# *LONGITUDINAL*

# *PLANE*

***Lecture 5***
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# *Terminology*

- **The beam will be described with reference to a *synchronous ion* that follows a particular *space-time trajectory*. The 'space' part is the central orbit of the transverse motion and the 'time' part is defined by initial conditions.**

- **When crossing a cavity, the synchronous ion receives a kick in momentum ($\Delta_s p$). Non-synchronous ions receive slightly different kicks ($\Delta_s p + \Delta p$ ).**

- **The motion of the non-synchronous ions is then expressed in terms of how much they lead or lag ($\Delta s$) the synchronous ion in their flight through the lattice and by how much they deviate from the synchronous ion in momentum ($\Delta p/p$).**

- **$\Delta s$-$\Delta p/p$ defines the *longitudinal phase space*.**

- **A large number of ions concentrated around a synchronous ion are referred to as a *bunch*.**

- **Without longitudinal focusing, a bunch of ions will progressively spread out and be lost.**

- **A focusing region in longitudinal phase-space around the synchronous ion is known as a *RF bucket*.**

# *Terminology continued*

- A ***stationary RF bucket*** is one that does not alter the momentum of the synchronous ion ($\Delta_s p = 0$), but does modify the momenta of the non-synchronous ions ($\Delta p \neq 0$).

- An ***accelerating bucket*** applies a positive momentum kick to the synchronous ion ($\Delta_s p > 0$). A ***decelerating bucket*** does the converse.

- RF cavities are usually (but not always) configured to bring non-synchronous ions closer to the synchronous ion.

- In transfer lines, this is called ***longitudinal focusing***.

- In a ring, it is called ***phase stability***.

**NOTE:**

'$\Delta_s$' refers to the synchronous ion.

'$\Delta$' refers to the difference with respect to the synchronous ion.

'd' and '$\partial$' used for mathematical differentials.

# *Fields in RF devices*

- **RF devices require a more understanding than the uniform blocks of field used in the 'hard-edge' model.**

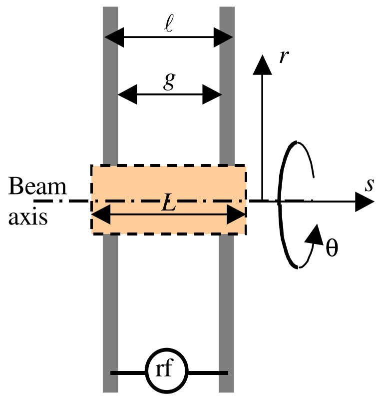
- **This section is devoted to *accelerating structures of the standing-wave type* with rotational symmetry excited by a TM010 mode. In the TM010 mode, only the $E_r$, $E_s$ and $B_\Theta$ components are non-zero.**

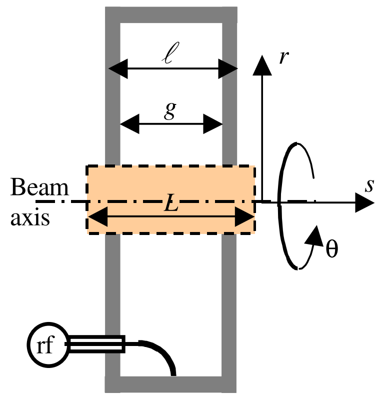
- **Whether the standing-wave structure is called a gap, a cavity, or a tank with drift tubes depends on the external geometry, see next slide.**

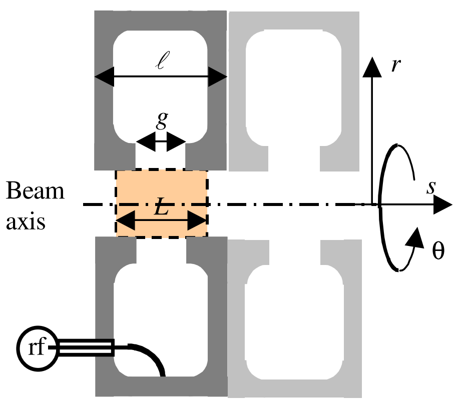
- **The basic modules can be used individually or in periodic arrays.**

- **Arrays can operate in the so-called *$\pi$-mode,* in which the fields of adjacent cells are $\pi$ out of phase, or the *$2\pi$-mode* for drift tubes in a tank when all the gaps are in phase.**

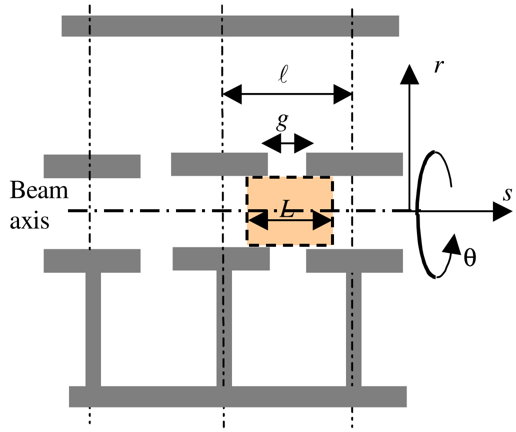
# RF standing-wave structures




Cavities with noses

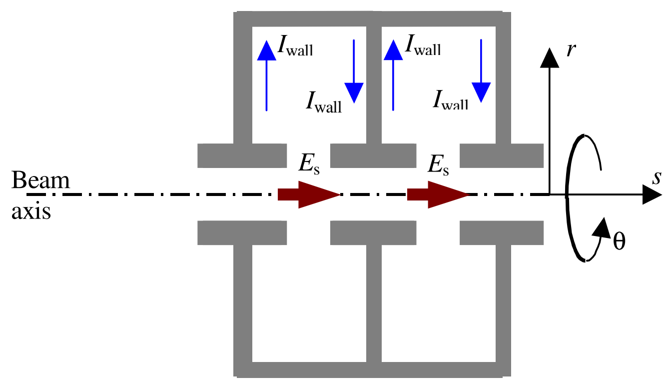



Drift tubes in a tank

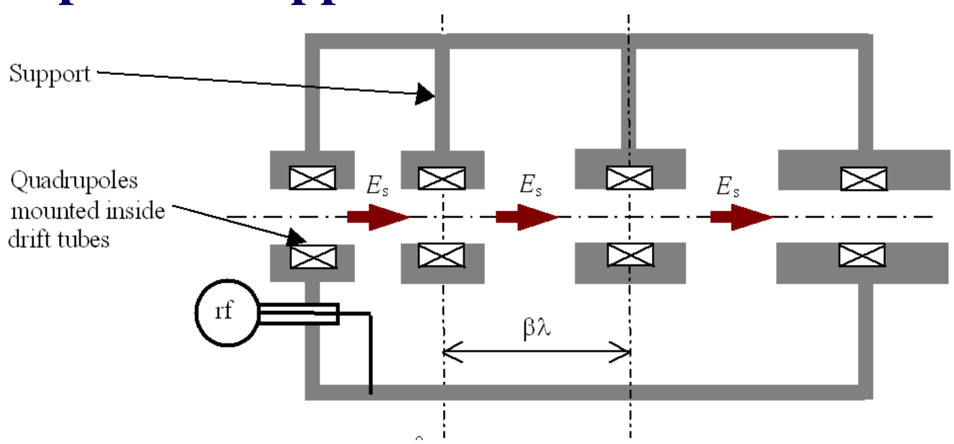
**The beige colour shows the 'useful' RF field region.**

# *Alvarez linac (non-relativistic)*

- **Start with a series of cavities with 'noses' or drift tubes and excite all cavities in phase (2π-mode). Note that the wall currents cancel.**

