Transverse Beam Dynamics

JUAS 2020 - Tutorials

1 Exercise: Wien Filter

A Wien Filter is a device that allows to select particles in a beam according to their velocity.

1. Write down the expression of the Lorentz force.
2. How should we orient an electric field $\vec{E}$ if we want to compensate the force of an uniform magnetic field $\vec{B}$?
3. Assuming the magnetic field is 2 mT, what would be the required electric field (V/m) to select protons travelling with a velocity of $0.15c$?
4. Assuming that the particles move along the z axis and $\vec{B} = \begin{pmatrix}0, & B_y, & 0\end{pmatrix}$, write the equations of motion.
5. [Optional] Could we use a Wien filter with a neutral beam (eg. neutron)? What other techniques could be employed to create a velocity filter?

2 Exercise: Understanding the phase space concept

1. Phase Space Representation of a Particle Source:
   - Consider a source at position s_0 with radius w emitting particles. Make a drawing of this setup in the configuration space and in the phase space. Which part of the phase space can be occupied by the emitted particles?
   - Any real beam emerging from a source like the one above will be collimated. This can be modelled by assuming that a distance d away from the source there is an iris with opening radius $R = w$. Draw this setup in the configuration space and in the phase space. Which part of the phase space is occupied by the beam, right after the collimator?
2. Sketch the emittance ellipse of a particle beam in:

   (I) horizontal x-x' phase space at the position of a transverse waist,

   (II) when the beam is divergent, and

   (III) when the beam is convergent.

3 Exercise: Local radius, rigidity

We wish to design a proton ring with a radius of $R = 200$ m. Let us assume that only 50% of the circumference is occupied by bending magnets:

- What will be the local radius of bend ρ in these magnets if they all have the same strength?
- If the kinetic energy of the protons is 2 GeV, calculate the beam rigidity $B\rho$ and the field in the dipoles.

4 Exercise: Thin-lens approximation

1. Compute and compare the matrices of the thick and thin-lens approximation of a quadrupole with $k_q = 0.01$ m^{-2}; $L_q = 5.5$ m
2. Verify if the stability condition is valid
3. Would the answer be the same, if the quadrupole was defocusing?

5 Exercise: Hill's equation

Solve the Hill's equation:

$$y'' + k(s)y = 0$$

by substituting:

$$y = A\sqrt{\beta(s)}\cos[\phi(s) + \phi_0] \text{ with } \phi' = \frac{1}{\beta(s)}, \text{ and where } A \text{ and } \phi_0 \text{ are constants,}$$

demonstrating that a necessary condition is:

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + k(s)\beta^2 = 1$$

6 Exercise: Particle momentum, geometry of a storage ring and thin lenses

The LHC storage ring at CERN collides proton beams with a maximum momentum of $p = 7$ TeV/c per beam. The main parameters of this machine are:

Circumference	$C_0 = 26658.9$ m	
Particle momentum	$p = 7$ TeV/c	
Main dipoles	$B = 8.392$ T	$l_B = 14.2$ m
Main quadrupoles	$G = 235$ T/m	$l_q = 5.5$ m

1. Calculate the magnetic rigidity of the design beam, the bending radius of the main dipole magnets in the arc and determine the number of dipoles that is needed in the machine.
2. Calculate the k-strength of the quadrupole magnets and compare its focal length to the length of the magnet. Can this magnet be treated as a thin lens?

7 Exercise: Stability condition

Consider a lattice composed by a single 2 meters long quadrupole, with $f = 1$ m

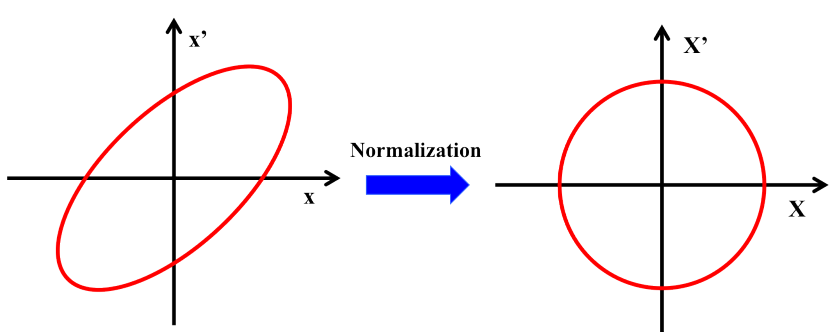
- Prove that if the quadrupole is defocusing, then a lattice is not stable
- Prove that if the quadrupole is focusing, then the lattice is stable

