Wake Fields and Instabilities

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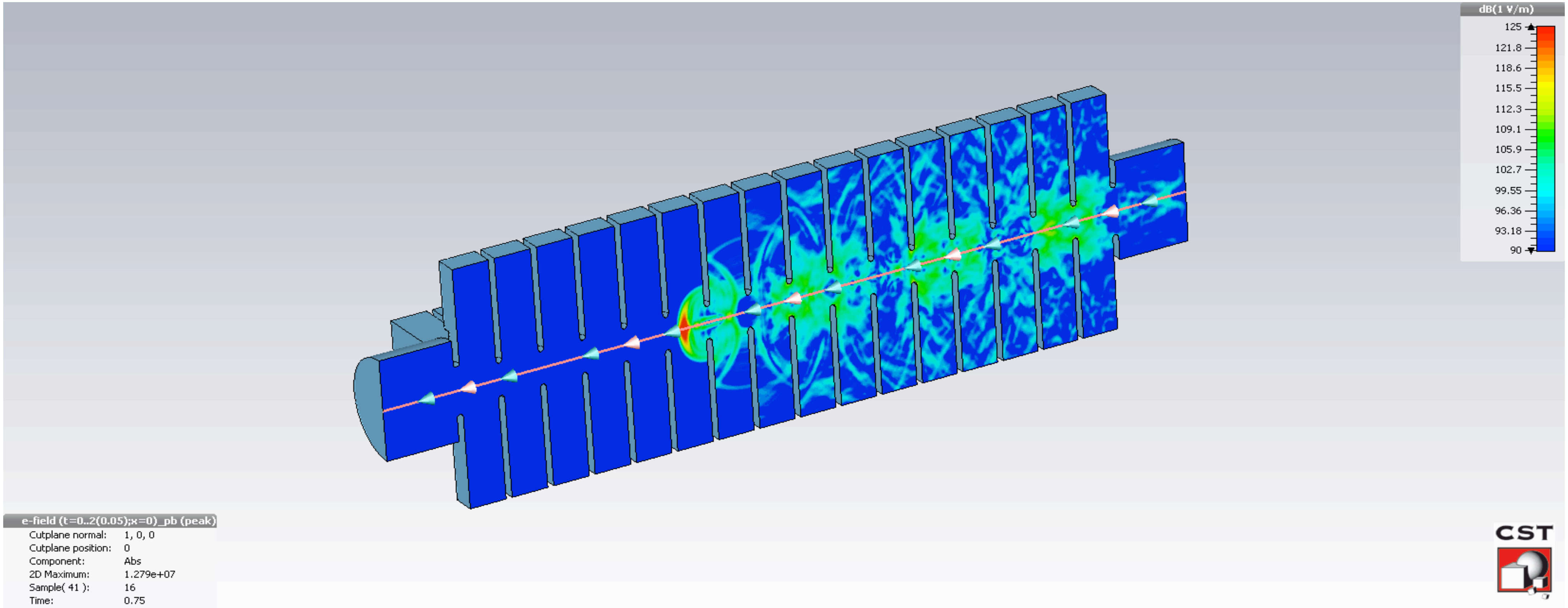
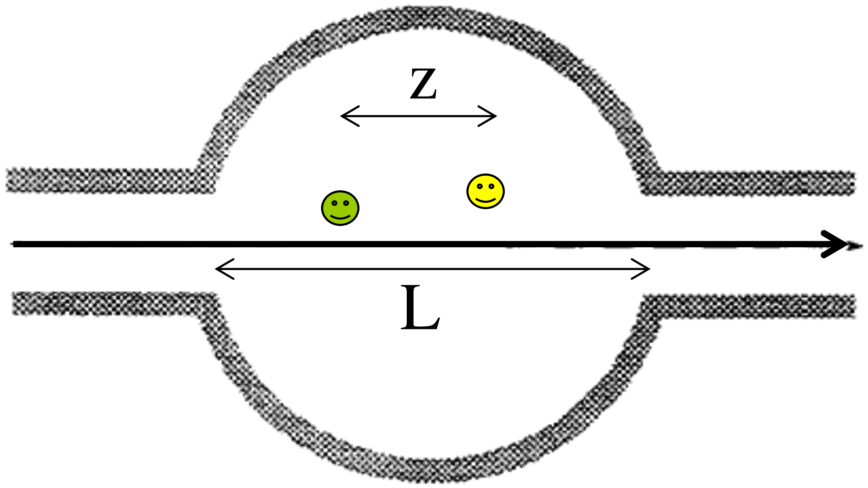
- **Introduction to wake fields/potentials**
- **Instability mechanism**
- **Instability in Linacs**
- **Instability in Circular Accelerators**

JUAS 2020


dB(1 V/m)
125
121.8
118.6
115.5
112.3
109.1
105.9
102.7
99.55
96.36
93.18
90
e-field (t=0..2(0.05);x=0)_pb (peak)
Cutplane normal: 1, 0, 0
Cutplane position: 0
Component: Abs
2D Maximum: 1.279e+07
Sample(41): 16
Time: 0.75
CST

Wake Fields and Wake Potentials



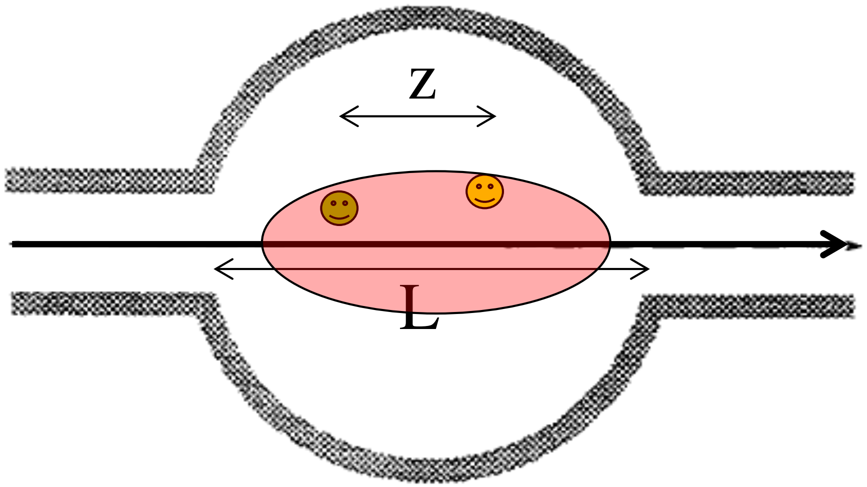
$$\vec{F} = q[E_z\hat{z} + (E_x - cB_y)\hat{x} + (E_y + cB_x)\hat{y} = \vec{F}_{\parallel} + \vec{F}_{\perp}$$

