Particle Optics – part II

N.Biancacci

With many thanks to J. De Conto, A.Latina, P.Lebrun and E.Mètral for all the input in the preparation!

For any question (also later on) e-mail: nicolo.biancacci@cern.ch

Outline of part II

- Recall of special relativity

- Beam optics
  - Lorentz force
  - Particle in a magnetic field (cartesian reference system)
  - Beam rigidity
  - Particle in a magnetic field (curvilinear reference system)
  - Dipole and quadrupole fields and their effect on particle motion
  - Solution of Hill's equation with and without dispersion
  - Transport matrix for simple accelerator elements
    - Drift
    - Quadrupole (thick vs thin)
    - Dipole (sector magnet)
  - Examples of optical systems and their treatment: spectrometer
    - Sector magnet
    - Edge effects

Recall of special relativity

Short recap, more details given already in H.Henke's lectures!

juas

JUAS - TIMETABLE 2020 - WEEK 1

Schedule 2020	Monday Jan 13	Tuesday Jan 14	Wednesday Jan 15	Thursday Jan 16	Friday Jan 17
:00		Relativity *H. Henke*	Electro-magnetism *H. Henke*	Intro. to Accelerator Design *Ph. Bryant*	Intro. to the Mini-Workshop *Ph. Bryant*
10:00		Coffee Break	Coffee Break	Coffee Break	Coffee Break
10:15		Relativity *H. Henke*	Electro-magnetism *H. Henke*	Intro. to Accelerator Design *Ph. Bryant*	Intro. to the Mini-Workshop *Ph. Bryant*
11:15	12:00 OFFICIAL OPENING (welcome & building visit)	Relativity *H. Henke*	Electro-magnetism *H. Henke*	Intro. to Accelerator Design *Ph. Bryant*	*Bus leaves at 11:15 from JUAS (Lunch at CERN, R3, offered by ESI)*
12:15	13:00 WELCOME LUNCH	BREAK	BREAK	BREAK	13:30 Visit of LHC Magnets Test Hall *M. Bajko*
14:00	14:00 Presentation of JUAS & Introduction of students *P. Lebrun*	Relativity *H. Henke*	Electro-magnetism *H. Henke*	Intro. to Accelerator Design *Ph. Bryant*	15:00 Introduction to CERN & its Accelerator Network Seminar - *R. Alemany*
15:00	History of particle accelerators Seminar *V. Vaccaro*	Particle optics N. Biancacci	Particle optics N. Biancacci	Intro. to Accelerator Design *Ph. Bryant*	16:30 Visit of CERN Control Center *R. Alemany* Bus leaves at 17:30 from CERN
16:00		Coffee Break	Coffee Break	Coffee Break	
16:15	CHECK-IN AT THE RESIDENCE & SHOPPING FOR GROCERIES	Particle optics N. Biancacci	Particle optics N. Biancacci	Intro. to Accelerator Design *Ph. Bryant*	
17:15		Particle optics N. Biancacci			
18:15		AFTER WORK AT ESI			

Recall of special relativity

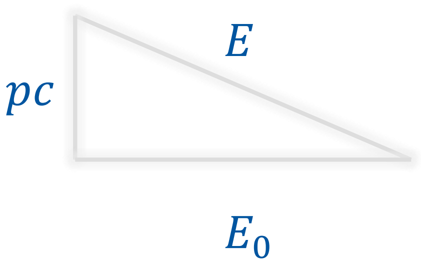
The energy-momentum relation is defined in special relativity as

$$E^2 = E_0^2 + (pc)^2$$

with E total energy of a particle, $p = mv$ the particle momentum, $E_0 = m_0 c^2$ its rest-energy.

The relativistic mass of a particle is given by $m = \gamma m_0$ from which the momentum can be expressed as $p = \gamma m_0 \beta c$ defining $\beta = {}^{v}/_{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

Substituting we get



$$E^2 = E_0^2 + (pc)^2 = E_0^2(1 + \beta^2\gamma^2) = \gamma^2 E_0^2$$

which means $E = \gamma E_0$

Recall of special relativity

For a particle at rest ($v = 0$):

$$E = \gamma E_0 = E_0$$

For a particle at low velocity ($v \ll c$, $\gamma \simeq 1$) we have:

$$E = \sqrt{E_0^2 + (pc)^2} = E_0 \sqrt{1 + \left(\frac{p}{m_0 c}\right)^2}$$

Expanding the square root in Taylor series at 2nd order:

$$\sqrt{1 + x^2} \simeq 1 + x^2/2$$

$$E = E_0 + \frac{1}{2} E_0 \left(\frac{p}{m_0 c}\right)^2 = E_0 + \frac{1}{2} \beta^2 \gamma^2 E_0 \simeq E_0 + \frac{1}{2} m_0 v^2$$

which reduces to the classical $E = E_0 + T$ with kinetic energy $T = 1/2\, m_0 v^2$.

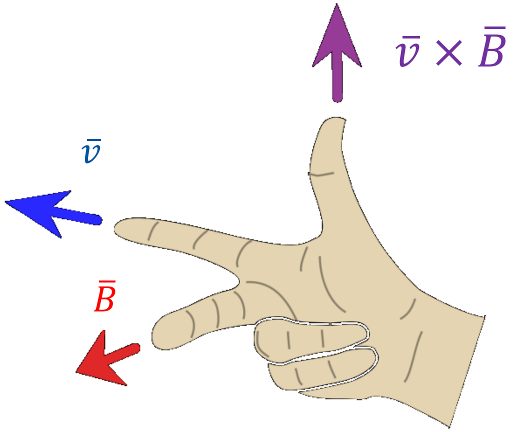
Lorentz force

- To guide charged particles we need to use electric or magnetic fields, i.e. apply the Lorentz force:

  $$\bar{F} = q(\bar{E} + \bar{v} \times \bar{B})$$

  with q charge of the particle, $\bar{v}$ its velocity, and $\bar{E}, \bar{B} = \mu\bar{H}$ are applied external electric and magnetic fields.

- $\bar{E}$ field accelerates/bends towards its same direction.
- $\bar{B}$ field bends only following the right hand rule*!



Source: wikipedia

In general solve $\bar{v} \times \bar{B} = \begin{vmatrix} \hat{x}_0 & \hat{y}_0 & \hat{z}_0 \\ v_x & v_y & v_z \\ B_x & B_y & B_z \end{vmatrix}$

* Also known as cavaturaccio’s rule in Italy (only in happy new year time)

Electric vs Magnetic force

Both transverse electric and magnetic fields can bend… what is better?

- At $v \simeq c$ and for typical values of $\bar{B} = 1T$ (=1Vs/m^2), for normal conducting magnets, determines a force of $3e8$ N/q.
- Same is achievable with 300 MV/m far beyond standard RF cavities gradients.

Therefore:

- High energy machines ($v \simeq c$) → magnetic bending/focusing
- Low energy machines ($v \ll c$) → also electric bending/focusing

We will now on concentrate mainly on magnetic focusing!

Put an eye to Maxwell's equations

The guiding magnets/electrodes are generally flat functions ($\frac{d}{dt} = 0 \rightarrow \omega = 0$).
Maxwell equations refer now to static fields and are decoupled!

Gauss' law $\nabla \cdot \bar{E} = \rho / \varepsilon$

Gauss' law for magnetism $\nabla \cdot \bar{H} = 0$

Faraday's law $\nabla \times \bar{E} = \cancel{-j\omega\mu\bar{H}} = 0$

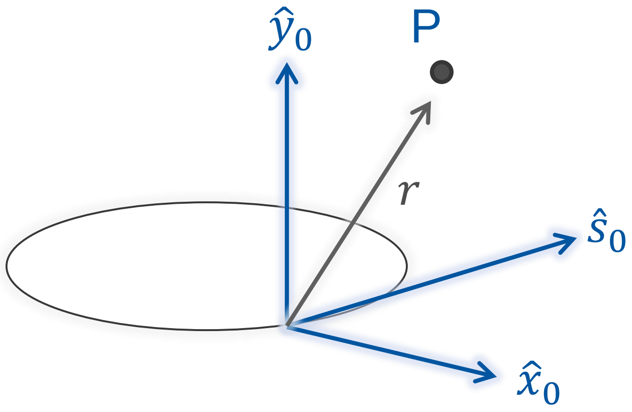
Ampère-Maxwell law $\nabla \times \bar{H} = \cancel{j\omega\varepsilon\bar{E}} + \bar{J}$ ← This can be the current flowing into the coil of a magnet

NB: This is false if you think to the beam as a point-like current source…. You will see this in the space-charge course of M.Migliorati on WEEK4.

