SPACE CHARGE EFFECTS AND INSTABILITIES

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1. SELF FIELDS

1.1 Introduction

Charged particles in a transport channel or in a circular accelerator are confined, guided and accelerated by external electromagnetic (e.m.) fields. Acceleration is usually provided by the electric field of the RF cavity, while magnetic fields in the dipole magnets are used for guiding the beam on the reference trajectory (orbit), in the quadrupoles for the transverse confinement, in the sextupoles for the chromaticity correction.

The motion of a single charge is governed by the Lorentz force through the equation:

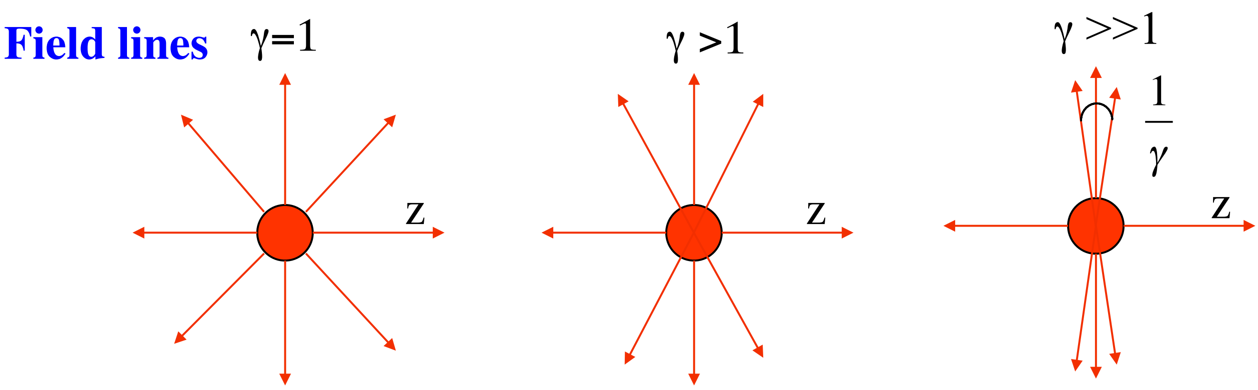
$$\frac{d\left(m_0\gamma\vec{v}\right)}{dt}=\vec{F}^{ext}=e\left(\vec{E}+\vec{v}\times\vec{B}\right)$$

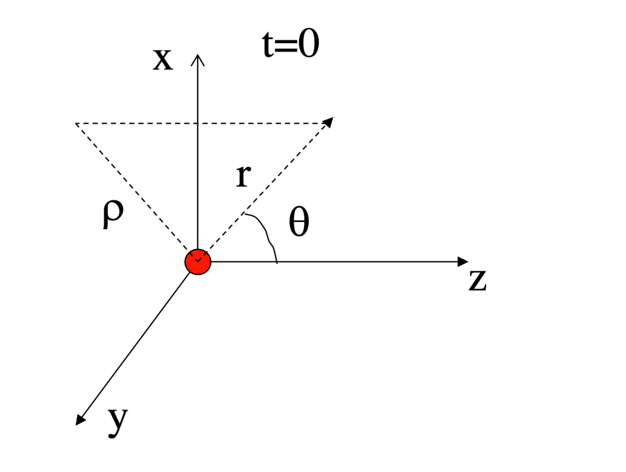
where m_0 is the rest mass, γ is the relativistic factor and $\vec{v}$ is the particle velocity. The external fields used for the beam transport do not depend on the beam current. With the above equation we can in principle calculate the trajectory of the charge moving through any e.m. field.

In a real accelerator, however, there is another important source of fields to be considered: the beam itself, which circulating inside the pipe produces additional e.m. fields called "self-fields". These fields depend on the beam pipe geometry, the surrounding materials, the beam charge and its velocity. They are responsible of many phenomena occurring in the beam dynamics: energy loss, betatron tunes shift, synchronous phase and tune shift, instabilities. It is customary to divide the self-fields in ***space charge fields*** and ***wake-fields***. The space charge forces are those generated by the charge distribution, including the image currents circulating on the walls of the smooth perfectly conducting pipe. The wake fields are produced by the finite conductivity of the walls, resonant devices, transitions of the beam pipe.

1.2 Direct space charge forces

Let us consider a relativistic charge moving with constant velocity **v**. It is well known that the electrostatic field is modified because of the relativistic Lorentz contraction along the direction of motion. For an ultra-relativistic charge $\gamma\to\infty$ the field lines are confined on a plane perpendicular to the direction of motion. In fact, from the Lorentz equations we can derive the electric field as function of the coordinates r (distance from the charge) and θ (angle with respect the motion axis):

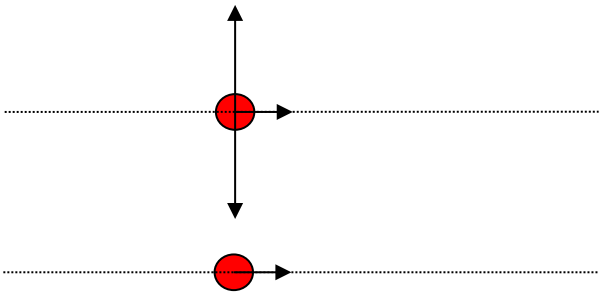




$$\boxed{\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{\gamma^2\left(1-\beta^2\sin^2\theta\right)^{3/2}}\frac{\vec{r}}{r^3}}$$

$$\boxed{\begin{array}{l} z = r\cos\theta \\ \rho^2 = x^2+y^2 = r^2\sin^2\theta \end{array}}$$

It is easy to see that for θ=π/2, the radial electric field is proportional to the charge energy, while for θ=0, the electric field vanishes as $(\text{energy})^{-2}$.



$$E_r\left(r,\theta=\frac{\pi}{2}\right) = \frac{q_o}{4\pi\,\varepsilon_o}\,\frac{\gamma}{r^2}$$

$$B_\theta\left(r,\theta=\frac{\pi}{2}\right) = \frac{q_o\beta}{4\pi\varepsilon_o c}\,\frac{\gamma}{r^2}$$

$$E_z(z) \propto \frac{1}{\gamma^2}$$

It is interesting to calculate the e.m. forces between two ultra-relativistic charges travelling on parallel trajectories. The longitudinal force is proportional to the electric field E_z, and vanishes as $1/\gamma^2$, while the transverse force vanishes because electric and magnetic fields effects have opposite sign:

$$F_r = q\left(E_r - \beta c B_\theta\right) = \frac{\gamma q_o}{4\pi\,\varepsilon_o r^2}\left(1-\beta^2\right) = \frac{q_o}{4\pi\,\varepsilon_o\gamma r^2}$$

For γ→∞, two charges travelling close by don't see each other.

In the case of a charge distribution, when γ→∞ the electric field lines are perpendicular to the direction of motion. The transverse fields intensity can be computed like in the static case, applying the Gauss's and Ampere's laws:

$$\int_S \boldsymbol{E}\cdot \boldsymbol{n}dS = \frac{q}{\varepsilon_o}, \qquad \oint \boldsymbol{B}\cdot d\boldsymbol{l} = \mu_o I$$

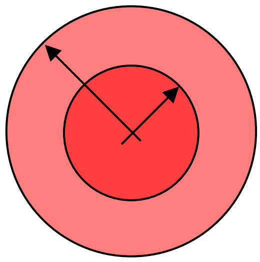
Example 1: Compute the transverse space charge forces on an electron inside a relativistic cylinder of radius a with a uniform charge density.

$$\lambda(r) = \lambda_o\left(\frac{r}{a}\right)^2; \int_S E_r(2\pi r)\Delta z = \frac{\lambda(r)\Delta z}{\varepsilon_o}$$

$$E_r = \frac{\lambda(r)}{2\pi\varepsilon_o r}; \;\; \mathrm{B}_\theta = \frac{\beta}{c}E_r$$

$$E_r(\boldsymbol{r}) = \frac{\lambda_o}{2\pi\,\varepsilon_o}\frac{r}{a^2}; \quad B_\theta(\boldsymbol{r}) = \frac{\lambda_o\beta}{2\pi\varepsilon_o c}\frac{r}{a^2}$$

$$F_\perp(r) = e(E_r - \beta c\, B_\theta) = e(1-\beta^2)E_r$$



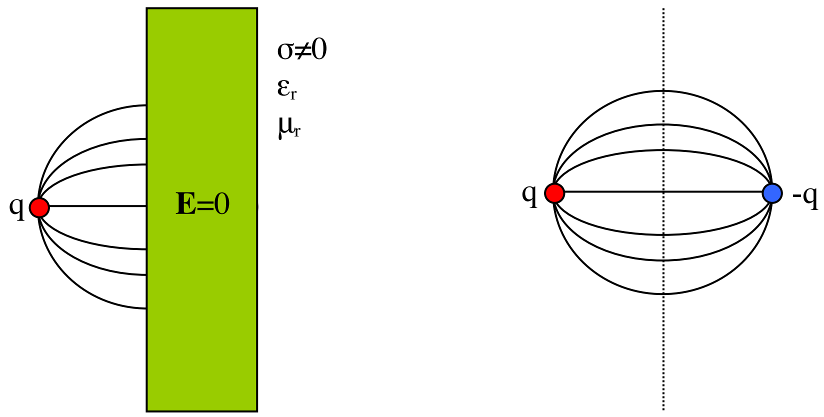
Inside a rigid cylinder with a uniform charge density, travelling with ultra-relativistic speed, the transverse space charge forces vanish as $1/\gamma^2$.

1.3 Effects of conducting or magnetic screens

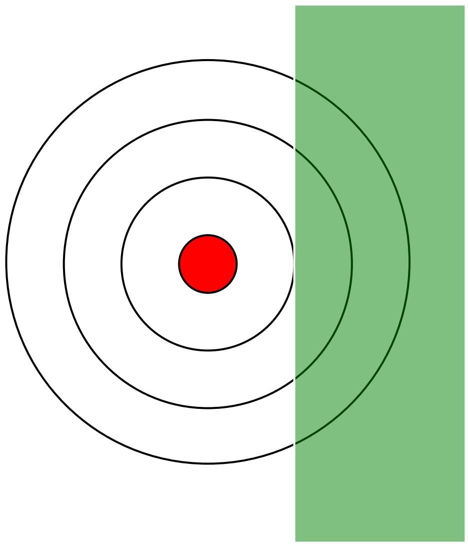
In an accelerator, the beams travel inside a vacuum pipe generally made of metallic material (aluminium, copper, stainless steel, etc.). The pipe goes through the coils of the magnets (dipoles, quadrupoles, sextupoles). Its cross section may have a complicated shape, as in the case of special devices like RF cavity, kickers, diagnostics and controls; however, most part of the beam pipe has a cross section with a simple shape: circular, elliptic or quasi-rectangular. For the moment we consider only the effect of a smooth pipe on the e.m. fields of the beam.

Before going into this problem, it is necessary to remind the basic features of fields close to metallic or magnetic materials. A detailed discussion of the boundary conditions is given in the Appendix 1.

Let us consider a point charge close to a conducting screen. The electrostatic field can be derived through the "image method". Since the metallic screen is an equi-potential plane, it can be removed provided that a "virtual" charge is introduced such that the potential is constant at the position of the screen.

