Problem 1 solution

Derive an expression for the resulting magnetic field when the closed orbit in a normal sextupole is displaced by **δx** from its center position. What are the resulting field components? Do the same for an octupole. What is the leading order multi-pole field error when displacing a general **2(n+1)-pole** magnet?

- The vertical field of a sextupole is $B_y = B_2 x^2$
- Considering a displacement $x \mapsto x + \delta x$ the field is written as

$$B_y = B_2(x+\delta x)^2 = B_2(\underbrace{x^2}_{\text{sextupole}} + \underbrace{2(\delta x)x}_{\text{quadrupole}} + \underbrace{(\delta x)^2}_{\text{dipole}})$$

- For an octupole

$$B_y = B_3(x+\delta x)^3 = B_3(\underbrace{x^3}_{\text{octupole}} + \underbrace{3(\delta x)x^2}_{\text{sextupole}} + \underbrace{3(\delta x)^2 x}_{\text{quadrupole}} + \underbrace{(\delta x)^3}_{\text{dipole}})$$

- The vertical field for a 2(n+1)-pole is

$$B_y(y=0) = B_n \bar{x}^n$$

Problem 1 solution

- By displacing it $x \mapsto x + \delta x$, the vertical field is

$$B_y(y{=}0) = B_n \bar{x}^n = B_n(x + \delta x)^n = B_n(x^n + n\delta x x^{n-1} + \frac{n(n-1)}{2}\delta x^2 x^{n-2} + \cdots + (\delta x)^n)$$

- So the leading order feed-down is a **2n-pole**

Problem 2 solution

Three correctors are placed at locations with phase advance of $\pi/4$ between them and beta functions of **12, 2** and **12 m**. How are the corrector kicks related to each other in order to achieve a closed **3-corrector bump** (i.e. what is the relative strength between the three kicks)?

- The relations for achieving a 3-bump are

$$\frac{\sqrt{\beta_1}}{\sin\psi_{23}}\theta_1 = \frac{\sqrt{\beta_2}}{\sin\psi_{31}}\theta_2 = \frac{\sqrt{\beta_3}}{\sin\psi_{12}}\theta_3$$

- The phase advances are $\psi_{12} = \psi_{23} = \pi/4$ and $\psi_{13} = \psi_{12} + \psi_{23} = \pi/2$ which gives $\psi_{31} = -\pi/2$

- So $\theta_1 = \theta_3$ and $\theta_2 = -\theta_1\sqrt{12}$

Problem 3 solution

SNS: A **proton** ring with kinetic energy of 1 GeV and a **circumference of 248 m** has **18, 1 m-long** focusing quads with **gradient of 5 T/m**. In one of the quads, the horizontal and vertical **beta function** are **12 m** and **2 m** respectively. The **rms beta** function in both planes on the focusing quads is **8 m**. With a horizontal tune of **6.23** and a vertical of **6.2**, compute the expected horizontal and vertical orbit distortions on a single focusing quad given by **horizontal** and **vertical misalignments of 1 mm in all the quads**. What happens to the horizontal and vertical orbit distortions if the horizontal tune drops to **6.1** and **6.01**?

- The rms orbit distortion is given by

$$u_{\rm rms}(s) = \frac{\sqrt{N\beta(s)\beta_{\rm rms}}}{2\sqrt{2}|\sin(\pi Q)|}\theta_{\rm rms}$$

- We need to determine the rms kick angle, which in the case of a quadrupole displacement is given by

$$\theta_{\rm rms} = \frac{Gl}{B\rho}(\delta u)_{\rm rms}$$

- The magnetic rigidity is

$$B\rho \ \ [\mathrm{T\ m}] = \frac{1}{0.2998}\beta_r E \ \ [\mathrm{GeV}]$$

Problem 3 solution

- We need to compute the total energy which is $E = T + E_0 = 1.938\ \mathrm{GeV}$
- Now we need to compute the relativistic beta. First we compute the relativistic gamma

$$\gamma_r = \frac{E}{E_0} = 2.07$$ and the relativistic beta is $$\beta_r = \sqrt{1 - 1/\gamma_r^2} = 0.875$$