Particle in a magnetic field

We want to study the trajectory of a particle in a magnetic field B.

Let's consider a fixed Cartesian reference system and consider a particle at location P = (y, x, s) moving with velocity v.



Particle in a magnetic field

$\dot{v} = (\dot{y}, \dot{x}, \dot{s})$ where *dot* derivative is with respect to time ($\dot{y} = dy/dt$).

As $\bar{B}$ is constant and only over y direction, the particle feels a Lorentz force of:

$$\begin{aligned}\overline{F_l} = q\bar{v} \times \bar{B} = \ & \hat{y}_0\,(v_x\,B_s\ - v_s\,B_x) \\ & + \hat{x}_0\left(v_y\,B_s\ - v_s\,B_y\right) \\ & + \hat{s}_0\left(v_y\,B_x\ - v_x\,B_y\right)\end{aligned}$$

$$\bar{v} \times \bar{B} = \begin{vmatrix} \hat{y}_0 & \hat{x}_0 & \hat{s}_0 \\ v_y & v_x & v_s \\ B_y & B_x & B_s \end{vmatrix}$$

The momentum $\bar{p} = m\bar{v}$ of the particle will change as:

$$\dot{\bar{p}} = m\,\dot{\bar{v}} = \overline{F_l}$$

Comparing the two expressions we have:

$$\dot{v}_y = \frac{q}{m}(v_x\,B_s - v_s\,B_x)$$

$$\dot{v}_x = \frac{q}{m}\left(v_y\,B_s\ - v_s\,B_y\right)$$

$$\dot{v}_s = \frac{q}{m}\left(v_y\,B_x\ - v_x\,B_y\right)$$

Particle in a magnetic field

We are interested to know the trajectory of the particle, i.e. how the transverse position changes along its motion, namely $x' = dx/ds$ and $y' = dy/ds$.

Changing the differentiation we have:

$$\dot{y} = \frac{dy}{ds}\frac{ds}{dt} = y'\dot{s} \qquad \ddot{y} = y''\dot{s}^2 + y'\ddot{s} \qquad \ddot{s} = \dot{v}_s = \frac{q}{m}\left(v_y\, B_x - v_x\, B_y\right)$$

$$\dot{x} = \frac{dx}{ds}\frac{ds}{dt} = x'\dot{s} \qquad \ddot{x} = x''\dot{s}^2 + x'\ddot{s} \qquad v^2 = \dot{s}^2(1 + y'^2 + x'^2)$$

Substituting we have:

$$\ddot{y} = \frac{q}{m}(\dot{x}B_s - \dot{s}B_x)$$

$$y''\dot{s}^2 + y'\ddot{s} = \frac{q}{m}(x'\dot{s}B_s - \dot{s}B_x)$$

$$\boxed{y'' = \frac{q}{m}\frac{1}{\dot{s}}\left(x'B_s - (1 + y'^2)B_x + y'x'\, B_y\right)}$$

Particle in a magnetic field

In analogous way we have the x''.

$$\boxed{y'' = \frac{q}{m}\frac{1}{\dot{s}}\left(x'B_s - (1 + y'^2)B_x + y'x'\,B_y\right)}$$

$$\boxed{x'' = -\frac{q}{m}\frac{1}{\dot{s}}\left(y'B_s - (1 + x'^2)B_y + y'x'\,B_x\right)}$$

Let us consider the simple case of a vertical B_y field only:

$$y'' = \frac{q}{m}\frac{1}{\dot{s}}\left(y'x'\,B_y\right) \qquad x'' = \frac{q}{m}\frac{1}{\dot{s}}(1 + x'^2)B_y$$

Looking at horizontal plane and replacing $\dot{s}$ we have:

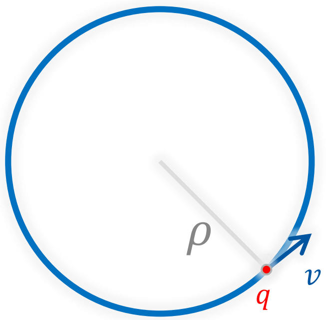
$$\boxed{\frac{x''}{(1 + x'^2)^{3/2}} = \frac{q}{m}\frac{1}{v}B_y = \frac{1}{\rho}}$$

Which describes circular trajectories of radius ρ.
(see Appendix for derivation)

Particle in constant B_y

We could have also derived it simply equating centripetal force to Lorentz force:

$$\frac{mv^2}{\rho} = q\ v\ B_y$$



Rearranging we have:

$$\boxed{\frac{p}{q} = B_y\rho}$$

where $B_y\rho$ (or simply $B\rho$) is called *beam rigidity* as it reflects the stiffness of the circulating beam under magnetic guiding forces.

For an ion X_A^Q, $m \simeq Am_u$ with m_u mass of the nucleon and $q = Qe$ with Q charge state of the ion.

Beam rigidity

The higher the momentum the higher the magnetic field to keep the beam on the same radius of curvature.

If higher momentum is wanted within technological By limits, then a larger circumference is needed -> staged accelerators chains.

We can make explicit the radius and derive a common handy formulation:

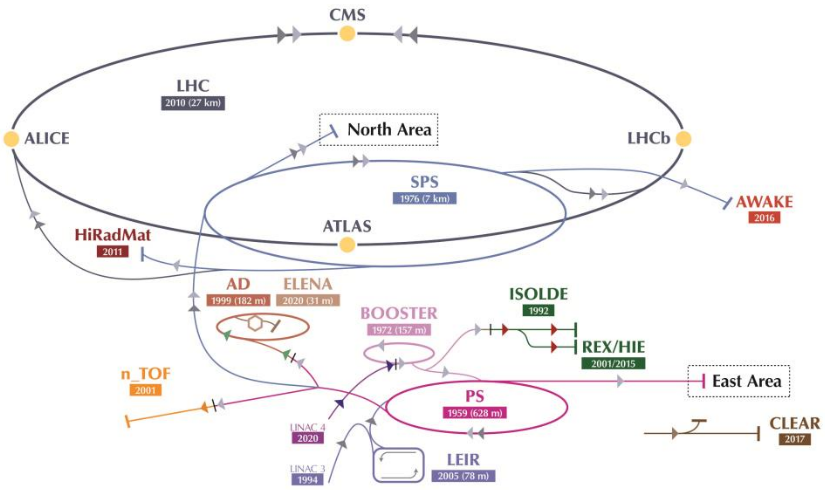
$$\frac{1}{\rho}[m^{-1}] = 0.2998\,\frac{Q}{A}\,\frac{B[T]}{p_u[\frac{GeV}{c}/n]}$$

For more details and examples see also A.Latina (week 2) and E.Métral (week 4) lectures on transverse and longitudinal beam dynamics!

Following the particle

So far we have derived the trajectory of a particle in a simple Cartesian system, which is ok to describe trajectory in a simple ring for example.

But the reality is much more complex…. Just have a look at the CERN accelerator complex to have an idea!



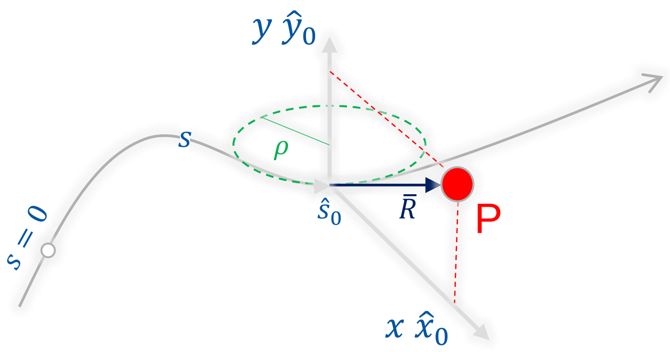
Better to have a reference system which is following the beam, i.e. a curvilinear reference system.

Particle in a magnetic field

Suppose a particle moves on a plane following a reference trajectory. We define a right hand reference system as $\hat{y}_0, \hat{x}_0, \hat{s}_0$.

