Transverse Beam Dynamics

JUAS 2020 - Tutorials (solutions)

1 Exercise: Wien Filter

A Wien Filter is a device that allows to select particles in a beam according to their velocity.

1. Write down the expression of the Lorentz force.

   Answer.

   $$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

2. How should we orient an electric field $\vec{E}$ if we want to compensate the force of an uniform magnetic field $\vec{B}$?

   Answer. The electric field must be oriented perpendicularly to both $\vec{v}$ and $\vec{B}$.

3. Assuming the magnetic field is 2 mT, what would be the required electric field (V/m) to select protons travelling with a velocity of $0.15c$?

   Answer.

   $$E = vB = 0.15c \ 2 \ \mathrm{mT} \approx 90'000 \ \mathrm{V/m}$$

4. Assuming that the particles move along the z axis and $\vec{B} = \begin{pmatrix} 0, & B_y, & 0 \end{pmatrix}$, write the equations of motion.

   Answer. If $\vec{B} = \begin{pmatrix} 0, & B_y, & 0 \end{pmatrix}$ and the particles move along the z axis, then $\vec{E} = \begin{pmatrix} E_x, & 0, & 0 \end{pmatrix}$, and the equations of motion read:

   $$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = q\left(E_x - B_y\frac{\mathrm{d}z}{\mathrm{d}t}\right)$$
   $$m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 0$$
   $$m\frac{\mathrm{d}^2z}{\mathrm{d}t^2} = qB_y\frac{\mathrm{d}x}{\mathrm{d}t}$$

   with $\frac{\mathrm{d}z}{\mathrm{d}t} = v_z$.

   - Particles with $v = v_z$ are not deflected if

   $$\left(E_x - B_y\frac{\mathrm{d}z}{\mathrm{d}t}\right) = 0 \quad \Rightarrow (E_x - B_y v_z) \quad \Rightarrow E_x = B_y v_z$$

   See: "CONSTRUCTION OF A WIEN FILTER HEAVY ION ACCELERATOR", K. Jensen and E. Veje, Nuclear Instruments and Methods 122 (1974)

5. [Optional] Could we use a Wien filter with a neutral beam (eg. neutron)? What other techniques could be employed to create a velocity filter?

   Answer. A Wien filter will not work with a neutral beam, but still it is possible to filter it by velocity. A possibility consists in having two rotating disks made of an absorbing material, presenting a radial slit. The velocity can be selected tuning the distance, the rotation frequency and the phase between the the discs.

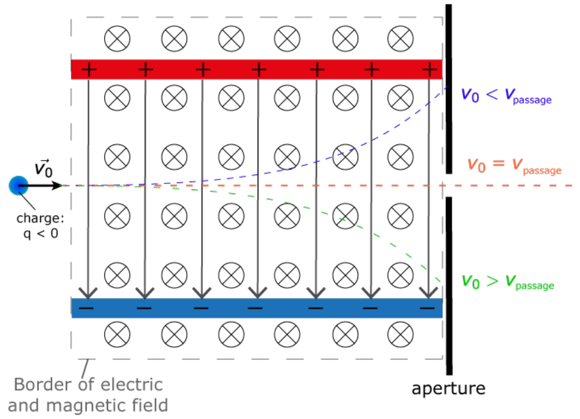
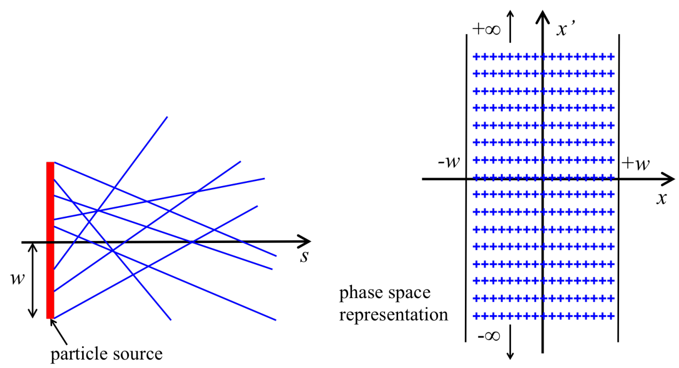


Figure 1: Sketch of a Wien filter

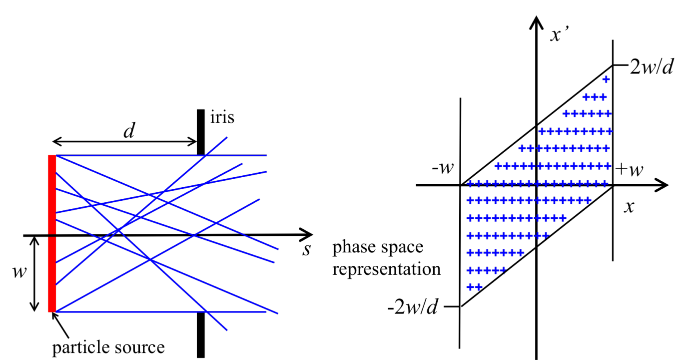
2 Exercise: Understanding the phase space concept

1. Phase Space Representation of a Particle Source:

   - Consider a source at position s_0 with radius w emitting particles. Make a drawing of this setup in the configuration space and in the phase space. Which part of the phase space can be occupied by the emitted particles?
   Answer. Particles are emitted from the entire source surface $x \in [-w, +w]$ with a trajectory slope $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, i.e. the particles can have any $x' \in \mathbb{R}$. The occupied phase space area is infinite.



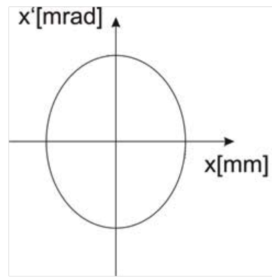
   - Any real beam emerging from a source like the one above will be collimated. This can be modelled by assuming that a distance d away from the source there is an iris with opening radius $R = w$. Draw this setup in the configuration space and in the phase space. Which part of the phase space is occupied by the beam, right after the collimator?
   Answer. Particles with angle $x' = 0$ are emitted from the entire source surface $x \in [-w, +w]$ and arrive behind the iris opening. For $x = \pm w$ there is a maximum angle $x' = \pm 2w/d$ that will still be accepted by the iris. This leads to a parallelogram in phase space. Such a beam has a specific emittance given by the occupied phase space area.



2. Sketch the emittance ellipse of a particle beam in:

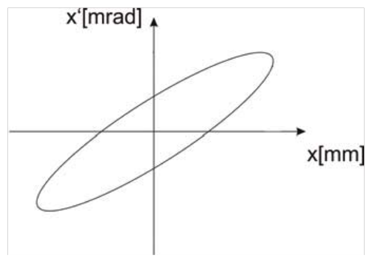
(I) horizontal x-x' phase space at the position of a transverse waist,

Answer. Beam at the position of a transverse (x) waist



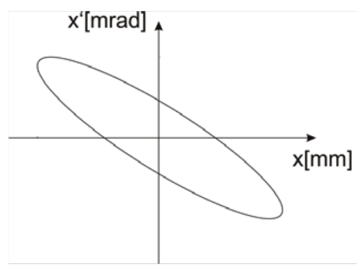
(II) when the beam is divergent, and

Answer. Divergent beam (positive slope):



(III) when the beam is convergent.

Answer. Convergent beam (negative slope):



3 Exercise: Local radius, rigidity

We wish to design a proton ring with a radius of $R = 200$ m. Let us assume that only 50% of the circumference is occupied by bending magnets:

- What will be the local radius of bend ρ in these magnets if they all have the same strength?

  Answer.
  $$2\pi\rho = 50\% \cdot 2\pi R \longrightarrow \rho = 100 \text{ m}.$$

- If the kinetic energy of the protons is 2 GeV, calculate the beam rigidity $B\rho$ and the field in the dipoles.

  Answer. The total energy is given by
  $$E = \sqrt{m_0^2 c^4 + p^2 c^2},$$
  where $m_0 = 938 \text{ MeV}/c^2$ is the rest mass of the proton. Knowing that the kinetic energy is
  $$E_k = E - m_0 c^2 = \sqrt{m_0^2 c^4 + p^2 c^2} - m_0 c^2$$
  then

$$p = 2.78 \text{ GeV}/c.$$

The magnetic rigidity is:

$$B\rho \approx \frac{1}{0.3} p \text{ [GeV/c]=9.27 T} \cdot \text{m}$$

and therefore $B \approx 0.09$ T.