8 Exercise: Normalised phase space

Let us consider the following phase space vector: (x, x'). The transformation to a *normalised phase space* (X, X') is given by:

$$\begin{pmatrix} X \\ X' \end{pmatrix} = \begin{pmatrix} 1/\sqrt{\beta_x} & 0 \\ \alpha_x/\sqrt{\beta_x} & \sqrt{\beta_x} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

The normalisation process of the phase space is illustrated in the figure below:



If we know that the transfer matrix between two points 1 and 2 (with phase advance ϕ_x between them) in the phase space (x, x') is given by:

$$M_{1\to 2} = \begin{pmatrix} \sqrt{\frac{\beta_{x2}}{\beta_{x1}}}(\cos\phi_x + \alpha_{x1}\sin\phi_x) & \sqrt{\beta_{x1}\beta_{x2}}\sin\phi_x \\ \frac{(\alpha_{x1}-\alpha_{x2})\cos\phi_x-(1+\alpha_{x1}\alpha_{x2})\sin\phi_x}{\sqrt{\beta_{x2}\beta_{x1}}} & \sqrt{\frac{\beta_{x1}}{\beta_{x2}}}(\cos\phi_x - \alpha_{x2}\sin\phi_x) \end{pmatrix}$$

Obtain the transfer matrix between two points 1 and 2 in the normalised phase space.

9 Exercise: beam size and luminosity

An e^+e^- collider has an interaction Point (IP) with $\beta_x^* = 0.5$ m and $\beta_y^* = 0.1$ cm. The peak luminosity available by a e^+e^- collider can be written as:

$$L = \frac{N_\mathrm{b} N_{e^-} N_{e^+} f_\mathrm{rev}}{4\pi\sigma_x^*\sigma_y^*} \ [\mathrm{cm}^{-2}\mathrm{s}^{-1}]$$

where $N_\mathrm{b} = 80$ is the number of bunches per beam (we assume the same number of bunches for both the e^- and the e^+ beams), $N_{e^-} = N_{e^+} = 5 \times 10^{11}$ is the number of particles per bunch (we assume the same number for both e^- and e^+ bunches), and f_rev is the revolution frequency. The horizontal and vertical normalised beam emittances are respectively: $\epsilon_{x,N} = 2.2$ mm and $\epsilon_{y,N} = 4.7$ μm.

- Compute the revolution frequency f_rev, knowing that the circumference is 80 km and that the beam moves nearly at the speed of light
- Calculate the beam transverse beam sizes σ_x^* and σ_y^* at the IP, and the luminosity L for two different beam energies: 45 GeV and 120 GeV
- What are the beam divergences (horizontal and vertical) at the IP for the 45 GeV case?
- What is the value of the betatron function at position $s = 0.5$ m from the IP?

10 Exercise: Basics of lattice design

Design a FODO cell such that it has: phase advance $\mu = 90$ degrees, a total length of 10 m, and a total bending angle of 5 degrees. What are β_{max}, β_{min}, D_{max}, D_{min}?

11 Exercise: Bump and Orbit Control

Given two kickers located at the two ends of a FODO cell with phase advance 45 degrees (the two kickers are located at L_cell distance from each other), compute the strengths of such kickers (in radians) in order to give the beam, initially at $(x_i,\, x_i') = (0,\, 0)$, an arbitrary offset at the end of the cell while preserving its angle, $\left(x_f,\, x_f'\right) = (x_\mathrm{arbitrary},\, 0)$.

12 Exercise: Measurement of Twiss parameters

One of the possible ways to determine experimentally the Twiss parameters at a given point makes use of a so-called quadrupole scan. One can measure the transverse size of the beam in a profile monitor, called Wire Beam Scanner (WBS), located at a distance L downstream a focusing quadrupole, as a function of the normalised gradient in this quadrupole. This allows to compute the emittance of the beam, as well as the β and the α functions at the entrance of the quadrupole.

Let's consider a quadrupole Q with a length of $l = 20$ cm. This quadrupole is installed in an electron transport line where the particle momentum is 300 MeV/c. At a distance $L = 10$ m from the quadrupole the transverse beam size is measured with a WBS, for various values of the current I_Q. The maximum value of the quadrupole gradient G is obtained for a current of 100 A, and is $G = 1$ T/m.