First let's start with the axis:

$$\dot{\hat{x}}_0 = \frac{\dot{s}}{\rho}\hat{s}_0,\ \dot{\hat{s}}_0 = -\frac{\dot{s}}{\rho}\hat{x}_0,\ \dot{\hat{y}}_0 = 0$$



where $\dot{s}$ is the velocity of the particle projection on the reference trajectory.

$$\bar{R} = y\hat{y}_0 + x\hat{x}_0$$

$$\bar{v} = \dot{\bar{R}} + \dot{s}\hat{s}_0 = \dot{y}\hat{y}_0 + \dot{x}\hat{x}_0 + \dot{s}\left(1 + \frac{x}{\rho}\right)\hat{s}_0$$

$$\dot{\bar{v}} = \ddot{y}\hat{y}_0 + \left(\ddot{x} - \frac{\dot{s}^2}{\rho}\left(1 + \frac{x}{\rho}\right)\right)\hat{x}_0 + \left(\frac{2\dot{x}\dot{s}}{\rho} + \ddot{s}\left(1 + \frac{x}{\rho}\right)\right)\hat{s}_0$$

Particle in a magnetic field

As before:

$$\dot{y} = \frac{dy}{ds}\frac{ds}{dt} = y'\dot{s} \qquad \ddot{y} = y''\dot{s}^2 + y'\ddot{s} \qquad \ddot{s} = \dot{v}_s = \frac{q}{m}\left(v_y B_x - v_x B_y\right)$$

$$\dot{x} = \frac{dx}{ds}\frac{ds}{dt} = x'\dot{s} \qquad \ddot{x} = x''\dot{s}^2 + x'\ddot{s} \qquad v^2 = \dot{s}^2(1 + y'^2 + x'^2)$$

Following the same steps done for the Cartesian case we have:

$$\boxed{y'' + \frac{\ddot{s}}{\dot{s}^2}y' = \frac{q}{m}\frac{1}{\dot{s}}\left(x'B_s - \left(1 + \frac{x}{\rho}\right)B_x\right)}$$

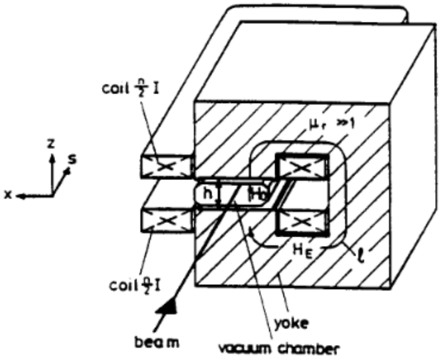
$$\boxed{x'' + \frac{\ddot{s}}{\dot{s}^2}x' - \frac{1}{\rho}\left(1 + \frac{x}{\rho}\right) = -\frac{q}{m}\frac{1}{\dot{s}}\left(y'B_s - \left(1 + \frac{x}{\rho}\right)B_y\right)}$$

Let’s specialize these equations to some common magnets in accelerators.

Dipole

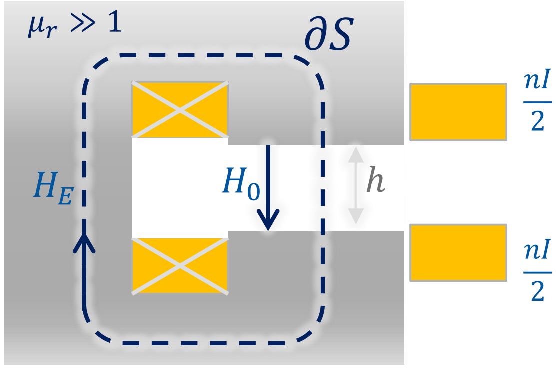
A *dipole* is a type of magnet that generates constant magnetic field across the poles.

The magnetic field is confined in the iron core and generated by the current flowing in the coils.



Courtesy J.Rossbach

The field across the gap can be computed using Ampère low on the surface S:



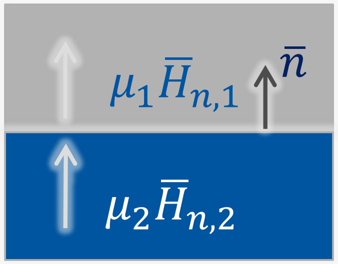
$$\oint_{\partial S} \bar{H} \cdot dl = \int_S \bar{J} \cdot dS$$

$$H_E l + H_0 h = nI$$

Recalling the boundary condition for the magnetic field

$$\mu_r \mu_0 H_E = \mu_0 H_0$$

Iron based magnet cores have μ_r of several thousands



$$H_E = \frac{1}{\mu_r} H_0 \sim 0$$

We have therefore:

$$H_0 h = nI \qquad \text{or} \qquad B_0 = \frac{\mu_0 nI}{h}$$

And we define the *bending strength* as: $\boxed{k_0 = \frac{qB_0}{p} = \frac{1}{\rho} \; [m^{-1}]}$

Quadrupole

In a similar way we can compute the field for a quadrupole:

$$\oint_{\partial S} \overline{H} \cdot dl = \int_S \bar{J} \cdot dS$$

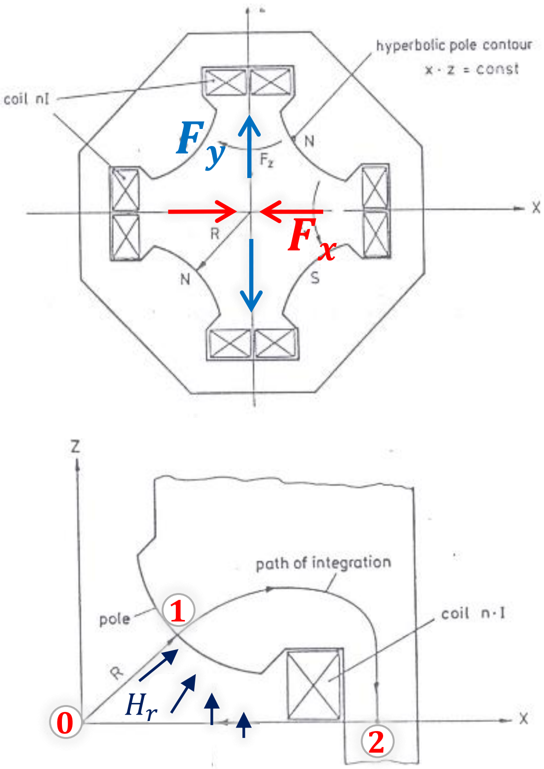
$$\int_0^1 H_r R dr = \int_0^R \frac{1}{\mu_0} g r dr = \frac{g R^2}{2\mu_0} = nI$$

Since $\mu_{rE} \gg 1$ we neglect the contribution of H_E and the $2 \rightarrow 0$ as the field is perpendicular to the path.

$$g = \frac{2\mu_0 n I}{R^2}$$

And we define the *focusing strength* as:

$$\boxed{k_1 = \frac{qg}{p} [m^{-2}]}$$

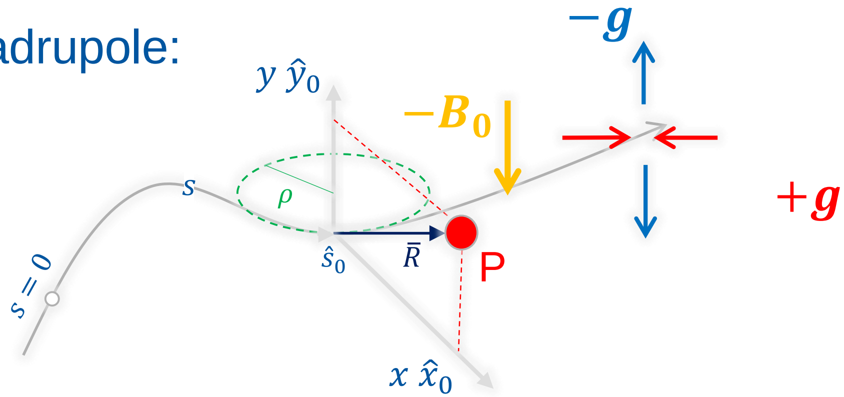


Courtesy J.Rossbach

NB: a quadrupole focuses in one plane and defocuses in the other!