4 Exercise: Thin-lens approximation

1. Compute and compare the matrices for the quadrupole of the previous exercise for the thin and thick cases.

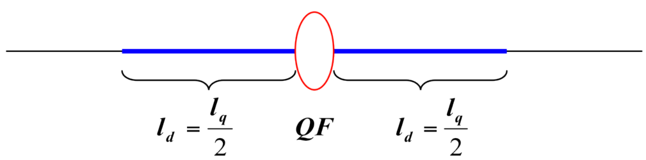
   Answer. The matrix of a focusing quadrupole is given by

$$M_{QF} = \begin{pmatrix} \cos\left(\sqrt{|k|}l_q\right) & \frac{1}{\sqrt{|k|}}\sin\left(\sqrt{|k|}l_q\right) \\ -\sqrt{|k|}\sin\left(\sqrt{|k|}l_q\right) & \cos\left(\sqrt{|k|}l_q\right) \end{pmatrix} = \begin{pmatrix} 0.8525 & 5.22 \\ -0.0522 & 0.8525 \end{pmatrix}$$

   In thin lens approximation we replace the matrix above by the expression

$$M_{QF} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{|f|} & 1 \end{pmatrix} \text{ with focal length } f = \frac{1}{|kl_q|} = 18.2 \text{ m}$$

   The thin lens description has to be completed by the matrix of a drift space of half the quadrupole length in front and after the thin lens quadrupole. The appropriate description is therefore



   So we write

$$M_{thinlens} = \begin{pmatrix} 1 & \frac{l_q}{2} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{l_q}{2} \\ 0 & 1 \end{pmatrix}$$

   Multiplying out we get

$$M_{thinlens} = \begin{pmatrix} 1 + \frac{l_q}{2}kl_q & \frac{l_q}{2}\left(2 + kl_q\frac{l_q}{2}\right) \\ kl_q & 1 + kl_q \end{pmatrix}$$

   With the parameters in the example we get finally

$$M_{thinlens} = \begin{pmatrix} 0.848 & 5.084 \\ -0.055 & 0.848 \end{pmatrix}$$

   which is still quite close to the result of the exact calculation above.

2. Verify if the stability condition is valid

3. Would the answer be the same, if the quadrupole was defocusing?

- **Answer.** The stability condition is:

$$|\mathrm{Tr}\,(M)| \leq 2$$

Indeed,

$$0.848 + 0.848 \leq 2.$$

If the quadrupole was defocusing, then

$$\mathrm{Tr}\,(M) = \cosh\left(\sqrt{|k|}l_q\right) + \cosh\left(\sqrt{|k|}l_q\right)$$

which is always > 2.

5 Exercise: Hill's equation

Solve the Hill's equation:

$$y'' + k(s)y = 0$$

by substituting:

$$y = A\sqrt{\beta(s)}\cos[\phi(s) + \phi_0] \text{ with } \phi' = \tfrac{1}{\beta(s)}, \text{ and where } A \text{ and } \phi_0 \text{ are constants,}$$

demonstrating that a necessary condition is:

$$\tfrac{1}{2}\beta\beta'' - \tfrac{1}{4}\beta'^2 + k(s)\beta^2 = 1$$

The first and second derivative of y:

$$\begin{aligned} y' &= \tfrac{A}{\sqrt{\beta(s)}}(\tfrac{\beta'}{2}\cos[\phi(s)+\phi_0] - \sin[\phi(s)+\phi_0]) \\ y'' &= \tfrac{A}{\sqrt{\beta(s)}}(\tfrac{\beta''}{2} - \tfrac{\beta'^2}{4\beta} - \tfrac{1}{\beta})\cos[\phi(s)+\phi_0] \end{aligned}$$

Substituting in the Hill's equation

$$\tfrac{A}{\sqrt{\beta(s)}}(\tfrac{\beta''}{2} - \tfrac{\beta'^2}{4\beta} + k(s)\beta - \tfrac{1}{\beta})\cos[\phi(s)+\phi_0] = 0$$

Since the phase $\phi(s)$ has a different value at every point around the orbit and the amplitude $A \neq 0$, the previous equation can only be satisfied if

$$\tfrac{\beta\beta''}{2} - \tfrac{\beta'^2}{4} + k(s)\beta^2 - 1 \quad = \quad 0$$

and therefore

$$\tfrac{1}{2}\beta\beta'' - \tfrac{1}{4}\beta'^2 + k(s)\beta^2 = 1$$

Q.E.D.!

6 Exercise: Particle momentum, geometry of a storage ring and thin lenses

The LHC storage ring at CERN collides proton beams with a maximum momentum of $p = 7$ TeV/c per beam. The main parameters of this machine are:

Circumference	$C_0 = 26658.9$ m	
Particle momentum	$p = 7$ TeV/c	
Main dipoles	$B = 8.392$ T	$l_B = 14.2$ m
Main quadrupoles	$G = 235$ T/m	$l_q = 5.5$ m

1. Calculate the magnetic rigidity of the design beam, the bending radius of the main dipole magnets in the arc and determine the number of dipoles that is needed in the machine.

   Answer. The beam rigidity is obtained in the usual way by the golden rule:

   $$B\rho = \frac{p}{e} = \frac{1}{0.299792} \cdot p[\text{GeV}/c] = 3.3356 \cdot p[\text{GeV}/c] = 3.3356 \cdot 7000 \text{ Tm} = 23349 \text{ T}\cdot\text{m}$$

   and knowing the magnetic dipole field we get

   $$\rho = \frac{3.3356 \cdot 7000\text{Tm}}{8.392\text{T}} = 2782\text{m}$$

   The bending angle for one LHC dipole magnet:

   $$\theta = \frac{l_B}{\rho} = \frac{14.2\text{m}}{2782\text{m}} = 5.104 \text{ mrad}$$

   and as we want to have a closed storage ring we require an overall bending angle of 2π:

   $$N = \frac{2\pi}{\theta} = 1231 \text{ Magnets}$$

2. Calculate the k-strength of the quadrupole magnets and compare its focal length to the length of the magnet. Can this magnet be treated as a thin lens?

   Answer. We can use the beam rigidity (or the particle momentum) to calculate the normalised quadrupole strength:

   $$k = \frac{G}{B\rho} = \frac{G}{p/e} = 0.299792 \cdot \frac{G}{p[\text{GeV/c}]} = 0.299792 \cdot \frac{235\text{T/m}}{7000\text{GeV/c}} = 0.01 \text{ m}^{-2}$$

   an the focal length:

   $$f = \frac{1}{k \cdot l_q} = 18.2 \text{ m} > l_q$$

   The focal length of this magnet is still quite bigger than the magnetic length l_q. So it is valid to treat that quadrupole in thin lens approximation.

7 Exercise: Stability condition

Consider a lattice composed by a single 2 meters long quadrupole, with $f = 1$ m

- Prove that if the quadrupole is defocusing, then a lattice is not stable
- Prove that if the quadrupole is focusing, then the lattice is stable

Solution:

- Let's work in thin-lens approximation

  $$M_{\text{QD}} = \begin{pmatrix} 1 & L_{\text{quad}}/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_{\text{quad}}/2 \\ 0 & 1 \end{pmatrix}$$

  which can be computed to be

  $$M_{\text{QD}} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

  has trace $\text{Tr}(M_{\text{QD}}) = 4$, which does not fulfill the stability requirement:

  $$|\text{Tr}(M_{\text{QD}})| \leq 2$$

- In the case of a focusing quadrupole:

  $$M_{\text{QF}} = \begin{pmatrix} 1 & L_{\text{quad}}/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_{\text{quad}}/2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

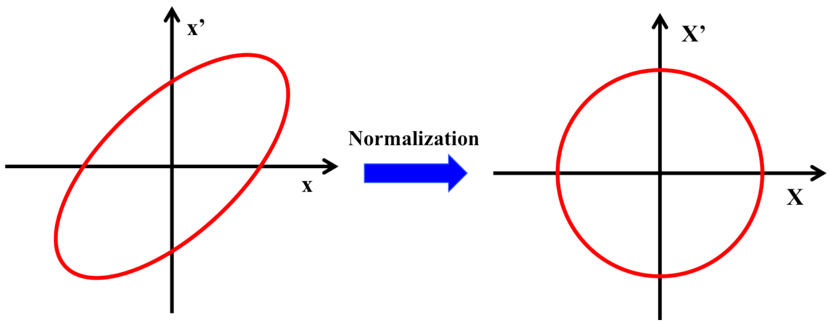
  which clearly satisfies the stability criterion.

8 Exercise: Normalised phase space

Let us consider the following phase space vector: (x, x'). The transformation to a *normalised phase space* (X, X') is given by:

$$\begin{pmatrix} X \\ X' \end{pmatrix} = \begin{pmatrix} 1/\sqrt{\beta_x} & 0 \\ \alpha_x/\sqrt{\beta_x} & \sqrt{\beta_x} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

The normalisation process of the phase space is illustrated in the figure below:



If we know that the transfer matrix between two points 1 and 2 (with phase advance ϕ_x between them) in the phase space (x, x') is given by:

$$M_{1\to 2} = \begin{pmatrix} \sqrt{\frac{\beta_{x2}}{\beta_{x1}}}(\cos\phi_x + \alpha_{x1}\sin\phi_x) & \sqrt{\beta_{x1}\beta_{x2}}\sin\phi_x \\ \frac{(\alpha_{x1}-\alpha_{x2})\cos\phi_x-(1+\alpha_{x1}\alpha_{x2})\sin\phi_x}{\sqrt{\beta_{x2}\beta_{x1}}} & \sqrt{\frac{\beta_{x1}}{\beta_{x2}}}(\cos\phi_x - \alpha_{x2}\sin\phi_x) \end{pmatrix}$$

Obtain the transfer matrix between two points 1 and 2 in the normalised phase space.