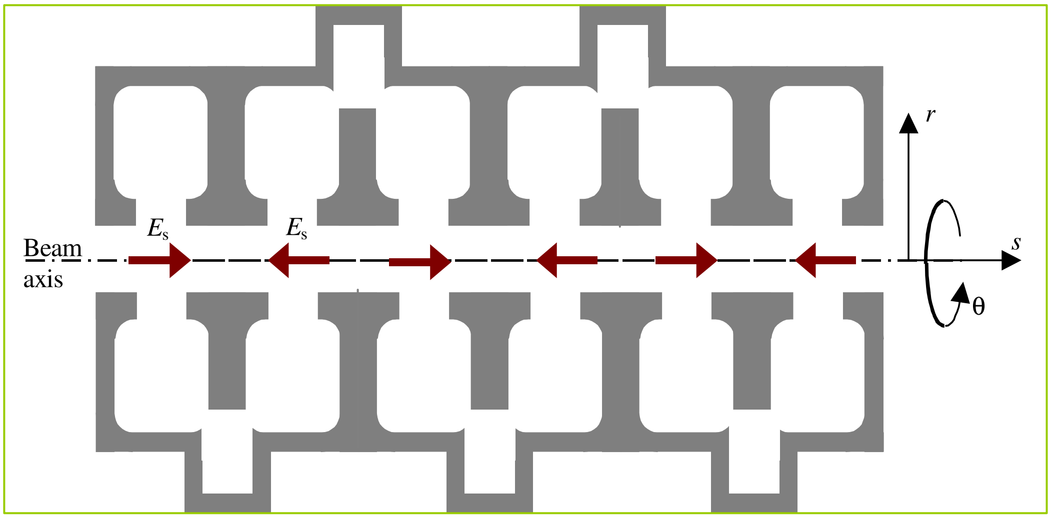



- **Since the wall currents cancel, remove the walls except for a support column for the drift tubes.**




- **Now adjust the drift tube lengths for the ion's velocity.**
- **Note there are quadrupoles lodged inside the drift tubes for transverse focusing.**
- **You have an Alvarez linac.**

# Coupled-cavity linac (relativistic)




- In a fully relativistic beam, the velocity is virtually that of light, so the cavities are identical.
- In the above diagram, the cavities are coupled to be excited in the π-mode. This saves having a RF source for each cavity and synchronizing them.

# *All different, but all the same*

- **The electric field on the axis always has the form,**

$$E_s(r,s,t) = E_s(r,s)\sin(\omega t + \phi_p)$$

**where $\omega$ is the angular frequency of the standing wave and $\phi_p$ is the RF phase at $t = 0$.**

- **It is too complicated and unnecessary to follow the full derivation, but it can be shown that the linearised fields are,**

$$E_r(r,s,t) = E_0 \sum_n A_n \frac{n\pi}{2L} r \sin\left(\frac{n\pi}{L}s\right)\sin\phi$$

$$E_s(s,t) = E_0 \sum_n A_n \cos\left(\frac{n\pi}{L}s\right)\sin\phi$$

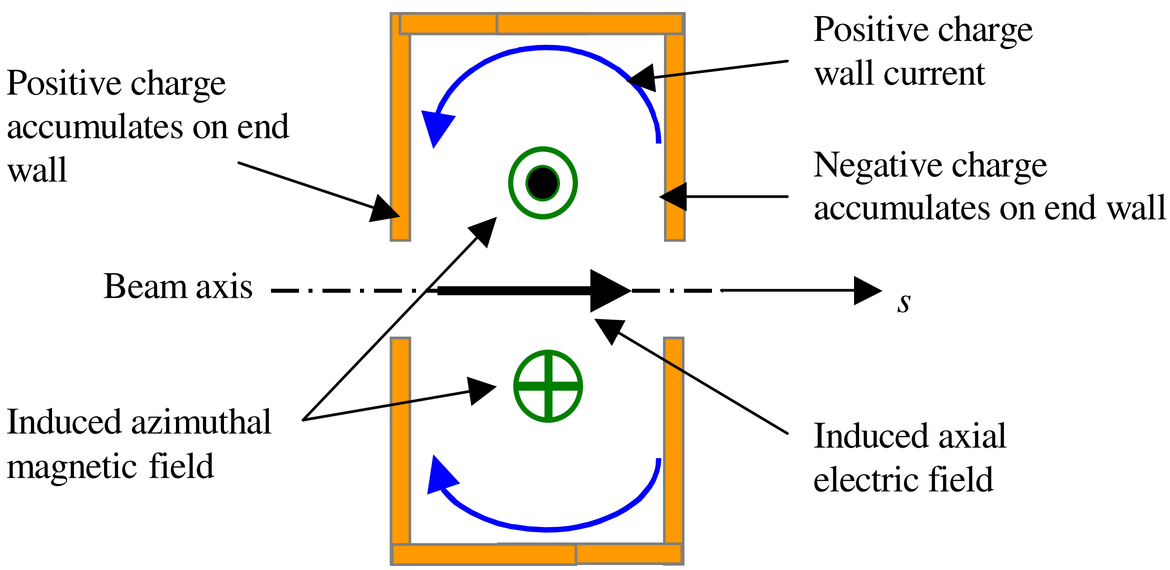
$$B_\theta(r,s,t) = E_0 \sum_n A_n \frac{\pi}{c\lambda} r \cos\left(\frac{n\pi}{L}s\right)\cos\phi \qquad (5.1)$$

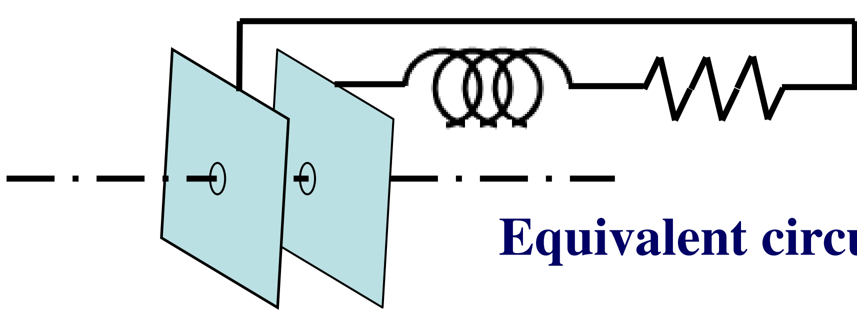
**where $n$ is odd, the amplitudes $A_n$ depend on the mechanical shape of the cavity, $E_0$ is the average electric field across the 'useful' region at the time of peak field, $L$ is the length of the active region, $\lambda$ is the free-space wavelength at the RF frequency $\omega$ and $\phi = \omega t + \phi_p$.**

# *Qualitative action of a cavity*

- **Wall currents flow back and forth between the end plates that store the charge.**
- **The current flow supports an azimuthal magnetic field.**
- **The charge accumulation on the end plates drives an electric field that acts on the beam.**
- **To relate the azimuthal magnetic field to the induced axial electric field use Faraday's law.**