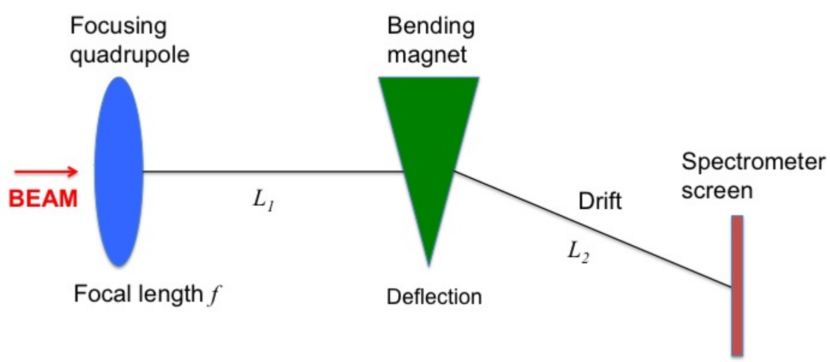
Hint: G is proportional to the current. **Advice:** use thin-lens approximation.

1. How does the normalised focusing strength K vary with I_Q?

2. Give the expression Σ_2 as function of α_1, β_1, and γ_1
3. Show that β_2 can be written in the form: $\beta_2 = A_2 (Kl)^2 + A_1 (Kl) + A_0$, and express A_0, A_1, and A_2 as a function of L, α_1, β_1, and γ_1.
4. Express the final beam size, σ_2, as a function of Kl, and find its minimum, which will correspond to $(Kl)_{\text{min}}$.
5. How does σ_2 vary with Kl when $|Kl - (Kl)_{\text{min}}| \gg 1/\beta_1$?
6. Deduce the values of α_1, β_1, and γ_1 from the measurement σ_2, as a function of the quadrupole current I_Q.

13 Exercise: The spectrometer line of CTF3

The CTF3 (CLIC Test Facility 3) experiment at CERN consists of a linac that injects very short electron bunches into an isochronous ring. A spectrometer line made of one quadrupole and one bending magnet is located at the end of the linac where the particle momentum is 350 MeV/c. The goal of the spectrometer is to measure the energy before injecting the electrons in the ring. The spectrometer line is sketched on the figure below. It is made of a focusing quadrupole of focal length f, a drift space of length L_1, a bending magnet of deflection angle θ in the horizontal plane, and a drift space of length L_2. We assume that the spectrometer line starts at the quadrupole and ends at the end of the second drift. We neglect the focusing effect of the dipole.



1. If the effective length of the dipole is $l_B = 0.43$ m, what should be the magnetic field (in Tesla) inside the dipole to deflect the electrons by an angle of 35 degrees?
2. Starting from the general horizontal 3×3 transfer matrix of a sector dipole of deflection angle θ, show that the transfer matrix of a dipole in the thin-lens approximation is

$$M_{dipole} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

   Which approximations are done?

   Hint: A sector dipole has the following 3×3 transfer matrix:

$$M_{\text{dipole}} = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{\sin\theta}{\rho} & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

3. In the thin-lens approximation, derive the horizontal extended 3×3 transfer matrix of the spectrometer line. Show that it is:

$$M_{spectro} = \begin{pmatrix} \frac{f-L_1-L_2}{f} & L_1 + L_2 & L_2\theta \\ -\frac{1}{f} & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

4. Assuming $D = D' = 0$ at the entrance of the quadrupole, what is the dispersion and its derivative at the end of the spectrometer line? Give the numerical value of D' at the end of the spectrometer line for the angle of 35 degrees.

5. What is the difference between a periodic lattice and a beam transport lattice (or transfer line) as concerns the betatron function ?

6. Derive the betatron function β_2 at the end of the spectrometer line in terms of L_1, L_2, f and β_1, assuming $\alpha_1 = 0$.

   Hint 1. Remember from the lecture:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

   An alternative way to transport the Twiss parameters is through the σ matrix:

$$\sigma_i = \begin{pmatrix} \beta_i & -\alpha_i \\ -\alpha_i & \gamma_i \end{pmatrix}$$

   This matrix multiplied by the emittance ϵ gives the so-called beam matrix (which has already been introduced during the lecture):

$$\Sigma_i = \begin{pmatrix} \beta_i\epsilon & -\alpha_i\epsilon \\ -\alpha_i\epsilon & \gamma_i\epsilon \end{pmatrix}$$

   If σ_1 is the matrix at the entrance of the transfer line, the matrix σ_2 at the exit of the transfer line is given by

$$\sigma_2 = M\sigma_1 M^T$$

   where M is the 2×2 transfer matrix of the line extracted from the extended 3×3 transfer matrix (see question 3), and M^T the transpose matrix of M.