Answer. If one writes

$$M_{1\to 2} = U_2^{-1} \cdot R \cdot U_1$$

with U_1 the transformation into normalised coordinates for the Twiss parameters at 1, and U_2 its inverse for the Twiss parameters at 2: i.e.,

$$U_1 = \begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ \frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix}; \qquad U_2^{-1} = \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix}$$

It can be shown that the matrix M_{12} can be written as:

$$M_{12} = \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix} \begin{pmatrix} \cos\Delta\phi & \sin\Delta\phi \\ -\sin\Delta\phi & \cos\Delta\phi \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ \frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix}$$

with

$$R = \begin{pmatrix} \cos\Delta\phi & \sin\Delta\phi \\ -\sin\Delta\phi & \cos\Delta\phi \end{pmatrix}.$$

9 Exercise: beam size and luminosity

An e^+e^- collider has an interaction Point (IP) with $\beta_x^* = 0.5$ m and $\beta_y^* = 0.1$ cm. The peak luminosity available by a e^+e^- collider can be written as:

$$L = \frac{N_b N_{e^-} N_{e^+} f_{\rm rev}}{4\pi\sigma_x^*\sigma_y^*} \; [\rm cm^{-2}s^{-1}]$$

where $N_b = 80$ is the number of bunches per beam (we assume the same number of bunches for both the e^- and the e^+ beams), $N_{e^-} = N_{e^+} = 5 \times 10^{11}$ is the number of particles per bunch (we assume the same number for both e^- and e^+ bunches), and $f_{\rm rev}$ is the revolution frequency. The horizontal and vertical normalised beam emittances are respectively: $\epsilon_{x,N} = 2.2$ mm and $\epsilon_{y,N} = 4.7$ μm.

- Compute the revolution frequency $f_{\rm rev}$, knowing that the circumference is 80 km and that the beam moves nearly at the speed of light

Solution. The revolution period is given by T_{rev} = circumference/c = 80km/c, and therefore the revolution frequency is:

$$f_{rev} = 1/T_{rev} = c/80 \text{ km} \simeq 3.75 \text{ kHz}$$

- Calculate the beam transverse beam sizes σ_x^* and σ_y^* at the IP, and the luminosity L for two different beam energies: 45 GeV and 120 GeV

Solution. For 45 GeV beam energy: in this case the Lorentz factor is $\gamma = 88062.622$, and $\sigma_x^* = \sqrt{\beta_x^* \epsilon_{x,N}/\gamma} \simeq 111.76$ μm and $\sigma_y^* = \sqrt{\beta_y^* \epsilon_{y,N}/\gamma} \simeq 0.23$ μm, and the luminosity is $L \simeq 2.32 \times 10^{34}$ cm^{-2}s^{-1}

For 120 GeV beam energy: in this case the Lorentz factor is $\gamma = 234833.66$, and $\sigma_x^* = \sqrt{\beta_x^* \epsilon_{x,N}/\gamma} \simeq 68.56$ μm and $\sigma_y^* = \sqrt{\beta_y^* \epsilon_{y,N}/\gamma} \simeq 0.14$ μm, and the luminosity is $L \simeq 6.22 \times 10^{34}$ cm^{-2}s^{-1}

- What are the beam divergences (horizontal and vertical) at the IP for the 45 GeV case?

Solution. Where $\alpha = 0$ we have

$$\sigma_{x'}^* = \sqrt{\frac{\epsilon_{x,N}}{\gamma \beta_x^*}} = ...$$

- What is the value of the betatron function at position $s = 0.5$ m from the IP?

Solution. We know that the betatron function in the drift space of a low beta region (where we have the interaction point) depends on the longitudinal coordinate as follows:

$$\beta(s) = \beta^* + \frac{s^2}{\beta^*}$$

Therefore, $\beta_x(0.5\text{m}) = 1$ m, and $\beta_y(0.5\text{m}) = 250$ m

10 Exercise: Basics of lattice design

Design a FODO cell such that it has: phase advance $\mu = 90$ degrees, a total length of 10 m, and a total bending angle of 5 degrees. What are β_{max}, β_{min}, D_{max}, D_{min}?

Answer.

Lattice parameters: $L = 10$ m, $\theta = 5$ degrees= 0.087266 rad, $f = \frac{1}{\sqrt{2}} \frac{L}{2} = 3.535$ m.

Maximum and minimum betatron functions:

$$\beta_{max} = \frac{L + \frac{L^2}{4f}}{\sin \mu} = L + \frac{L^2}{4f} = 17.07 \text{ m}, \quad \beta_{min} = \frac{L - \frac{L^2}{4f}}{\sin \mu} = L - \frac{L^2}{4f} = 2.93 \text{ m}$$

Maximum and minimum dispersion:

$$D_{max} = \frac{L\theta \left(1 + \frac{1}{2} \sin \frac{\mu}{2}\right)}{4 \sin^2 \frac{\mu}{2}} = \frac{f}{L} \left(4f + \frac{L}{2}\right) \theta = 0.59060 \text{ m}, \quad D_{min} = \frac{L\theta \left(1 - \frac{1}{2} \sin \frac{\mu}{2}\right)}{4 \sin^2 \frac{\mu}{2}} = \frac{f}{L} \left(4f - \frac{L}{2}\right) \theta = 0.28207 \text{ m}$$

11 Exercise: Bump and Orbit Control

Given two kickers located at the two ends of a FODO cell with phase advance 45 degrees (the two kickers are located at L_{cell} distance from each other), compute the strengths of such kickers (in radians) in order to give the beam, initially at $(x_i, x_i') = (0, 0)$, an arbitrary offset at the end of the cell while preserving its angle, $\left(x_f, x_f'\right) = (x_{\text{arbitrary}}, 0)$.

Solution

The transfer matrix of a periodic cell is:

$$M = \begin{pmatrix} \cos\varphi + \alpha \sin\psi & \beta \sin\varphi \\ -\gamma \sin\varphi & \cos\varphi - \alpha \sin\varphi \end{pmatrix}$$

Substituting the value for the phase advance we get the matrix to apply to the beam right after the first kick k_1:

$$\begin{pmatrix} x_f \\ x'_f \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1+\alpha & \beta \\ -\gamma & 1-\alpha \end{pmatrix} \begin{pmatrix} 0 \\ k_1 \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} \beta k_1 \\ (1-\alpha) k_1 \end{pmatrix}$$

From this we see that to achieve an arbitrary x_f we need:

$$k_1 = \frac{\sqrt{2} x_f}{\beta}$$

The second kick, k_2, has only to remove the final tilt:

$$k_2 = -x'_f = -\frac{(1-\alpha)}{\sqrt{2}} k_1$$

Notice that one can reduce the strength of the kickers by placing them close to a focusing quadrupoles, where β is maximum.

12 Exercise: Measurement of Twiss parameters

One of the possible ways to determine experimentally the Twiss parameters at a given point makes use of a so-called quadrupole scan. One can measure the transverse size of the beam in a profile monitor, called Wire Beam Scanner (WBS), located at a distance L downstream a focusing quadrupole, as a function of the normalised gradient in this quadrupole. This allows to compute the emittance of the beam, as well as the β and the α functions at the entrance of the quadrupole.

Let's consider a quadrupole Q with a length of $l = 20$ cm. This quadrupole is installed in an electron transport line where the particle momentum is 300 MeV$/c$. At a distance $L = 10$ m from the quadrupole the transverse beam size is measured with a WBS, for various values of the current I_Q. The maximum value of the quadrupole gradient G is obtained for a current of 100 A, and is $G = 1$ T/m.

Hint: G is proportional to the current. **Advice:** use thin-lens approximation.