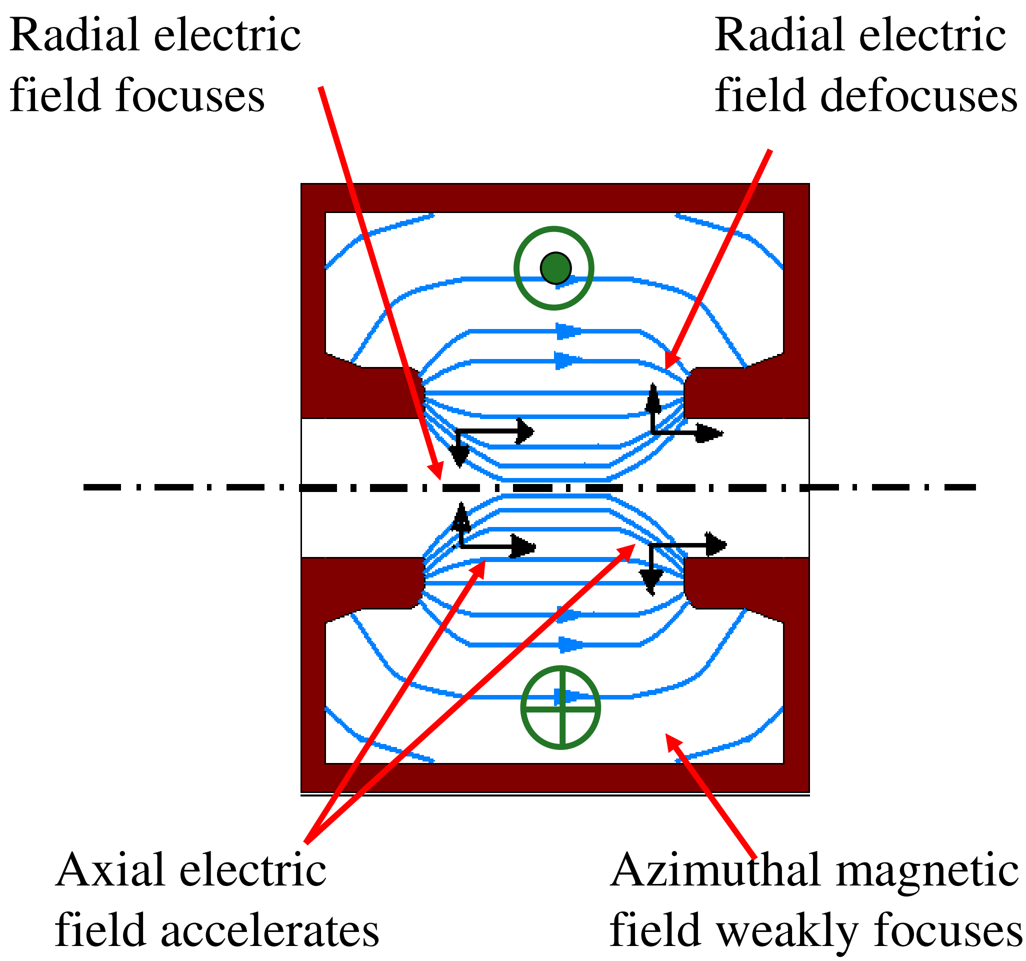
$$\nabla \times E = -\partial B / \partial t$$



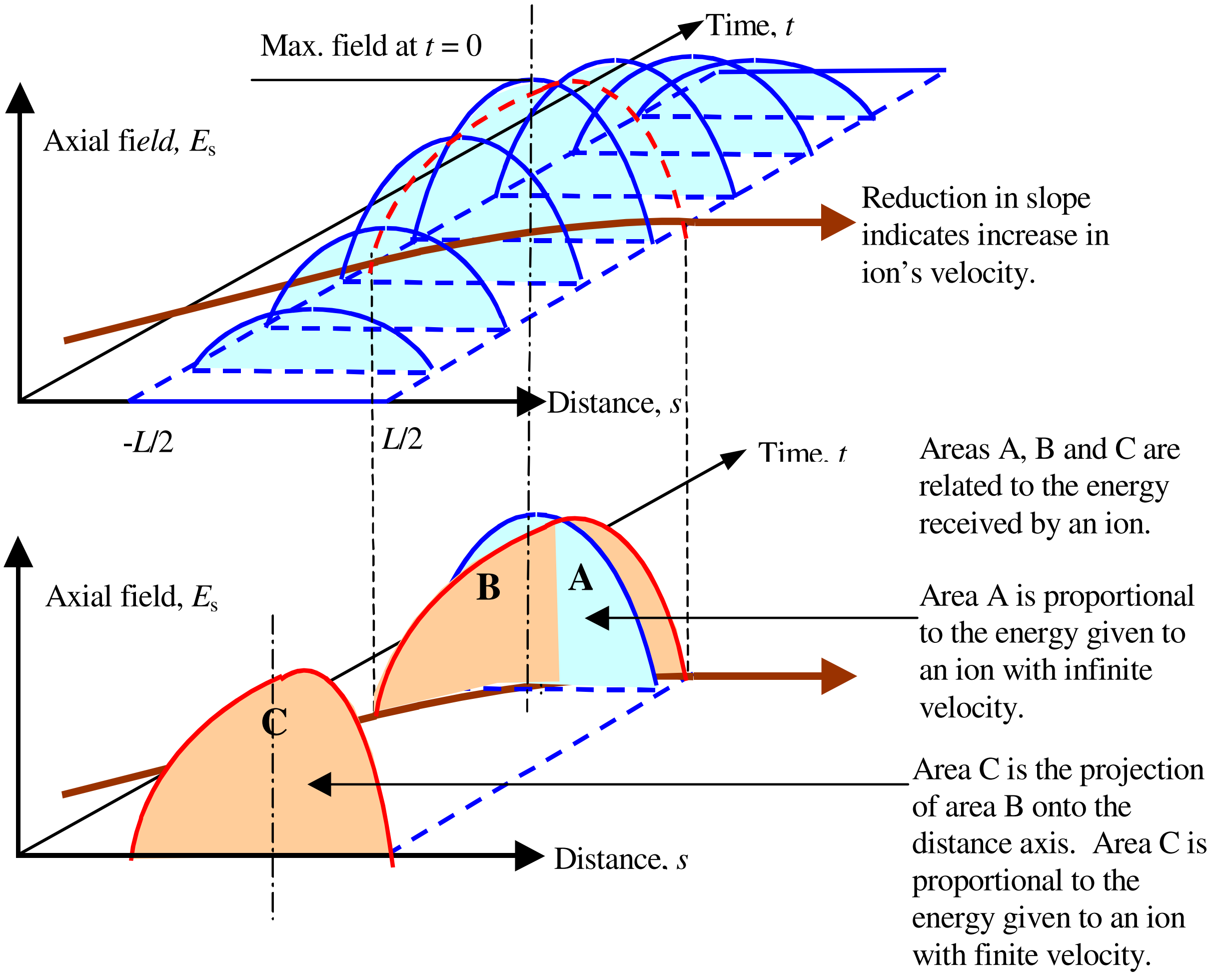
**Equivalent circuit**

# *Qualitative action of the cavity fields*




- **The 'noses' shield the incoming ion from the axial field.**
- **As the ion enters the gap, the axial field rises with a cosine-like form.**
- **The radial focusing at the entry slightly exceeds the defocusing at the exit because the ion has a higher energy and is stiffer.**

# Crossing a cavity




- **The blue half waves show the axial cavity field as it changes sinusoidally with time.**
- **The brown curve shows the ion's trajectory in space and time.**
- **The beige areas are defined by the position of the ion in space and time and the corresponding axial field.**
- **Beige area 'C' is the projection on the *s*-axis of the axial field 'seen' by the ion. The area of 'C' is the energy given to the ion.**

# *Transit time factor*

- The exact solution for the transit of an ion in an RF cavity is complicated.

- The problem is partially avoided by defining something called the ***Transit Time Factor, T***.

- *T* is defined as the ratio of the maximum integral of the axial electric field that can be 'seen' by a ion traversing an RF cavity with velocity $v_s(s)$ to the maximum integral that can be 'seen' by a particle traversing with infinite velocity.

- To obtain the maximum integral the ion must enter shortly before the peak field is reached and exit shortly after.