This force depends on the longitudinal and transverse position of the two particles. It is useful to distinguish two effects on the **test charge** :

1) a longitudinal force which changes its energy,

2) a transverse force which deflects its trajectory.

Two approximations



At high energies, the particle beam is rigid and two approximations apply:

1) The rigid beam approximation, which says that the beam traverses the discontinuity of the vacuum chamber rigidly and the electromagnetic field is a perturbation that does not affect the motion of the beam during the traversal of the discontinuity. *This implies that the distance 'z' between the two charges does not change.*

2) The impulse approximation: although the test charge sees a force coming from the electromagnetic field all along the structure, what it cares is the impulse

$$\Delta\vec{p} = \int_{-\infty}^{\infty} \vec{F}\, dt$$

as the charge completes the traversal through the discontinuity at its fixed velocity v.

If we consider a device of length L, we can perform the integral of the force acting on the test charge along the longitudinal path and get:

the Energy Gain (J):

$$U(r, r_0, z) = \int_0^L F_{\parallel} ds \simeq U(z)$$

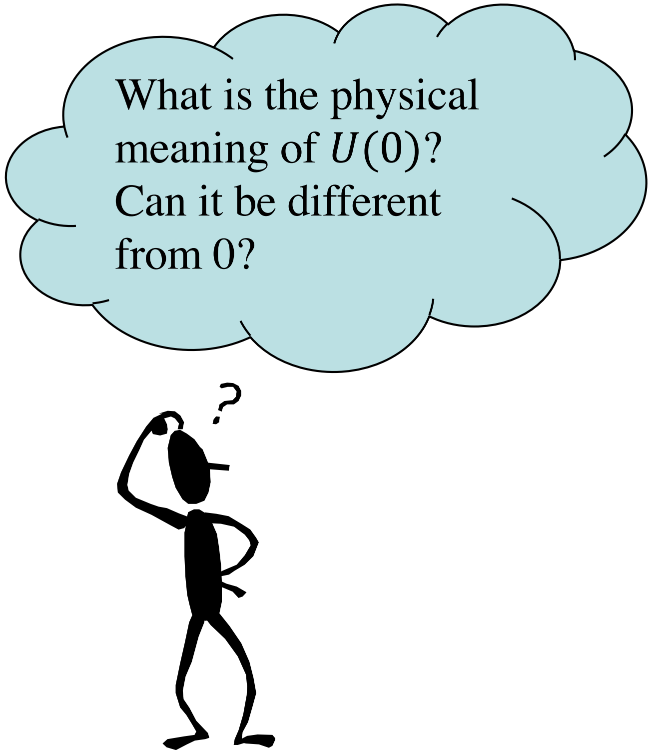
the Transverse Deflecting Kick (N · m):
(dipolar)

$$\vec{M}(r, r_0, z) = \int_0^L \vec{F}_{\perp} ds \simeq r_0 \vec{M}(z)$$

These quantities are both function of the distance z between the two particles. The transverse deflecting kick depends also on r_0, the transverse position of the source charge.

Note that the integration is performed over a given path of the trajectory.

These quantities, normalised to the charges, are called ***wake fields***



Longitudinal wake field
(Volt/Coulomb)

$$w_{\parallel}(z) = -\frac{U(z)}{q^2}$$

Transverse dipole wake field
(Volt/Coulomb/meter)

$$\vec{w}_{\perp}(z) = \frac{\vec{M}(z)}{q^2}$$

The minus sign in the longitudinal wake field means that the test charge loses energy when the wake is positive.

Positive transverse wake means that the transverse force is defocusing.

The wake fields are the important quantities to study the beam dynamics.

Coupling Impedance

The wake fields are generally useful to study the beam dynamics in the time domain (for example instabilities in a LINAC). If we take the equation of motion in the frequency domain (a trick generally used to study instabilities in circular accelerators), we need the Fourier transforms of the wake fields. Since these quantities have ohms units they are called ***coupling impedances***:

Longitudinal impedance (Ω)

$$Z_{\parallel} = \frac{1}{c}\int_{-\infty}^{\infty} w_{\parallel}(z) e^{i\frac{\omega z}{c}} dz$$

Transverse dipolar impedance (Ω/m)