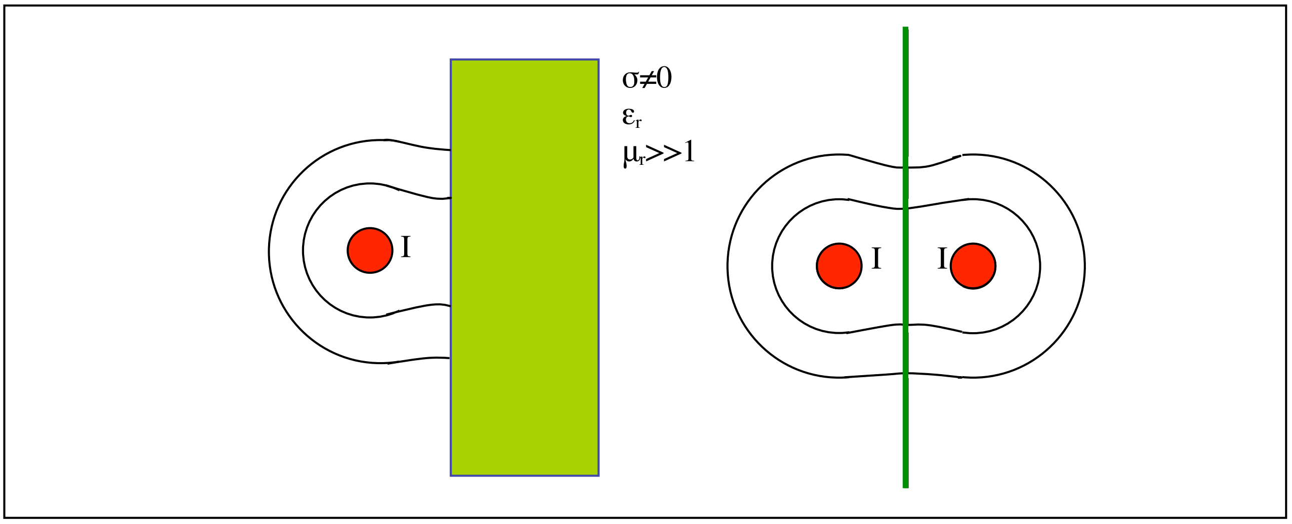


A constant current in the free space produces circular magnetic field. If $\mu_r \approx 1$, the material, even in the case of a good conductor, does not affect the field lines.

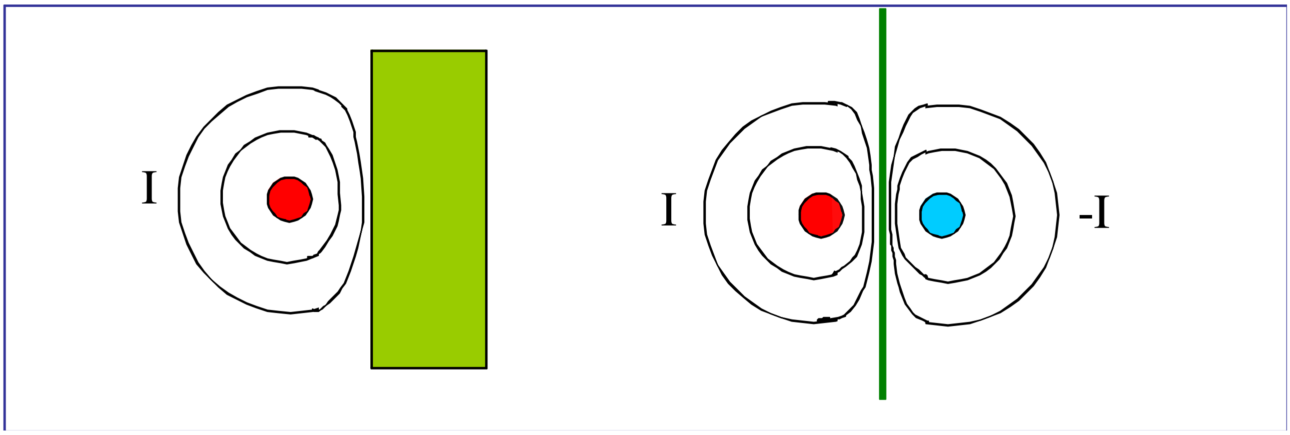


However, if the material is of ferromagnetic type, with $\mu_r >> 1$, due to its magnetisation, the magnetic field lines are strongly affected, inside and outside the material. In particular a very high magnetic permeability makes the tangential magnetic field zero at the boundary so that the magnetic field is perpendicular to the surface, just like the electric field lines close to a conductor.

In analogy with the image method for charges close to conducting screens, we get the magnetic field, in the region outside the material, as superposition of the fields due to two symmetric equal currents flowing in the same direction. It easy to see that these currents make zero the tangential component of the magnetic field.



As discussed in Appendix 1, the scenario changes when we deal with time-varying fields for which it is necessary to compare the wall thickness and the skin depth (region of penetration of the e.m. fields) in the conductor. If the fields penetrate and pass through the material, we are practically in the static boundary conditions case. Conversely, if the skin depth is very small, fields do not penetrate, the electric field lines are perpendicular to the wall, as in the static case, while the magnetic field line are tangent to the surface. In this case, the magnetic field lines can be obtained by considering two linear charge distributions with opposite sign, flowing in the same direction (opposite charges, opposite currents).

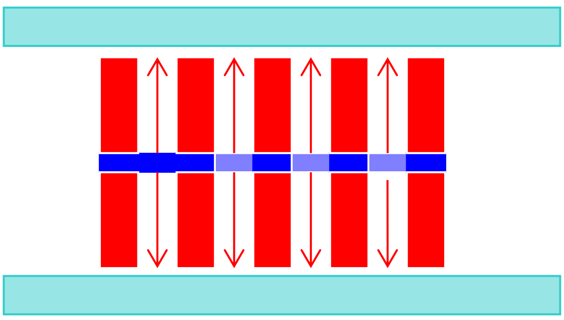


We are ready now to analyse some simple cases.

CIRCULAR PERFECTLY CONDUCTING PIPE (Beam at center)

Due to the symmetry, the transverse fields produced by an ultra-relativistic charge inside the pipe are the same as in the free space. This implies that for a distribution with cylindrical symmetry, in ultra-relativistic regime, there is a cancellation of the electric and magnetic forces. Therefore, the uniform beam of Example 1, produces exactly the same forces as in the free space. It is interesting to note that this result does not depend on the longitudinal distribution of the beam.

Example 2: Compute the fields and the space charge forces produced by a relativistic charged cylinder of radius a, moving inside a circular pipe of radius b. Consider a uniform radial distribution and a longitudinal linear density $\lambda(z)$.



Because of the symmetry, the fields are the same as in the free space. The transverse fields intensity reproduce longitudinally the charge density.

The transverse force on a test charge inside the cylinder is:

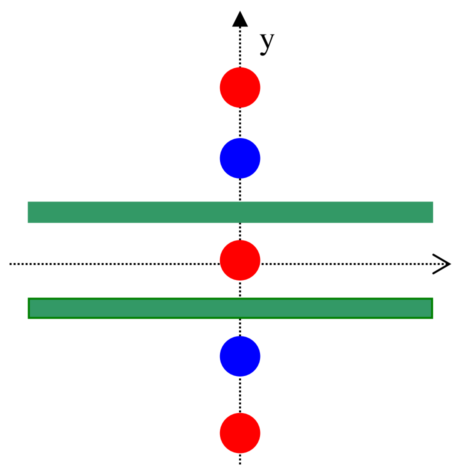
$$F_{\perp}(\boldsymbol{r},z) = \frac{e\lambda(z)}{2\pi\,\varepsilon_o\gamma^2}\frac{r}{a^2}$$

Exercise 1: Compute the transverse space charge forces for the following longitudinal distributions: parabolic, sinusoidal, gaussian.

Exercise 2: Compute the transverse space charge forces for a bunch with the following distribution:

$$\rho(r,z) = \frac{q_o}{\left(\sqrt{2\pi}\right)^3 \sigma_z \sigma_r^2} e^{\frac{-z^2}{2\sigma_z^2}} e^{\frac{-r^2}{2\sigma_r^2}}$$

PARALLEL PLATES (Beam at center)



In some cases, the beam pipe cross section is such that we can consider only the surfaces closer to the beam, which behave like two parallel plates. In this case, we use the image method to a charge distribution of radius *a* between two conducting plates *2h* apart. By applying the superposition principle we get the total image field at a position *y* inside the beam.

$$E_y^{im}(z,y) = \frac{\lambda(z)}{2\pi\,\varepsilon_o} \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{2nh+y} - \frac{1}{2nh-y}\right]$$

$$E_y^{im}(z,y) = \frac{\lambda(z)}{2\pi\,\varepsilon_o} \sum_{n=1}^{\infty} (-1)^n \frac{-2y}{(2nh)^2 - y^2} \cong \frac{\lambda(z)}{4\pi\,\varepsilon_o h^2} \frac{\pi^2}{12} y$$

Where we have assumed *h>>a>y*. For d.c. or slowly varying currents, the boundary condition imposed by the conducting plates does not affect the magnetic field. As a consequence there is no cancellation effect for the fields produced by the "image" charges.

From the divergence equation we derive also the other transverse component:

$$\frac{\partial}{\partial x} E_x^{im} = -\frac{\partial}{\partial y} E_y^{im} \Rightarrow E_x^{im}(z,x) = \frac{-\lambda(z)}{4\pi\,\varepsilon_o h^2} \frac{\pi^2}{12} x$$

Including also the direct space charge force, we get:

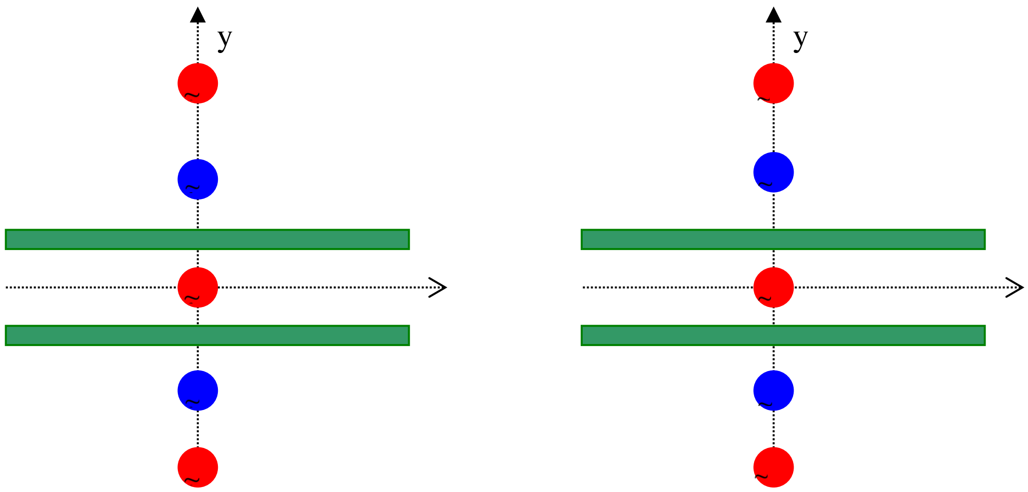
$$F_x(z,x) = \frac{e\lambda(z)x}{\pi\,\varepsilon_o}\left(\frac{1}{2a^2\gamma^2} - \frac{\pi^2}{48h^2}\right)$$

$$F_y(z,y) = \frac{e\lambda(z)y}{\pi\,\varepsilon_o}\left(\frac{1}{2a^2\gamma^2} + \frac{\pi^2}{48h^2}\right)$$

Therefore, for γ>>1, and for d.c. or slowly varying currents the cancellation effect applies only for the direct space charge forces. There is no cancellation of the electric and magnetic forces due to the "image" charges.