- The general energy gain, $\Delta_s E$, is defined as,

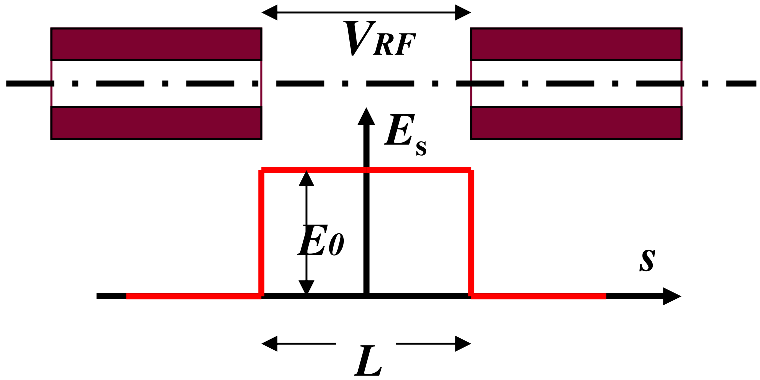
$$\Delta E_s = qE_0 LT \sin\phi_p = qV_{RF} T \sin\phi_p \qquad (5.2)$$

where $E_0$ is the average of the field distribution across the gap at peak field, $L$ is the 'active' gap length, $V_{RF}$ is the peak voltage and $\phi_p$ is the phase of the cavity field as the ion crosses the centre point (due to a different choice of origin you will also see cos $\phi_p$).

- All the problems are now hidden in *T*, which can be estimated and a numerical calculation can be put off until really necessary.

# First approximation for T

- **Let the field in the gap have an amplitude $E_0 = V_{RF}/L$. Assume the field is constant with respect to distance $s$ and is zero in the drift tubes. Assume the velocity is constant.**




- **The accelerating field is $E(t) = (V_{RF}/L)\cos(\omega t)$.**
- **For an ion passing the centre of the gap at $t = 0$ and with an average velocity $v_0$, its position is $s = v_0 t$ and its energy gain will be,**

$$\Delta E_v = q\int_{-L/2}^{+L/2}\frac{V_{RF}}{L}\cos(\omega_{RF}t)\,\mathrm{d}s = q\frac{V_{RF}}{L}\int_{-L/2}^{+L/2}\cos\left(\frac{\omega_{RF}}{v_0}s\right)\mathrm{d}s$$

$$\Delta E_v = qV_{RF}\frac{\sin(\theta/2)}{(\theta/2)} \quad \text{where } \theta = \frac{\omega_{RF}L}{v_0}, \text{Transit Angle}$$

- **The maximum energy that can be extracted by an ion with infinite velocity is $qV_{RF}$ so that,**

$$\text{Transit time, } T = \frac{\Delta E_v}{\Delta E_\infty} = \frac{\sin(\theta/2)}{(\theta/2)} \qquad (5.3)$$

# Second approximation for T

- The field distribution with distance in the gap is close to a cosine [Equation (5.1) 1st harmonic only], so we can improve our approximation,

$$\Delta E_v = q\frac{V_{\mathrm{RF}}}{L}A_1\int_{-L/2}^{+L/2}\cos\left(\frac{\pi}{L}s\right)\cos(\omega_{\mathrm{RF}}t)\,\mathrm{d}s$$

$$\Delta E_v = \frac{qV_{\mathrm{RF}}}{2L}A_1\int_{-L/2}^{+L/2}\cos\left(\frac{\pi}{L}s+\frac{\omega_{\mathrm{RF}}}{v_0}s\right)+\cos\left(\frac{\pi}{L}s-\frac{\omega_{\mathrm{RF}}}{v_0}s\right)\mathrm{d}s$$

I leave you to do the mathematics,

$$\Delta E_v = \frac{\pi}{2}qV_{\mathrm{RF}}A_1\left(\frac{\cos(\theta/2)}{(\pi/2)^2-(\theta/2)^2}\right)\quad \text{where } \theta = \frac{\omega_{\mathrm{RF}}L}{v_0}$$

- The ion crossing with infinite velocity receives,

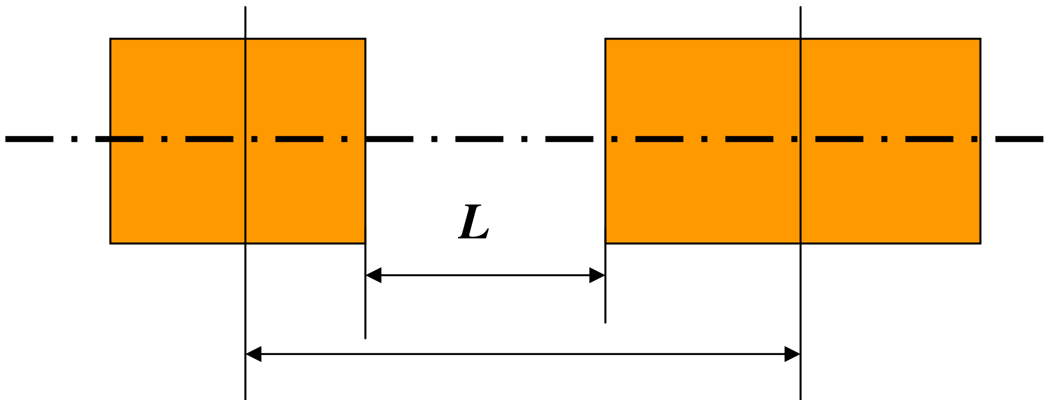
$$\Delta E_\infty = q\frac{V_{\mathrm{RF}}}{L}A_1\int_{-L/2}^{+L/2}\cos\left(\frac{\pi}{L}s\right)\mathrm{d}s = \frac{2qV_{\mathrm{RF}}}{\pi}$$

- So finally,

$$\text{Transit time, } T = \frac{\Delta E_v}{\Delta E_\infty} = \frac{\cos(\theta/2)}{1-(\theta/\pi)^2} \qquad (5.4)$$

# *Transit time factor continued*

- **Consider an Alvarez structure operating in the 2π-mode and let the gap length be 0.4 of the cell length.**




Cell length = 2.5*L* = v/$f_{RF}$ = βc(2π/$\omega_{RF}$)

- **The Transit Angle is,**

$$\theta = \frac{\omega_{RF} L}{v_0} = \frac{\omega_{RF}}{\beta c}\frac{\beta c\, 2\pi}{2.5\,\omega_{RF}} = 0.8\pi$$

- **First approximation gives, *T* = 0.757**
- **Second approximation gives, *T* = 0.858**

**Note that the first and second approximation have singular points at $\theta = 0$ and $\theta = \pi$, respectively.**

# *Linacs and Rings*

- **If you are interested in linacs, then the fields, the Transit Time Factor and the Transit Angle will be important for you.**
- **If you are interested in rings, then it is likely that you will be able to take a very simplified view of the cavities.**
- **In linacs, the RF period will be a few times the gap transit time.**
- **In a ring, the RF period will be related to the revolution period by a factor called the harmonic number, *h*, (which can be unity)**

$$h = \frac{\text{Revolution period}}{\text{RF period}} \qquad (5.5)$$

- **In most cases, the time to cross the gap in a ring will be very small compared to the RF period and the ion will be fully relativistic, so that the Transit Time Factor will be close to unity.**
- **In this case, the energy gained by the beam will be,**

$$\Delta_{\mathrm{s}} E = q V_{\mathrm{RF}} \sin\phi_{\mathrm{s}} \qquad (5.6)$$

**$\phi_s$ refers to the synchronous ion.**

# *Synchronous RF acceleration*

- **From basic theory for bending in a dipole (omit sign),**

$$qv_0B = \frac{mv_0^{\,2}}{\rho_0} \quad \Rightarrow \quad p_0 = q\rho_0 B$$

**so that,**
$$\frac{\mathrm{d}p}{\mathrm{d}t} = q\rho_0 \frac{\mathrm{d}B}{\mathrm{d}t}.$$

- **Re-writing for one turn,**

$$\Delta p_{\mathrm{Turn}} = q\rho_0 \dot{B}\, T_{\mathrm{Rev}}$$

- **Let *R* be the average machine radius, then,**

$$\Delta p_{\mathrm{Turn}} = q\rho_0 \dot{B}\, \frac{2\pi R}{\beta c}$$

- **Now, $\Delta E = \beta c \Delta p$ , so,**

$$\Delta_{\mathrm{s}} E_{\mathrm{Turn}} = 2\pi q R \rho_0 \dot{B}$$

- **but we already have for *N* cavities,**

$$\Delta_{\mathrm{s}} E_{\mathrm{Turn}} = q V_{\mathrm{RF}} N \sin\phi_{\mathrm{s}}$$