- The magnetic rigidity is then $B\rho = 5.657\ \mathrm{Tm}$ and the rms angle in both planes is

$$\theta_{\mathrm{rms}} = 8.8 \times 10^{-4}\ \mathrm{rad}$$

- Now we can calculate the rms orbit distortion on the single focusing quad

$$x_{\mathrm{rms}}(s) = \frac{\sqrt{N\beta_x(s)\beta_{x\mathrm{rms}}}}{2\sqrt{2}|\sin(\pi Q_x)|}\theta_{x\mathrm{rms}} = \frac{\sqrt{18 \times 12 \times 8}}{2\sqrt{2}|\sin(6.23\pi)|} 8.8 \times 10^{-4} = 19.6\mathrm{mm}$$

- The vertical is

$$y_{\mathrm{rms}}(s) = \frac{\sqrt{N\beta_y(s)\beta_{y\mathrm{rms}}}}{2\sqrt{2}|\sin(\pi Q_y)|}\theta_{y\mathrm{rms}} = \frac{\sqrt{18 \times 2 \times 8}}{2\sqrt{2}|\sin(6.20\pi)|} 8.8 \times 10^{-4} = 9\mathrm{mm}$$

- For $Q_x = 6.1$ the horizontal orbit distortion becomes $x_{\mathrm{rms}}(s) = 41.9\mathrm{mm}$
- For $Q_x = 6.01$ we have $x_{\mathrm{rms}}(s) = 0.41\ \mathrm{m}$
- The vertical remains unchanged…

Problem 4 solution

The SPS is a **400 GeV proton synchrotron** with a FODO lattice consisting of **108** focusing and **108** defocusing quadrupoles of length **3.22 m** and a gradient of **15 T/m**, with a **horizontal and vertical beta of 108 m and 30 m** in the focusing quads (**30 m and 108 m** for the defocusing ones). The tunes are **Q_x=20.13** and **Q_y=20.18**. Due to a mechanical problem, a focusing quadrupole was sinking down in 2016, resulting in an increasing closed orbit distortion compared to a reference taken earlier in the year.

- By how much the quadrupole had shifted down when the maximum vertical closed orbit distortion amplitude in defocusing quadrupoles reached 4 mm?
- Why was there no change of the horizontal orbit measured?
- How big would have been the maximum closed orbit distortion amplitude if it would have been a defocusing quadrupole?

- The magnetic rigidity is $B\rho \;\; [\mathrm{T\ m}] = \frac{1}{0.2998}\beta_r E \;\; [\mathrm{GeV}]$
- For 400GeV, the relativistic beta is almost 1 and then the magnetic rigidity is

$$B\rho = 1334 \;\; \mathrm{T\ m}$$

- The focusing normalized gradient is $K_F = \frac{G_F}{B\rho} = \frac{15}{1334} = 0.011\,\mathrm{m}^{-2}$
- The defocusing one is just the same with opposite sign $K_D = -0.011\,\mathrm{m}^{-2}$

Problem 4 solution

- The closed orbit distortion from a single dipole error is given by

$$u(s) = \theta \frac{\sqrt{\beta(s)\beta_0}}{2\sin(\pi Q)} \cos(\pi Q - |\psi(s) - \psi_0|)$$

- We are interested in the peak orbit distortion

$$\hat{y} = \theta \frac{\sqrt{\hat{\beta}_y \beta_0}}{2\sin(\pi Q)}$$

- From this we can calculate the required kick

$$\theta = \frac{\hat{y} 2\sin(\pi Q)}{\sqrt{\hat{\beta}_y \beta_0}} = \frac{0.004 \times 2\sin(\pi 20.18)}{\sqrt{108 \times 30}} = 75\ \mu\text{rad}$$