   Hint 2. For the calculations, write M as $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ and replace the values of the matrix elements only at the end.

7. Given the numerical values $L_1 = 1$ m, $L_2 = 2$ m, $\beta_1 = 10$ m, $\alpha_1 = 0$, compute the value of the focal length f such that the betatron function at the end of the spectrometer line is minimum.

8. For an off-momentum particle, compute the deviation from the design orbit? Why is it important to minimise the β function in the spectrometer?

14 Exercise: Transfer matrix of a dipole magnet

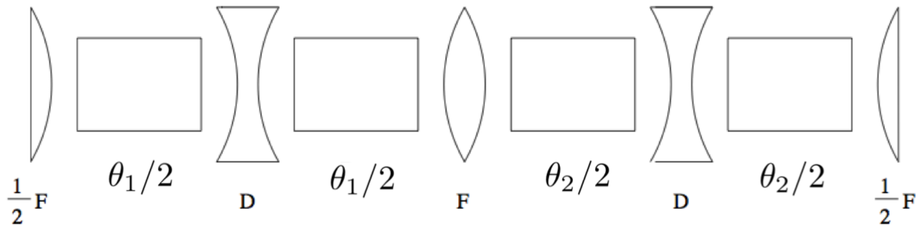
- Remember weak focusing:
  $K = \frac{1}{\rho^2}$:

$$M_{\text{Dipole}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix} = \begin{pmatrix} \cos\frac{L}{\rho} & \rho\sin\frac{L}{\rho} \\ -\frac{1}{\rho}\sin\frac{L}{\rho} & \cos\frac{L}{\rho} \end{pmatrix}$$

- Compute the 3×3 matrix of a sector dipole including the dispersion terms.

15 Exercise: Dispersion suppressor

In several straight sections of the accelerator, like the ones hosting RF cavities, extraction systems or other devices such as detectors, it is preferable to have no dispersion $\eta(s) = \eta'(s) = 0$. For example, in big colliders, such as the LHC, where small spot sizes are required at the interaction points, the dispersion is reduced to zero at the detector positions. The most common dispersion suppressors consists of two FODO cells of equal length L and equal quadrupole strengths. Bending magnets are placed in the space between the quadrupoles with a different bending field in each FODO. Figure below shows a typical dispersion suppressor.



1. Considering two FODO cells with different total bend angles, $\theta_1 \neq \theta_2$, calculate the relation between the angles θ_1 and θ_2 which must be satisfied to cancel the dispersion at the end of the lattice.

   Hint:

   For each FODO cell, $M_{\text{FODO}} = M_{1/2\text{F}} \cdot M_{\text{dipole}} \cdot M_{\text{D}} \cdot M_{\text{dipole}} \cdot M_{1/2\text{F}}$, in thin-lens approximation we have the following 3×3 matrix:

$$M_{\text{FODO }j} = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L\left(1 + \frac{L}{4f}\right) & \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_j \\ -\frac{L}{4f^2}\left(1 - \frac{L}{4f}\right) & 1 - \frac{L^2}{8f^2} & \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_j \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \cos\mu & \beta\sin\mu & \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_j \\ -\frac{\sin\mu}{\beta} & \cos\mu & \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_j \\ 0 & 0 & 1 \end{pmatrix}$$

   where $j = 1, 2$ (1=first cell, 2=second cell).

   The following condition must be satisfied:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = M_{\text{suppressor}} \begin{pmatrix} \eta_0 \\ 0 \\ 1 \end{pmatrix} \tag{1}$$

   where η_0 is the initial dispersion (at the middle of the first focusing quadrupole). It can be demonstrated that for a FODO lattice the dispersion has its maximum at the middle of the focusing quadrupole:

$$\eta_0 = \frac{4f^2}{L}\left(1 + \frac{L}{8f}\right)\theta \tag{2}$$

   with $\theta = \theta_1 + \theta_2$ the total bend in the suppressor.