PARALLEL PLATES (Beam at center, a.c. currents)

We have seen that close to a conductor the e.m. fields have a different behaviour, depending on the skin depth of the material (Appendix 1). Usually, the frequency beam spectrum is quite rich of harmonics, especially for bunched beams. It is convenient to decompose the current into a d.c. component, $I = \lambda v$, for which $\delta_w >> \Delta_w$, and an a.c. component, $\tilde{I} = \tilde{\lambda} v$, for which $\delta_w << \Delta_w$. While the d.c. component of the magnetic field does not perceives the presence of the material, its a.c. component is obliged to be tangent at the wall. We can see that this current produces a magnetic field able to cancel the effect of the electrostatic force.



$$\tilde{E}_y(z,y) = \frac{\tilde{\lambda}(z) y}{\pi \varepsilon_o} \frac{\pi^2}{48 h^2}; \quad \tilde{B}_x(z,y) = -\frac{\beta}{c} \tilde{E}_y(z,y)$$

$$\tilde{F}_y(z,y) = \frac{e \tilde{\lambda}(z) y}{\pi \varepsilon_o \gamma^2} \frac{\pi^2}{48 h^2}$$

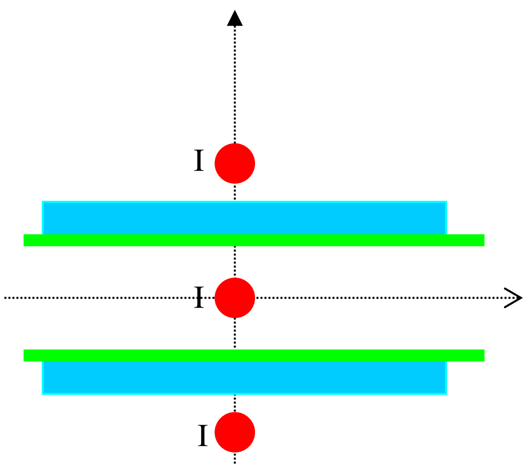
Combining with the direct space charge force, we get for the a.c. component:

$$\tilde{F}_y(z,x) = \frac{e \tilde{\lambda}(z) y}{2\pi \varepsilon_o \gamma^2} \left(\frac{1}{a^2} + \frac{\pi^2}{24 h^2} \right)$$

and analogously:

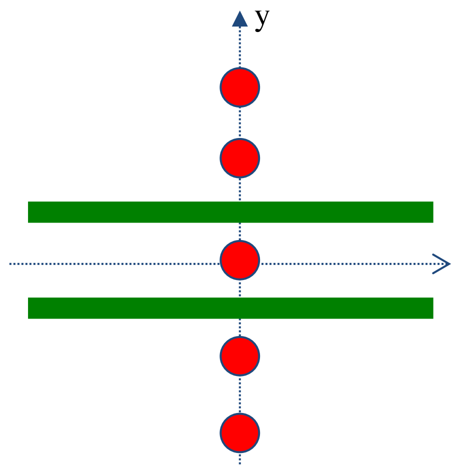
$$\tilde{F}_x(z,x) = \frac{e \tilde{\lambda}(z) x}{2\pi \varepsilon_o \gamma^2} \left(\frac{1}{a^2} - \frac{\pi^2}{24 h^2} \right)$$

PARALLEL PLATES (Beam at center, d.c. currents, ferromagnetic case)



Let us consider now the case where outside the metallic pipe, there is a dipole magnet with a gap g. The magnetic field produced by the d.c. current doesn't see the conducting pipe, while it is strongly affected by ferromagnetic material. In fact, as seen in the section about the boundary conditions, the magnetic field lines must be orthogonal to the pole surface. We have also seen that the total magnetic field can be obtained by considering image currents with the same sign.

Proceeding analogously to the section of parallel plates (beam at center), we get:



$$B_x^{im}(z,y) = \frac{\mu_o \beta c \lambda(z)}{2\pi} \sum_{n=1}^{\infty} \left[\frac{1}{2ng - y} - \frac{1}{2ng + y} \right]$$

$$B_x^{im}(z,y) \cong \frac{\mu_o \beta c \lambda(z) y}{4\pi g^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\mu_o \beta c \lambda(z) \pi^2 y}{24\pi g^2}$$

From the relation $\nabla\times\mathbf{B}=0$, we get:

$$B_y^{im}(z,x) \cong \frac{\mu_o \beta c \lambda(z) \pi^2}{24\pi g^2} x$$

and

$$F_x^{im}(z,x) \cong -\frac{e\beta^2 \lambda(z) \pi^2}{24\pi \varepsilon_o g^2} x$$

<u>PARALLEL PLATES (General expression of the force)</u>

Taking into account all the boundary conditions for d.c. and a.c. currents, we can write the expression of the force as:

$$F_u = \frac{e}{2\pi\,\varepsilon_o} \left[\frac{1}{\gamma^2} \left(\frac{1}{a^2} \mp \frac{\pi^2}{24 h^2} \right) \lambda \mp \beta^2 \left(\frac{\pi^2}{24 h^2} + \frac{\pi^2}{12 g^2} \right) \bar{\lambda} \right] u$$

where λ is the total current divided by βc, and $\bar{\lambda}$ its d.c. part. We take the sign (+) if u=y, and the sign (–) if u=x.

One finds often the above expression in terms of the "Laslett " form factors:

$$F_u = \frac{e}{\pi\,\varepsilon_o} \left[\frac{1}{\gamma^2} \left(\frac{\varepsilon_0}{a^2} \mp \frac{\varepsilon_1}{h^2} \right) \lambda \mp \beta^2 \left(\frac{\varepsilon_1}{h^2} + \frac{\varepsilon_2}{g^2} \right) \bar{\lambda} \right] u$$

where the "Laslett " form factors ε_0, $\varepsilon 1$ and ε_2 can be obtained for several beam pipe geometries. For example, for parallel plates we get $\varepsilon_0=1/2$, $\varepsilon 1=\pi^2/48$, $\varepsilon_2=\pi^2/24$. It is interesting to note that these forces are anyway linear in the transverse displacement x and y.

1.4 Longitudinal forces

What about the longitudinal forces?

Using the results of Appendix 2, putting $E_s=0$ at wall, we get:

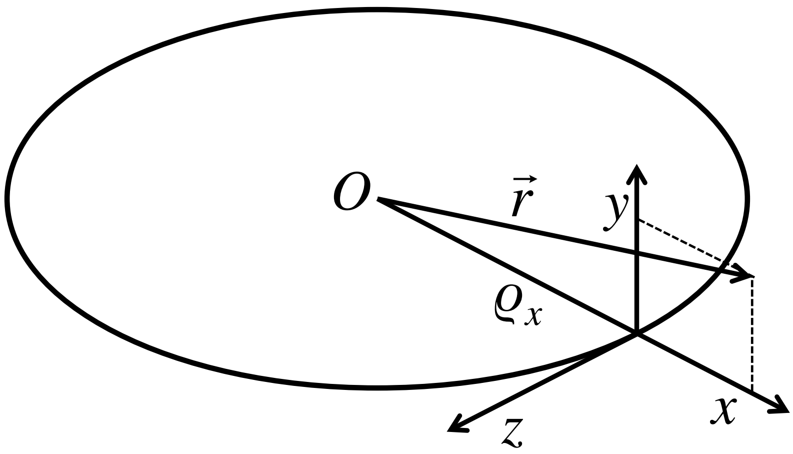
$$F_s(r,z) = \frac{-e}{4\pi\varepsilon_o \gamma^2} \left(1 - \frac{r^2}{a^2} + 2\ln\frac{b}{a} \right) \frac{\partial\,\lambda(z)}{\partial\, z}$$

Thus, the longitudinal force acting on the test charge is positive (negative) in the region with negative (positive) density slope.

Exercise 3: Compute the longitudinal direct space charge forces for the following longitudinal distributions: parabolic, sinusoidal, Gaussian.

2. BETATRON TUNE SHIFT

We consider a perfectly circular accelerator with radius ρ_x. The beam circulates inside the beam pipe. The transverse single particle motion in the linear regime, is derived from the equation of motion.



Including the self field forces in the equation of motion, we have

$$\frac{d(m\gamma\boldsymbol{v})}{dt} = \boldsymbol{F}^{ext}(\vec{r}) + \boldsymbol{F}^{self}(\vec{r})$$

which at constant energy becomes:

$$\frac{d\boldsymbol{v}}{dt} = \frac{\boldsymbol{F}^{ext}(\vec{r}) + \boldsymbol{F}^{self}(\vec{r})}{m\gamma}$$

According to the coordinate system of the figure, indicating with $\vec{r}$ the charge position and with x and y the transverse displacements with respect to the reference trajectory, we write:

$$\vec{r} = (\rho_x + x)\hat{e}_x + y\hat{e}_y$$
$$\vec{v} = \dot{x}\hat{e}_x + \dot{y}\hat{e}_y + \omega_o(\rho_x + x)\hat{e}_z$$
$$\vec{a} = \left[\ddot{x} - \omega_o^2(\rho_x + x)\right]\hat{e}_x + \ddot{y}\hat{e}_y + \left[\dot{\omega}_o(\rho_x + x) + 2\omega_o\dot{x}\right]\hat{e}_z$$

Where the dots mean derivative with respect to time.

For the motion along x:

$$\ddot{x} - \omega_o^2(\rho_x + x) = \frac{1}{m\gamma}\left(F_x^{ext} + F_x^{self}\right)$$

which is rewritten with respect to the azimuthal position s=v_zt:

$$\ddot{x} = v_z^2 x'' = \omega_o^2(\rho_x + x)^2 x''$$
$$x'' - \frac{1}{\rho_x + x} = \frac{1}{m v_z^2\gamma}\left(F_x^{ext} + F_x^{self}\right)$$

We assume now small transverse displacements x with respect to the machine radius ρ_x, so that we can linearly expand the above equation into

$$x'' - \frac{1}{\rho_x} + \frac{1}{\rho_x^2} x = \frac{1}{m_0 v_z^2 \gamma}\left(F_x^{ext} + F_x^{self}\right)$$

In addition, we have that the external force is due to the magnetic guiding fields. We suppose to have only dipoles and quadrupoles, or, equivalently, we expand the external guiding fields in a Taylor series up to the quadrupole component

$$-F_x^{ext} = qv_z B_y = qv_z B_{y0} + qv_z \left(\frac{\partial B_y}{\partial x}\right)_0 x + \ldots$$

and recognize that the dipolar magnetic field B_{y0} is responsible of the circular motion along the reference trajectory of radius ρ_x according to the equation

$$qv_z B_{y0} = \frac{m_0 \gamma v_z^2}{\rho_x}$$

we finally obtain

$$x'' + \left[\frac{1}{\rho_x^2} + \frac{q}{m_0 v_z \gamma}\left(\frac{\partial B_y}{\partial x}\right)\right] x = \frac{1}{m_0 v_z^2 \gamma} F_x^{self}$$

which can also be written as

$$x'' + \left[\frac{1}{\rho_x^2} - k\right] x = \frac{1}{m_0 v_z^2 \gamma} F_x^{self}$$

where we have introduced the normalized gradient

$$k = \frac{g}{p/q} = -\frac{q}{m_0 v_z \gamma}\left(\frac{\partial B_y}{\partial x}\right)$$

with g the quadrupole gradient in [T/m] and p the charge momentum.

It is important to note that both the curvature radius and the normalized gradient depend on the azimuthal position ‘s’. By using the focusing constant $K_x(s)$ we then should write

$$x''(s) + K_x(s) x(s) = \frac{1}{m_0 v_z^2 \gamma} F_x^{self}(x,s)$$

In absence of self fields, the solution of the free equation (Hill’s equation) gives the well known betatron oscillation:

$$x(s) = a_x \sqrt{\beta_x(s)} \cos\left[\mu_x(s) - \varphi_x\right]$$

where a_x and φ_x depend on the initial conditions, and

$$\begin{aligned}
&\frac{1}{2}\beta_x \beta_x'' - \frac{1}{4}\beta_x'^2 + K_x(s)\beta_x^2 = 1 \\
&\mu_x'(s) = 1/\beta_x(s) \\
&Q_x = \frac{\omega_x}{\omega_o} = \frac{1}{2\pi}\int_0^L \frac{ds'}{\beta_x(s')}
\end{aligned}$$

In the analysis of the motion in presence of the self induced fields, however, we adopt a simplified model where particles execute simple harmonic oscillations around the reference trajectory. This is equivalent to have the focusing term K_x constant around the machine. Although this case is never fulfilled in a real accelerator, however it provides a reliable model for the description of the beam instabilities. Under this approximation we can write

$$x''(s) + K_x x(s) = \frac{1}{m_0 v_z^2 \gamma} F_x^{self}(x,s)$$

which is a linear differential equation. The homogeneous solution is given by

$$x(s) = A_x \cos\left[\sqrt{K_x}\, s - \varphi_x\right]$$

with

$$a_x \sqrt{\beta_x} = A_x$$

$$\beta_x = \frac{1}{\mu_x'} = \frac{1}{\sqrt{K_x}}$$

$$\mu_x(s) = \sqrt{K_x}\, s$$

$$Q_x = \frac{1}{2\pi}\int_0^L \frac{ds'}{\beta_x(s')} = \frac{L}{2\pi\beta_x} = \rho_x\sqrt{K_x} \Rightarrow K_x = \left(\frac{Q_x}{\rho_x}\right)^2$$

The differential equation of motion then becomes

$$x''(s) + \left(\frac{Q_x}{\rho_x}\right)^2 x(s) = \frac{1}{m_0 v_z^2 \gamma} F_x^{self}(x,s)$$

By expanding the self-forces around the ideal orbit, we have a constant term which changes the equilibrium orbit and a linear term (proportional to the displacement) which changes the focussing strength and therefore induces and a shift of the betatron frequencies.