1. How does the normalised focusing strength K vary with I_Q?

   Answer. The quadrupole gradient G is proportional to the current flowing through the coils I_Q

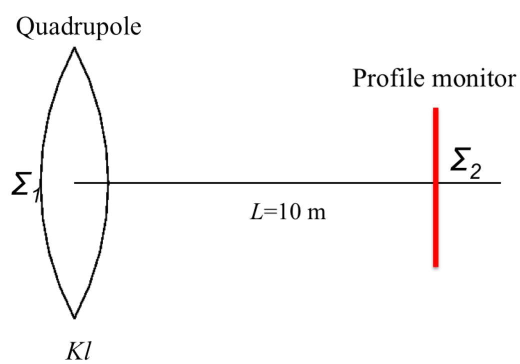
   $$G = C \cdot I_Q,$$

   C is the proportionality coefficient. We know that $G = 1$ T/m when $I_Q = 100$ A, therefore $C = 0.01$ T/(A·m). The normalised focusing strength is

   $$K = \frac{G}{P/q} \quad \text{therefore} \quad K = \frac{C \cdot I_Q}{P/q}$$

2. Give the expression Σ_2 as function of α_1, β_1, and γ_1

   Answer. Let Σ_1 and Σ_2 be the 2×2 matrices with the twiss parameters, $\Sigma = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$, at the quadrupole entrance and at the wire scanner, respectively.



It is worth explaining that the matrix Σ multiplied by the emittance ϵ is the covariance matrix of the beam distribution:

$$\Sigma\epsilon = \begin{pmatrix} \beta\epsilon & -\alpha\epsilon \\ -\alpha\epsilon & \gamma\epsilon \end{pmatrix} = \begin{pmatrix} \langle x^2\rangle & \langle xx'\rangle \\ \langle x'x\rangle & \langle x'^2\rangle \end{pmatrix}$$

The transverse beam size of the beam is given by $\sigma_x = \sqrt{\langle x^2\rangle} = \sqrt{\beta_x\epsilon_x}$ (horizontal beam size), and $\sigma_y = \sqrt{\langle y^2\rangle} = \sqrt{\beta_y\epsilon_y}$ (vertical beam size). Here we will simply use the following notation: $\sigma_1 = \sqrt{\beta_1\epsilon}$ for the beam size (horizontal or vertical) at position 1, and $\sigma_2 = \sqrt{\beta_2\epsilon}$ for the beam size (horizontal or vertical) at position 2.The matrix Σ propagates from position 1 to position 2 as follows:

$$\Sigma_2 = M\Sigma_1 M^T$$

where M is the transfer matrix of the system and M^T its transpose. We have:

$$\begin{aligned}\Sigma_2 = \begin{pmatrix} \beta_2 & -\alpha_2 \\ -\alpha_2 & \gamma_2 \end{pmatrix} &= \begin{pmatrix} 1-KlL & L \\ -Kl & 1 \end{pmatrix}\begin{pmatrix} \beta_1 & -\alpha_1 \\ -\alpha_1 & \gamma_1 \end{pmatrix}\begin{pmatrix} 1-KlL & -Kl \\ L & 1 \end{pmatrix} \\ &= \begin{pmatrix} \beta_1 L^2(Kl)^2 + 2L(\alpha_1 L - \beta_1)Kl + \beta_1 - 2\alpha_1 L + \gamma_1 L^2 & \beta_1 L(Kl)^2 + (2\alpha_1 L - \beta_1)Kl + \gamma_1 L - \alpha_1 \\ \beta_1 L(Kl)^2 + (2\alpha_1 L - \beta_1)Kl + \gamma_1 L - \alpha_1 & \beta_1(Kl)^2 + 2\alpha_1 Kl + \gamma_1 \end{pmatrix}\end{aligned} \tag{1}$$

3. Show that β_2 can be written in the form: $\beta_2 = A_2\,(Kl)^2 + A_1\,(Kl) + A_0$, and express A_0, A_1, and A_2 as a function of L, α_1, β_1, and γ_1.

   Answer. We can see from Eq. (1) that:

   $$\beta_2 = \beta_1 L^2(Kl)^2 + 2L(\alpha_1 L - \beta_1)Kl + \beta_1 - 2\alpha_1 L + \gamma_1 L^2$$

   and therefore:

   $$\begin{aligned} A_2 &= \beta_1 L^2 \\ A_1 &= 2L(\alpha_1 L - \beta_1) \\ A_0 &= \beta_1 - 2\alpha_1 L + \gamma_1 L^2 \end{aligned}$$

   Hint for the next questions: show that if one expresses β_2 as

   $$\beta_2 = B_0 + B_1\,(Kl - B_2)^2$$

   one has:

   $$\begin{aligned} B_0 &= A_0 - A_1^2/4A_2^2 = L^2/\beta_1 \\ B_1 &= A_2 = L^2\beta_1 \\ B_2 &= -A_1/A_2 = 1/L - \alpha_1/\beta_1 \end{aligned}$$

4. Express the final beam size, σ_2, as a function of Kl, and find its minimum, which will correspond to $(Kl)_{\mathrm{min}}$.

   Answer. The transverse r.m.s. beam size is $\sigma = \sqrt{\epsilon\beta}$, where ϵ is the transverse (geometric) emittance. As we have seen in the previous questions β_2 depends quadratically on Kl: $\beta_2 = B_0 + B_1\,(Kl - B_2)^2$. Since ϵ is constant, if we want to minimise σ_2, we have to minimise β_2:

   $$\frac{\mathrm{d}\beta_2}{\mathrm{d}(Kl)} = 0 \longrightarrow 2B_1(Kl - B_2) = 0 \longrightarrow (Kl)_{min} = B_2 = \frac{1}{L} - \frac{\alpha_1}{\beta_1} \tag{2}$$

   We can write:

   $$\sigma_2^2 = \beta_2\epsilon = \frac{L^2}{\beta_1}\left(1 + \beta_1^2(Kl - (Kl)_{min})^2\right)\epsilon$$

   Why is this useful? By means of a quadrupole scan (i.e. changing the quadrupole strength) we identify the strength Kl which minimises the value σ_2^2. We fit a parabola to the measurements σ_2^2 vs. Kl, and select then $\sigma_2^2((Kl)_{min})$. The minimum beam size is given by:

   $$\mathrm{Min}(\sigma_2) = L\sqrt{\frac{\epsilon}{\beta_1}} = \sqrt{B_0\epsilon} \tag{3}$$

5. How does σ_2 vary with Kl when $|Kl - (Kl)_{\min}| \gg 1/\beta_1$?

Answer. Under this condition:

$$\sigma_2^2 = \frac{L^2}{\beta_1}\left(1 + \beta_1^2(Kl - (Kl)_{min})^2\right)\epsilon \longrightarrow \sigma_2 \simeq L\sqrt{\beta_1\epsilon}(Kl - (Kl)_{min})$$

For $|Kl - (Kl)_{\min}| \gg 1/\beta_1$, σ_2 depends linearly on Kl, with slope

$$\frac{\mathrm{d}\sigma_2}{\mathrm{d(Kl)}} = \frac{L^2\beta_1}{\sigma_2}(Kl - (kl)_{min})\epsilon.$$

6. Deduce the values of α_1, β_1, and γ_1 from the measurement σ_2, as a function of the quadrupole current I_Q.

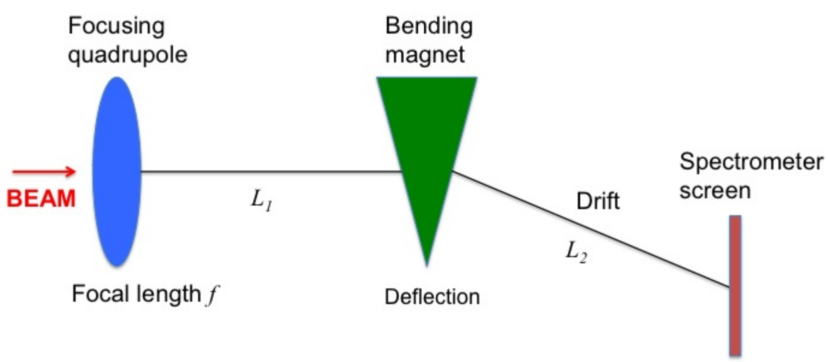
Answer. We know that

$$Kl = \frac{G \cdot l}{p/e} = \frac{C \cdot l \cdot I_Q}{p/e} = \frac{0.01[\mathrm{T/(Am)}]\cdot 0.2[\mathrm{m}]}{(0.3[\mathrm{GeV}]/0.3)[\mathrm{Tm}]} \cdot I_Q = 2 \times 10^{-3} \cdot I_Q$$

If we measure σ_2 as a function of the quadrupole current I_Q, from the minimum value we can get β_1 (Eq. (3)), and since from the measurement we obtain $(Kl)_{min} = 2 \times 10^{-3}(I_Q)_{min}$, using Eq. (2) we can calculate α_1. Once we know β_1 and α_1, it is then straightforward to calculate $\gamma_1 = (1 + \alpha_1^2)/\beta_1$.

13 Exercise: The spectrometer line of CTF3

The CTF3 (CLIC Test Facility 3) experiment at CERN consists of a linac that injects very short electron bunches into an isochronous ring. A spectrometer line made of one quadrupole and one bending magnet is located at the end of the linac where the particle momentum is 350 MeV/c. The goal of the spectrometer is to measure the energy before injecting the electrons in the ring. The spectrometer line is sketched on the figure below. It is made of a focusing quadrupole of focal length f, a drift space of length L_1, a bending magnet of deflection angle θ in the horizontal plane, and a drift space of length L_2. We assume that the spectrometer line starts at the quadrupole and ends at the end of the second drift. We neglect the focusing effect of the dipole.



