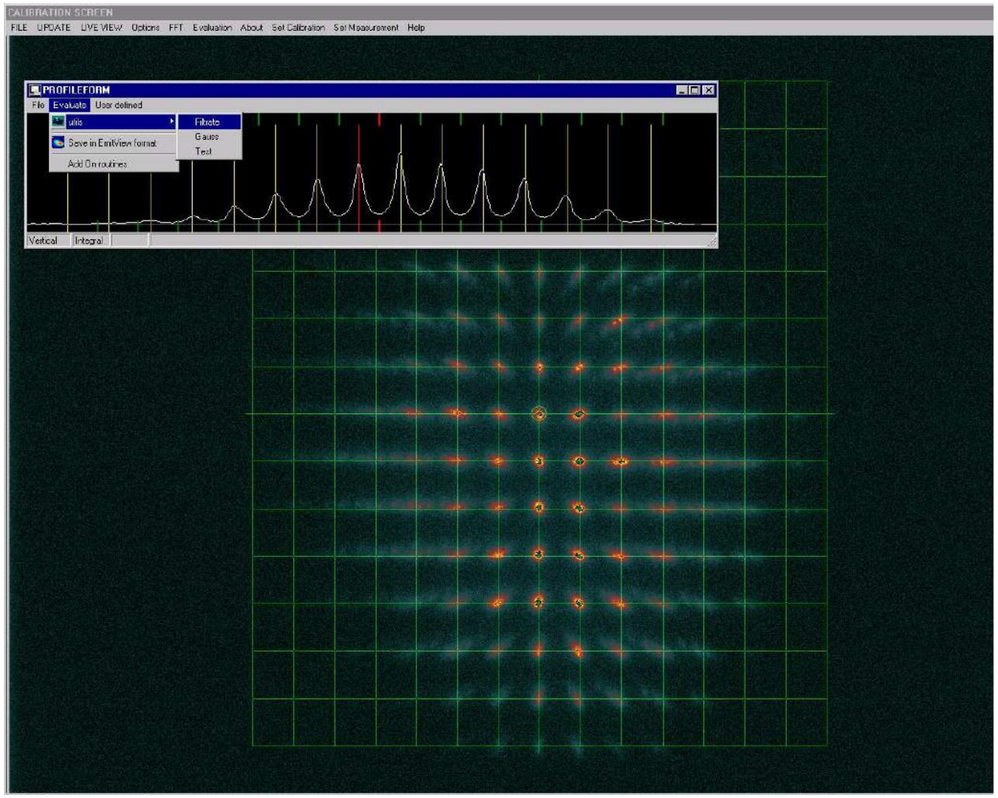
Good **spatial** resolution if many holes are illuminated.

Good **angle** resolution ***only*** if spots do not overlap.

Result of a Pepperpot Emittance Measurement

Example: Ar $^{1+}$ ion beam at 1.4 MeV/u,

screen image from single shot at GSI:

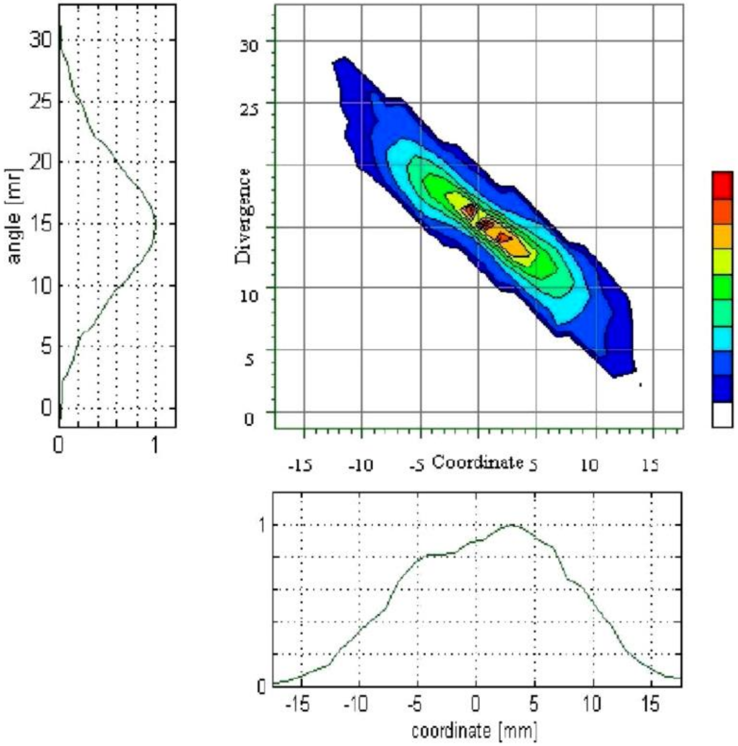


Data analysis:

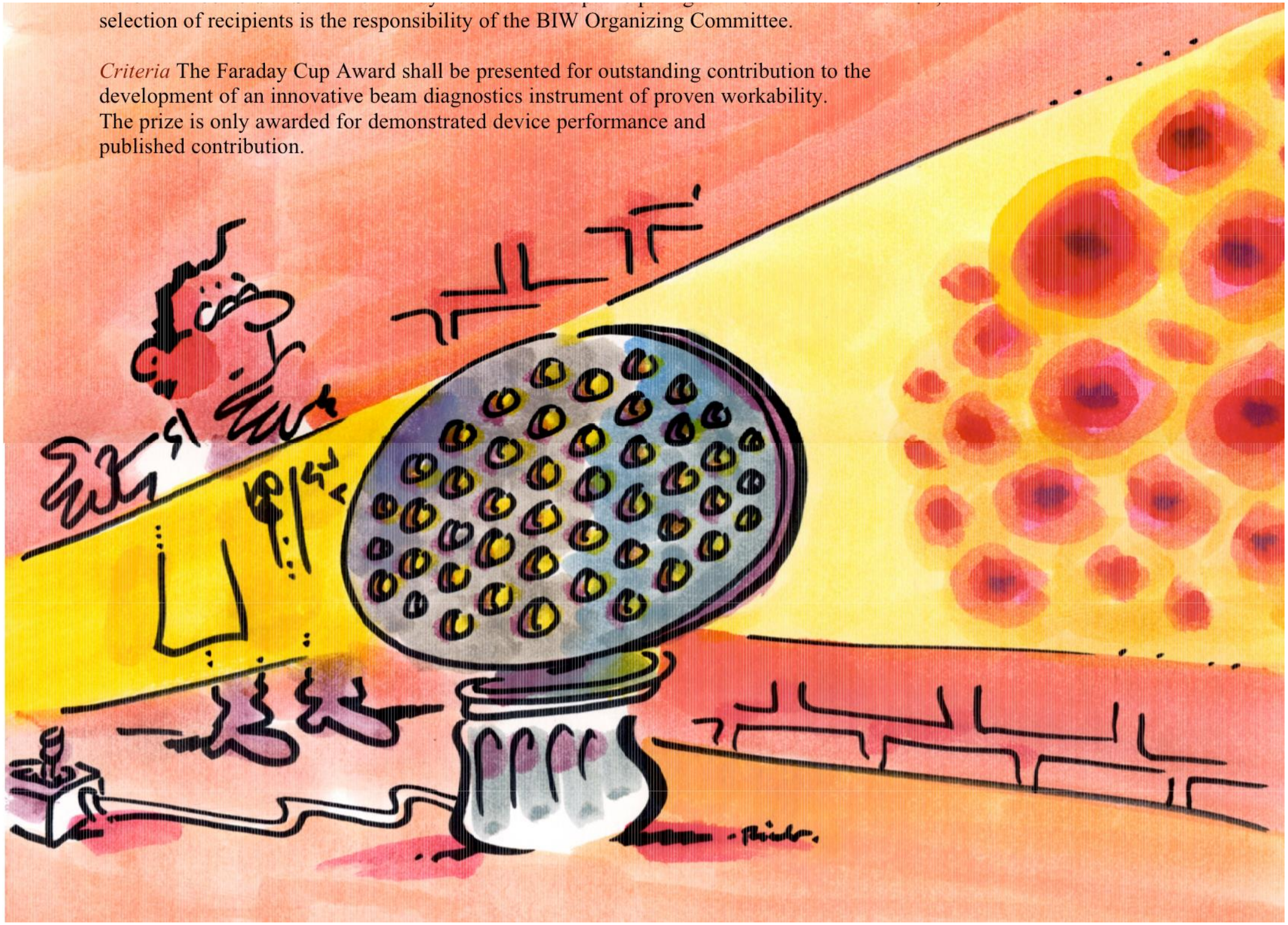
Projection on

horizontal and vertical plane

→ analog to slit-grid.



The Artist View of a Pepperpot Emittance Device



selection of recipients is the responsibility of the BIW Organizing Committee.

Criteria The Faraday Cup Award shall be presented for outstanding contribution to the development of an innovative beam diagnostics instrument of proven workability. The prize is only awarded for demonstrated device performance and published contribution.

Excurse: Some Properties of the Transfer Matrix

- ➢ The transformation can be done successive: with

with $\mathbf{R_1 = R(} s_0 \rightarrow s_1 \mathbf{)} ,\ldots, \mathbf{R_n = R(} s_{n-1} \rightarrow s_n \mathbf{)}$

It is $\mathbf{R = R_n \cdot R_{n-1} \cdot \ldots \cdot R_1}$

- ➢ The elements describe the coupling between the components

$R_{11}= (x \mid x),\ R_{12}= (x \mid x'),\ R_{13}= (x \mid y),\ R_{14}=(x \mid y'),\ R_{15}= (x \mid l),\ R_{16} =(x \mid \delta)\ldots.$

- ➢ **If** all forces are symmetric along the reference orbit than the horizontal and vertical plane are decoupled:

$\Rightarrow$ sub-matrix is sufficient

$$\mathrm{R} = \begin{pmatrix} (x|x) & (x|x') & 0 & 0 & 0 & (x|\delta) \\ (x'|x) & (x'|x') & 0 & 0 & 0 & (x'|\delta) \\ 0 & 0 & (y|y) & (y|y') & 0 & 0 \\ 0 & 0 & (y'|y) & (y'|y') & 0 & 0 \\ (l|x) & (l|x') & 0 & 0 & 1 & (l|l) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- ➢ It is **det(R) = 1** (Liouville's Theorem)

  i.e. **R** is invertible

- ➢ Sub-matrix

  here shown for drift ***L***:

$$\mathrm{R}_x = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \quad \mathrm{R}_y = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \quad \mathrm{R}_s = \begin{pmatrix} 1 & L/\gamma^2 \\ 0 & 1 \end{pmatrix}$$