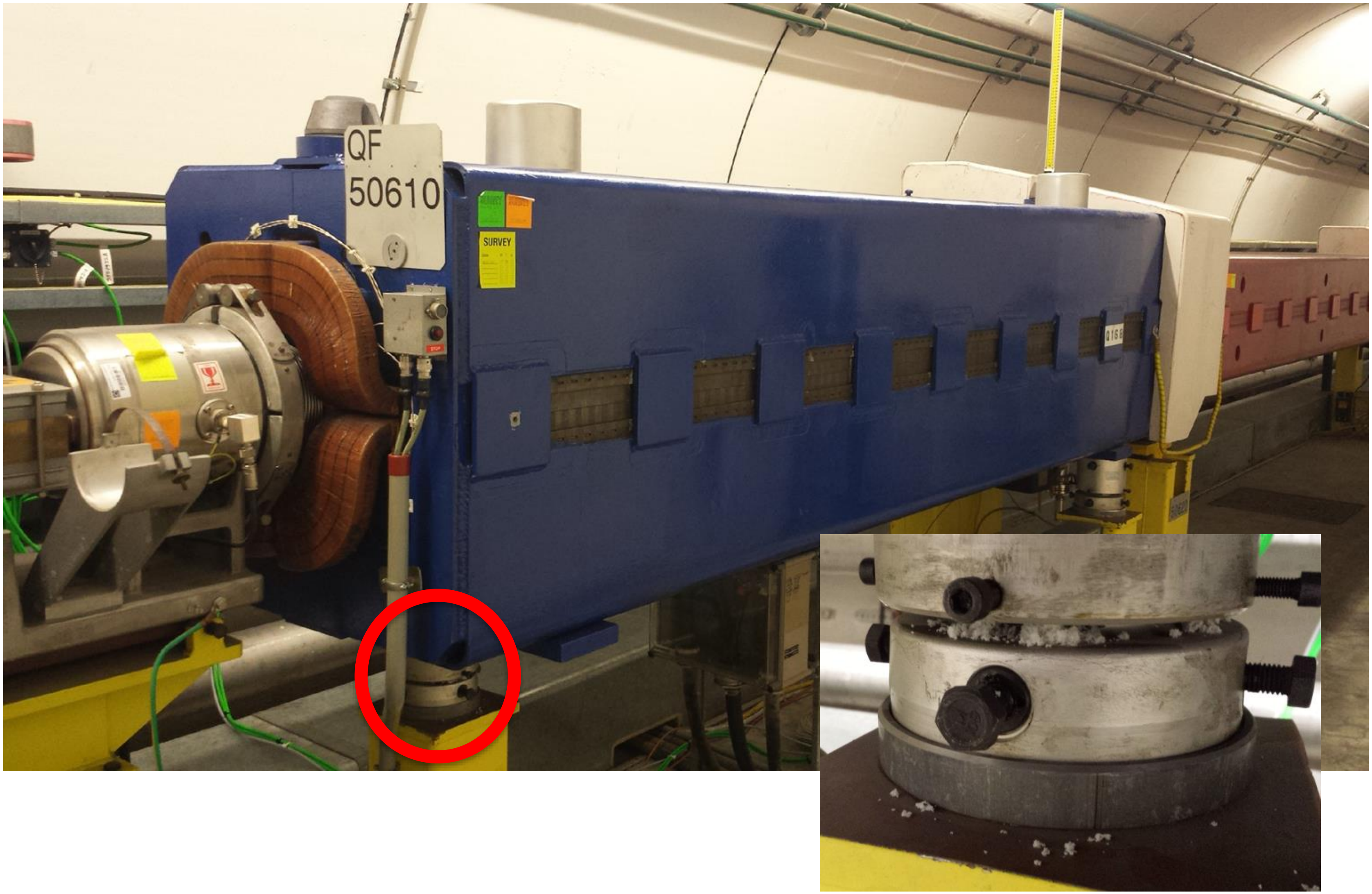
- And finally the required quadrupole displacement to produce this deflection

$$\theta = \frac{G l \delta y}{B\rho} = K_F l_F \delta y$$

$$\delta y = \frac{\theta}{K_F l_F} = \frac{75 \times 10^{-6}}{0.011 \times 3.22}\text{m} = 2\,\text{mm}$$

Problem 4 solution

- In fact what was found in the machine was that one of the supports of the quadrupole was broken and on that end it was shifted down by 4 mm, resulting in 2 mm average shift



Problem 4 solution

- No horizontal orbit change was observed, because the quadrupole shifted only in the vertical plane resulting in a pure vertical kick.
- If it would have been a defocusing quadrupole, the kick would have been the same but with opposite sign. However, the impact on the closed orbit would have been bigger since the vertical β-function is bigger in the defocusing quadupole, such that the peak orbit distortion would reach

$$\hat{y} = \theta \frac{\sqrt{\hat{\beta}_y \beta_0}}{2\sin(\pi Q)} = \theta \frac{\sqrt{\hat{\beta}_y \hat{\beta}_y}}{2\sin(\pi Q)} = 75 \times 10^{-6} * \frac{\sqrt{108 \times 108}}{2\sin(\pi 20.18)} \,\mathrm{m} = 7.5\,\mathrm{mm}$$

Problem 5 solution

The SPS is a **400 GeV proton synchrotron** with a FODO lattice consisting of **108** focusing and **108** defocusing quadrupoles of length **3.22 m** and a gradient of **15 T/m**, with a **horizontal and vertical beta of 108 m and 30 m** in the focusing quads (**30 m and 108 m** for the defocusing ones). The tunes are **Q_x=20.13** and **Q_y=20.18**.