1. If the effective length of the dipole is $l_B = 0.43$ m, what should be the magnetic field (in Tesla) inside the dipole to deflect the electrons by an angle of 35 degrees?

Answer. One has $\theta = \frac{l}{\rho}$ and $B\rho = 3.356\,p$: $B = \dfrac{3.356\,p\,\theta}{l} = 1.66$ T.

2. Starting from the general horizontal 3×3 transfer matrix of a sector dipole of deflection angle θ, show that the transfer matrix of a dipole in the thin-lens approximation is

$$M_{dipole} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

Which approximations are done?

Hint: A sector dipole has the following 3×3 transfer matrix:

$$M_{\text{dipole}} = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{\sin\theta}{\rho} & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

Answer. We need to compute the limit for $l \to 0$ while keeping $\theta = \frac{l}{\rho} = \text{const}$. Remember that, if θ is a small angle, $\cos\theta \approx 1$, $\sin\theta \approx \theta$. Besides the trivial elements, such as m_{11}, m_{22}, and m_{23}, the others read:

$$m_{12}: \quad \lim_{l\to 0} \rho\sin\theta = \lim_{l\to 0} \frac{\sin\theta}{\frac{1}{\rho}} = \lim_{l\to 0} l \cdot \underbrace{\frac{\sin\frac{l}{\rho}}{\frac{l}{\rho}}}_{\text{const}} = 0$$

$$m_{13}: \quad \lim_{l\to 0} \rho\,(1-\cos\theta) = \lim_{l\to 0} \rho\left(1 - \cos\frac{l}{\rho}\right) = \lim_{l\to 0} l \cdot \underbrace{\frac{1-\cos\frac{l}{\rho}}{\frac{l}{\rho}}}_{\text{const}} = 0$$

$$m_{21}: \quad \lim_{l\to 0} -\frac{\sin\theta}{\rho} = \lim_{\rho\to\infty} -\frac{\sin\theta}{\rho} = 0$$

therefore, in thin-lens approximation the matrix of a dipole magnet, is

$$M_{\text{dipole}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}.$$

3. In the thin-lens approximation, derive the horizontal extended 3×3 transfer matrix of the spectrometer line. Show that it is:

$$M_{spectro} = \begin{pmatrix} \frac{f-L_1-L_2}{f} & L_1+L_2 & L_2\theta \\ -\frac{1}{f} & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

Answer. For the spectrometer line, one has

$$M_{\text{spectro}} = M_{\text{Drift2}} \cdot M_{\text{Dipole}} \cdot M_{\text{Drift1}} \cdot M_{\text{Quad}}$$

therefore:

$$M_{\text{spectro}} = \begin{pmatrix} 1 & L_2 & L_2\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1-\frac{L_1}{f} & L_1 & 0 \\ -\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4. Assuming $D = D' = 0$ at the entrance of the quadrupole, what is the dispersion and its derivative at the end of the spectrometer line? Give the numerical value of D' at the end of the spectrometer line for the angle of 35 degrees.

   Answer. At the entrance of the line, $D = 0$ and $D' = 0$. If M is the transfer matrix of a system the dispersion D at exit is the element m_{13} of M, whereas D' is the element m_{23}:

$$D = L_2\theta,$$
$$D' = \theta = 35 \text{ degrees} = 0.61.$$

5. What is the difference between a periodic lattice and a beam transport lattice (or transfer line) as concerns the betatron function ?

   Answer. In a periodic lattice the β-functions are periodic and contained in the (periodic) transfer matrix of the lattice. In transfer line one needs to know the initial conditions in order to calculate the β-functions at any point (using the transfer matrix).

6. Derive the betatron function β_2 at the end of the spectrometer line in terms of L_1, L_2, f and β_1, assuming $\alpha_1 = 0$.

Hint 1. Remember from the lecture:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

An alternative way to transport the Twiss parameters is through the σ matrix:

$$\sigma_i = \begin{pmatrix} \beta_i & -\alpha_i \\ -\alpha_i & \gamma_i \end{pmatrix}$$

This matrix multiplied by the emittance ϵ gives the so-called beam matrix (which has already been introduced during the lecture):

$$\Sigma_i = \begin{pmatrix} \beta_i\epsilon & -\alpha_i\epsilon \\ -\alpha_i\epsilon & \gamma_i\epsilon \end{pmatrix}$$

If σ_1 is the matrix at the entrance of the transfer line, the matrix σ_2 at the exit of the transfer line is given by

$$\sigma_2 = M\sigma_1 M^T$$

where M is the 2×2 transfer matrix of the line extracted from the extended 3×3 transfer matrix (see question 3), and M^T the transpose matrix of M.

Hint 2. For the calculations, write M as $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ and replace the values of the matrix elements only at the end.

Answer. One has $\sigma_2 = M\sigma_1 M^T$. If $\alpha_1 = 0$, then $\sigma_1 = \begin{pmatrix} \beta_1 & 0 \\ 0 & 1/\beta_1 \end{pmatrix}$

$$\sigma_2 = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \beta_1 & 0 \\ 0 & 1/\beta_1 \end{pmatrix} \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} \beta_1 m_{11}^2 + m_{12}^2/\beta_1 & \beta_1 m_{11}m_{21} + m_{12}m_{22}/\beta_1 \\ \beta_1 m_{11}m_{21} + m_{12}m_{22}/\beta_1 & \beta_1 m_{21}^2 + m_{22}^2/\beta_1 \end{pmatrix}$$

Therefore:

$$\beta_2 = \beta_1 m_{11}^2 + m_{12}^2/\beta_1$$

Since $m_{11} = \frac{f - L_1 - L_2}{f}$ and $m_{12} = L_1 + L_2$, one has:

$$\beta_2 = \beta_1 \left(1 - \frac{L_1 + L_2}{f}\right)^2 + \frac{(L_1 + L_2)^2}{\beta_1}.$$

7. Given the numerical values $L_1 = 1$ m, $L_2 = 2$ m, $\beta_1 = 10$ m, $\alpha_1 = 0$, compute the value of the focal length f such that the betatron function at the end of the spectrometer line is minimum.

Answer. If $L_1 = 1$ m, $L_2 = 2$ m, and $\beta_1 = 10$ m, then $\beta_2 = 0.9 + 10\left(1 - \frac{3}{f}\right)^2$. To have β_2 minimum one needs $\left(1 - \frac{3}{f} = 0\right)$.Therefore, $f = 3$ m.

8. For an off-momentum particle, compute the deviation from the design orbit? Why is it important to minimise the β function in the spectrometer?

Answer. With dispersion, the deviation from the design orbit is $\Delta x = D\frac{\Delta P}{P_0}$. Measuring Δx allows to determine ΔP and therefore P, if one has calibrated the spectrometer at P_0. It is important to minimise β_2 (at the screen location) in order to have the best possible resolution for Δx: a smaller β_2 will result in a smaller transverse beam size on the screen, which favours an accurate measurement of the momentum.

14 Exercise: Transfer matrix of a dipole magnet

- Remember weak focusing:
  $K = \frac{1}{\rho^2}$:

$$M_{\text{Dipole}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix} = \begin{pmatrix} \cos\frac{L}{\rho} & \rho\sin\frac{L}{\rho} \\ -\frac{1}{\rho}\sin\frac{L}{\rho} & \cos\frac{L}{\rho} \end{pmatrix}$$

- Compute the 3×3 matrix of a sector dipole including the dispersion terms.
  Remembering that:

$$D(s) = S(s)\int_0^s \frac{1}{\rho(t)}C(t)\,\mathrm{d}t - C(s)\int_0^s \frac{1}{\rho(t)}S(t)\,\mathrm{d}t$$

  one can easily find that:

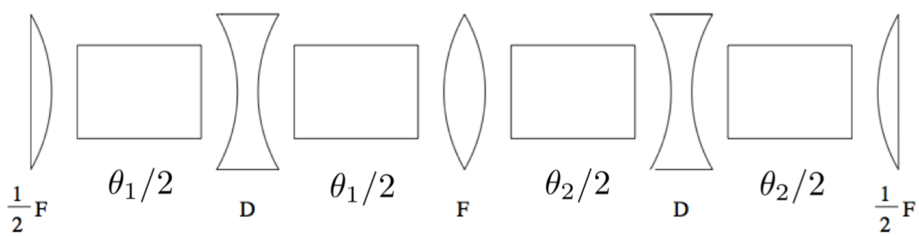
$$D(L) = \rho\left(1 - \cos\frac{L}{\rho}\right)$$
$$D'(L) = \sin\frac{L}{\rho}$$

  which allows to write M_{dipole} as 3×3 matrix in the form:

$$M_{\text{Dipole}} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix}$$

15 Exercise: Dispersion suppressor

In several straight sections of the accelerator, like the ones hosting RF cavities, extraction systems or other devices such as detectors, it is preferable to have no dispersion $\eta(s) = \eta'(s) = 0$. For example, in big colliders, such as the LHC, where small spot sizes are required at the interaction points, the dispersion is reduced to zero at the detector positions. The most common dispersion suppressors consists of two FODO cells of equal length L and equal quadrupole strengths. Bending magnets are placed in the space between the quadrupoles with a different bending field in each FODO. Figure below shows a typical dispersion suppressor.