- **Finally,**

$$V_{\mathrm{RF}} N \sin\phi_s = 2\pi\rho_0 R\, \dot{B} \qquad (5.7)$$

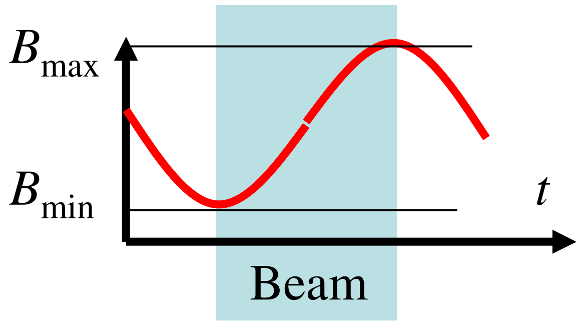
# *Frequency swing*

- The harmonic number, $h$ sets the number of RF oscillations in one revolution. There will be one stable RF bucket per RF oscillation, i.e. $h$ buckets and correspondingly up to $h$ bunches in the machine.

$$f_{\mathrm{Rev}} = \frac{1}{T_{\mathrm{Rev}}} = \frac{\beta c}{2\pi R}$$

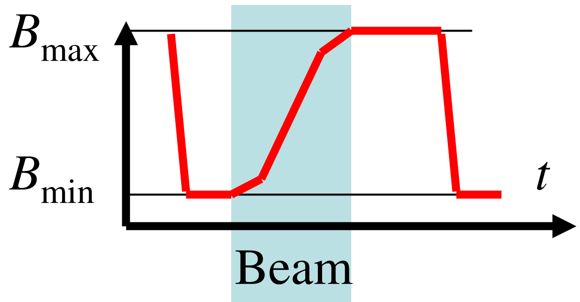
$$f_{\mathrm{RF}} = \frac{\beta c h}{2\pi R} = \frac{ch}{2\pi R}\sqrt{1-\frac{1}{(1+T/E_0)^2}}$$

- The magnetic field ramp is the 'driving' parameter behind the RF programs for $f_{\mathrm{RF}}$, $V_{\mathrm{RF}}$ and $\phi_s$.
- Fast cycling machines have resonant power supplies.




Maximum d$B$/d$t$ at mid-cycle

- Slow cycling machines are 'ramp and hold'




d$B$/d$t$ mostly constant with 'round in' and 'round-out' curves

# *RF acceleration with radiation*

- The energy lost per electron per turn due to synchrotron radiation is,

$$U_\gamma \approx \frac{2}{3} r_c m_0 c^2 \gamma^4 \oint \frac{\mathrm{d}s}{\rho^2}$$

where
$$r_c = \frac{e^2}{4\pi\varepsilon_0 m_0 c^2}$$

This will be covered in detail later in the course.

- The RF cavity (ies) must also compensate this loss. In a machine like the CERN LEP, the RF cavities can occupy hundreds of metres.

- Thus equation ( 5.7) has to be upgraded for electron machines to

$$V_{\mathrm{RF}} N \sin\phi_s = 2\pi\rho_0 R\,\dot{B} + \frac{2}{3}\frac{r_c m_0 c^2 \gamma^4}{e}\oint \frac{\mathrm{d}s}{\rho^2} \qquad (5.8)$$

In proton machines, the synchrotron radiation loss can be safely ignored in most cases. An exception is the CERN LHC where the few watts that are generated are a significant load for the cryogenic system.

# *Transition energy*

- The angular revolution frequency is, $\Omega_{\mathrm{Re}\,v} = \dfrac{2\pi v}{C}$

  which gives,
$$\frac{\Delta\Omega_{\mathrm{Re}\,v}}{\Omega_{\mathrm{Re}\,v}} = \frac{\Delta v}{v} - \frac{\Delta C}{C}. \qquad \text{(A)}$$

- From relativity,
$$\frac{\Delta v}{v} = \frac{1}{\gamma^2}\frac{\Delta p}{p}. \qquad \text{(B)}$$

- Define α as the ***momentum compaction function***.
  **α depends only on the lattice,**
$$\frac{\Delta C}{C} = \alpha\frac{\Delta p}{p}. \qquad \text{(C)}$$

- Substituting (B) and (C) into (A) gives,

$$\frac{\Delta\Omega_{\mathrm{Re}\,v}}{\Omega_{\mathrm{Re}\,v}} = \frac{\Delta p}{p}\left(\frac{1}{\gamma^2} - \alpha\right) = \frac{\Delta p}{p}\left(\frac{1}{\gamma^2} - \frac{1}{\gamma_{\mathrm{tr}}{}^2}\right) \qquad (5.9)$$

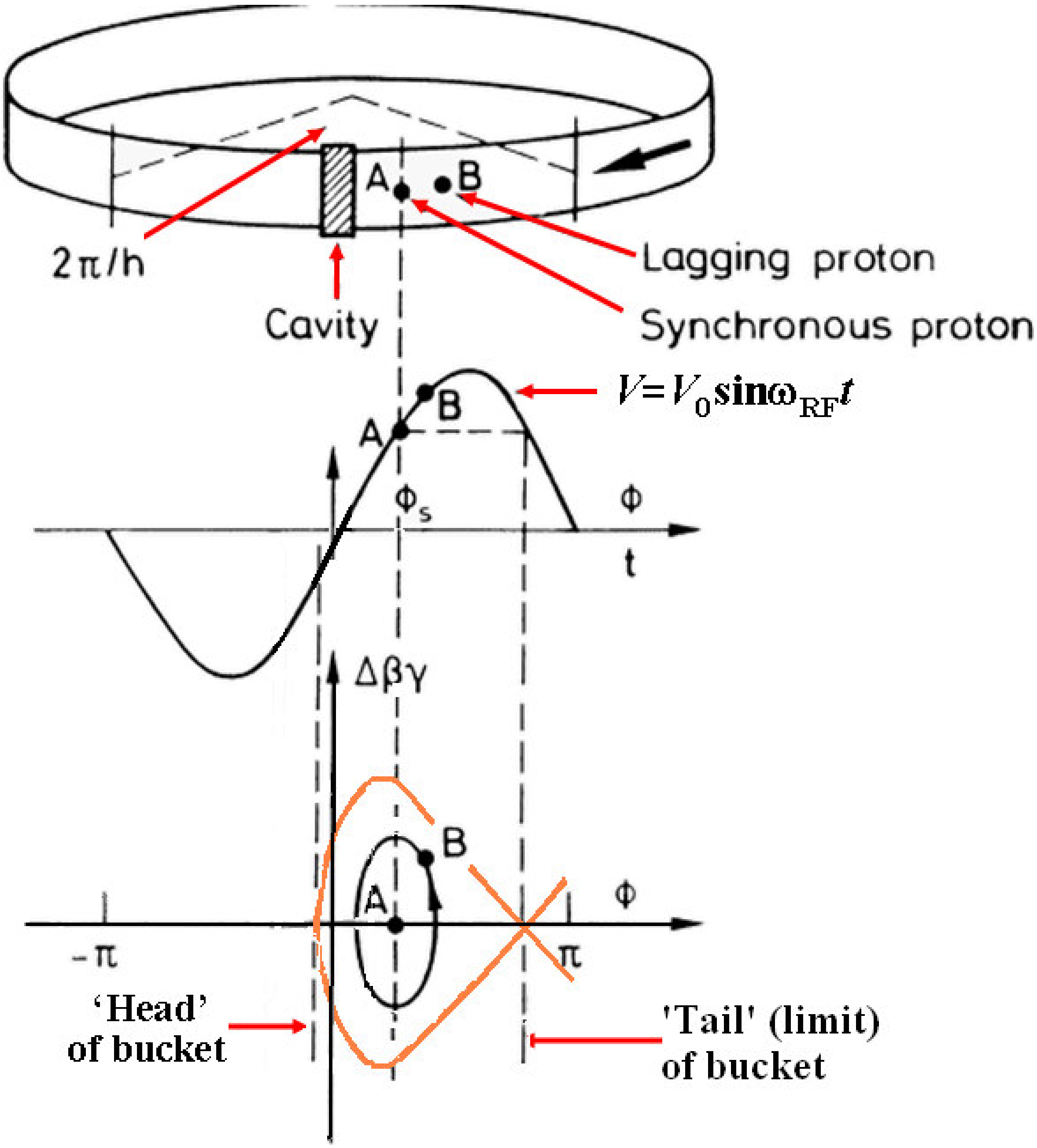
  where the ***transition energy,*** $\gamma_{\mathrm{tr}}$ is defined as