This can happen either in the motion of individual particles inside the beam (incoherent motion), or in the oscillation of the whole beam (coherent motion) around the closed orbit.

2.1 Shift and spread of the incoherent tunes

When the beam is located at the centre of symmetry of the pipe, the e.m. forces due to space charge and images cannot affect the motion of the centre of mass (coherent), but change the trajectory of individual charges in the beam (incoherent). These forces may have a complicate dependence on the charge position. A simple analysis is done considering only the linear expansion of the self-fields forces around the equilibrium trajectory.

TRANSVERSE INCOHERENT EFFECTS

We take the linear term of the transverse force in the betatron equation:

$$F_x^{self}(x,s) \cong \left(\frac{\partial F_x^{self}}{\partial x}\right)_{x=0} x$$

In case of uniform transverse beam distribution, either in a circular pipe and between parallel plates, the linearity of the force is not an approximation. For other kinds of distribution, we can always suppose that the transverse displacement r of a charge is much smaller than the transverse bunch dimension σ_r, so that we can always expand the force at first order r. In any case we end up with a self induced force linearly dependent on the particle displacement. As a consequence, by considering, for example, the motion along the x direction, we get

$$x''(s)+\left(\frac{Q_x}{\rho_x}\right)^2 x(s)=\frac{1}{\beta^2 E_0}\left(\frac{\partial F_x^{self}}{\partial x}\right)_{x=0} x(s)$$

The linear additional term produces a shift of the betatron tune. Indeed we can write:

$$x''(s)+\left[\left(\frac{Q_x}{\rho_x}\right)^2-\frac{1}{\beta^2 E_0}\left(\frac{\partial F_x^{self}}{\partial x}\right)_{x=0}\right]x(s)=0$$

We recognize in the brackets a term proportional to the square of the new betatron tune, shifted, with respect to the initial one Q_x, by the self induced forces. This term can be written as $\left(Q_x+\Delta Q_x\right)^2/\rho_x^2$. For small perturbations, the shift ΔQ_x can then be obtained by:

$$\left[\left(\frac{Q_x}{\rho_x}\right)^2-\frac{1}{\beta^2 E_0}\left(\frac{\partial F_x^{self}}{\partial x}\right)_{x=0}\right]=\frac{\left(Q_x+\Delta Q_x\right)^2}{\rho_x^2}\cong\frac{Q_x^2+2Q_x\Delta Q_x}{\rho_x^2}$$

thus giving

$$\Delta Q_x=-\frac{\rho_x^2}{2\beta^2 E_0 Q_x}\left(\frac{\partial\, F_x^{self}}{\partial x}\right)$$

A similar expression is found in the y direction. The betatron shift is negative since the space charge forces are defocusing on both planes. Notice that the tune shift is in general function of "z", therefore there is a tune spread inside the beam. This conclusion is generally true also for more realistic non-uniform transverse beam distributions, which are characterized by a tune shift dependent also on the betatron oscillation amplitude. In these cases, instead of tune shift the effect is called tune spread.

Example: Incoherent shift of the betatron tune for a uniform electron beam of radius *a* and length l_o, inside a circular pipe. ($N=10^{10}$, $\rho_x=20$ *m*, $E_o=1$ *GeV*, $l_o=.1$ *m*, $Q_x=10$, $a=.1$ *mm*).

electron rest mass: 0.51 MeV $\rightarrow$ $\gamma=2000$ (relativistic charge)

$$\Delta Q_x=-\frac{\rho_x^2}{2\beta^2 E_o Q_x}\left(\frac{\partial\, F_x^{s.c.}}{\partial u}\right)=-\frac{\rho_x^2 N e^2}{4\pi\varepsilon_o a^2 l_o \beta^2\gamma^2 E_o Q_x}$$

or expressed in terms of the classical radius of electron:

$$\Delta Q_x=-\frac{r_o\rho_x^2 N}{a^2 l_o\beta^2\gamma^3 Q_x}$$

Exercise 4: Compute the incoherent tune spread of a proton beam with uniform radial density and longitudinal parabolic density $\lambda(z)$, moving on the axis of a circular pipe.

Exercise 5: Compute the incoherent tune shift for a bi-gaussian electron beam in the linear approximation.

Exercise 6: Compute the incoherent betatron tune shift for a uniform proton beam inside two parallel plates with h=2 cm.

In the general case of non-uniform focusing, the betatron tune shift is given by the following formula:

$$\Delta Q_u = \frac{1}{4\pi}\oint \beta_u(s)\Delta K_u(s)ds = \frac{-1}{4\pi\beta^2 E_0}\oint \beta_u(s)\left(\frac{\partial F_u^{self}}{\partial u}\right)ds$$

Example: Incoherent shift of the betatron tune for a uniform electron beam of radius *a* and length l_o, non-uniform focusing: ($N=10^{10}$, $\rho_x=20$ *m*, $E_o=1$ *GeV*, $l_o=.1$, $\varepsilon_x= 5\ 10^{-7}$).

$$\Delta Q_x = -\frac{r_0 N_p}{2\pi\beta^2\gamma^3 l_0}\oint \frac{\beta_x(s)}{a^2(s)}ds = -\frac{r_0 N_p}{2\pi\beta^2\gamma^3 l_0}\frac{2\pi\rho_x}{\varepsilon_x}$$

2.2 Consequences of the space charge tune shifts

In order to avoid beam instabilities, in circular accelerators, the values of the betatron tunes should not cross the linear and non-linear resonances of the tune diagram. The tune spread induced by the space charge force may result in the impossibility to accommodate the beam in the space between the resonant lines making it hard to satisfy this basic requirement.

Typically, in order to avoid major resonances the stability requires

$$\left|\Delta Q_u\right| < 0.5$$

If the tune spread exceeds this limit, it is possible to reduce the effect of space charge tune spread by increasing the injection energy.

It is worth noting that, provided that the coherent tune remains inside the incoherent tune spread, this last one produces a beneficial effect, Landau damping, which can cure the coherent instabilities.

In a LINAC or a beam transport line, the space charge forces cause an energy spread and perturb the equilibrium beam size. It is required that the defocusing space charge forces must not be larger than the external forces.

2.3 The synchrotron motion

In the longitudinal case the motion is governed by the RF voltage:

$$V(z) = \hat{V}\sin\left[\frac{h\omega_0}{c}z + \varphi_s\right]$$

Where *h* is the harmonic number, ω_0 is the revolution frequency, φ_s is the synchronous phase. In the linear approximation, the equation of motion is that of a simple harmonic oscillator:

$$z'' + \left(\frac{Q_z}{\rho_x}\right)^2 z = 0$$

the particles oscillate around the synchronous phase □s with a synchronous tune given by:

$$Q_z = \frac{\omega_z}{\omega_o} = \sqrt{\frac{e\eta h\hat{V}\cos\varphi_s}{2\pi\beta^2 E_o}} \quad ; \qquad \text{with} \quad \eta = \frac{1}{\gamma^2} - \alpha_c$$

The factor η accounts for the increase of the speed with energy ($1/\gamma^2$), and the length of the real orbit due to the dispersion (α_c).

The interaction of the beam with the surroundings, may induces longitudinal e.m. forces which are to be included in the equation of motion:

$$z'' + \left(\frac{Q_z}{\rho_x}\right)^2 z = \frac{\eta\, F_z^{self}(s)}{\beta^2 E_o}$$

In the linear approximation, the longitudinal force produces the shift of both the synchronous phase and the synchronous tune.

2.4 Longitudinal incoherent effects

The constant term leads to a shift of the synchronous phase and to the displacement of the closed orbit. We analyse the motion around the new equilibrium:

$$F_z^{self}(r,z) \cong \left(\frac{\partial F_z^{self}}{\partial z}\right)_{z=0} z$$

$$z'' + \left(\frac{Q_z}{\rho_x}\right)^2 z = \frac{\eta}{\beta^2 E_o}\left(\frac{\partial F_z^{self}}{\partial z}\right)_{z=0} z$$

We consider the same approximation used in the transverse case and we get

$$\Delta Q_z = \frac{-\eta\rho_x^2}{2\beta^2 E_o Q_z}\left(\frac{\partial\, F_z^{self}}{\partial z}\right)$$

The synchrotron shift can be either positive or negative and changes with the position of the charge.

Example: Beam of radius a and length l_o, inside a circular pipe. Uniform transverse and parabolic longitudinal. ($N=10^{10}$, $\rho_x=20\ m$, $E_o=10\ GeV$, $l_o=.1\ m$, $Q_z=10^{-3}$, $b/a=5$).

$$\Delta Q_z = -6\frac{\eta\rho_x^2 r_0 N_p}{\beta^2\gamma^3 Q_z}\left(1-\frac{r^2}{a^2}+2\ln\frac{b}{a}\right)\left(\frac{1}{l_0^3}\right)$$

Exercise 7: Compute the longitudinal incoherent tune spread of a proton beam with uniform radial density and Gaussian longitudinal density $\lambda(z)$, in a circular pipe. (Same data of example).

3. WAKE FIELDS and POTENTIALS

Parasitic forces are generated by the beam, which interacts with all the components of the vacuum chamber. These components may have a complex geometry: kickers, bellows, RF cavities, diagnostics components, special devices, etc. The study of the fields requires to solve the Maxwell equations in a given structure taking the beam current as source of fields. This is a quite complicated task for which it has been necessary to develop dedicated computer codes, which solve the e.m. problem in the frequency or in the time domain. There are several useful codes for the design of accelerator devices: CST Studio Suite, GDFIDL, ACE3P, ABCI, and others.

Here we first discuss the general features of the parasitic fields, and then show a few simple examples in cylindrical geometry: perfectly conducting pipe and the resonant modes of a RF cavity. Although the space charge forces have been studied separately, they can be seen as a particular case of wake-fields.

3.1 Wake potentials

The parasitic fields depend on the particular charge distribution of the beam. It is therefore desirable to know what is the effect of a single charge (i.e. find the Green function) in order to reconstruct the fields produced by any charge distribution.

The e.m. fields created by a point charge, act back on the charge itself and on any other charge of the beam. We focus, therefore, our attention on the source charge q_o, and on the test charge q, assuming that both are moving with the same constant velocity $\mathbf{v} = \beta c$ on trajectories parallel to the axis.

Let **E** and **B** be the fields generated by q_0 inside the structure, ($s_0 = vt, r_0$) be the position of the source charge and ($s = s_0 + z, r$) be the position of the test charge q.