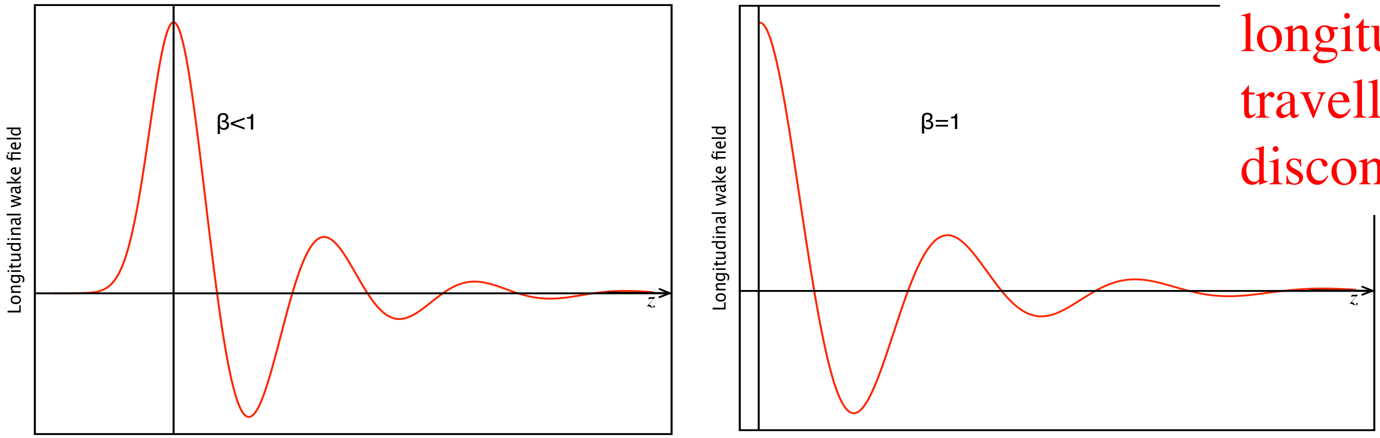
$$\vec{Z}_{\perp} = -\frac{i}{c}\int_{-\infty}^{\infty} \vec{w}_{\perp}(z) e^{i\frac{\omega z}{c}} dz$$

It is also useful to define the ***loss factor*** as the normalised energy lost by the **source charge q**

$$k = -\frac{U(z=0)}{q^2} = w_{\parallel}(z=0)$$

Although in general the loss factor is given by the longitudinal wake at z=0, for charges travelling with the speed of light, the longitudinal wake field is discontinuous at z=0

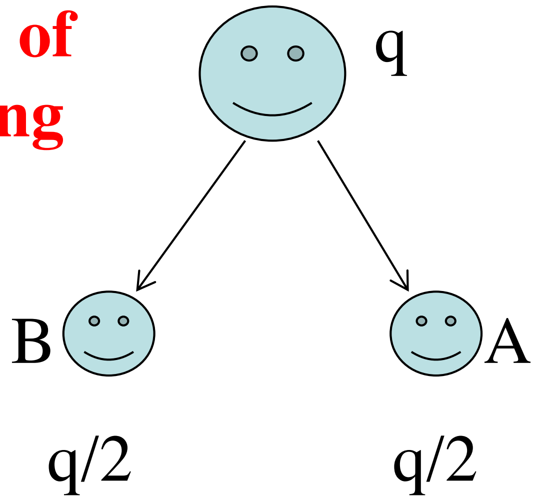
Causality requires that the longitudinal wake field of a charge travelling with the speed of light is discontinuous in the origin.



The exact relationship between k and $w(z \to 0)$ is given by the ***beam loading theorem***:

$$k = \frac{w_{\parallel}(z \to 0^+)}{2}$$

Demonstration of the beam loading theorem



$$U_A = q_A^2 k = \frac{q^2}{4} k$$

$$U_B = q_B^2 k + q_A q_B w_{//}(z)$$

$$= \frac{q^2}{4} k + \frac{q^2}{4} w_{//}(z)$$

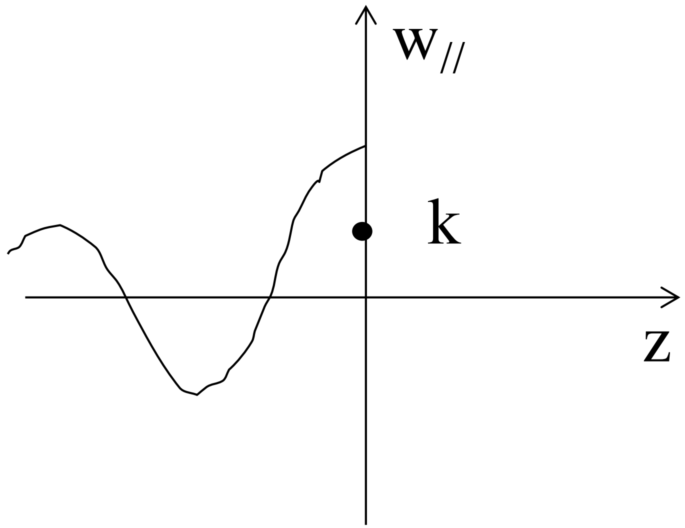
$$U_A + U_B = \frac{q^2}{2} k + \frac{q^2}{4} w_{//}(z)$$