Effect on particle motion

Consider a particle between dipole and quadrupole:



$$B_y = -B_0 - gx$$
$$B_x = +gy$$
$$B_s = 0$$

Let's consider only the linear terms of the particle motion:

$$\boxed{x'' + \frac{\ddot{s}}{\dot{s}^2}x' - \frac{1}{\rho}\left(1 + \frac{x}{\rho}\right) = \frac{q}{p}\frac{v}{\dot{s}}\left(1 + \frac{x}{\rho}\right)(-B_0 - gx)}$$

$\ddot{s} \simeq 0$ (particle do not accelerate)

$\frac{v}{\dot{s}} \simeq (1 + x/\rho)$ (small angles with respect to the trajectory) → Recall the paraxial approximation!

$$x'' - \frac{1}{\rho}\left(1 + \frac{x}{\rho}\right) = \frac{q}{p}\left(1 + \frac{x}{\rho}\right)^2 (-B_0 - gx)$$

Effect on particle motion

$$x'' - \frac{1}{\rho}\left(1 + \frac{x}{\rho}\right) = \frac{q}{p}\left(1 + \frac{x}{\rho}\right)^2 (-B_0 - gx)$$

$p \simeq p_0 + \delta p$ (small particle momentum deviations)

$$\frac{1}{p} \simeq \frac{1}{p_0} - \frac{\delta p}{p_0^2} = \frac{1}{p_0}\left(1 - \frac{\delta p}{p_0}\right)$$

$$x'' - \frac{1}{\rho}\left(1 + \frac{x}{\rho}\right) = \frac{q}{p_0}\left(1 - \frac{\delta p}{p_0}\right)\left(1 + \frac{x}{\rho}\right)^2 (-B_0 - gx)$$

Expanding and keeping only the linear terms we have:

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = \left(1 - \frac{\delta p}{p_0}\right)\left(1 + \frac{x}{\rho}\right)^2 \left(-\frac{1}{\rho} - k_1 x\right) = \frac{1}{\rho}\frac{\delta p}{p_0} - k_1 x - \frac{1}{\rho}$$

Effect on particle motion

Finally we have:

$$x'' + \left(k_1 + \frac{1}{\rho^2}\right)x = \frac{1}{\rho}\frac{\delta p}{p_0}$$

$$\boxed{x'' + k_x x = \frac{1}{\rho}\frac{\delta p}{p_0}}$$

with $k_x = k_1 + \frac{1}{\rho^2}$

For the vertical plane we have:

$$y'' = \frac{q}{p_0}\left(1 - \frac{\delta p}{p_0}\right)\left(1 + \frac{x}{\rho}\right)^2 gy = \frac{qgy}{p_0} = k_1 y$$

Which leads to:

$$\boxed{y'' - k_y y = 0}$$

with $k_y = k_1$

This is called the Hill's equation

Solution of Hill's equation

In a generic trajectory, the focusing strength are function of the path variable s:

$$\boxed{x''(s) + k(s)x(s) = 0}$$

The general solution can be expressed as:

$$\boxed{x(s) = C(s)x_0 + S(s)x'_0}$$

$$\boxed{x'(s) = C'(s)x_0 + S'(s)x'_0}$$

$C(s)$ is solution of the homogeneous equation with $C(s_0) = 1$ and $S(s_0) = 0$.
$S(s)$ is solution of the homogeneous equation with $C'(s_0) = 0$ and $S'(s_0) = 1$.

For this choice, $C(s), S(s)$ are called *cosine-like, sine-like* functions or *principal trajectories.*

In a compact form we can write, for x and x':

$$\boxed{\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{bmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{bmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}}$$

Transfer map

Solution of Hill's equation with dispersion

$$\boxed{x''(s) + k(s)x(s) = \frac{1}{\rho}\frac{\delta p}{p_0}}$$

To the homogeneous Hill's equation we need to add a particular solution.

We introduce the dispersion $D(s)$ as:

$$\boxed{x(s) = C(s)x_0 + S(s)x_0' + D(s)\delta p/p}$$

$$\boxed{x'(s) = C'(s)x_0 + S'(s)x_0' + D'(s)\delta p/p}$$

We look for a solution of $D(s)$ with $D_0 = 0$ and $D_0' = 0$ with this shape:

$$D(s) = S(s)\int_0^s \frac{1}{\rho(\sigma)}C(\sigma)d\sigma - C(s)\int_0^s \frac{1}{\rho(\sigma)}S(\sigma)d\sigma$$

Let's check it verifies the Hill's equation:

Solution of Hill's equation with dispersion

$$D = S\int_0^s \frac{1}{\rho} C d\sigma - C\int_0^s \frac{1}{\rho} S d\sigma$$

$$D' = S'\int_0^s \frac{1}{\rho} C d\sigma + \frac{S'C}{\rho} - C'\int_0^s \frac{1}{\rho} S d\sigma - \frac{C'S}{\rho}$$

$$D'' = S''\int_0^s \frac{1}{\rho} C d\sigma - C''\int_0^s \frac{1}{\rho} S d\sigma + \frac{1}{\rho}(S'C - C'S)$$

How to compute the quantity $(S'C - C'S)$?

$$\frac{d}{ds}(S'C - C'S) = S''C - C''S = -kSC + kCS = 0$$ as S and C satisfy the Hill's equation

This means $(S'C - C'S)$ is a constant and to satisfy the initial conditions it is $= 1$

$$D'' = S''\int_0^s \frac{1}{\rho} C d\sigma - C''\int_0^s \frac{1}{\rho} S d\sigma + \frac{1}{\rho}$$

Solution of Hill's equation with dispersion

In turns we substitute D'' and D into Hill's equation to get:

$$D''(s)\frac{\delta p}{p} + k(s)D(s)\frac{\delta p}{p} = \frac{1}{\rho}\frac{\delta p}{p}$$

$$S''\int_0^s \frac{1}{\rho}C d\sigma - C''\int_0^s \frac{1}{\rho}S d\sigma + \frac{1}{\rho} + k\left(S\int_0^s \frac{1}{\rho}C d\sigma - C\int_0^s \frac{1}{\rho}S d\sigma\right) = \frac{1}{\rho}$$

$$\int_0^s \frac{1}{\rho}C d\sigma \underbrace{(S'' + kS)}_{=0} - \int_0^s \frac{1}{\rho}S d\sigma \underbrace{(C'' + kC)}_{=0} = 0$$

as S and C satisfy the Hill's equation

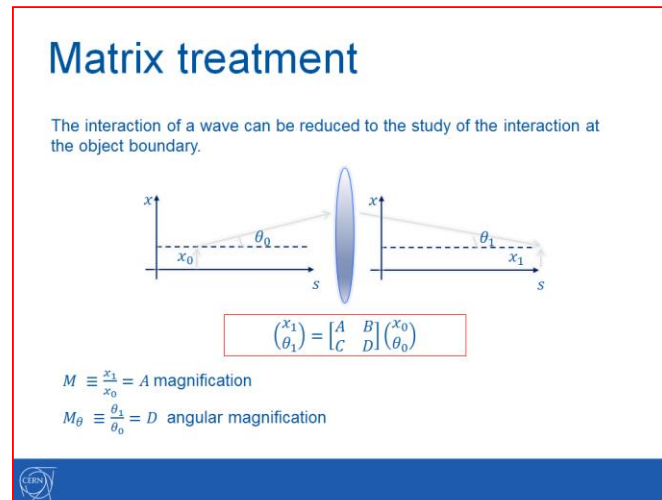
which verifies the validity of our assumption.