Since the velocity of both charges is along **z**, the Lorentz force has the following components:

$$\boldsymbol{F} = q\left[E_z\hat{z} + \left(E_x - vB_y\right)\hat{x} + \left(E_y + vB_x\right)\hat{y}\right] \equiv \boldsymbol{F}_{//} + \boldsymbol{F}_{\perp}$$

Thus, there can be two effects on the test charge: a longitudinal force which changes its energy, and a transverse force which deflects its trajectory. If we consider a device of length L, the energy gain (joule) is:

$$U = \int_0^L F_z ds$$

and the transverse deflecting kick (newton ·meter) is :

$$\boldsymbol{M} = \int_0^L \mathbf{F}_{\perp} ds$$

These quantities, normalised to the two charges, are called ***wake-fields*** (volt/coulomb) and are both function of the distance z. Note that the integration is performed over a given path of the trajectory. In many cases, we deal with structures having particular symmetric shapes, generally cylindrical. It is possible to demonstrate that with a multipole expansion of the wake fields, the dominant term in the longitudinal wake field depends only on the distance z between the two charges, while the dominant one in the transverse wake field is still function of the distance z, but it is also linear with the transverse position of the source charge r_0. If we then divide the transverse wake field by r_0 we obtain the transverse dipole wake field, which is the transverse wake per unit of transverse displacement, depending only on z:

Longitudinal wake field (volt/coulomb)

$$w_{//} = -\frac{U}{q_o q}$$

Transverse dipole wake field (volt/coulomb meter)

$$\boldsymbol{w}_{\perp} = \frac{1}{r_o}\frac{\boldsymbol{M}}{q_o q}$$

The sign minus in the longitudinal wake-field means that the test charge loses energy when the wake is positive. Positive transverse wake means that the transverse force is defocusing.

Example : Longitudinal wake field of "space charge".

Even if in the ultra-relativistic limit with $\gamma \longrightarrow \infty$, there is no space charge effect, we can still define a wake field by considering a moderately relativistic beam with γ>>1 but not infinite. It turns out that the space charge forces can fit into the definition of wake field, and when that is done, we find that the wake depends on beam properties such as the transverse beam radius a and the beam energy γ. Let us consider a relativistic beam with cylindrical symmetry and uniform transverse distribution. The longitudinal force acting on a charge of the beam travelling inside a cylindrical pipe of radius b is given by [13]:

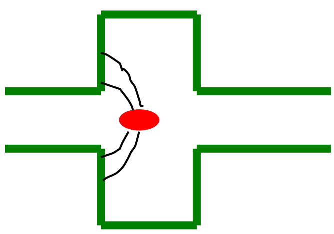
$$F_{//}(r,z) = \frac{-q}{4\pi\varepsilon_0\gamma^2}\left(1 - \frac{r^2}{a^2} + 2\ln\frac{b}{a}\right)\frac{\partial\,\lambda(z)}{\partial\, z}$$

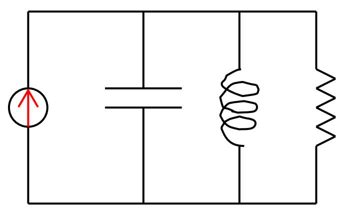
with $\lambda(z)$ the longitudinal distribution ($z > 0$ at the bunch head). Note that, since the space charge forces move together with the beam, they are constant along the accelerator if the beam pipe remains constant. We can therefore define the longitudinal wake field per unit length (V/Cm). To get the longitudinal wake field of a piece of pipe, we just multiply by the pipe length. Assuming $r \rightarrow 0$ (particle on axis), and a charge line density given by $\lambda(z) = q_0\delta(z)$ we obtain:

$$\frac{dw_{//}(z)}{ds} = \frac{1}{4\pi\varepsilon_0\gamma^2}\left(1 + 2\ln\frac{b}{a}\right)\frac{\partial}{\partial\, z}\delta(z)$$

Example : Longitudinal wake field of a resonant HOM

When a charge crosses a resonant structure, it excites the fundamental mode and high order modes (HOM). Each mode can be treated as an electric RLC circuit loaded by an impulsive current.





Just after the charge passage, the capacitor is charged with a voltage $V_o=q_o/C$ and the electric field is $E_{so}= V_o/l_o$. The time evolution of the electric field is governed by the same differential equation of the voltage:

$$V_C = V_L = V_R \equiv V$$

$$V_R = RI_R$$

$$V_L = L\dot{I}_L$$

$$V_C = \frac{1}{C}\int I_c dt$$

$$I = I_C + I_L + I_R$$

$$\dot{I} = C\ddot{V} + \frac{V}{L} + \frac{\dot{V}}{R}$$

$$\ddot{V} + \frac{1}{RC}\dot{V} + \frac{1}{LC}V = \frac{1}{C}\dot{I}$$

The passage of the impulsive current charges only the capacitor, which changes its potential by an amount $V_c(0)$. This potential will oscillate and decay producing a current flow in the resistor and inductance. For t>0 the potential satisfies the following equation and boundary conditions:

$$\ddot{V} + \frac{1}{RC}\dot{V} + \frac{1}{LC}V = 0$$

$$V(t = 0^+) = \frac{q}{C} \equiv V_0$$

$$\dot{V}(t = 0^+) = \frac{\dot{q}}{C} = \frac{I(0^+)}{C} = \frac{V_0}{RC}$$

$$V(t) = V_0 e^{-\gamma t}\left[\cos(\varpi z/c) - \frac{\gamma}{\varpi}\sin(\varpi t)\right]$$

$$\varpi^2 = \omega_r^2 - \gamma^2$$

putting z= -ct (z is assumed negative behind the charge), we get the longitudinal wake field of a resonant HOM:

$$w_{//}(z) = \frac{-V(z)}{q_o} = \frac{R\omega_r}{Q}e^{\gamma z/c}\left[\cos(\varpi z/c) + \frac{\gamma}{\varpi}\sin(\varpi z/c)\right] = w_o e^{\gamma z/c}\left[\cos(\varpi z/c) + \frac{\gamma}{\varpi}\sin(\varpi z/c)\right]$$

In an analogous way, it is possible to obtain the transverse wake field of a HOM

$$w_\perp(z) = \frac{R_\perp\omega_r}{Q}e^{-\Gamma z/c}\sin(\varpi z/c)$$

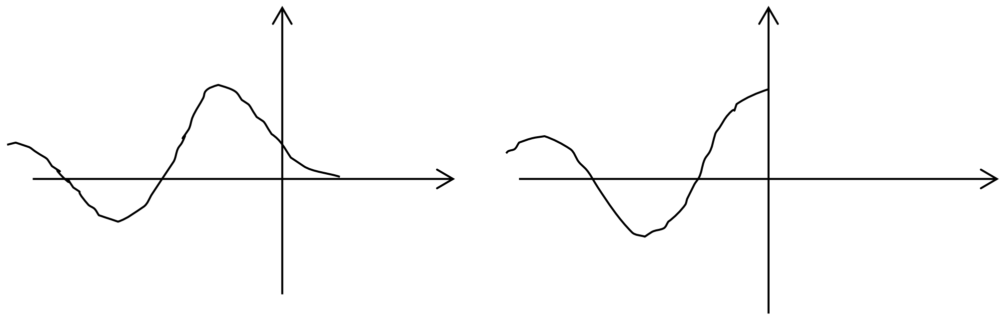
3.2 Loss factor

It is also useful to define the ***loss factor*** as the normalised energy lost by the source charge q_o:

$$k = -\frac{U(z=0)}{q_o^2}$$

Although in general the loss factor is given by the longitudinal wake at z=0, for charges travelling with the light velocity the longitudinal wake potential is discontinuous at z=0.

The exact relationship between *k* and *w(z→0)* is given by the ***beam loading theorem.***



3.3 Beam loading theorem

When the source charge travels with the light velocity v=c, it leaves the e.m. fields mainly on the back, reason why we call these fields "wake fields". Any e.m. perturbation produced by the charge cannot overtake the charge itself. This means that the longitudinal wake field vanishes in the region z>0. This property is a consequence of the ***"causality principle"***. Causality requires that the longitudinal wake field of a charge travelling with the velocity of light is discontinuous at the origin.

The beam loading theorem states that:

$$k = \frac{w_{//}(z \to 0)}{2}$$

Example : Show that the beam loading theorem is fulfilled by the wake field of the resonant mode.

The energy lost by the charge q_o loading the capacitor is:

$$U = \frac{CV_o^2}{2} = \frac{q_o^2}{2C} \quad \Rightarrow \quad k = \frac{1}{2C}$$

$$\text{compare with } \; w_{//}(z \to 0) = \frac{1}{C}$$

3.4 Relationship between transverse and longitudinal forces

Another important feature worth mentioning, is the differential relationship existing between longitudinal and transverse forces:

$$\nabla_\perp F_{//} = \frac{\partial}{\partial z} \boldsymbol{F}_\perp$$

$$\nabla_\perp w_{//} = \frac{\partial}{\partial z} \boldsymbol{w}_\perp$$

The above relations are known as "Panofsky-Wenzel theorem".

3.5 Coupling Impedance

The wake potentials are used for to study the beam dynamics in the time domain (*s*=*vt*). If we take the equation of motion in the frequency domain, we need the Fourier transform of the wake potentials. Since these quantities have ohms units they are called ***coupling impedances***:

Longitudinal impedance (Ω)

$$Z_{//}(\omega)=\frac{1}{v}\int_{-\infty}^{\infty} w_{//}(z)e^{i\frac{\omega z}{v}}\,dz$$

Transverse dipole impedance (Ω/m)

$$\boldsymbol{Z}_{\perp}(\omega)=-\frac{i}{v}\int_{-\infty}^{\infty} \boldsymbol{w}_{\perp}(z)e^{i\frac{\omega z}{v}}\,dz$$

Example :   Coupling impedances (Ω/m) of the " space charge"

$$\frac{\partial Z_{//}(\omega)}{\partial s}=\frac{1}{v}\int_{-\infty}^{\infty}\frac{\partial w_{//}(z)}{\partial s}e^{i\frac{\omega z}{v}}\,dz=\frac{\left(1+2\ln\frac{b}{a}\right)}{v4\pi\varepsilon_o\gamma^2}\int_{-\infty}^{\infty}\frac{d}{dz}\delta(z)e^{i\frac{\omega z}{v}}\,dz$$

$$\text{use}\rightarrow\int_{-\infty}^{\infty}\delta'(z)f(z)dz=f'(0)$$

$$\frac{\partial Z_{//}(\omega)}{\partial s}=\frac{i\omega Z_o}{4\pi c\beta^2\gamma^2}\left(1+2\ln\frac{b}{a}\right)$$

Example: Coupling impedances of a HOM resonant

Longitudinal Impedance:

$$Z_{//}(\omega)=\frac{R_s}{1+iQ_r\left(\frac{\omega_r}{\omega}-\frac{\omega}{\omega_r}\right)}$$

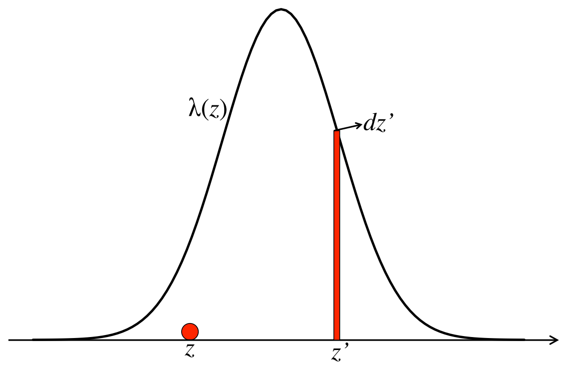
where $R_s=\frac{w_0}{2\gamma}$ is the shunt impedance, and $Q_r=\frac{\omega_r}{2\gamma}$ is the quality factor; quantities that we can obtain with the computer codes. Note that the loss factor is also determined: $k=\frac{\omega_r R_s}{2Q_r}$

Transverse Impedance:

$$Z_{\perp}(\omega)=\frac{c}{\omega}\frac{R_{\perp}}{1+iQ\left(\frac{\omega_r}{\omega}-\frac{\omega}{\omega_r}\right)}$$

3.6 Wake potentials and energy loss of a bunched distribution.

When we have a bunch with density λ(z), such that $q=\int_{-\infty}^{\infty}\lambda(z')dz'$ we may wonder what is the amount of energy lost or gained by a single unit charge *e* in the beam.



To this end we calculate the effect on the charge from the whole bunch by means of the convolution integral:

$$U(z) = -e\int_{-\infty}^{\infty} w_{//}\left(z'-z\right)\lambda(z')dz'$$

which allows to define the ***wake potential of a distribution***:

$$W_{//}(z) = -\frac{U(z)}{qe}$$

The total energy lost by the bunch is computed summing up the losses of all particles:

$$U_{bunch} = \frac{1}{e}\int_{-\infty}^{\infty} U\left(z'\right)\lambda(z')dz'$$

4. WAKE FIELDS EFFECTS IN LINEAR ACCELERATORS

We study the effect of the wake fields on the dynamics of a beam in a LINAC, assuming a high energy such that the motion is “frozen” in the longitudinal phase space.