$$z \to 0 \quad U_A + U_B = q^2 k$$

$$\frac{q^2}{2} k + \frac{q^2}{4} w_{//}(0) = q^2 k$$

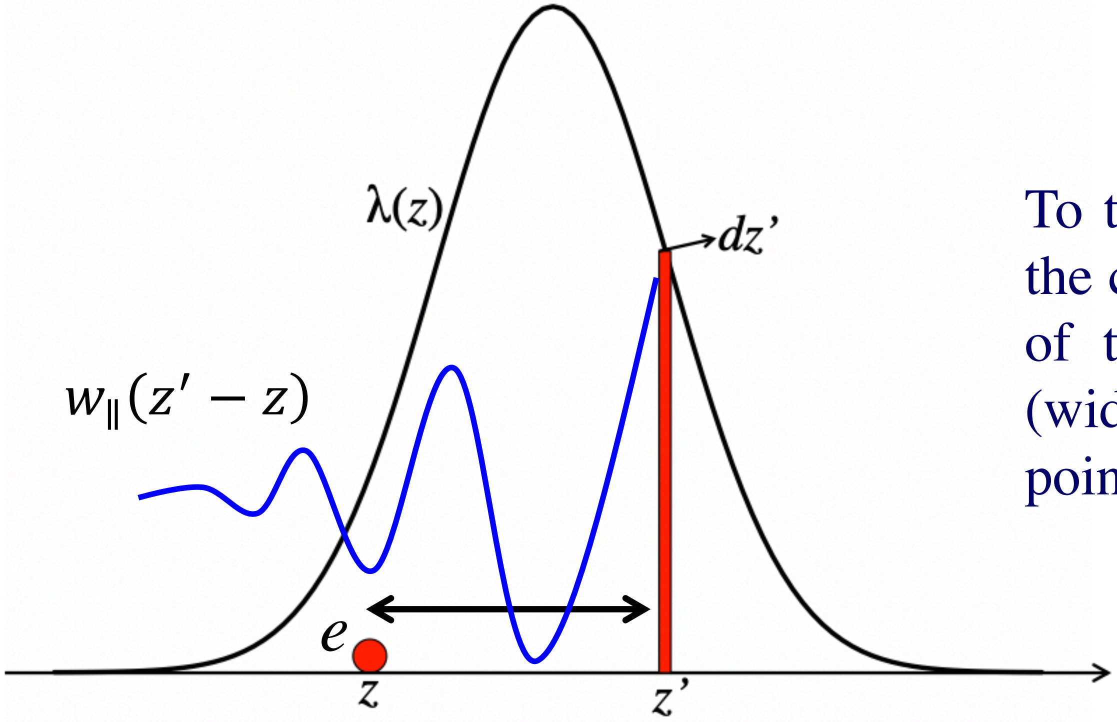
$$\frac{w_{//}(0)}{4} = \frac{k}{2}$$

$$k = \frac{w_{//}(0)}{2}$$



Wake potential and energy loss of a bunched distribution

When we have a bunch with longitudinal charge density $dq/dz = \lambda(z)$, we may want to get the amount of energy lost or gained by a single charge *e* in the beam.



To this end let us evaluate the effect on the charge *e* in a posizion *z* due to a slice of the bunch in a position *z*' so thin (width *dz*') that it can be considered as a point charge:

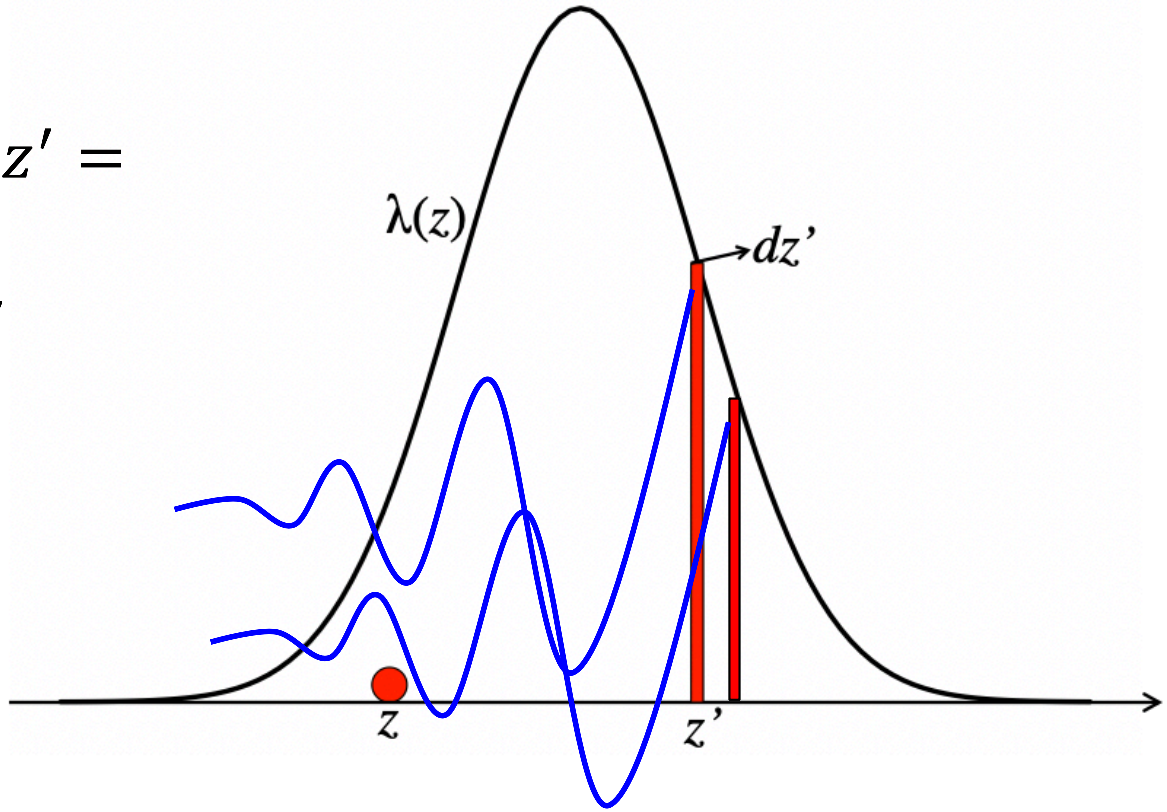
$$dU(z) = -edq(z')w_{\parallel}(z'-z) = -ew_{\parallel}(z'-z)\lambda(z')dz'$$

We now use the superposition principle to obtain the energy lost or gained by the charge *e* due to the entire distribution.

Wake potential and energy loss of a bunched distribution

$$U(z) = -e\int_{-\infty}^{\infty} w_{\parallel}(z'-z)\lambda(z')dz' =$$

$$= -e\int_{z}^{\infty} w_{\parallel}(z'-z)\lambda(z')dz'$$



NB: we have $q = \int_{-\infty}^{\infty} \lambda(z)dz$

The energy lost allows to define the ***longitudinal wake potential of a distribution***

$$W_{\parallel}(z) = -\frac{U(x)}{qe} = \frac{1}{q}\int_{-\infty}^{\infty} w_{\parallel}(z'-z)\lambda(z')dz'$$

The total energy lost by the bunch is computed summing up the losses of all the particles:

$$U_{bunch} = \frac{1}{e}\int_{-\infty}^{\infty} U(z)\lambda(z)dz = -q\int_{-\infty}^{\infty} W_{\parallel}(z)\lambda(z)dz$$

Some comments on the wake potential

$$W_{\parallel}(z) = -\frac{U(x)}{qe} = \frac{1}{q}\int_{-\infty}^{\infty} w_{\parallel}(z'-z)\lambda(z')dz'$$

- Observe that if we know the wake field, we can obtain the wake potential of any distribution, but if we know the wake potential, we are limited to a particular beam distribution.
- In a LINAC, with particles moving at the speed of light, the longitudinal distribution does not change, and the wake potential can be used to evaluate the energy variation of particles inside the bunch (energy spread). In this situation, the knowledge of the wake potential can be sufficient to study the beam dynamics.
- In a circular accelerator the longitudinal position of a charge depends on its energy through the slippage factor, and this energy is modified by the wake potential. As a consequence the wake potential changes the longitudinal distribution which, on its turn, changes the wake potential. In this case we have to study the beam dynamics in a self consistent way, and the knowledge of the wake potential is not sufficient.