- Find the tune change for systematic gradient errors of 1% in the focusing and 0.5% in the defocusing quads.
- What is the natural chromaticity of the machine (without gradient errors)?

- The magnetic rigidity is $B\rho\ \ [\mathrm{T\ m}] = \frac{1}{0.2998}\beta_r E\ \ [\mathrm{GeV}]$
- For 400GeV, the relativistic beta is almost 1 and then the magnetic rigidity is

$$B\rho = 1334\ \ \mathrm{T\ m}$$

- The focusing normalized gradient is $K_F = \frac{G_F}{B\rho} = \frac{15}{1334} = 0.011\,\mathrm{m}^{-2}$
- The defocusing one is just the same with opposite sign $K_D = -0.011\,\mathrm{m}^{-2}$
- Now, the tune change is given by

$$\delta Q_u = \frac{1}{4\pi}\sum_i \beta_u K_i \left(\frac{\delta K}{K}\right)_i l_i$$

Problem 5 solution

- By splitting the focusing and defocusing quads, we have

$$\delta Q_u = \frac{1}{4\pi}\left(N_F \beta_u^F K_F \left(\frac{\delta K}{K}\right)_F l_F + N_D \beta_u^D K_D \left(\frac{\delta K}{K}\right)_D l_D\right)$$

- As $N_F = N_D = N$, $l_F = l_D = l$ and $K_F = -K_D = K$ the tune shift can be rewritten as

$$\delta Q_{x,y} = \frac{1}{4\pi} N l K \left(\pm\beta_{x,y}^F \left(\frac{\delta K}{K}\right)_F \mp \beta_{x,y}^D \left(\frac{\delta K}{K}\right)_D\right)$$

- This gives a horizontal and vertical tune shift of

$$\delta Q_x = \frac{108 \times 3.22 \times 0.011}{4\pi}(+108 \times 0.01 - 30 \times 0.005) = 0.3$$

$$\delta Q_y = \frac{108 \times 3.22 \times 0.011}{4\pi}(-30 \times 0.01 + 108 \times 0.005) = 0.07$$

- The chromaticity of the machine is

$$\xi_{x,y} = -\frac{1}{4\pi}\sum_i \beta_{x,y}^i K_{x,y}^i l^i$$

Problem 5 solution

- By splitting again the focusing and defocusing quads' contribution, we have

$$\xi_{x,y} = -\frac{1}{4\pi} NlK(\pm\beta^F_{x,y} \mp \beta^D_{x,y})$$

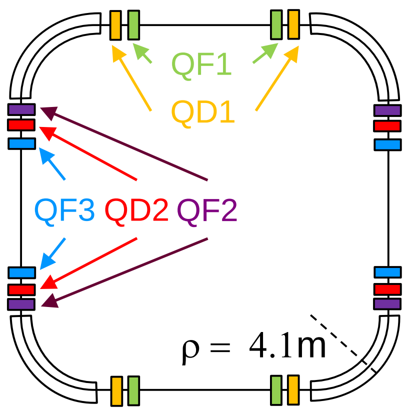
- This gives in both planes

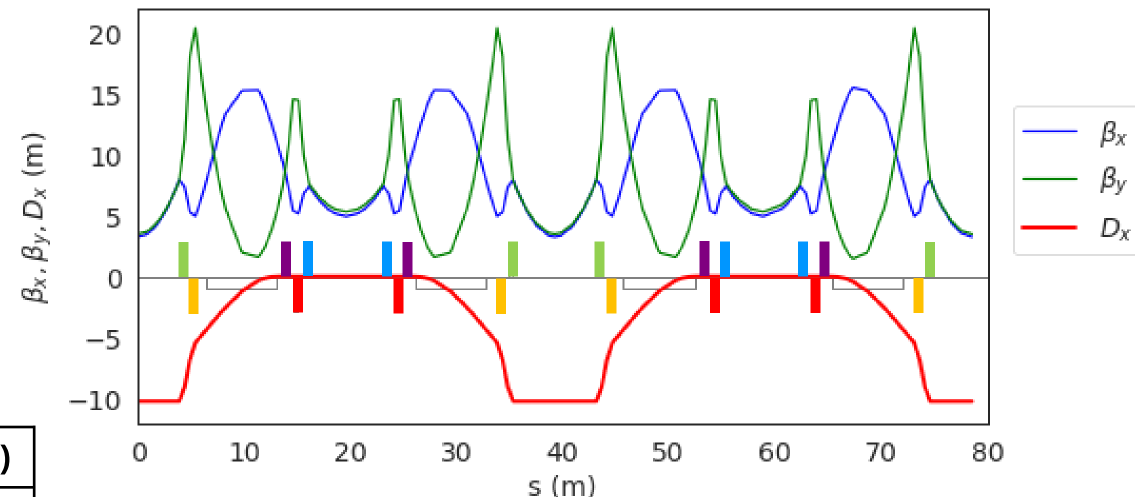
$$\xi_{x,y} = -\frac{108 \times 3.22 \times 0.011}{4\pi}(108 - 30) = -24$$