$$\gamma_{\mathrm{tr}} = \alpha^{-1/2} \qquad (5.10)$$

- For many accelerators, $\gamma = \gamma_{\mathrm{tr}}$ falls within the operating range. This point is called ***transition***.

  Below transition $\dfrac{\Delta\Omega_{\mathrm{Re}\,v}}{\Omega_{\mathrm{Re}\,v}}$ is positive and above it is negative.

# Phase stability

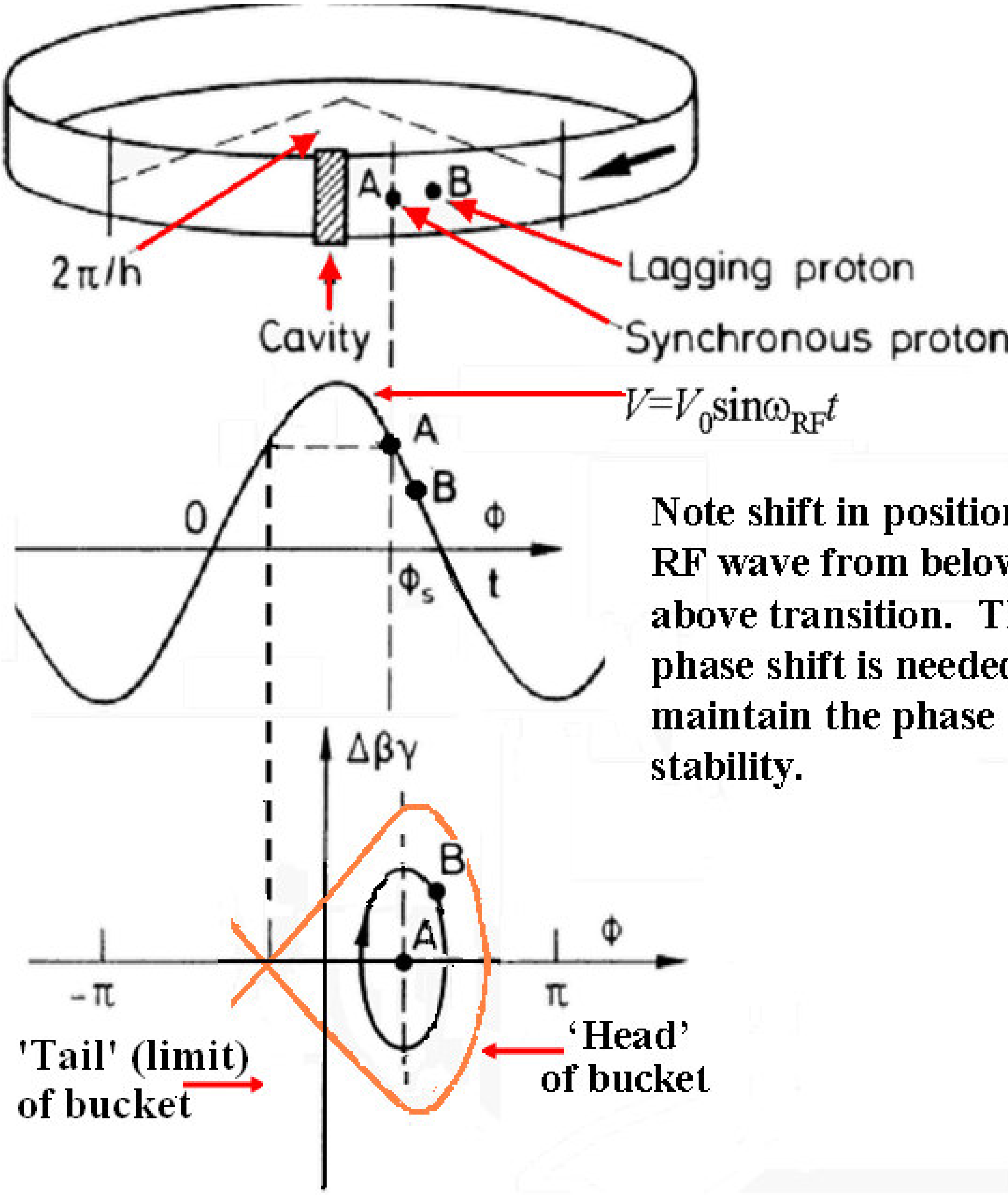



**BELOW TRANSITION and ACCELERATING**

**This case is intuitive as Δ*v* dominates.**

- **Lag behind -get more energy - catch up.**
- **Get ahead - get less energy - fall back.**

# *Phase stability continued*




Note shift in position of RF wave from below to above transition. This phase shift is needed to maintain the phase stability.

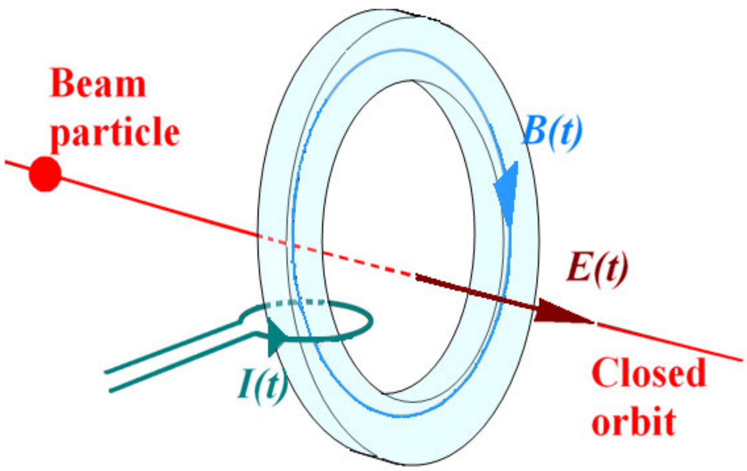
**ABOVE TRANSITION and ACCELERATING**

**This case is not intuitive as $\Delta C$ dominates.**

- **Lag behind –get less energy - catch up.**
- **Get ahead - get more energy - fall back.**