Numerical Analysis

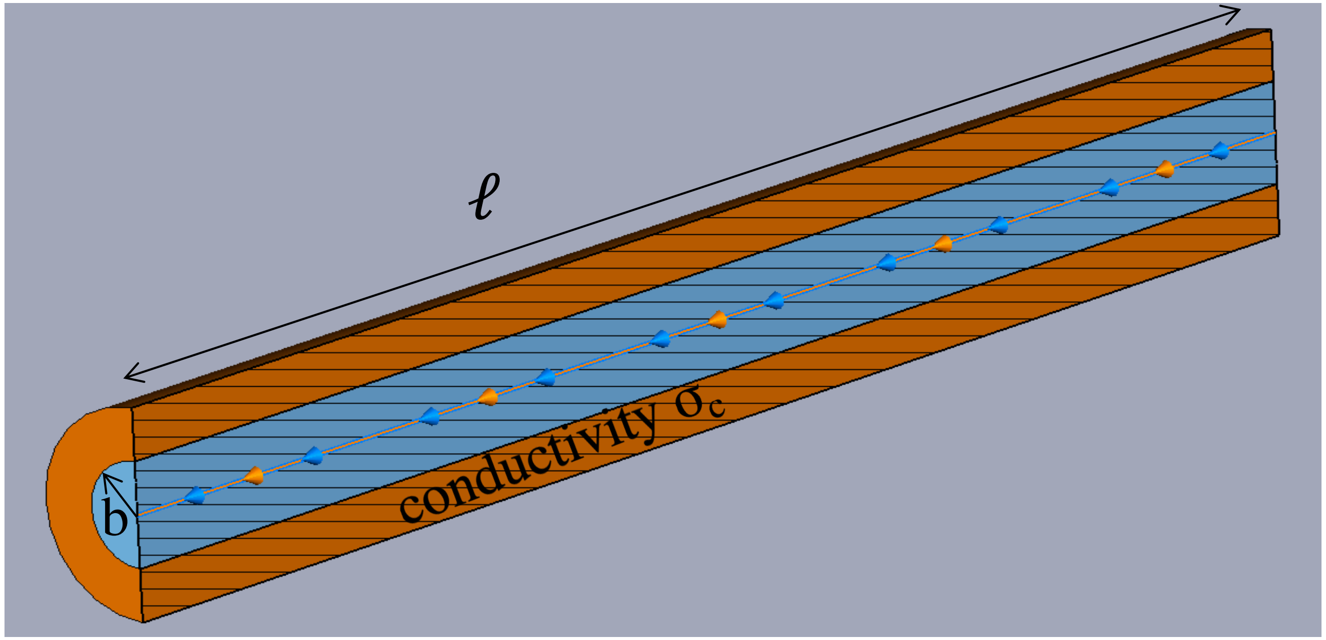
The study of the em fields requires to solve the Maxwell's equations in a given structure taking the beam current as source of fields. This is a quite complicated task for which it has been necessary to develop dedicated computer codes, which solve the e.m. problem in the frequency or in the time domain. There are several useful codes for the em design of accelerator devices, and new ones are developed. Examples of codes: **CST STUDIO SUITE, GDFIDL, ACE3P, ABCI, …**

However, the result of the codes is a wake potential and not a wake field …

Theoretical Analysis

The wake potentials given by numerical codes depend on the particular charge distribution of the beam. It is therefore desirable to know what is the effect produced by a single charge, i.e. **find the Green function** (wake field), in order to reconstruct the fields produced by any charge distribution.

Example of longitudinal wake field and coupling impedance: finite conductivity of a circular pipe wall (resistive wall)



Hp: high conductivity such that the skin depth is much smaller than the wall thickness and

$$c\chi\,/b \ll \omega \ll c\chi^{-1/3}/b$$

$$\chi^{1/3}b \ll z \ll b/\chi$$

with $\chi = \dfrac{1}{Z_0\sigma_c b}$

Example: aluminum σ_c=3.5x10^7 [Ωm]$^{-1}$, b=5 cm:

$9 \ll \omega \ll 5.2\times10^{12}$ [rad/s] $5.7\times10^{-5} \ll z \ll 3.3\times10^{7}$ [m]