Problem 6

LEIR: Consider a heavy-ion **synchrotron** with **5 families of quadrupoles** (no FODO structure) and the optical functions from the plot and table below.

- What is the natural chromaticity of the machine? (quad length l_q=0.5m for all families, dipole length l_d=6.44m)
- What is the optimum placement for the sextupole magnets to correct the chromaticity? Give their estimated k$_2$ value to get $\xi_{x,y}^{tot} = 0$? (assume l_s = l_q)





	k$_1$ (m^{-2})	β_x (m)	β_y (m)	D_x (m)
QF1	0.9041	7.9	7.9	-10.9
QD1	-1.1303	5.3	18.0	-7.3
QF2	0.3088	7.2	10.8	0
QD2	-1.3181	5.4	14.5	0
QF3	0.7167	7.4	7.6	0

Problem 6 solution

What is the natural chromaticity of the machine?

- The natural chromaticity created by the quadrupoles is:

$$\xi_{x,y} = \mp \frac{1}{4\pi} \sum_i k_1^i l_q^i \beta_{x,y}^i$$

- There are N=4 quads per family, with l =0.5 m:

$$\xi_x = -\frac{1}{4\pi} N l_q (k_1^{QF1}\beta_x^{QF1} + k_1^{QD1}\beta_x^{QD1} + k_1^{QF2}\beta_x^{QF2} + k_1^{QD2}\beta_x^{QD2} + k_1^{QF3}\beta_x^{QF3}) = -0.25$$

$$\xi_y = +\frac{1}{4\pi} N l_q (k_1^{QF1}\beta_y^{QF1} + k_1^{QD1}\beta_y^{QD1} + k_1^{QF2}\beta_y^{QF2} + k_1^{QD2}\beta_y^{QD2} + k_1^{QF3}\beta_y^{QF3}) = -3.74$$

However, when we calculate the natural chromaticity from the LEIR lattice using MADX we get instead:

$$\xi_x = -2.19$$
$$\xi_y = -3.74$$

What could we be missing in the **horizontal case**?

Problem 6 solution

Remember the Equations of motion:

$$\begin{aligned} x'' + K_x(s)\, x &= 0 \\ y'' + K_y(s)\, y &= 0 \end{aligned} \quad \text{with} \quad K_x(s) = \left(k(s) + \frac{1}{\rho(s)^2} \right) \qquad K_y(s) = -k(s)$$

In a small ring like LEIR the contribution from the dipoles to the focusing (weak focusing) cannot be neglected, i.e. the **natural chromaticity is generated by dipoles and quadrupoles**

General expression of natural chromaticity: $$\xi_{x,y} = \mp \frac{1}{4\pi} \sum_i K^i_{x,y} l^i \beta^i_{x,y}$$

Thus, for the horizontal plane:

$$\xi_x = -\frac{1}{4\pi} \sum_i (k_1^i l_q^i + \frac{l_d^i}{\rho_i^2}) \beta_x^i$$

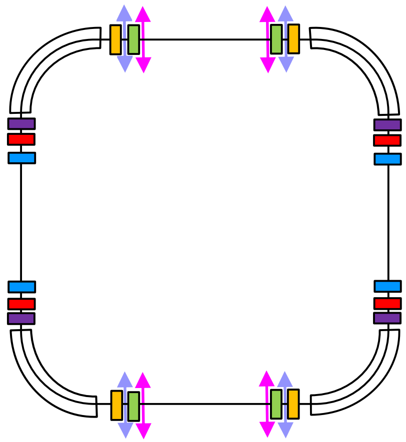
From the plot: we have 4 dipoles, with ρ = 4.1m, l_d ~6.44m, and $<\beta_x>$ ~10m along the dipoles