2. Obtain the relation between the angles for the cases of phase-advance per cell $\mu = \pi/3$ and $\pi/2$

16 Exercise: Double-Bend Achromat (DBA) lattice

A Double-Bend Achromat (DBA) can be made from two dipoles with a horizontally focusing quadrupole between them. The transfer matrix through the achromat is:

$$M_{\text{DBA}} = M_{\text{bend}} M_{\text{drift}} M_{1/2\text{F}} M_{1/2\text{F}} M_{\text{drift}} M_{\text{bend}}$$

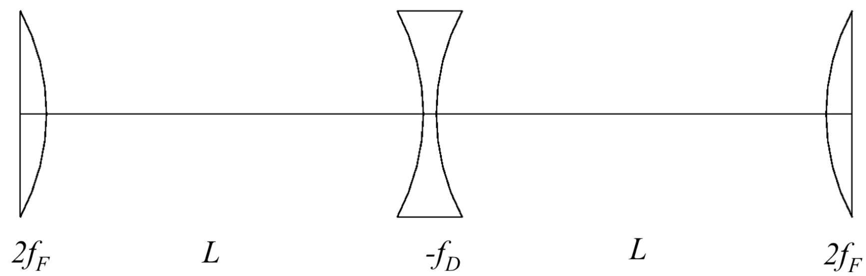
Note that this magnet configuration does not produce vertical focusing, therefore it will not be enough to create a stable lattice. A full DBA typically comprises additional quadrupole doublets before and after the bending section. For sake of simplicity, we will neglect them.

1. Use the thin-lens approximation for quadrupoles and small-angle approximation for bends to find the dispersion in the middle of the quadrupole. Write the focal length in terms of the drift and bend parameters.
2. Show that the dispersion vanishes after the bend.
3. Compute the parameters L, f for a 10-meter long DBA which bends the beam by an angle of 1 radians. What is the dispersion in the centre? Given a particle with 1% energy deviation, compute the displacement at the centre of the cell.

17 Exercise: Chromaticity in a FODO cell

Consider a ring made of N_{cell} identical FODO cells with equally spaced quadrupoles. Assume that the two quadrupoles are both of length l_q, but their strengths may differ.

1. Calculate the maximum and the minimum betatron function in the FODO cell. (*Use the thin-lens approximations*)



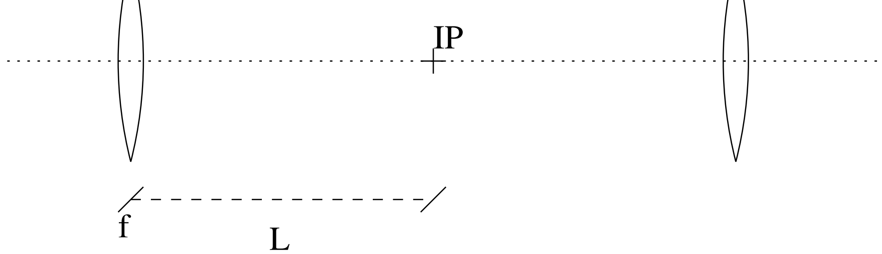
2. Calculate the natural chromaticities for this ring.
3. Show that for short quadrupoles, if $f_F \simeq f_D$,

$$\xi_N \simeq -\frac{N_{cell}}{\pi} \tan \frac{\mu}{2}.$$

4. Design the FODO cell such that it has: phase advance $\mu = 90$ degrees, a total length of 10 m, and a total bending angle of 5 degrees. What are β_{max}, β_{min}, D_{max}, D_{min}?
5. Add two sextupoles at appropriate locations to correct horizontal and vertical chromaticities. (hints: use 1 sextupole for the horizontal plane and 1 for the vertical plane; do not consider geometric aberrations).
6. If the gradient of all focusing quadrupoles in the ring is wrong by $+10\%$, how much is the tune-shift with and without sextupoles?

18 Exercise: Low-Beta Insertion

Consider the following low-beta insertion around an interaction point (IP). The quadrupoles are placed with mirror-symmetry with respect to the IP:



The beam enters the quadrupole with Twiss parameters $\beta_0 = 20$ m and $\alpha_0 = 0$. The drift space has length $L = 10$ m.

(i) Determine the focal length of the quadrupole in order to locate the waist at the IP.

(ii) What is the value of $\beta^{\star}$?

(iii) What is the phase advance between the quadrupole and the IP?