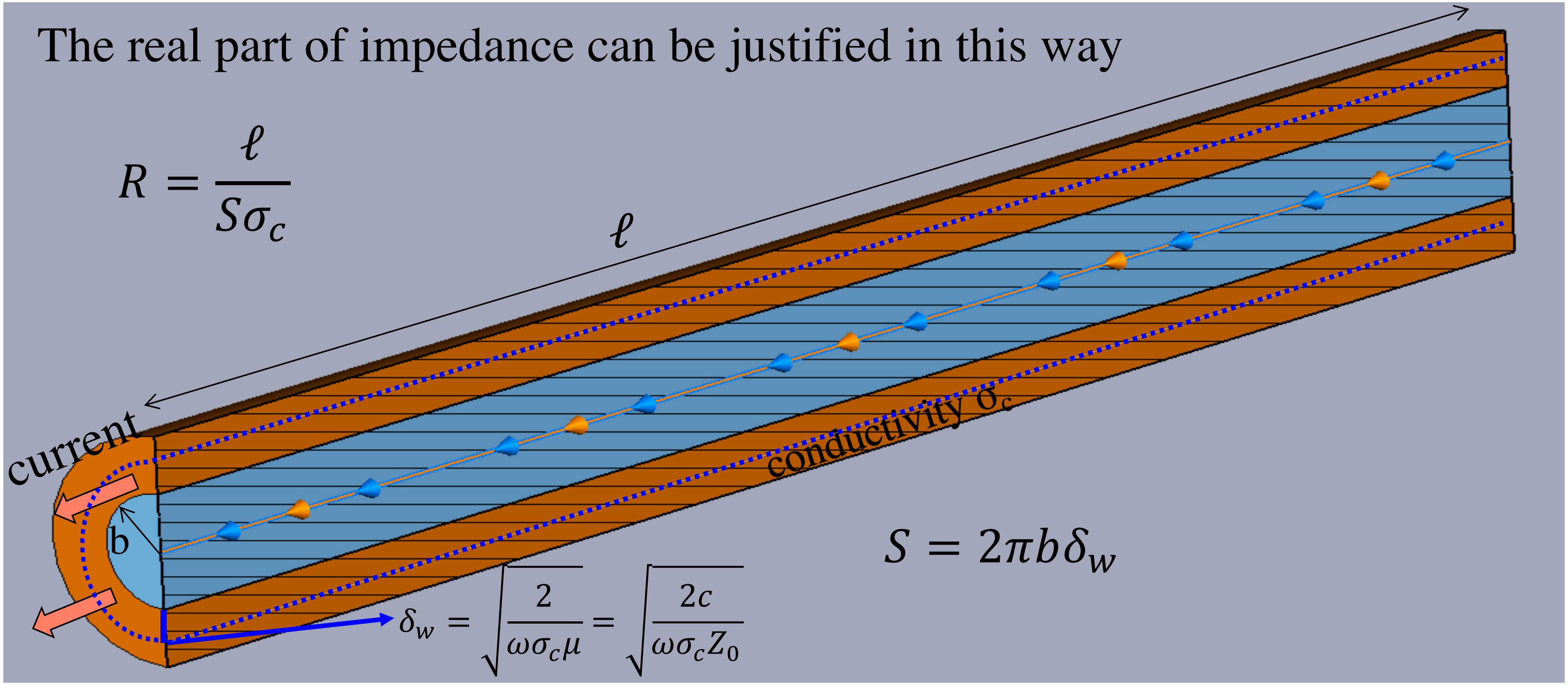
$$Z_{\parallel}(\omega) = [1 - i\,\mathrm{sgn}(\omega)]\,\frac{\ell}{2\pi b}\sqrt{\frac{Z_0|\omega|}{2c\sigma_c}}$$

$$w_{\parallel}(z) = \frac{\ell c}{4\pi b}\sqrt{\frac{Z_0}{\pi\sigma_c}}\frac{1}{|z|^{3/2}}$$

Example of longitudinal wake field and coupling impedance: finite conductivity of a circular pipe wall (resistive wall)

The real part of impedance can be justified in this way

$$R = \frac{\ell}{S\sigma_c}$$



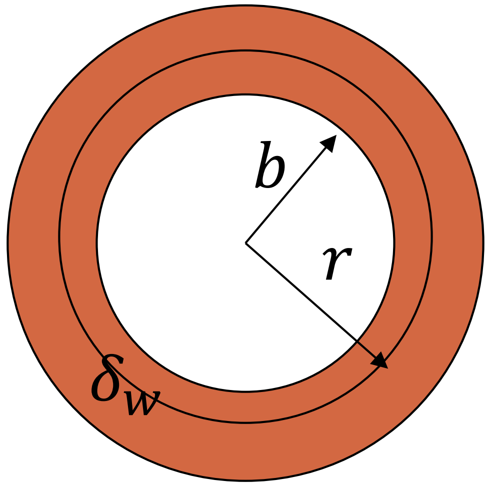
$$S = 2\pi b\delta_w$$

$$\delta_w = \sqrt{\frac{2}{\omega\sigma_c\mu}} = \sqrt{\frac{2c}{\omega\sigma_c Z_0}}$$

$$R = \frac{\ell}{2\pi b\delta_w\sigma_c} = \frac{\ell}{2\pi b}\sqrt{\frac{Z_0\omega}{2c\sigma_c}}$$

$$Z_\parallel(\omega) = [1 - i\,\mathrm{sgn}(\omega)]\frac{\ell}{2\pi b}\sqrt{\frac{Z_0|\omega|}{2c\sigma_c}}$$

Example of longitudinal wake field and coupling impedance: finite conductivity of a circular pipe wall (resistive wall)



The imaginary part of impedance can be justified in this way: the current is flowing through the brown area of thickness $\delta_w \ll b$. From the Ampere's law $(2\pi r \simeq 2\pi b)$