4.1 Energy Spread

The longitudinal wake forces change the energy of individual particles depending on their position in the beam. As consequence the wake can induce an energy spread in the beam.

Example: Energy spread induced by the space charge force in a Gaussian bunch.

$$\frac{dU(z)}{ds} = -e\int_{-\infty}^{\infty} \frac{dw_{//}\left(z-z'\right)}{ds}\lambda(z')dz' =$$

$$= \frac{eq}{4\pi\varepsilon_o\gamma^2\sqrt{2\pi}\sigma_z^3}\left(1+2\ln\frac{b}{a}\right)z\, e^{-(z^2/2\sigma_z^2)}$$

The bunch head gains energy (z>0), while the tail loses energy:

Exercise : Compute the energy loss induced by a resonant HOM on the charges inside a rectangular uniform bunch.

$$U(z) = \frac{-eqw_o}{2} \frac{\sin\left[\frac{\omega_r}{c}\left(\frac{l_o}{2} - z\right)\right]}{\left(\frac{\omega_r l_o}{2c}\right)}$$

4.2 Parasitic loss

Because of the wake fields, the beam loses a certain amount of energy, called ***parasitic loss***. There is no loss in a perfectly conducting pipe, while the beam deposits energy in the pipe with resistive wall, in the resonant devices and in the high frequency fields propagating inside the pipe.

Exercise : Parasitic loss of a rectangular uniform bunch in a resonant HOM

$$U_{bunch} = \frac{-q^2 w_o}{2} \frac{\sin^2\left(\frac{\omega_r l_o}{2c}\right)}{\left(\frac{\omega_r l_o}{2c}\right)^2}$$

4.3 Beam Break Up

A beam injected off-center in a LINAC, because of the focusing quadrupoles, executes betatron oscillations. The displacement produces a transverse wake field in all the devices crossed during the flight, which deflects the trailing charges. In order to understand the effect, we consider a simple bunch model with only two charges $q_1=Ne/2$ (leading = half bunch) and $q_2=e$ (trailing = single charge).

■ the leading charge executes free betatron oscillations:

$$y_1(s) = \hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right)$$

■ the trailing charge, at a distance z behind, over a length L_w experiences a deflecting force proportional to the displacement y_1, and dependent on the distance z:

$$\left\langle F_y^{self}(z, y_1)\right\rangle = \frac{Ne^2}{2L_w} w_\perp(z) y_1(s)$$

Notice that L_w is the length of the device for which the transverse wake has been computed. For example, in the case of a cavity cell L_w is the length of the cell. This force drives the motion of the trailing charge:

$$y_2'' + \left(\frac{\omega_y}{c}\right)^2 y_2 = \frac{Ne^2 w_\perp(z)}{2E_o L_w} \hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right)$$

This is the typical equation of a resonator driven at the resonant frequency.

The solution is given by the superposition of the "free" oscillation and a "forced" oscillation which, being driven at the resonant frequency, grows linearly with *s*:

$$y_2(s) = \hat{y}_2 \cos\left(\frac{\omega_y}{c} s\right) + y_2^{forced}$$

$$y_2^{forced} = \frac{cNe^2 w_\perp(z) s}{4\omega_y E_o L_w} \hat{y}_1 \sin\left(\frac{\omega_y}{c} s\right)$$

At the end of the LINAC of length L_L, the oscillation amplitude is grown by ($\hat{y}_1 = \hat{y}_2$):

$$\left(\frac{\Delta\hat{y}_2}{\hat{y}_2}\right)_{\max} = \frac{cNew_\perp(z) L_L}{4\omega_y (E_o/e) L_w}$$

If the transverse wake is given per cell, the relative displacement of the tail with respect to the head of the bunch, depends on the number of cells. It depends, of course, on the focusing strength through the frequency ω_y.

4.4 BNS Damping

The BBU instability is quite harmful and hard to take under control even at high energy with a strong focusing, and after a careful injection and steering. A simple method to cure it has been proposed observing that the strong oscillation amplitude of the bunch tail is mainly due to the "resonant" driving. If the tail and the head move with a different frequency, this effect can be significantly removed.

Let us assume that the tail oscillates with a frequency $\omega_y+\Delta\omega_y$; the equation of motion reads:

$$y_2'' + \left(\frac{\omega_y + \Delta\omega_y}{c}\right)^2 y_2 = \frac{Ne^2 w_\perp(z)}{2E_o L_w} \hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right)$$

the solution of which is:

$$y_2(s) = \hat{y}_2 \cos\left(\frac{\omega_y + \Delta\omega_y}{c} s\right) - \frac{c^2 Ne^2 w_\perp(z)}{4\omega_y \Delta\omega_y E_o L_w} \hat{y}_1 \left[\cos\left(\frac{\omega_y + \Delta\omega_y}{c} s\right) - \cos\left(\frac{\omega_y}{c} s\right)\right]$$

where the amplitude of the oscillation is limited. Furthermore, by a suitable choice of $\Delta\omega_y$, it is possible to fully depress the oscillations of the tail. In fact, by setting:

$$\Delta\omega_y = \frac{c^2 Ne^2 w_\perp(z)}{4\omega_y E_o L_w}$$

if $\hat{y}_2 = \hat{y}_1$, we get:

$$y_2(s) = \hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right)$$

The extra focusing at the bunch tail can be obtained by:

1. Using an RFQ, where head and tail see a different focusing strength,

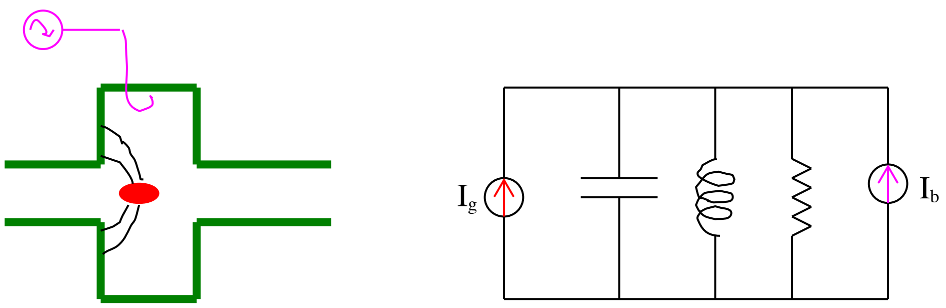
2. Exploit the energy spread across the bunch which, because of the chromaticity, induces a spread in the betatron frequency. An energy spread correlated with the position is attainable with the external accelerating voltage, or with the wake fields.

5. WAKE FIELDS EFFECTS IN STORAGE RINGS

We study here some effects in the longitudinal beam dynamics of bunched beams in storage rings.

5.1 Beam loading

When a charge crosses a resonant structure, it excites the fundamental mode and high order modes (HOMs). We limit our analysis to the fundamental accelerating mode in order to see how the voltage induced by the beam interferes with the accelerating voltage. To this end, we schematise the cavity with an RLC parallel circuit, fed by the generator current I_g and by the beam current I_b.



The voltage in the cavity is given by a combined effect of the two currents. The beam current is impulsive, and repeats itself at each beam passage. The generator current feeding the cavity is an alternating signal at the RF frequency:

$$I_g(t) = \hat{I}_g \cos\left(\omega_g t\right)$$

$$I_b(t) = q_b \sum_{-\infty}^{\infty} \delta(t - t_s - nT_o)$$

The time evolution of the voltage in the cavity is governed by the same differential equation found in chapter 3:

$$\ddot{V} + \frac{1}{RC}\dot{V} + \frac{1}{LC}V = \frac{1}{C}\left(\dot{I}_g + \dot{I}_b\right)$$

It is possible to solve separately the equation for the two sources, and apply the superposition principle. The steady state voltage induced by the generator current is of the type:

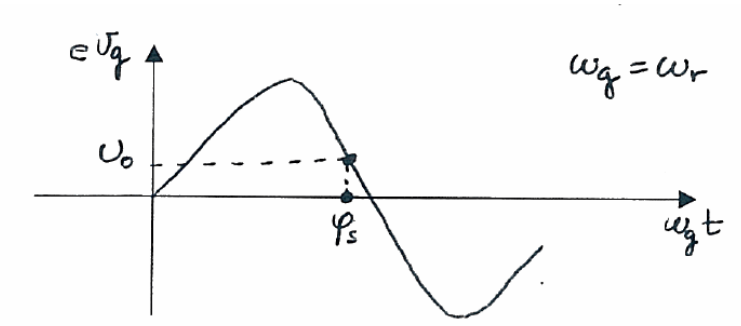
$$V_g(t) = \hat{V}_g \sin(\omega_g t - \varphi_g)$$

where the amplitude and the phase depend on the amplitude of the driving current and on the circuit parameters:

$$\hat{V}_g = \hat{I}_g \frac{R_S}{1 + Q_r^2\left(\frac{\omega_g}{\omega_r} - \frac{\omega_r}{\omega_g}\right)^2} \equiv \hat{I}_g Z_r(\omega_g)$$

$$\varphi_g = arctan\left[Q_r\left(\frac{\omega_g}{\omega_r} - \frac{\omega_r}{\omega_g}\right)\right]$$

Notice that if $\omega_g=\omega_r$, i.e. when the generator drives the cavity at the resonance, the phase of the generator voltage is zero, and the amplitude is just given by the current amplitude times the shunt impedance. The phase φ_s is called "detuning angle" and it is usually chosen in order to ensure the stability of the beam and to maximise the power transfer from the generator to the cavity.



The synchronous phase of the particle without the effect of beam loading is given by:

$$\varphi_s = \arcsin\left(\frac{U_o}{eI_g Z_r(\omega_g)}\right) \equiv \arcsin\left(\frac{U_o}{e\hat{V}_g}\right)$$

For the moment, we consider the generator operating at the resonance $\omega_g=\omega_r$, and we want to evaluate the effect of the beam loading on the accelerating field in the cavity and on the synchronous phase.

The expression of the beam current can be expanded in Fourier harmonics:

$$I_b(t) = \frac{q_b}{T_o}\left[1 + 2\cos\left(n\omega_o(t - t_s)\right)\right]$$

Since the cavity is a high Q resonator, the term at n=h (harmonic number) is much stronger than the others, and we approximate the beam current with the harmonic signal at the resonant frequency:

$$I_b(t) \approx \frac{2q_b}{T_o}\cos\left(\omega_r(t - t_s)\right)$$

This current oscillates at the resonant frequency and therefore induces a voltage

$$V_b \approx R_s I_b(t) \approx \frac{2q_b R_s}{T_o}\cos\left(\omega_r(t - t_s)\right)$$

The total voltage in the cavity is therefore:

$$V_T = V_g + V_b \approx \hat{I}_g R_s \sin(\omega_r t) + \frac{2q_b R_s}{T_o}\cos\left(\omega_r(t - t_s)\right)$$

The synchronous phase:

$$\varphi_s = h\omega_o t_s = \omega_r t_s$$

is obtained by imposing that the charge gains the energy lost per turn:

$$eV_T(t_s) = U_o$$

$$\hat{I}_g R_s \sin(\varphi_s) + \frac{2q_b R_s}{T_o} = \frac{U_o}{e}$$

$$\sin(\varphi_s) = \frac{\frac{U_o}{e} - \frac{2q_b R_s}{T_o}}{\hat{I}_g R_s}$$

Both the total voltage amplitude and the synchronous phase are affected by the voltage induced by the beam. In the real life, the operation on the cavity is more complex, since the frequency of the generator is usually detuned with respect to the resonant frequency, and the amplitude of the total voltage is also kept under control with a feedback system.