# *Betatron core*

- **A betatron core is a closed magnetic circuit in the form of a ferromagnetic ring through which the beam passes. A coil wound on the ring controls the flux inside the ring. Variations in this flux induce an electric field on the axis that changes the kinetic energy of the circulating beam.**

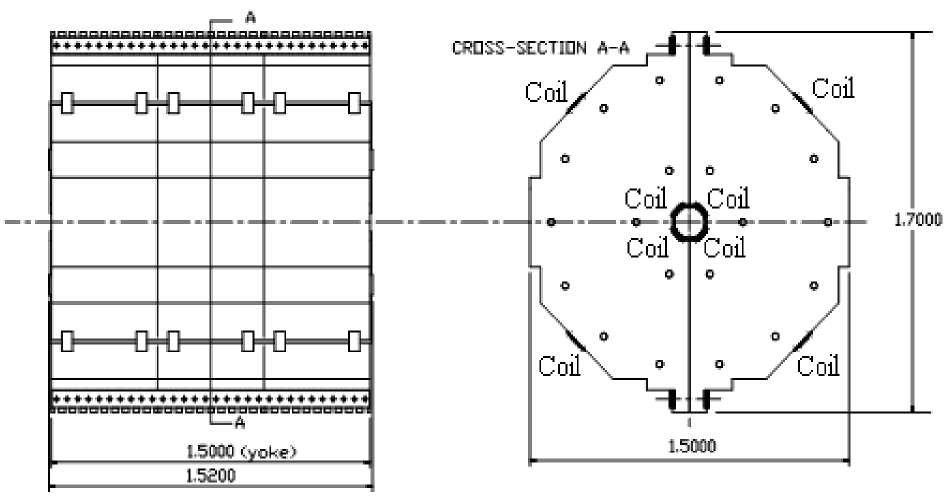



- ***The betatron core can smoothly accelerate or decelerate an unbunched beam* without varying any optical parameters in the ring. The core does not require exact dimensions and it is a very robust piece of equipment.**

**The betatron core inverts the configuration of the betatron and so that the magnetic field forms a loop around the beam rather than the beam forming a loop about the magnetic field.**

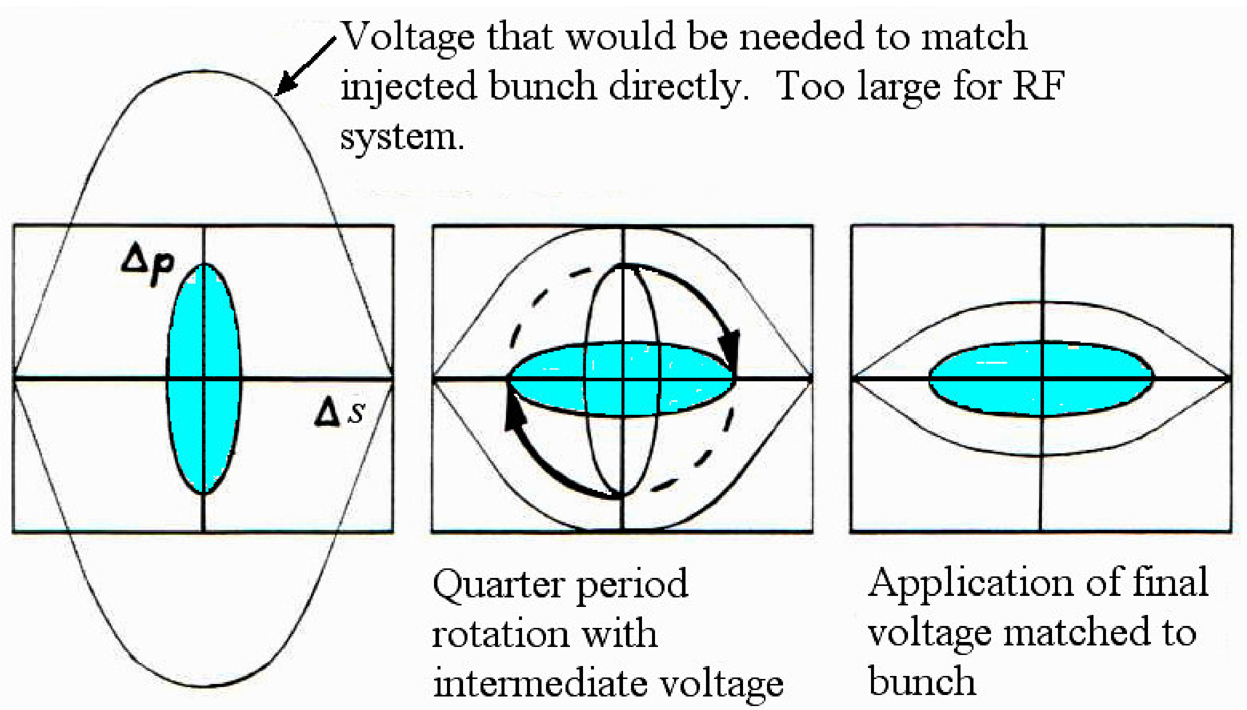
# Betatron core continued

- Preferably the core should comprise a stack of thin annular laminations,
  - To limit eddy currents and
  - To avoid air gaps in the magnetic circuit.
  - The lamination thickness determines the frequency response.
- However, engineering requirements often prevail and the core is made in 2 halves.
  - The air gaps increase the magnetic resistance and tend to make it unpredictable.
  - The air gaps also cause highly non-linear fringe fields to permeate the outer parts of the aperture.
- The PIMMS core.