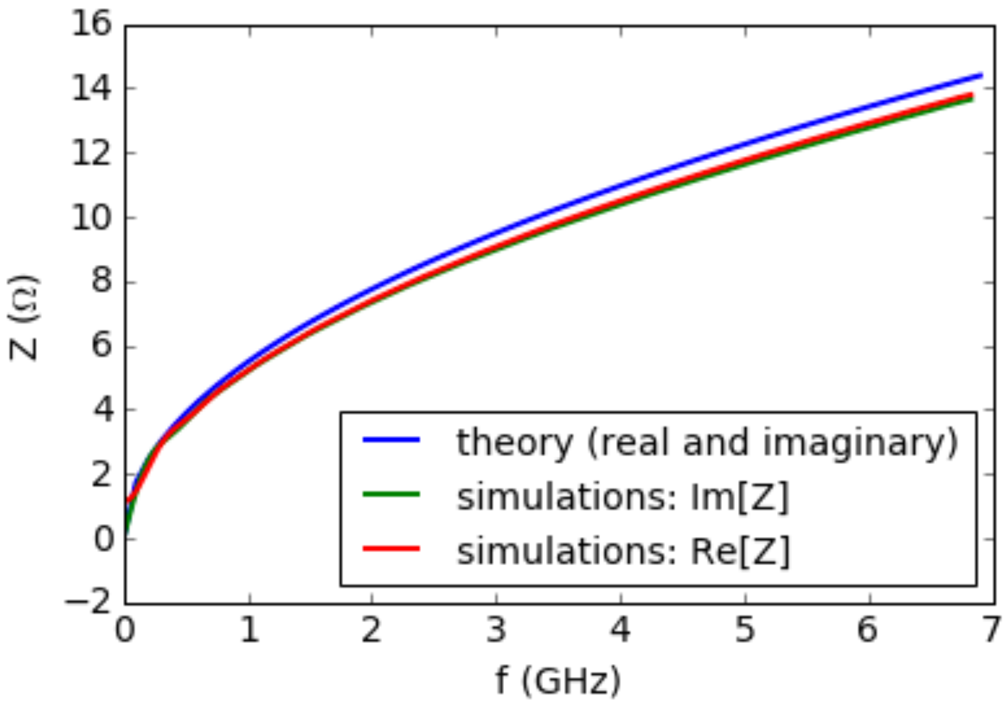
$$2\pi b B = \mu J 2\pi b(r-b) \;\; \rightarrow \;\; B = \mu J (r-b)$$

$$\Phi(B) = \ell \mu J \int_b^{b+\delta_w} (r-b)dr = \frac{\ell \mu J}{2}(r-b)^2 \Big|_b^{b+\delta_w} = \frac{\ell \mu J}{2}\delta_w^2$$

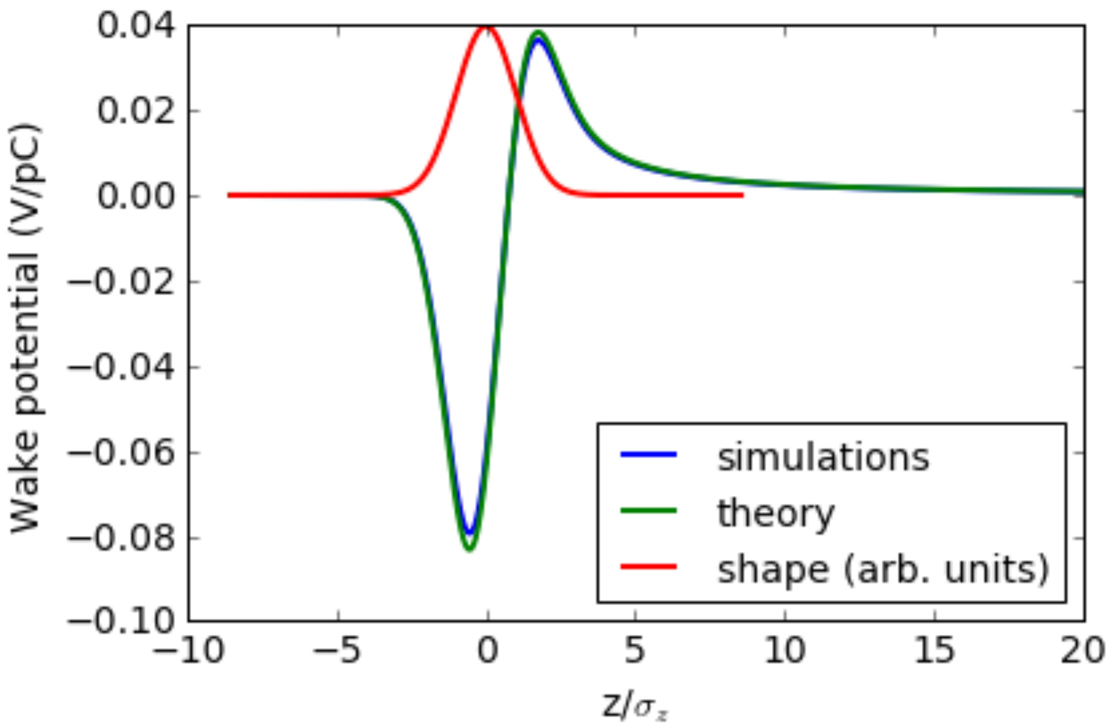
$$I = J 2\pi b \delta_w \;\; \rightarrow \;\; \Phi(B) = \frac{\ell \mu I \delta_w}{4\pi b} \qquad\qquad L = \frac{\Phi}{I} = \frac{\ell \mu \delta_w}{4\pi b}$$

$$Z_{im} = \omega L = \frac{\ell}{2\pi b}\frac{\omega \mu \delta_w}{2} = \mathrm{sgn}(\omega)\,\frac{\ell}{2\pi b}\sqrt{\frac{2\mu^2\omega^2}{4\omega\sigma_c\mu}} = \boxed{\mathrm{sgn}(\omega)\,\frac{\ell}{2\pi b}\sqrt{\frac{Z_0|\omega|}{2c\sigma_c}}}$$

Example of longitudinal wake field and coupling impedance: finite conductivity of a circular pipe wall (resistive wall)



Impedance comparison



Wake potential comparison

Example of longitudinal wake field and coupling impedance: space charge

Even if in the ultra-relativistic limit with $\gamma \longrightarrow \infty$, we have seen that there is no space charge effect, we can still define a wake field by considering a moderately relativistic beam with γ>>1 but not infinite. It turns out that the space charge forces can fit into the definition of wake field, and when that is done, we find that the wake depends on beam properties such as the transverse beam radius a and the beam energy γ. Let us consider a relativistic beam with cylindrical symmetry and uniform transverse distribution. We have already obtained the longitudinal force acting on a charge of the beam travelling inside a cylindrical pipe of radius b:

$$F_{\parallel}(r,z) = \frac{-q}{4\pi\varepsilon_0\gamma^2}\left(1 - \frac{r^2}{a^2} + 2\ln\frac{b}{a}\right)\frac{\partial}{\partial z}\lambda(z)$$