1. Considering two FODO cells with different total bend angles, $\theta_1 \neq \theta_2$, calculate the relation between the angles θ_1 and θ_2 which must be satisfied to cancel the dispersion at the end of the lattice.
   Hint:
   For each FODO cell, $M_{\text{FODO}} = M_{1/2\text{F}} \cdot M_{\text{dipole}} \cdot M_{\text{D}} \cdot M_{\text{dipole}} \cdot M_{1/2\text{F}}$, in thin-lens approximation we have the following 3×3 matrix:

$$M_{\text{FODO }j} = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L\left(1 + \frac{L}{4f}\right) & \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_j \\ -\frac{L}{4f^2}\left(1 - \frac{L}{4f}\right) & 1 - \frac{L^2}{8f^2} & \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_j \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \cos\mu & \beta\sin\mu & \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_j \\ -\frac{\sin\mu}{\beta} & \cos\mu & \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_j \\ 0 & 0 & 1 \end{pmatrix}$$

where $j = 1, 2$ (1=first cell, 2=second cell).

The following condition must be satisfied:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = M_{\text{suppressor}} \begin{pmatrix} \eta_0 \\ 0 \\ 1 \end{pmatrix} \tag{4}$$

where η_0 is the initial dispersion (at the middle of the first focusing quadrupole). It can be demonstrated that for a FODO lattice the dispersion has its maximum at the middle of the focusing quadrupole:

$$\eta_0 = \frac{4f^2}{L}\left(1 + \frac{L}{8f}\right)\theta \tag{5}$$

with $\theta = \theta_1 + \theta_2$ the total bend in the suppressor.

Answer.

Performing the corresponding matrix multiplication yields

$$M_{\text{suppressor}} = \begin{pmatrix} \cos 2\mu & \beta \sin 2\mu & D_x \\ -\frac{\sin 2\mu}{\beta} & \cos 2\mu & D'_x \\ 0 & 0 & 1 \end{pmatrix}$$

where:

$$\begin{aligned}
\cos 2\mu &= 1 - \frac{L^2}{2f^2} + \frac{L^4}{34f^4} \\
\beta \sin 2\mu &= 2L\left(1 - \frac{L^2}{8f^2}\right)\left(1 + \frac{L}{4f}\right) \\
\frac{\sin 2\mu}{\beta} &= \frac{L}{2f^2}\left(1 - \frac{L^2}{8f^2}\right)\left(1 - \frac{L}{4f}\right) \\
D_x &= \cos\mu \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_1 + \beta \sin\mu \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_1 + \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_2 \\
D'_x &= -\frac{\sin\mu}{\beta}\frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_1 + \cos\mu\left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_1 + \left(1 - \frac{L}{8f^2} - \frac{L^2}{32f^2}\right)\theta_2
\end{aligned} \tag{6}$$

Taking into account:

$$\cos\mu = 1 - \tfrac{L^2}{8f^2};\ \beta \sin\mu = L + \tfrac{L^2}{4f} \text{ and } \tfrac{\sin\mu}{\beta} = \tfrac{1}{4f^2}\left(1 - \tfrac{L}{4f}\right)$$

the elements D_x and D'_xmay also be written as

$$\begin{aligned}
D_x &= \frac{L}{2}\left(1 + \frac{L}{8f}\right)\left[\left(3 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2\right] \\
D'_x &= \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\left[\left(1 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2\right]
\end{aligned} \tag{7}$$

From the condition Eq. (4) we have

$$\begin{aligned}
\eta_0 \cos 2\mu + D_x &= 0 \\
-\eta_0 \frac{\sin 2\mu}{\beta} + D'_x &= 0
\end{aligned} \tag{8}$$

Substituting Eq. (5) in Eq. (8) one obtains:

$$\left(3 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2 = \left(4 - \frac{L^2}{4f^2} - \frac{8f^2}{L^2}\right)\theta$$
$$\left(1 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2 = \left(2 - \frac{L^2}{4f^2}\right)\theta$$

In terms of phase advance μ this can be written as:

$$\begin{aligned} \theta_1 &= \left(1 - \frac{1}{4\sin^2\frac{\mu}{2}}\right)\theta \\ \theta_2 &= \frac{1}{4\sin^2\frac{\mu}{2}}\theta \end{aligned} \tag{9}$$

where $\theta_1 + \theta_2 = \theta$.

2. Obtain the relation between the angles for the cases of phase-advance per cell $\mu = \pi/3$ and $\pi/2$

   Answer.

   - For $\mu = \pi/3 \longrightarrow 4\sin^2\frac{\mu}{2} = 1$ and therefore (using Eq. (9)) $\theta_1 = 0$ and $\theta_2 = \theta$. This corresponds to a dispersion suppressor with missing magnets.
   - For $\mu = \pi/2 \longrightarrow 4\sin^2\frac{\mu}{2} = 2$ and therefore $\theta_1 = \theta_2 = \theta/2$.

16 Exercise: Double-Bend Achromat (DBA) lattice

A Double-Bend Achromat (DBA) can be made from two dipoles with a horizontally focusing quadrupole between them. The transfer matrix through the achromat is:

$$M_{\mathrm{DBA}} = M_{\mathrm{bend}} M_{\mathrm{drift}} M_{1/2\mathrm{F}} M_{1/2\mathrm{F}} M_{\mathrm{drift}} M_{\mathrm{bend}}$$

Note that this magnet configuration does not produce vertical focusing, therefore it will not be enough to create a stable lattice. A full DBA typically comprises additional quadrupole doublets before and after the bending section. For sake of simplicity, we will neglect them.

1. Use the thin-lens approximation for quadrupoles and small-angle approximation for bends to find the dispersion in the middle of the quadrupole. Write the focal length in terms of the drift and bend parameters.

   Answer. Let us consider the 3×3 transfer matrices of each element of the lattice (using the thin lens approximation and small angle approximation for the bending magnets) for the beam coordinates x, $x'$and $\Delta p/p$:

$$M_{\mathrm{bend}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{\mathrm{drift}} = \begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{1/2\mathrm{F}} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   Assuming the initial dispersion vector $(\eta_0, \eta'_0, 1) = (0, 0, 1)$ and propagating it to the centre of the quadrupole:

$$\begin{pmatrix} \eta_c \\ 0 \\ 1 \end{pmatrix} = M_{1/2\mathrm{F}} M_{\mathrm{drift}} M_{\mathrm{bend}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

   Here we take into account that η'=0 at the centre of a quadrupole. After matrix multiplication we obtain:

$$\begin{pmatrix} \eta_c \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & L & L\theta \\ -\frac{1}{2f} & 1 - \frac{L}{2f} & \theta\left(1 - \frac{L}{2f}\right) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

   Therefore, one obtains:

$$\eta_c = L\theta$$

$$1 - \frac{L}{2f} = 0 \Rightarrow L = 2f$$

2. Show that the dispersion vanishes after the bend.

   Answer. Propagate the dispersion vector from the centre of the quadrupole to the end of the lattice:

$$\begin{pmatrix} \eta_{end} \\ \eta'_{end} \\ 1 \end{pmatrix} = M_{\text{bend}} M_{\text{drift}} M_{1/2\text{F}} \begin{pmatrix} \eta_c \\ 0 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} \eta_{end} \\ \eta'_{end} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{2f} & L & 0 \\ -\frac{1}{2f} & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_c \\ 0 \\ 1 \end{pmatrix}.$$

   Taking into account $L = 2f$, we obtain:

$$\eta_{end} = (2f - L)\theta = 0,$$

$$\eta'_{end} = \theta - \frac{1}{2f}\eta_c = \theta - \frac{1}{2f}(2f\theta) = 0$$

3. Compute the parameters L, f for a 10-meter long DBA which bends the beam by an angle of 1 radians. What is the dispersion in the centre? Given a particle with 1% energy deviation, compute the displacement at the centre of the cell.