Solution of Hill’s equation with dispersion

In a compact form we can write, for x, x' and $\delta p/p$:

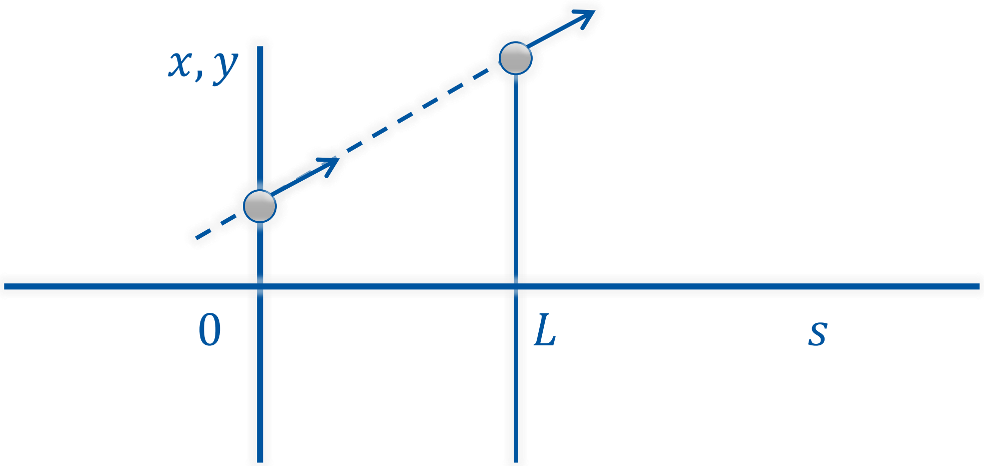
$$\begin{pmatrix} x \\ x' \\ \delta p/p \end{pmatrix} = \begin{bmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \delta p_0/p \end{pmatrix}$$

Nested in we recognize the ABCD matrix used in light optics, together with a dispersive term which accounts for off-momentum particles.



Particle in a drift

Let's consider a drift of length L. We have $\left(\frac{1}{\rho} = 0, K = 0\right)$



$$\begin{cases} x'' = 0 \\ x(s=0) = x_0 \\ x'(s=0) = x'_0 \end{cases} \rightarrow \begin{cases} x = x_0 + x'_0 L \\ x' = x'_0 \end{cases}$$

$$\rightarrow M_{x,y} = \begin{bmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$D(s) = D'(s) = 0$ as $\frac{1}{\rho} = 0$

Particle in a quadrupole

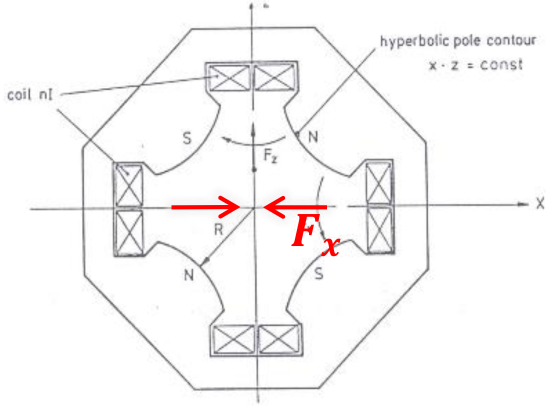
Let's consider a focusing quadrupole of length L: $\left(\frac{1}{\rho} = 0, \boldsymbol{k_x > 0}\right)$

$$\begin{cases} x''(s) + k_x x(s) = 0 \\ x(s=0) = x_0 \\ x'(s=0) = x'_0 \end{cases} \longrightarrow \begin{cases} x = x_0 \cos\left(\sqrt{k_x} L\right) + x'_0 \dfrac{1}{\sqrt{k_x}} \sin\left(\sqrt{k_x} L\right) \\ x' = -x_0 \sin\left(\sqrt{k_x} L\right) \sqrt{k_x} + x'_0 \cos\sqrt{k_x} L \end{cases}$$

$D(s) = D'(s) = 0$ as $\frac{1}{\rho} = 0$

which, in matrix form is:

$$\boldsymbol{M_x} = \begin{bmatrix} \cos\sqrt{k_x} L & \dfrac{1}{\sqrt{k_x}} \sin\sqrt{k_x} L & 0 \\ -\sqrt{k_x} \sin\sqrt{k_x} L & \cos\sqrt{k_x} L & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Particle in a quadrupole

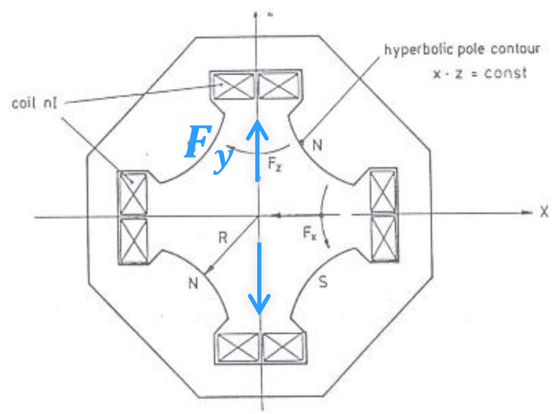
Let's consider a defocusing quadrupole of length L: $\left(\frac{1}{\rho} = 0, \boldsymbol{k_y < 0}\right)$

$$\begin{cases} y''(s) + k_y y(s) = 0 \\ y(s=0) = y_0 \\ y'(s=0) = y_0' \end{cases} \longrightarrow \begin{cases} y = y_0 \cosh\left(\sqrt{k_y L}\right) + y_0' \dfrac{1}{\sqrt{k_y}} \sinh\left(\sqrt{k_y L}\right) \\ y' = -y_0 \sinh\left(\sqrt{k_y L}\right)\sqrt{k_y} + y_0' \cosh\sqrt{k_y L} \end{cases}$$

$D(s) = D'(s) = 0$ as $\frac{1}{\rho} = 0$

which, in matrix form is:

$$\boldsymbol{M_y} = \begin{bmatrix} \cosh\sqrt{|k_y| L} & \dfrac{1}{\sqrt{|k_y|}} \sinh\sqrt{|k_y| L} & 0 \\ \sqrt{|K|} \sinh\sqrt{|k_y| L} & \cosh\sqrt{|k_y| L} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Particle in a thin quadrupole

If $L \to 0$ we can obtain the thin lens approximation:

$$\boxed{\boldsymbol{M_x} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{M_y} = \begin{bmatrix} 1 & 0 & 0 \\ +\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

where we define the focal length as $f = \lim_{L \to 0} \frac{1}{|K|L}$ with dimensions of $[m]$.

Thin quadrupole magnet $\longleftrightarrow$ thin magnetic lens

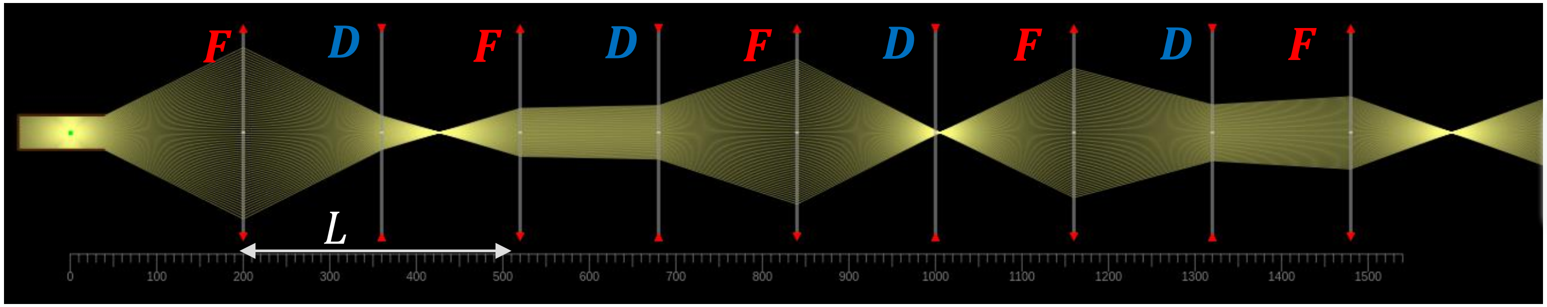
Like glass lenses in optics, charged particle focusing and defocusing lenses can be used to form images of an object (electron microscope), to transport a beam from one point to another, or to focus a beam onto a small target.

Particle in a FODO lattice

An arrangement of focusing defocusing thin quadrupole lenses allows to transport particles in a stable manner.

The principle is exactly the same as the one studied in light optics with the condition of having $L \leq 4f$ for stability.

You can look back at this picture, imagining the yellow trace to be a beam envelope: that's not the reality but quite close.