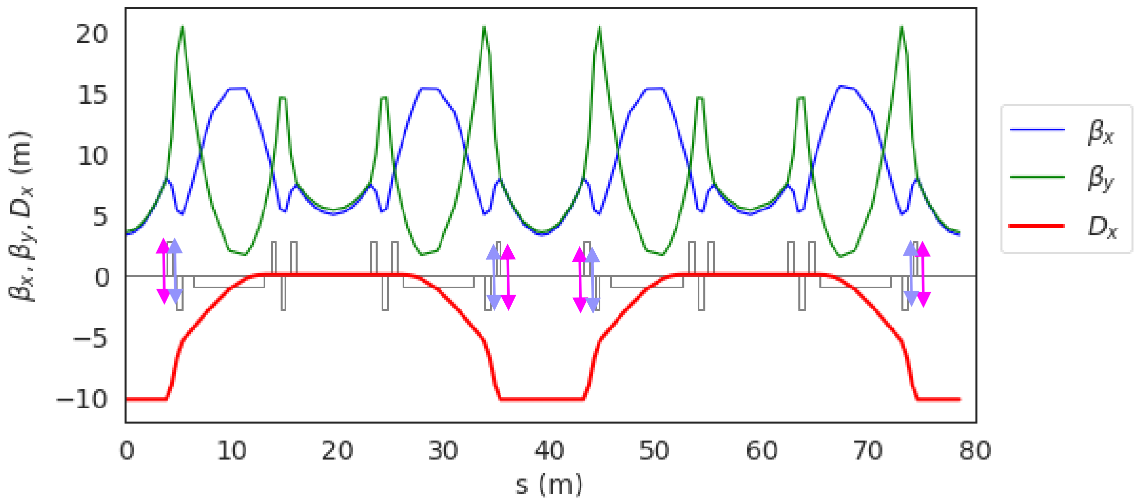
$$\xi_x = -\frac{1}{4\pi} \left[N_q l_q (k_1^{QF1} \beta_x^{QF1} + k_1^{QD1} \beta_x^{QD1} + k_1^{QF2} \beta_x^{QF2} + k_1^{QD2} \beta_x^{QD2} + k_1^{QF3} \beta_x^{QF3}) + N_d \frac{l_d}{\rho^2} \beta_x \right] = -1.47$$

Result closer to the one calculated using MADX but still smaller. The dipoles in the machine and also in the lattice model are not simply a one-piece dipole, but consist of many pieces with different angles and also edge focusing on certain pieces…

What is the optimum placement for the sextupole magnets to correct the chromaticity?

- Optimum placement for the sextupole magnets at positions with large D_x (however center of straight sections typically occupied by other systems, e.g. injection/extraction septa, RF cavities, etc)
- We put 2 families, to correct horizontal and vertical chromaticities





Problem 6 solution

Give their estimated k_2 value to get $\xi^{tot}_{x,y} = 0$ (assume $l_s = l_q$)

- To cancel the total chromaticity the sexupole-induced chromaticity has to be equal to the natural chromaticity but with opposite sign

$$\xi^S_{x,y} = \pm\frac{1}{4\pi}\sum_i k_2^i l_s^i D_x^i \beta^i_{x,y} = -\xi^{nat}_{x,y}$$

- Extract k_2^F and k_2^D from the equation system:

$$+\frac{1}{4\pi}\cdot 4\cdot 0.5(k_2^F D_x^F \beta_x^F + k_2^D D_x^D \beta_x^D) = 1.47$$
$$-\frac{1}{4\pi}\cdot 4\cdot 0.5(k_2^F D_x^F \beta_y^F + k_2^D D_x^D \beta_y^D) = 3.74$$

- To simplify let's assume the same beta and dispersion values for the sextupoles as for the quadrupoles

$$\left.\begin{array}{l}\frac{1}{4\pi}4\cdot 0.5\cdot(k_2^F\cdot(-10.9)\cdot 7.9 + k_2^D\cdot(-7.3)\cdot 5.3) = 1.47 \\ -\frac{1}{4\pi}4\cdot 0.5\cdot(k_2^F\cdot(-10.9)\cdot 7.9 + k_2^D\cdot(-7.3)\cdot 18) = 3.74\end{array}\right\} \quad \begin{array}{l} k_2^F = -0.266 \\ k_2^D = 0.353\end{array}$$

Usually this kind of problems is solved numerically in an optimization we call "matching"