   Answer.

$$L = 5 \text{ m}$$

$$f = 2.5 \text{ m}$$

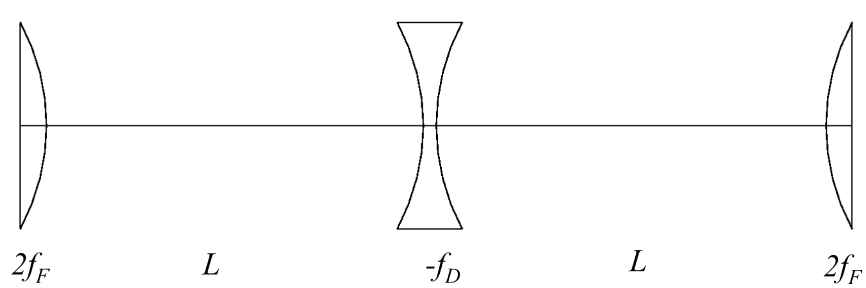
$$D = L \cdot \theta = 5 \text{ m}$$

$$x = \delta D = 0.01 * 5 \text{ m} = 5 \text{ cm}$$

17 Exercise: Chromaticity in a FODO cell

Consider a ring made of N_{cell} identical FODO cells with equally spaced quadrupoles. Assume that the two quadrupoles are both of length l_q, but their strengths may differ.

1. Calculate the maximum and the minimum betatron function in the FODO cell. (*Use the thin-lens approximations*)



Answer. First we calculate the transfer matrix for a FODO cell (see figure). We start from the centre of the focusing quadrupole where the betatron function is maximum. This exercise considers a general case where f_F is not necessarily equal to f_D. Using the thin lens approximation for the FODO cell with drifts of length L we get the following matrix:

$$\begin{aligned} M_{cell} &= \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_F} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f_D} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_F} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - L(\frac{1}{f_F} - \frac{1}{f_D} + \frac{L}{2f_F f_D}) & 2L + \frac{L^2}{f_D} \\ \frac{1}{f_D} - \frac{1}{f_F}(1 - \frac{L}{2f_F} + \frac{L}{f_D} - \frac{L^2}{4f_F f_D}) & 1 - L(\frac{1}{f_F} - \frac{1}{f_D} + \frac{L}{2f_F f_D}) \end{pmatrix} \end{aligned} \tag{10}$$

Remember that, in terms of betatron functions and phase advance, the matrix of a FODO cell is given by:

$$M_{cell} = \begin{pmatrix} cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix} \tag{11}$$

Since β has a maximum at the centre of the focusing quadrupole, then $\alpha = -\beta'/2 = 0$, and we can also write:

$$M_{cell} = \begin{pmatrix} cos\mu & \beta \sin\mu \\ -\frac{\sin\mu}{\beta} & \cos\mu \end{pmatrix}$$

Equating Eq. (10) to Eq. (12) we obtain:

$$\cos\mu = \frac{1}{2}\mathrm{tr}(M_{cell}) = 1 + \frac{L}{f_D} - \frac{L}{f_F} - \frac{L^2}{2f_D f_F} = 1 - 2\sin^2\frac{\mu}{2}$$

or

$$2\sin^2\frac{\mu}{2} = \frac{L}{f_F} - \frac{L}{f_D} + \frac{L^2}{2f_D f_F} \tag{12}$$

Where we have applied the following trigonometric identity: $\cos\mu = 1 - 2\sin^2\frac{\mu}{2}$.

The maximum for the betatron function β_{max} occurs at the focusing quadrupole. Since Eq. (10) is for a periodic cell starting at the centre of the focusing quadrupole, the m_{12} component of the matrix gives us

$$\beta_{max}\sin\mu = 2L + \frac{L^2}{f_D}$$

Rearranging:

$$\beta_{max} = \frac{2L + \frac{L^2}{f_D}}{\sin\mu} \tag{13}$$

On the other hand, the minimum for the betatron function occurs at the defocusing quadrupole position. Therefore, interchanging f_F with $-f_D$ for a FODO cell gives:

$$\beta_{min} = \frac{2L - \frac{L^2}{f_F}}{\sin\mu} \tag{14}$$

2. Calculate the natural chromaticities for this ring.

Answer. Let us remember the definition of natural chromaticity. The so-called "natural" chromaticity is the chromaticity that derives from the energy dependence of the quadrupole focusing, i.e. the chromaticity arising only from quadrupoles. The chromaticity is defined in the following way:

$$\xi = \frac{\Delta Q}{\Delta P/P_0} \tag{15}$$

where ΔQ is the tune shift due to the chromaticity effects and $\Delta P/P_0$ is the momentum offset of the beam or the particle with respect to the nominal momentum p_0.

The natural chromaticity is defined as (remember from Lecture 4):

$$\xi_N = -\frac{1}{4\pi}\oint \beta(s)k(s)ds \tag{16}$$

Sometimes, especially for small accelerators, the chromaticity is normalised to the machine tune Q and defined also as:

$$\xi' = \frac{\Delta Q/Q}{\Delta P/P_0} \tag{17}$$

$$\xi'_N = -\frac{1}{4\pi Q}\oint \beta(s)k(s)ds \tag{18}$$

For this exercise, either you decide to use Eq. (16) or Eq. (18) it is fine! From now on let us use Eq. (16):

$$\begin{aligned}
\xi_N &= -\frac{1}{4\pi}\oint \beta(s)k(s)ds \\
&= -\frac{1}{4\pi}\times N_{cell}\int_{cell}\beta(s)k(s)ds \\
&= -\frac{N_{cell}}{4\pi}\sum_{i\in\{quads\}}\beta_i(kl_q)_i
\end{aligned}$$

Here we have used the following approximation valid for thin lens:

$$\int_{cell}\beta(s)k(s)ds \simeq \sum_{i\in\{quads\}}\beta_i(kl_q)_i$$

where we sum over each quadrupole i in the cell. In the case of the FODO cell we have two half focusing quadrupoles and one defocusing quadrupole. Taking into account that $(kl_q)_i = 1/f_i$, we have:

$$\begin{aligned}
\xi_N &\simeq -\frac{N_{cell}}{4\pi}\sum_{i\in\{quads\}}\beta_i(kl_q)_i \\
&= -\frac{N_{cell}}{4\pi}\left[\beta_{max}\left(\frac{1}{2f_F}\right)+\beta_{min}\left(-\frac{1}{f_D}\right)+\beta_{max}\left(\frac{1}{2f_F}\right)\right] \\
&= -\frac{N_{cell}}{4\pi}\left[\beta_{max}\left(\frac{1}{f_F}\right)+\beta_{min}\left(-\frac{1}{f_D}\right)\right] \\
&= -\frac{N_{cell}}{4\pi\sin\mu}\left[\left(2L+\frac{L^2}{f_D}\right)\frac{1}{f_F}-\left(2L-\frac{L^2}{f_F}\right)\frac{1}{f_D}\right] \\
&= -\frac{N_{cell}L}{2\pi\sin\mu}\left[\frac{1}{f_F}-\frac{1}{f_D}+\frac{L}{f_Ff_D}\right]
\end{aligned}$$

Here we have used the expressions (13) and (14) for β_{max} and β_{min}.

3. Show that for short quadrupoles, if $f_F \simeq f_D$,

$$\xi_N \simeq -\frac{N_{cell}}{\pi}\tan\frac{\mu}{2}.$$

Answer. If $f_F \simeq f_D$, we have

$$\begin{aligned}
\xi_N &\simeq -\frac{N_{cell}}{2\pi\sin\mu}\frac{L^2}{f_Ff_D} \\
&= -\frac{N_{cell}}{4\pi\sin\frac{\mu}{2}\cos\frac{\mu}{2}}4\sin^2\frac{\mu}{2}
\end{aligned}$$

where we have used the trigonometric identity: $\sin\mu = 2\sin\frac{\mu}{2}\cos\frac{\mu}{2}$

Considering Eq. (12), we have

$$4\sin^2\frac{\mu}{2} = \frac{L^2}{f_F f_D}$$

which finally gives:

$$\xi_N \simeq -\frac{N_{cell}}{\pi}\tan\frac{\mu}{2}$$

Q.E.D.!

4. Design the FODO cell such that it has: phase advance $\mu = 90$ degrees, a total length of 10 m, and a total bending angle of 5 degrees. What are β_{max}, β_{min}, D_{max}, D_{min}?