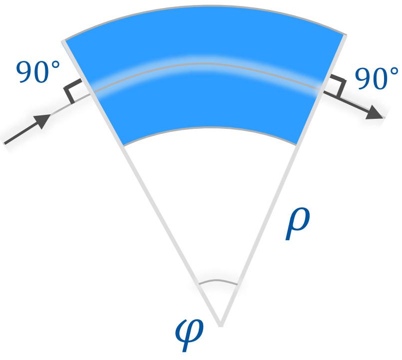
$L < 4f$

Particle in a sector magnet

A sector magnet allows a pure rotation of the particle, i.e. the particles enter/exit the magnet perpendicular to the input/output surfaces.

We have $\left(\frac{1}{\rho} \neq 0, K = 0\right)$.

$$M_x = \begin{bmatrix} \cos\varphi & \rho\sin\varphi & \boxed{\begin{matrix} \rho\,(1-\cos\varphi) \\ \sin\varphi \end{matrix}} \\ -\frac{1}{\rho}\sin\varphi & \cos\varphi & \\ 0 & 0 & 1 \end{bmatrix}$$



With $\varphi = L/\rho$.

- Note the weak focusing term $-\frac{1}{\rho}\sin\varphi = -\frac{L}{\rho^2}$
- In vertical plane can be considered as a simple drift of length L.
- Dispersion terms appear → spectrometry!

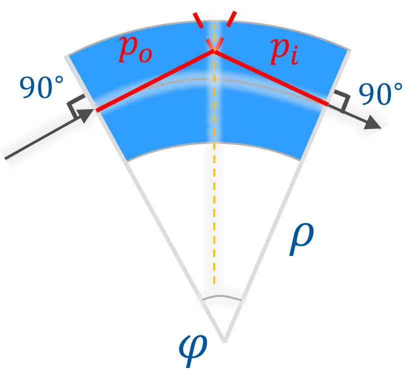
Sector magnet as a spectrometer

A sector magnet is the simplest spectrometer system thanks to the ability of differentiating particles with different momentum deviations*.

$$\begin{pmatrix} x \\ x' \\ \delta p/p \end{pmatrix} = \begin{bmatrix} \cos\varphi & \rho\sin\varphi & \rho\,(1-\cos\varphi) \\ -\frac{1}{\rho}\sin\varphi & \cos\varphi & \sin\varphi \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_0 \\ x_0' \\ \delta p_0/p \end{pmatrix}$$

To find the image made from the magnet we can use what we learnt from optics: we need to find effective focal length and principal planes:

$$\begin{cases} p_i = \dfrac{1-A}{C} \\ p_o = \dfrac{1-D}{C} \\ f_{eff} = -\dfrac{1}{C} \end{cases} \rightarrow \begin{cases} p_i = \rho\dfrac{\cos\varphi - 1}{\sin\varphi} = -\rho\tan\dfrac{\varphi}{2} \\ p_o = p_i \\ f_{eff} = \dfrac{\rho}{\sin\varphi} \end{cases}$$



*some authors report $1/2$ multiplying the dispersion matrix elements → this comes from $\frac{dp}{p} \sim \frac{1}{2}\frac{dK}{K}$ for low velocity ($\gamma \sim 1$)

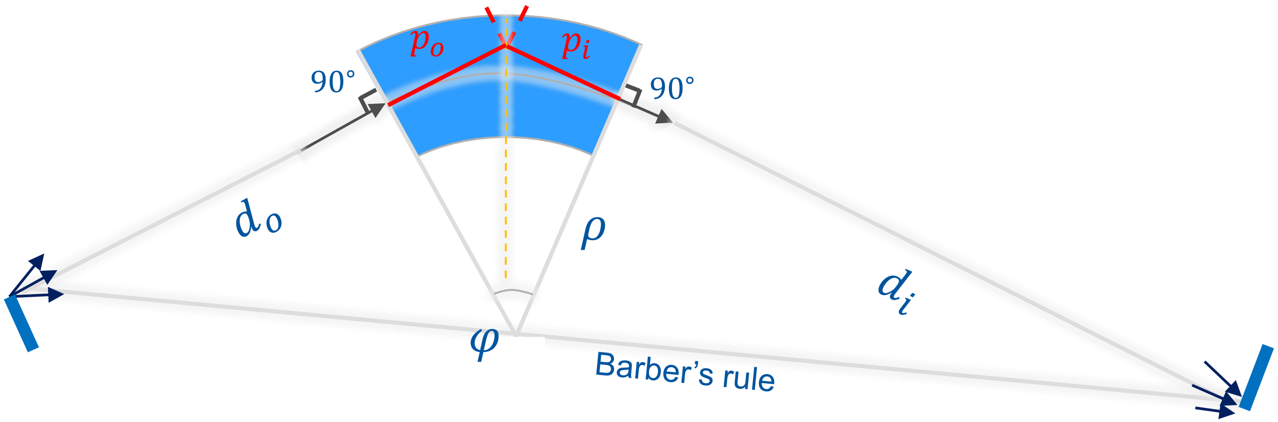
Sector magnet as a spectrometer

Next we apply the image relation to find the focusing point of an object placed in front of the spectrometer

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f_{eff}}$$

$$M = -\frac{d_i}{d_o}$$

$$M_\theta = -\frac{d_o}{d_i}$$



It can be shown (Barber's rule) that object, origin and image are on the same line.
To obtain the final transfer matrix we do:

This we knew it already

$$T = \begin{bmatrix} 1 & d_i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi & \rho\sin\varphi & \rho\,(1-\cos\varphi) \\ -\frac{1}{\rho}\sin\varphi & \cos\varphi & \sin\varphi \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d_o & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} M & 0 & \rho\,(1-\cos\varphi) + d_i\sin\varphi \\ -\frac{1}{f_{eff}} & M_\theta & \sin\varphi \\ 0 & 0 & 1 \end{bmatrix} \simeq \begin{bmatrix} M & 0 & d_i\sin\varphi \\ -\frac{1}{f_{eff}} & M_\theta & \sin\varphi \\ 0 & 0 & 1 \end{bmatrix}$$

$d_i \gg p_i$

Sector magnet as a spectrometer

The dispersion terms produce a spread in the output image. Let's consider a source with angular spread $\boldsymbol{\alpha}$, width $\boldsymbol{w}$ and natural velocity spread $\boldsymbol{\delta}$:

$$\begin{pmatrix} x \\ x' \\ \delta p/p \end{pmatrix} = \begin{bmatrix} M & 0 & d_i \sin\varphi \\ -\frac{1}{f_{eff}} & M_\theta & \sin\varphi \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} w \\ \alpha \\ \delta \end{pmatrix}$$

$$\frac{dp}{p} = \frac{dm}{m} + \frac{dv}{v} = \nu + \delta$$

The final spot size is given by: $\delta\,(d_i \sin\varphi) + w d_i/d_o$

A different mass would produce a different momentum error $\boldsymbol{\nu}$ which we would like to distinguish from the natural velocity spread of the beam. Equating to the spot size give us the spectrometer resolution:

$$\nu_{th}(d_i \sin\varphi) = \delta\,(d_i \sin\varphi) + \frac{w d_i}{d_o} \quad \rightarrow \quad \nu_{th} = \delta + \frac{w}{d_o(\sin\varphi)}$$

If $d_o = d_i = 2f_{eff}$ we have: $\boxed{\nu_{th} = \delta + \frac{w}{2\rho}}$

Mass resolution is:

- limited by beam momentum spread
- independent of angle φ
- can improve if ρ larger

Exercise

Consider a spectrometer made by a sector magnet of angle $\varphi = 90^\circ$ and 5π meters long. Consider a source of particles with a natural momentum spread of 0.1%.

Q1: If we place the source at $15\,\mathrm{m}$ from the entrance of the sector bend, where should the detector be located? Sketch it into a drawing.

Q2: If we require a spectrometer resolution of 0.2% what should be the size of the beam outgoing from the source?

Q3: What is the size of the detector opening in order to be 10 times above the resolution limit?