5.2 Robinson instability

Let us assume now that the beam oscillates around the synchronous phase:

$$\varphi(t) = \varphi_s + \varphi_o \cos(\omega_s t)$$

This is nothing else than a phase modulation of the signal of the beam around the harmonic $h\omega_o$

$$I_b(t) \approx \frac{2q_b}{T_o}\left[\cos\left(\omega_g t - \varphi_s\right) + \frac{\varphi_o}{2}\left\{\sin\left((\omega_g + \omega_s)t - \varphi_s\right) + \sin\left((\omega_g - \omega_s)t - \varphi_s\right)\right\}\right]$$

The voltage induced by such a signal is:

$$\begin{aligned} V_b(t) \approx \frac{2q_b}{T_o}\Big\{ & Z_r(\omega_g)\cos\left(\omega_g t - \varphi_s - \varphi_g\right) + \\ & + \frac{\varphi_o}{2} Z_r(\omega_g + \omega_s)\sin\left((\omega_g + \omega_s)t - \varphi_s - \varphi_g\right) + \\ & + \frac{\varphi_o}{2} Z_r(\omega_g - \omega_s)\sin\left((\omega_g - \omega_s)t - \varphi_s - \varphi_g\right)\Big\} \end{aligned}$$

Because of the synchronous oscillations, the energy gain per turn is

$$U_b \approx \frac{2q_b^2}{T_o}\left\{Z_r(\omega_g) + \frac{\varphi_o \sin\left(\omega_s t\right)}{2}\left[Z_r(\omega_g + \omega_s) - Z_r(\omega_g - \omega_s)\right]\right\}$$

where the low varying term $\sin(\omega_s t)$ has been considered constant in a revolution time. Since

$$\dot{\varphi}(t) = -\omega_s \varphi_o \sin(\omega_s t)$$

$$U_b \approx \frac{2q_b^2}{T_o}\left\{Z_r(\omega_g) - \frac{\dot{\varphi}}{2\omega_s}\left[Z_r(\omega_g + \omega_s) - Z_r(\omega_g - \omega_s)\right]\right\}$$

including this term of energy gain per turn in the equation of motion, we get:

$$\ddot{\varphi} + \left\{\frac{\eta q_b}{E T_o^2 \omega_s}\left[Z_r(\omega_g + \omega_s) - Z_r(\omega_g - \omega_s)\right]\right\}\dot{\varphi} + \frac{e\eta h \hat{V}\cos\varphi_s}{2\pi E_o}\varphi = 0$$

Where we recognise that the motion is damped or unstable depending on the sign of the coefficient of $d\varphi/dt$.. We can introduce a strong damping on the synchronous motion if we make this term negative and high. Above the transition energy this can be done by operating with a generator frequency higher than the resonant frequency.

6. LANDAU DAMPING

6.1 Driven oscillators

There is a fortunate stabilising effect against the collective instabilities called "Landau Damping". The basic mechanism relies on the fact that if the particles in the beam have a spread in their natural frequencies (synchrotron or betatron), their motion can't be coherent for a long time.

In order to understand the physical nature of this effect, we consider a simple harmonic oscillator, at rest for t<0, driven by an oscillatory force for t>0.

$$\frac{d^2x}{dt^2} + \omega^2 x = A\cos(\Omega t)$$

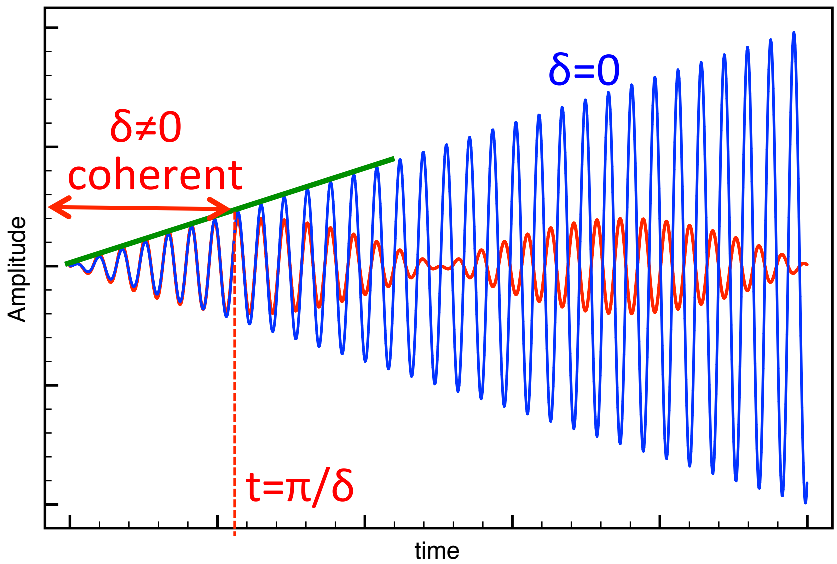
The general solution is given by the superposition of the free and forced solutions:

$$x(t) = \frac{A}{\omega^2 - \Omega^2}\left[\cos(\Omega t) - \cos(\omega t)\right]$$

Let us assume now that the external force is driving a particle population characterised by a spread of natural frequency of oscillation around a mean value ω_x. Furthermore, let the forcing frequency Ω be inside the spectrum such that $\delta \equiv \Omega - \omega \ll \omega_x$.

The motion of a given particle in the bunch can be approximated by:

$$x(t) \cong \frac{At}{2\omega_x}\sin(\omega_x t)\frac{\sin\left(\frac{\delta}{2}t\right)}{\left(\frac{\delta}{2}t\right)}$$



Let us observe now two particles in the bunch, one with $\delta=0$, and the other with $\delta\neq0$. Both are at rest, and at t=0 they start to oscillate with the same amplitude and phase (coherency). However, while the amplitude of the former charge growths indefinitely, (driven at resonance), the latter reaches a maximum amplitude (beating of two close frequencies). We say that the system of the particles has lost the coherency at the time when the beating amplitude is maximum, i.e. for $t=\pi/\delta$.

We can also say that at any time t^*, only those oscillators inside the bandwidth $|\delta| < \pi/ t^*$, oscillate coherently. The longer we wait, the narrower is the coherent bandwidth and therefore the less is the number of "coherent" particles.

6.2 Amplitude of oscillations

At any instant we can divide the bunch population in two groups: the coherent particles, oscillating all together with an amplitude growing linearly with time, and the "incoherent" particles which have different phases and a saturated amplitude of oscillation.

It is interesting to see that, albeit the amplitude of the coherent oscillators growths linearly with time, the average amplitude of the whole system remains bounded. The reason is that the number of coherent particles decreases inversely with time.

$$\langle x(t)\rangle^{\max} = \frac{1}{N}\left[\sum_{coh} x(t) + \sum_{incoh} x(t)\right]^{\max}$$

We consider the time when the coherent particles have the maximum amplitude. The amplitudes of the incoherent particles, being uncorrelated, have a zero average. For the coherent particles we have:

$$\langle x(t)\rangle^{\max} = \frac{N_{coh}}{N} x^{\max}(t) = \frac{N_{coh}}{N}\frac{A}{2\omega_x} t$$

On the other hand, the number of oscillators keeping the coherency decreases with time:

$$N_{coh} = \frac{N}{\Delta\omega}\frac{\pi}{t} \quad \Rightarrow \quad \langle x(t)\rangle^{\max} = \frac{\pi}{\Delta\omega}\frac{A}{2\omega_x}$$

6.3 Energy of the system

What happens to the energy of the system? Also in this case we distinguish between the coherent and incoherent particles. The energy of the coherent particles growths quadratically with time, while the energy of the incoherent particles is bounded. In this case, although the number of coherent oscillators decreases with time, the total energy still grows linearly.

$$\mathrm{E}(t) = \mathrm{E}_{coh}(t) + \mathrm{E}_{incoh}(t)$$

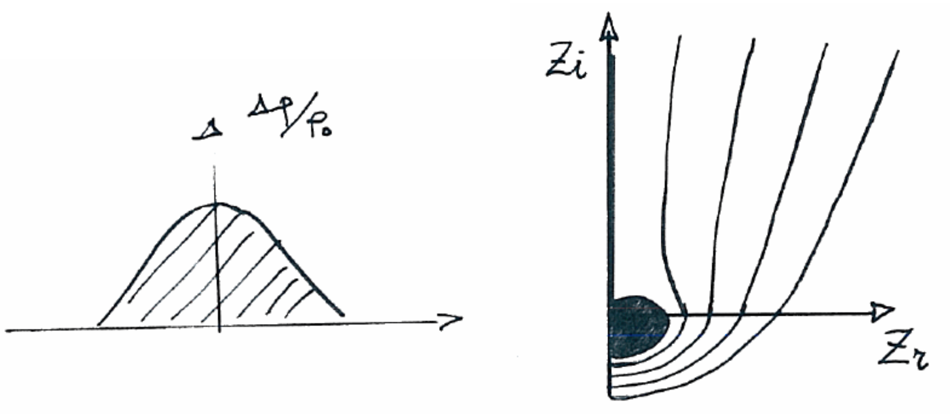
$$\mathrm{E}_{coh}(t) = N_{coh}\left[\frac{1}{2}k\, x_{coh}^2(t)\right] \propto \frac{N\pi}{\Delta\omega}\left(\frac{A}{2\omega_x}\right)^2 t$$

In conclusions when a force drives such a system, only at beginning the whole system follows the external force. Afterwards, fewer and fewer particles are driven at the resonance. The result is that although the system absorbs energy, the average amplitude remains bounded.

This mechanism works also when the driving force is produced by the bunch itself. In order to make the coherent instability to start, the rise time of the instability has to be shorter than the "de-coherency" time of the bunch:

$$\tau^{inst} < \tau^{decoh} = \frac{2\pi}{\Delta\omega}$$

A rigorous analysis of the beam dynamics in the frequency domain shows that, because of the Landau damping, a new stability region appears in the impedance plane, whose shape depends on distribution of the energy spread in the beam (form factor F). For instance, for a beam with a parabolic energy spread distribution we have the following figures:



Approximating the stable area with a circle, we get the ***stability criterion***:

$$\left|\frac{Z}{n}\right| \leq F\,\frac{\eta\left(E_o/e\right)}{\beta^2 I_o}\left(\frac{\Delta p}{p_o}\right)^2$$

REFERENCES

See, e.g. : "Transverse Beam Dynamics" and "Longitudinal Beam Dynamics", JUAS Lectures.

A. Hofmann " Tune Shift from Self-Fields and Images", CERN-94-01.

L. Palumbo "Wake Fields and Impedances", CERN-95-06

J. L. Laclare "Coasting Beam Coherent Instabilities", CERN 94-01.

A. Chao "Physics of Collective Beam Instabilities in High Energy Accelerators", Wiley-Interscience Pub.

J. Seeman "Observation of High Current Effects in Linear High Energy Accelerators", Joint US-CERN Accelerator School, 1990

K. Schlindl " Space Charge", CAS, 1996" and "Instabilities", CAS 1997.

USEFUL CONSTANTS

e	unit electric charge	1.602×10^{-19} (C)
c	speed of light in vacuum	2.997×10^{8} (m/s)
m_o	rest mass of electron	0.511 (MeV)
m_o	rest mass of proton	938.272 (MeV)
r_e	classical radius of electron	2.818×10^{-15} (m)
r_p	classical radius of proton	1.534×10^{-18} (m)
μ_o	permeability of vacuum	$4\pi \times 10^{-7}$ (H/m)
ε_o	permittivity of vacuum	8.854×10^{-1} (F/m)
Z_o	impedance of vacuum	120π (Ω)

$$r_{e,p} = \frac{\mathrm{e}^2}{4\pi\varepsilon_0 m_o c^2}$$

$$\varepsilon_0\mu_0 = 1/c^2$$

$$\varepsilon_0 Z_0 = 1/c$$

APPENDIX 1. BOUNDARY RELATIONS FOR CONDUCTORS.

STATIC FIELDS

When we have two materials with different relative permittivity, which we call ε_{r1} and ε_{r2}, in the passage from one material to another, the tangential electric field and the normal electric displacement are preserved, so that we have the boundary relations:

$$E_{t1} = E_{t2}$$
$$\varepsilon_{r1} E_{n1} = \varepsilon_{r2} E_{n2}$$

If one of the two materials is a conductor with a finite conductivity, then the electric field vanishes inside it, and the walls are equipotential surfaces. This implies that the electric field lines are orthogonal to the conductor surface, independently of the dielectric and magnetic properties of the material. The only condition is to have a finite conductivity.