   Answer. Lattice parameters: $L = 10$ m, $\theta = 5$ degrees$= 0.087266$ rad, $f = \frac{1}{\sqrt{2}}\frac{L}{2} = 3.535$ m

   Maximum and minimum betatron functions:

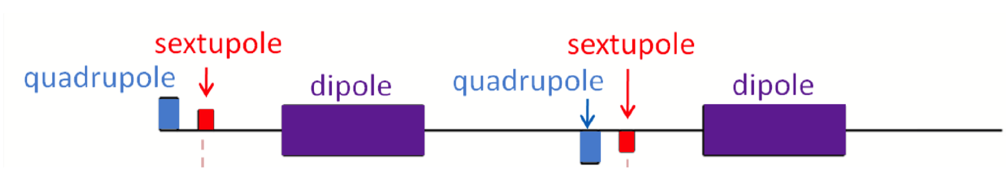
$$\beta_{max} = \frac{L + \frac{L^2}{4f}}{\sin\mu} = L + \frac{L^2}{4f} = 17.07 \text{ m}, \quad \beta_{min} = \frac{L - \frac{L^2}{4f}}{\sin\mu} = L - \frac{L^2}{4f} = 2.93 \text{ m}$$

   Maximum and minimum dispersion:

$$D_{max} = \frac{L\theta\left(1 + \frac{1}{2}\sin\frac{\mu}{2}\right)}{4\sin^2\frac{\mu}{2}} = \frac{f}{L}\left(4f + \frac{L}{2}\right)\theta = 0.59060 \text{ m}, \quad D_{min} = \frac{L\theta\left(1 - \frac{1}{2}\sin\frac{\mu}{2}\right)}{4\sin^2\frac{\mu}{2}} = \frac{f}{L}\left(4f - \frac{L}{2}\right)\theta = 0.28207 \text{ m}$$

5. Add two sextupoles at appropriate locations to correct horizontal and vertical chromaticities. (hints: use 1 sextupole for the horizontal plane and 1 for the vertical plane; do not consider geometric aberrations).

   Answer. By locating sextupoles with strength $K_s > 0$ where β_x is large and β_y is small, we can correct the horizontal chromaticity with relatively little impact on the vertical chromaticity. Similarly, by locating sextupoles with $K_s < 0$ where β_y is large and β_x is small, we can correct the vertical chromaticity with relatively little impact on the horizontal chromaticity. See figure below.



   Let us assume the case of a FODO lattice where $f_F = f_D = f$. Then the natural chromaticity of this FODO cell is given by the expression (exercise 1.3):

$$\xi_N \simeq -\frac{1}{\pi}\tan\frac{\mu}{2}$$

   For $\mu = 90$ it is $\xi_N \simeq -1/\pi$ in both horizontal and vertical plane. Therefore, we need to adjust the strength of the sextupoles to cancel this chromaticity:

$$-\frac{1}{4\pi}\left[K_{2F}D_{max}\beta_{max} + K_{2D}D_{min}\beta_{min}\right] \simeq -\frac{1}{\pi}$$

   where $K_{2F} = k_{2F}l_s$ is the normalised integrated strength of the sextupole located near the focusing quadrupole, and $K_{2D} = k_{2D}l_s$ the normalised integrated strength of the sextupole near the defocusing quadrupole (with l_s the effective length of the sextupole). For an effective cancellation of the chromaticity in both planes, usually $K_{2F} > 0$ and $K_{2D} < 0$. Substituting the values for the maximum and minimum dispersion and betatron function in terms of the total length of the lattice L and the focal length of the quadrupoles f, one obtains the following expression:

$$-\frac{1}{4\pi}\frac{f}{L}\theta\left[K_{2F}\left(4f+\frac{L}{2}\right)\left(L+\frac{L^2}{4f}\right)+K_{2D}\left(4f-\frac{L}{2}\right)\left(L-\frac{L^2}{4f}\right)\right]\simeq-\frac{1}{\pi}$$

Considering the same absolute value for the strength of the sextupoles, $K_{2F}=-K_{2D}=K_s$, we can write then:

$$\frac{3}{4\pi}K_sLf\theta=\frac{1}{\pi}$$

The strength of the sextupole is given then by:

$$K_s=\frac{4}{3Lf\theta}$$

Then, substituting all the numerical values for the lattice parameters:
$K_{2F}=0.865\ \mathrm{m}^{-2}$
$K_{2D}=-0.865\ \mathrm{m}^{-2}$

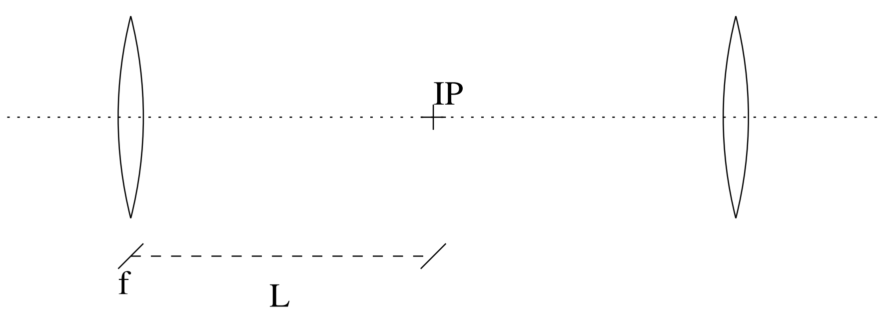
6. If the gradient of all focusing quadrupoles in the ring is wrong by $+10\%$, how much is the tune-shift with and without sextupoles?

Answer.

If the gradient of the focusing quadrupole has and error of 10%, then the corresponding quad. strength error is also 10%. We calculate the number of cells of a ring made of these FODO cells, $N_{cell}=72$ cells, and then we calculate the total tune-shift in both planes:
$\Delta Q_x=N_{cell}\frac{\Delta K_F\beta_{max}}{4\pi}=9.78$
$\Delta Q_y=N_{cell}\frac{\Delta K_F\beta_{min}}{4\pi}=1.68$
When the sextupoles correct for the chromaticity, the particles have, in principle, no tune-shift with energy. In real machines, one wants to have a non-zero residual chromaticity to stabilise the beam against resonant imperfections.

18 Exercise: Low-Beta Insertion

Consider the following low-beta insertion around an interaction point (IP). The quadrupoles are placed with mirror-symmetry with respect to the IP:



The beam enters the quadrupole with Twiss parameters $\beta_0=20$ m and $\alpha_0=0$. The drift space has length $L=10$ m.

(i) Determine the focal length of the quadrupole in order to locate the waist at the IP.

(ii) What is the value of $\beta^\star$?

(iii) What is the phase advance between the quadrupole and the IP?

Solution.

$$M = \begin{pmatrix} 1-\frac{L}{f} & L \\ -\frac{1}{f} & 1 \end{pmatrix}$$

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_{\mathrm{IP}} = M \cdot \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_0 \cdot M^T$$

$$\begin{pmatrix} \beta_{\mathrm{IP}} & 0 \\ 0 & 1/\beta_{\mathrm{IP}} \end{pmatrix} = M \cdot \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix} \cdot M^T$$

We get a system of equations:

$$\begin{cases} \beta_{\mathrm{IP}} = & \beta_0 \left(1-\frac{L}{f}\right)^2 + \frac{L^2}{\beta_0} \\ \frac{1}{\beta_{\mathrm{IP}}} = & \frac{\beta_0}{f^2} + \frac{1}{\beta_0} \end{cases}$$

multiplying them:

$$1 = \left(\beta_0 \left(1-\frac{L}{f}\right)^2 + \frac{L^2}{\beta_0}\right)\left(\frac{\beta_0}{f^2} + \frac{1}{\beta_0}\right)$$

and solving for f:

$$f = \frac{\beta_0 \sqrt{(\beta_0^2 - 4L^2)} + \beta_0^2}{2L}$$

from which one finds:

$$f = 20 \text{ m}$$

and substituting back into one of the equations in the system:

$$\beta_{\mathrm{IP}} = 10 \text{ m}.$$

The phase advance can be computed remembering that

$$M_{0\to s} = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\sin\psi_s \\ \frac{(\alpha_0-\alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}\left(\cos\psi_s - \alpha_s\sin\psi_s\right) \end{pmatrix}$$

In this case, $\alpha_0 = \alpha_{\mathrm{IP}} = 0$,

$$\mathrm{Trace}\,(M) = \frac{3}{2} = \left(\sqrt{\frac{\beta^\star}{\beta_0}} + \sqrt{\frac{\beta^0}{\beta^\star}}\right)\cos\Delta\mu$$

$$\Delta\mu = \arccos\left(\frac{3}{2}\cdot\frac{1}{\sqrt{\frac{\beta^\star}{\beta_0}} + \sqrt{\frac{\beta^0}{\beta^\star}}}\right) = \arccos\left(\frac{3}{2}\cdot\frac{1}{2.1213}\right) = 45 \text{ degrees}$$

Alternatively, given that the system:

$$M = Q \cdot D \cdot D \cdot Q$$

is indeed periodic, one can say:

$$M = \begin{pmatrix} 1-\frac{2L}{f} & 2L \\ \frac{2L}{f^2} - \frac{2}{f} & 1-\frac{2L}{f} \end{pmatrix}$$

$$\cos\Delta\mu_{\mathrm{twice}} = \frac{1}{2}\mathrm{Trace}\,(M) = \frac{1}{2}\mathrm{Trace}\left(2-\frac{4L}{f}\right) = 0$$

$$\Delta\mu_{\mathrm{twice}} = 90 \text{ degrees} \qquad \Rightarrow \Delta\mu = 45 \text{ degrees}$$