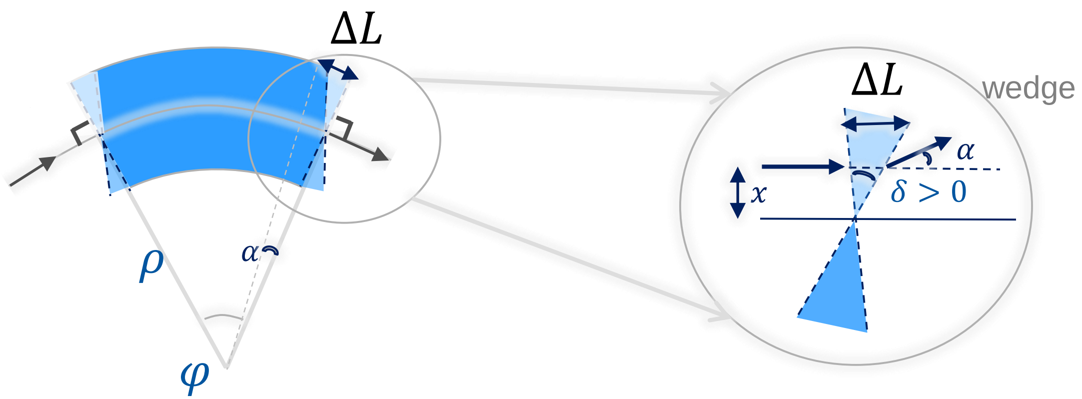
Q4: Make a drawing of the trajectory of 2 particles with reference mass and velocity (i.e. reference momentum) leaving the center of the source with zero and positive angle.

Q5: Same as Q4 but consider 2 particles with lower mass.

Edge effects

A sector magnet provides no focusing in y plane, while it is desirable to have focusing in both planes (reduce detector aperture, more intensity, etc..).

- A magnet with tilted entrance/exit faces can provide this.
- It can be seen as the superposition of a sector with a wedge of angle δ (suppose equal on both sides).



As the magnetic field is effectively less towards the outside, the particle will be less deflected.
Conversely it is larger towards the inside and the particle will experience additional bending.
Overall there is a defocusing effect in horizontal plane.

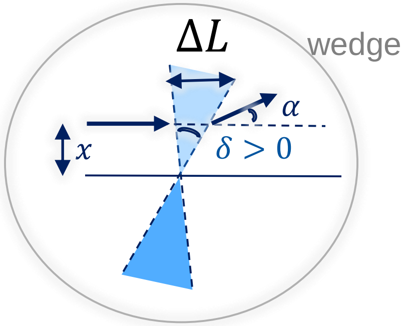
The change of path is

$$\Delta L = x \tan \delta$$

The angle acquired is $\alpha \simeq \frac{\Delta L}{\rho} = \frac{x \tan \delta}{\rho}$

Edge effects

Writing the transfer matrix of the wedge we have:



$$M_x^{wedge} = \begin{bmatrix} 1 & 0 & 0 \\ \dfrac{\tan\delta}{\rho} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Which is a **thin defocusing** lens of focal length $f_{x,eff} = -\rho/\tan\delta$

Since $\nabla \cdot B = 0$ at the wedge, an opposite magnetic field acts in the vertical plane → **focusing**!.

$$M_y^{wedge} = \begin{bmatrix} 1 & 0 & 0 \\ -\dfrac{\tan\delta}{\rho} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Edge effects

In the end the magnet is equivalent to a sector magnet followed/preceded by thin lenses of focal length $f = \rho/\tan\delta$, focusing in y and defocusing in x.

In x plane:

$$M_x = \begin{bmatrix} 1 & 0 & 0 \\ \frac{\tan\delta}{\rho} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\varphi & \rho\sin\varphi & \rho\,(1-\cos\varphi) \\ -\frac{1}{\rho}\sin\varphi & \cos\varphi & \sin\varphi \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ \frac{\tan\delta}{\rho} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

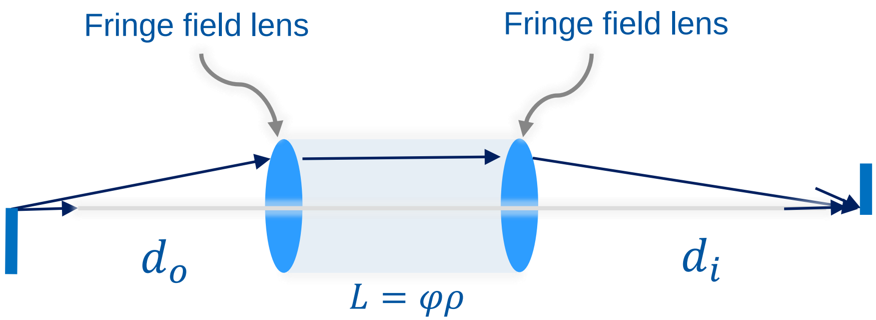
In y plane:

$$M_y = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{\tan\delta}{\rho} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ -\frac{\tan\delta}{\rho} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Edge effects

With this system it is possible to achieve stigmatic imaging, i.e. simultaneous focusing of both planes in the same position.

For the vertical plane:



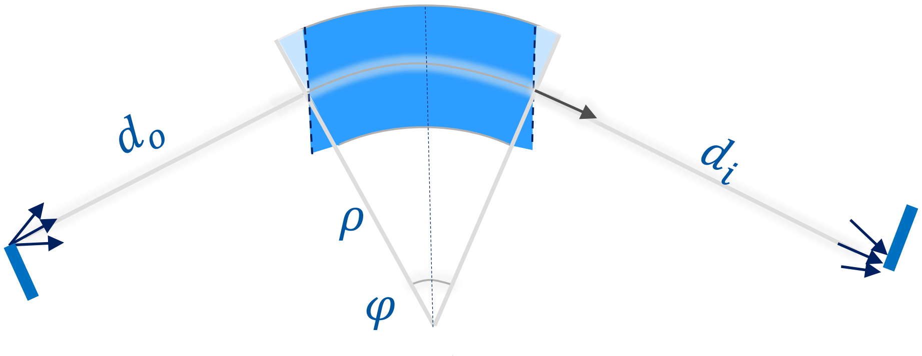
Placing the particle source at $d_o = f_{eff} = \frac{\rho}{\tan\delta}$ the particles will drift straight in the magnet (see practical drawing in slide 27, part I).

The second lens will then focus back in $d_i = f_{eff}$ where we can place the detector.

Edge effects

With this system it is possible to achieve stigmatic imaging, i.e. simultaneous focusing of both planes in the same position.

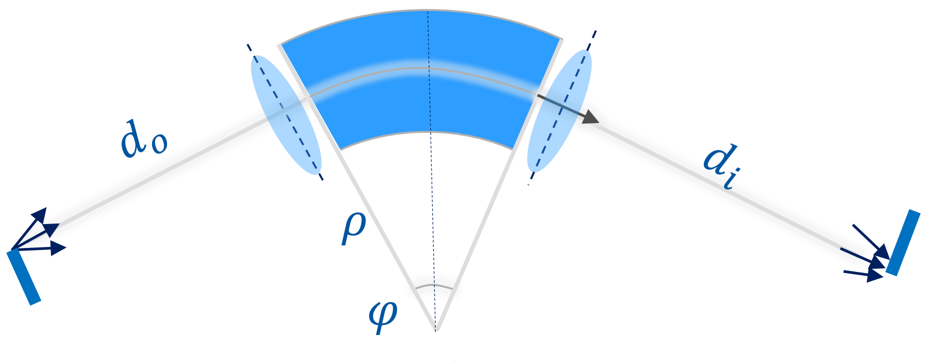
For the horizontal plane:



Edge effects

With this system it is possible to achieve stigmatic imaging, i.e. simultaneous focusing of both planes in the same position.

For the horizontal plane:



Exercise:
Show that for $\varphi = 90°$ and $\tan\delta = 0.5$ (i.e. $\delta \sim 26.6°$) the magnet focalizes at the same locations both vertical and horizontal planes.

Bibliography

[1] Justin Peatross, Michael Ware, “Physics of Light optics”, Brigham Young University, 2015

[2] M.Born, E.Wolf, “Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light”, Cambridge University Press, 13 October 1999.

[3] E. Wollnik, “Optics of Charged Particles”, Elsevier, 2012

[4] A. El-Kareh, “Electron Beams, Lenses, and Optics”, Elsevier, 2 dic 2012

[5] Light optics simulator webtool (really nice!) https://ricktu288.github.io/ray-optics/ also available as Chrome app.

[6] A.Chao, M.Tigner, F.Zimmermann, “Handbook Of Accelerator Physics And Engineering (2nd Edition)”, Sec. 2.2.1, World Scientific Publishing, 2013.