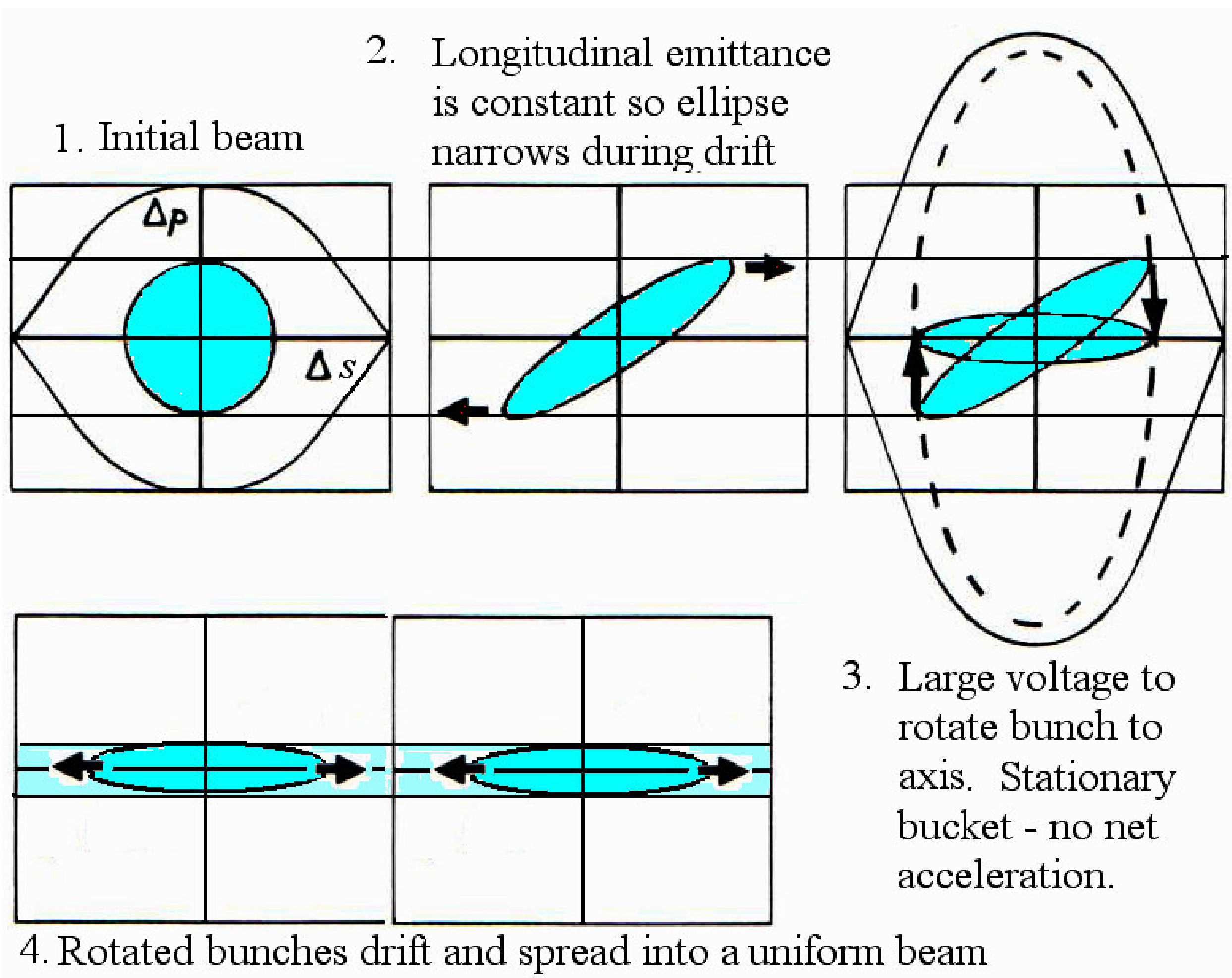
# *Voltage step matching*

- Often the momentum spread in a beam is too large for the RF system to trap directly with a matched ellipse, but the RF system is still powerful enough to enclose the bunch within the stable region of a bucket.
- As in the transverse case, the bunch will be mismatched and will tumble around the matched ellipse.
- This opens the possibility of matching by ***a voltage step*** or ***quarter-wave transformer*** technique, as shown below.



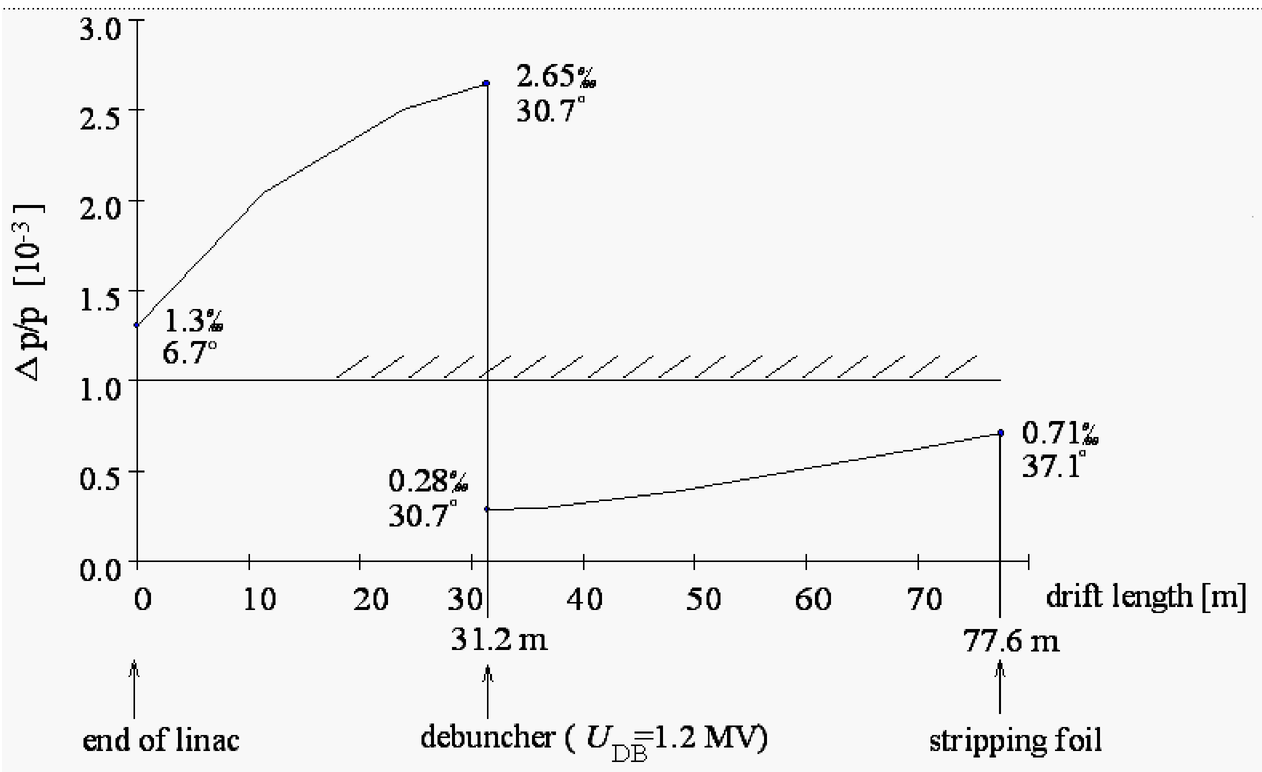
# RF debuncher

- High-intensity machines with a multi-turn injection work best with a continuous beam with the lowest possible $\Delta p/p$.
- The problem is to perform this debunching in the transfer line before injection.
- The basic idea is shown below.



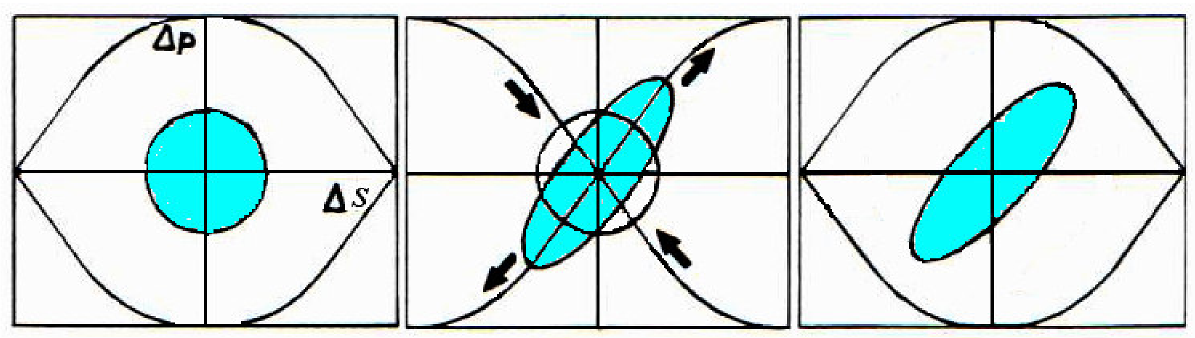
# RF debuncher continued

- The stretching, rotating and debunching look straightforward, but there is a hidden problem.
- Debunching is often required for low-energy, multi-turn injection schemes with high space charge.
- The longitudinal self-field from the space charge increases the $\Delta p/p$ during the drift periods.
- The example below shows the debunching scheme for a spallation source. Note that the rate of increase in $\Delta p/p$ after the debunching cavity is lower, since the effective space charge density has been reduced.



# *Phase jump*

- The longitudinal phase space ellipse can also be manipulated by jumping to the unstable fixed point.




- This manipulation could be used for a low momentum spread debunching in a ring before re-trapping with a second RF system.