If we have a charge close to a conductor, in order to obtain the electric field, we need to include the effects of the induced charges on the conducting surfaces, and we must know how they are distributed. Generally this task is not easy, but if we have an infinite conducting screen, the problem can be easily solved by making use of the method of images: we can remove the screen and put at a symmetric location a charge with opposite sign, as shown in figure.

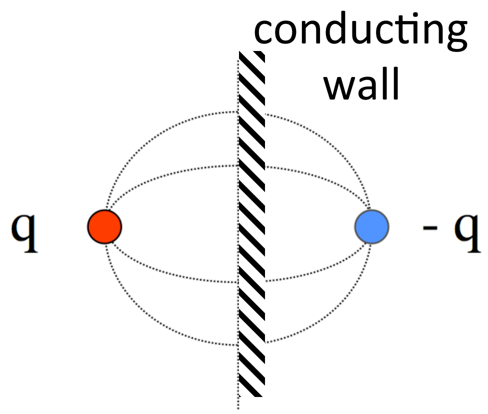


Fig. A1: Method of images.

The total electric field is the sum of the direct and the image field:

$$\vec{E}^{tot} = \vec{E}^{direct} + \vec{E}^{images}$$

For the static magnetic field between two materials with different permeability, the following boundary relations hold:

$$H_{t1} = H_{t2}$$
$$\mu_{r1} H_{n1} = \mu_{r2} H_{n2}$$

Thus, static magnetic fields do not perceive the presence of the conductor, if it has a magnetic permeability $\mu_r \approx 1$, as copper or aluminium, and the field lines behave as in the free space. However, a beam pipe in a real machine goes through many magnetic components (like dipoles and quadrupoles) made of ferromagnetic materials with high permeability (of the order of $10^3 - 10^5$). For these materials, due to the boundary conditions, the magnetic field lines are practically orthogonal to the surface. Similarly to electric field lines for a conductor, the total magnetic field can be derived by using the image method: we remove the magnetic wall and put a symmetric current with same sign, as shown in figure.

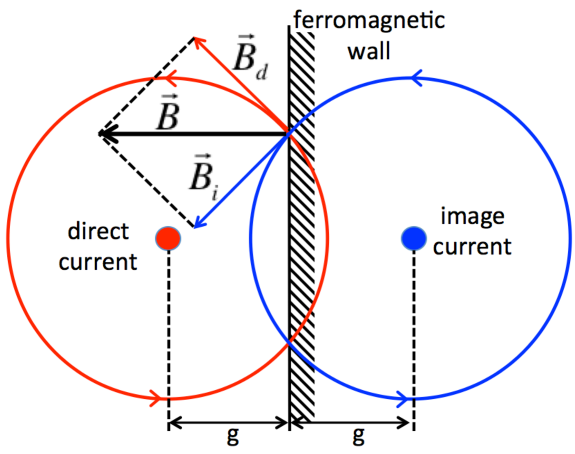


Fig. A2: Method of image current.

TIME VARYING FIELDS

Static electric fields vanish inside a conductor for any finite conductivity, while static magnetic fields pass through unless of high permeability. This is no longer true for time changing fields, which can penetrate inside the material in a region δ_w called skin depth. In order to write the skin depth as a function of the material properties, we write the following Maxwell's equations inside the conducting material together with the constitutive relations:

$$\begin{cases} \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t} \end{cases} \qquad \begin{cases} \vec{B} = \mu \vec{H} \\ \vec{D} = \varepsilon \vec{E} \\ \vec{J} = \sigma \vec{E} \end{cases}$$

Let us consider a plane wave linearly polarized with the electric field in the y direction propagating in the conducting material along x, as shown in Fig. A3.

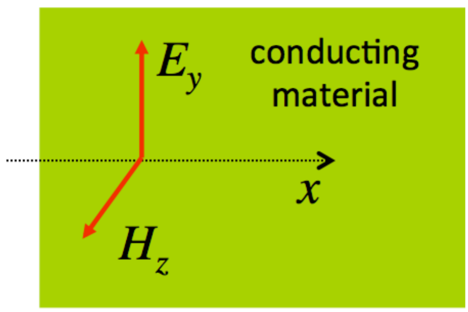


Fig. A3: Plane wave propagating inside a conducting material.

From the Maxwell's equations we get the wave equation for the electric field:

$$\frac{\partial^2 E_y}{\partial x^2} - \varepsilon\mu \frac{\partial^2 E_y}{\partial t^2} - \sigma\mu \frac{\partial E_y}{\partial t} = 0$$

In order to find the solution of the wave equation, we assume that the electric field propagates in the x direction with the law

$$E_y = \tilde{E}_0 e^{i\omega t - \gamma x}$$

If we substitute the above expression in the wave equation, we get the equation for the complex amplitude of the electric field $\tilde{E}_0$

$$(\gamma^2 + \varepsilon\mu\omega^2 - i\omega\mu\sigma)\tilde{E}_0 e^{i\omega t - \gamma x} = 0$$

An analogous equation holds for H_z. In order to have non-zero electric field, the term inside the parentheses must be zero. If $\sigma \gg \omega\varepsilon$ this reduces to

$$\gamma \cong (1+i)\sqrt{\frac{\sigma\mu\omega}{2}}$$

Under such a condition we say that the material behaves like a conductor. Since γ has a real part, fields propagating in the material are attenuated. The attenuation constant, measured in meters, is called skin depth δ_w:

$$\delta_w \cong \frac{1}{\Re(\gamma)} = \sqrt{\frac{2}{\omega\sigma\mu}}$$

The skin depth depends on the material properties and the frequency. Time varying fields pass through the conductor wall if the skin depth is larger than the wall thickness. This happens at relatively low frequency when δ_w is large, while at higher frequencies, for a good conductor, the skin depth is very small and much lower than the wall thickness, so that we can consider that both electric and magnetic fields vanish inside the wall. In this condition, the electric field lines are perpendicular to the wall surface, as in the static case, while the magnetic field lines are tangent to the wall. As a consequence, in order to obtain the electric field which is time varying close to a good conductor, we can still use the method of the images, while for the magnetic field it is easy to see that we can use the method shown in Fig. A2, by changing the direction of the image current.

Copper, for example, has a skin depth of

$$\delta_w \cong \frac{6.66}{\sqrt{f}}(cm)$$

If we assume a beam pipe 2 mm thick, we find that fields pass through the wall up to frequencies of 1 kHz.

APPENDIX 2. LONGITUDINAL FORCES

In order to derive the relationship between the longitudinal and transverse forces inside a beam, let us consider the case of cylindrical symmetry and ultra-relativistic bunches. We know from Faraday's law of induction that a varying magnetic field produces a rotational electric field:

$$\oint \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\int_S \vec{B}\cdot\hat{n}\,dS$$

In order to obtain the longitudinal electric field, we choose, as path for the circulation, a rectangle going through the beam pipe (a cylinder of radius b) and the beam, parallel to the z axis and with radius a, as shown in Fig. A4. For a generic position $r<a$, and by considering Δz small enough so that we can consider the electric field constant, we have:

$$E_z(r,z)\Delta z + \int_r^b E_r(r,z+\Delta z)dr - E_z(b,z)\Delta z - \int_r^b E_r(r,z)dr = -\Delta z\frac{\partial}{\partial t}\int_r^b B_\phi(r)dr$$

We now write $E_r(r,z+\Delta z) - E_r(r,z) = \frac{\partial E_r(r,z)}{\partial z}\Delta z$ so that from the above equation we get

$$E_z(r,z) = E_z(b,z) - \int_r^b \left[\frac{\partial E_r(r,z)}{\partial z} + \frac{\partial B_\phi(r,z)}{\partial t}\right]dr$$

By considering that $z = -\beta ct$, we can also write

$$E_z(r,z) = E_z(b,z) - \frac{\partial}{\partial z}\int_r^b \left[E_r(r,z) - \beta c B_\phi(r,z)\right]dr$$

Since the transverse electric field and the azimuthal magnetic field are related by $B_\phi = \frac{\beta}{c}E_r$, we finally obtain

$$E_z(r,z) = E_z(b,z) - \left(1-\beta^2\right)\frac{\partial}{\partial z}\int_r^b E_r(r,z)dr$$

Note that for perfectly conducting walls we have $E_z(b,z) = 0$, so that

$$E_z(r,z) = -\frac{1}{\gamma^2}\frac{\partial}{\partial z}\int_r^b E_r(r,z)dr$$

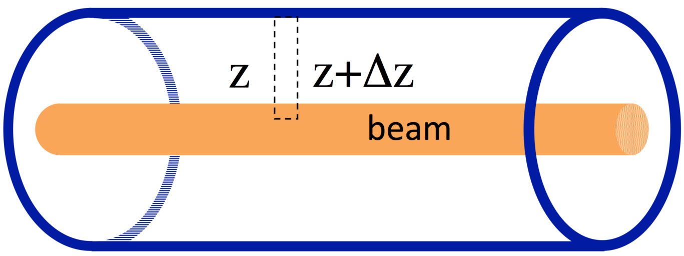


Fig. A4: Geometry for obtaining the longitudinal electric field due to space charge.

APPENDIX 3. DRIVEN OSCILLATORS

Consider a harmonic oscillator with natural frequency ω, with an external excitation at frequency Ω:

$$\ddot{x} + \omega^2 x = A\cos(\Omega t)$$

General solution:

$$x(t) = x^{free}(t) + x^{driven}(t)$$

$$\cos(\Omega t) \Rightarrow e^{i\Omega t}$$

$$x^{free}(t) = \tilde{x}_m^f e^{i\omega t}$$

$$x^{driven}(t) = \tilde{x}_m^d e^{i\Omega t}$$

The driven solution (steady state) is found by direct substitution in the differential equation:

$$(\omega^2 - \Omega^2)\tilde{x}_m^d e^{i\Omega t} = A e^{i\Omega t}$$

$$x^{driven}(t) = \frac{A}{(\omega^2 - \Omega^2)} e^{i\Omega t}$$

The general solution has to satisfy the initial condition at t=0. In our case we assume that the oscillator is at rest for t=0:

$$x^{free}(t=0) = -x^{driven}(t=0)$$

$$\tilde{x}_m^f = -\frac{A}{\omega^2 - \Omega^2}$$

thus we get:

$$x(t) = \frac{A}{\omega^2 - \Omega^2}\left[e^{i\Omega t} - e^{i\omega t}\right]$$

taking only the real part:

$$x(t) = \frac{A}{\omega^2 - \Omega^2}\left[\cos(\Omega t) - \cos(\omega t)\right]$$

This expression is suitable for deriving the response of the oscillator driven at resonance or at a frequency very close:

$$\omega = \Omega + \delta, \quad \delta \to 0$$

$$\overline{\omega} = (\omega + \Omega)/2; \; \omega = \overline{\omega} + \delta/2, \; \Omega = \overline{\omega} - \delta/2$$

$$x(t) = \frac{A}{2\overline{\omega}\delta}\ \Big\{ \left[\cos(\overline{\omega}t)\cos(\delta t/2) + \sin(\overline{\omega}t)\sin(\delta t/2)\right] +$$

$$- \ \left[\cos(\overline{\omega}t)\cos(\delta t/2) + \sin(\overline{\omega}t)\sin(\delta t/2)\right] \Big\}$$

$$x(t) = \frac{A}{\overline{\omega}\delta}\ \sin(\overline{\omega}t)\sin\left(\frac{\delta t}{2}\right) \equiv \frac{At}{2\overline{\omega}}\sin(\overline{\omega}t)\frac{\sin\left(\frac{\delta t}{2}\right)}{\frac{\delta t}{2}}$$

$$\lim_{\delta \to 0} x(t) = \frac{At}{2\overline{\omega}}\sin(\overline{\omega}t)$$