[7] J.Rossbach, Basic course on accelerator optics, CAS - CERN Accelerator School : 5th General Accelerator Physics Course, Jyväskylä, Finland, 7 - 18 Sep 1992, pp.17-88 (CERN-1994-001)

[8] P.Schmüser, “Basic course on accelerator optics”, CAS, Aarus, Denmark, 1986

[9] Steffen, K, “Basic course on accelerator optics”, CAS - CERN Accelerator School : General Accelerator Physics, v.1, Gif-sur-Yvette, France, 3 - 14 Sep 1984, pp.25-63 (CERN-1985-019-V-1)

[10] H.Liebel, “Applied Charged Particle Optics”, Springer, 2008

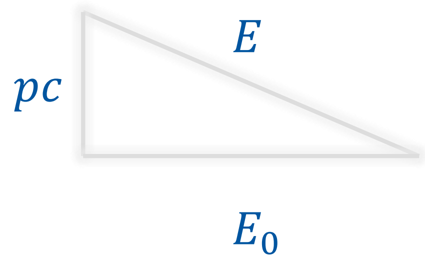
Appendix

Summary of relativistic relations

$$E_0 = m_0 c^2$$
$$E = E_0 + T$$
$$E = \gamma m_0 c^2$$

$$m = \gamma \cdot m_0$$
$$\beta = {}^{v}/_{c}$$
$$\gamma = \frac{E}{E_0} = \frac{E_0 + T}{E_0}$$

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}}$$
$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}$$



	cp	**T**	**E**	γ
$\beta =$	$\frac{1}{\sqrt{(\frac{E_0}{cp})^2+1}}$	$\sqrt{1 - \frac{1}{(1+\frac{T}{E_0})^2}}$	$\sqrt{1 - (\frac{E_0}{E})^2}$	$\sqrt{1 - \gamma^{-2}}$
$\mathbf{cp} =$	cp	$\sqrt{T(2E_0 + T)}$	$\sqrt{E^2 - E_0^2}$	$E_0\sqrt{\gamma^2 - 1}$
$E_0 =$	$\frac{cp}{\sqrt{\gamma^2-1}}$	$T/(\gamma - 1)$	$\sqrt{E^2 - c^2p^2}$	E/γ
$\mathbf{T} =$	$cp\sqrt{\frac{\gamma-1}{\gamma+1}}$	**T**	$E - E_0$	$E_0(\gamma - 1)$
$\gamma =$	$cp/E_0\beta$	$1 + T/E_0$	E/E_0	γ

	$\frac{d\beta}{\beta}$	$\frac{dp}{p}$	$\frac{dT}{T}$	$\frac{dE}{E} = \frac{d\gamma}{\gamma}$
$\frac{d\beta}{\beta} =$	$\frac{d\beta}{\beta}$	$\frac{1}{\gamma^2}\frac{dp}{p}$	$\frac{1}{\gamma(\gamma+1)}\frac{dT}{T}$	$\frac{1}{(\beta\gamma)^2}\frac{d\gamma}{\gamma}$
$\frac{dp}{p} =$	$\gamma^2\frac{d\beta}{\beta}$	$\frac{dp}{p}$	$[\gamma/(\gamma+1)]\frac{dT}{T}$	$\frac{1}{\beta^2}\frac{d\gamma}{\gamma}$
$\frac{dT}{T} =$	$\gamma(\gamma+1)\frac{d\beta}{\beta}$	$(1 + \frac{1}{\gamma})\frac{dp}{p}$	$\frac{dT}{T}$	$\frac{\gamma}{(\gamma-1)}\frac{d\gamma}{\gamma}$
$\frac{dE}{E} =$	$(\beta\gamma)^2\frac{d\beta}{\beta}$	$\beta^2\frac{dp}{p}$	$(1 - \frac{1}{\gamma})\frac{dT}{T}$	$\frac{d\gamma}{\gamma}$
$\frac{d\gamma}{\gamma} =$	$(\gamma^2 - 1)\frac{d\beta}{\beta}$	$\frac{dp}{p} - \frac{d\beta}{\beta}$	$(1 - \frac{1}{\gamma})\frac{dT}{T}$	$\frac{d\gamma}{\gamma}$

W.Herr, *special relativity*, CAS 1/06/2016, Hamburg, Germany

Circle differential equation

Start from:

$$(x - c_1)^2 + (s - c_2)^2 = \rho^2$$

Differential twice w.r.t. s:

$$(x - c_1)x' + (s - c_2) = 0 \qquad x'^2 + (x - c_1)x'' + 1 = 0$$

$$x'^2 + (x - c_1)x'' + 1 = 0$$

From which we find c_1 and $s - c_2$:

$$x - c_1 = -\frac{1 + x'^2}{x''} \qquad s - c_2 = x' \frac{1 + x'^2}{x''}$$

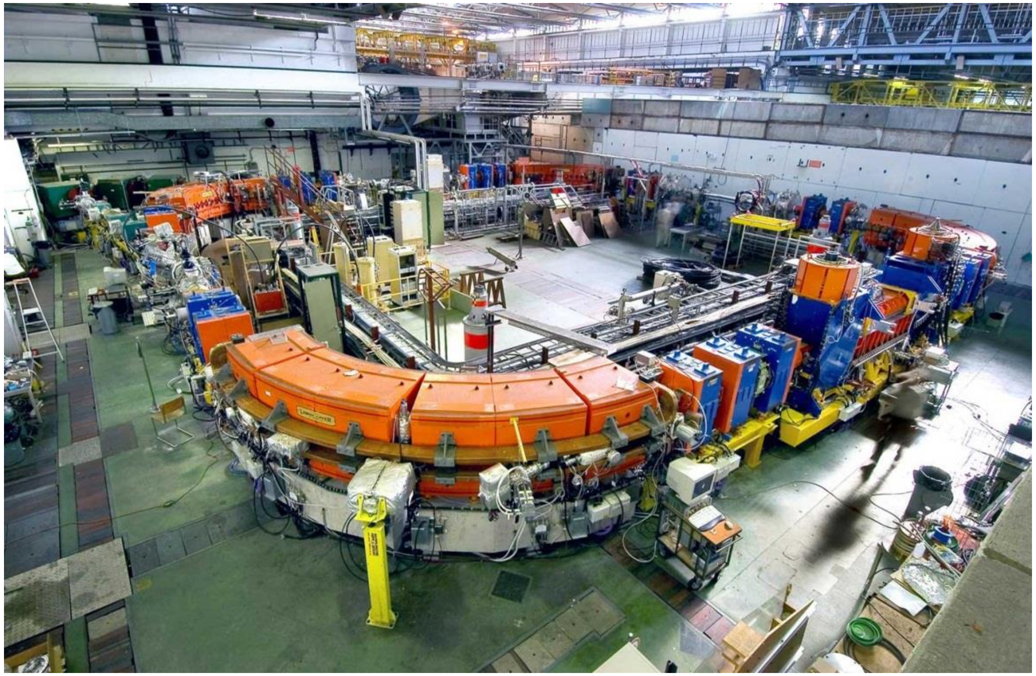
Substituting back:

$$\left(\frac{1 + x'^2}{x''}\right)^2 + x'^2 \left(\frac{1 + x'^2}{x''}\right)^2 = \rho^2 \rightarrow \frac{x''^2}{\left(1 + x'^2\right)^3} = \frac{1}{\rho^2} \rightarrow \frac{x''}{\left(1 + x'^2\right)^{\frac{3}{2}}} = \frac{1}{\rho}$$

Exercise 1

Consider the LEIR (Low Energy Ion Ring) at CERN. This is the first synchrotron of the CERN Pb_{208}^{+54} ion chain to the LHC.

The kinetic energy given from Linac 3 is 4.2 MeV/n and the rest energy of the nucleon is 0.938 GeV.



Exercise 1

Compute the relativistic γ and β factors, the total momentum in MeV/c/n.

After acceleration the beam is extracted at 72.2 MeV/n kinetic energy. What is now the β factor and momentum?

Exercise 2

Following the specifications of Exercise 1, compute the beam rigidity at LEIR injection and extraction energies.

What happens if instead of Pb_{208}^{+54}, we want to accelerate Xe_{129}^{+39}?