Example of longitudinal wake field and coupling impedance: space charge

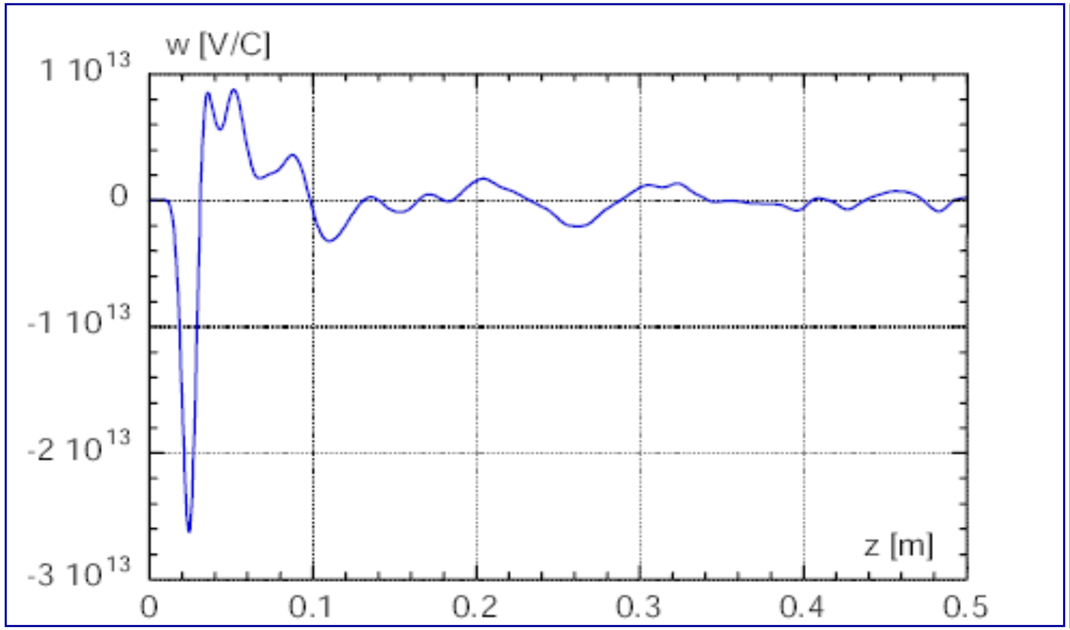
Since the space charge forces move together with the beam, they are constant along the accelerator if the beam pipe remains constant. We can therefore define the longitudinal wake field of a piece of pipe of length ℓ. Assuming r→0 (particle on axis), and a charge line density given by $\lambda(z) = q_0 \delta(z)$, we obtain

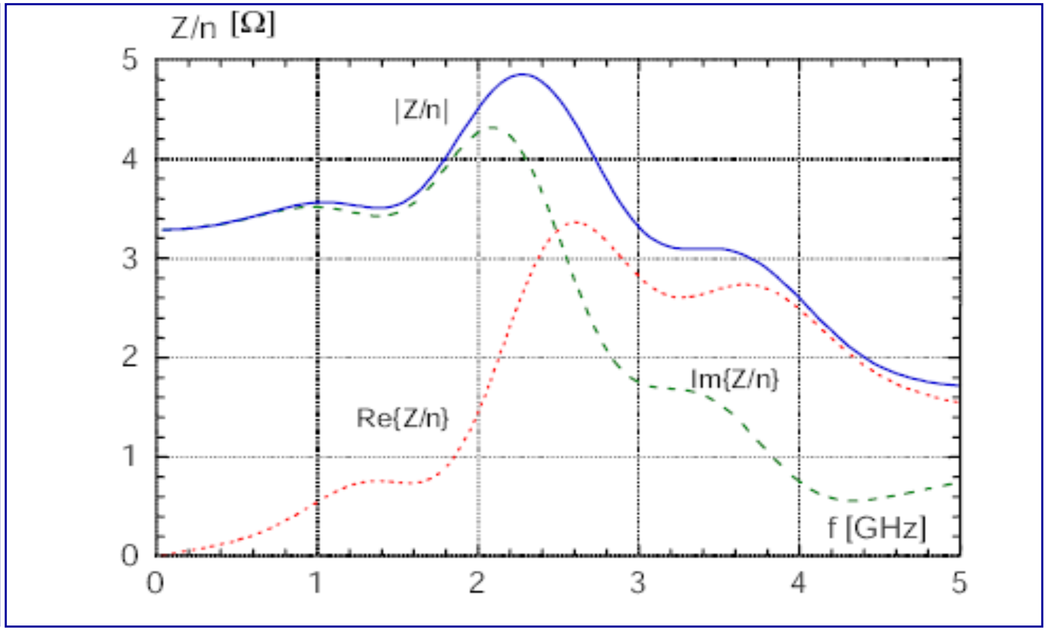
$$w_{\parallel}(z) = -\frac{1}{q q_0}\int_0^L F_{\parallel} ds \qquad w_{\parallel}(z) = \frac{\ell}{4\pi\varepsilon_0\gamma^2}\left(1 + 2\ln\frac{b}{a}\right)\frac{d}{dz}\delta(z)$$

$$Z_{\parallel}(\omega) = \frac{1}{v}\int_{-\infty}^{\infty} w_{\parallel}(z) e^{i\frac{\omega}{v}z} dz = \ell\,\frac{1 + 2\ln(b/a)}{v 4\pi\varepsilon_0\gamma^2}\int_{-\infty}^{\infty} e^{i\frac{\omega}{v}z}\left(\frac{d}{dz}\delta(z)\right) dz$$

$$\boxed{Z_{\parallel}(\omega) = \ell\,\frac{i\omega Z_0}{4\pi c\beta^2\gamma^2}\left(1 + 2\ln\frac{b}{a}\right)}$$

Example of wake potential and longitudinal coupling impedance for an entire machine: DAΦNE accumulator

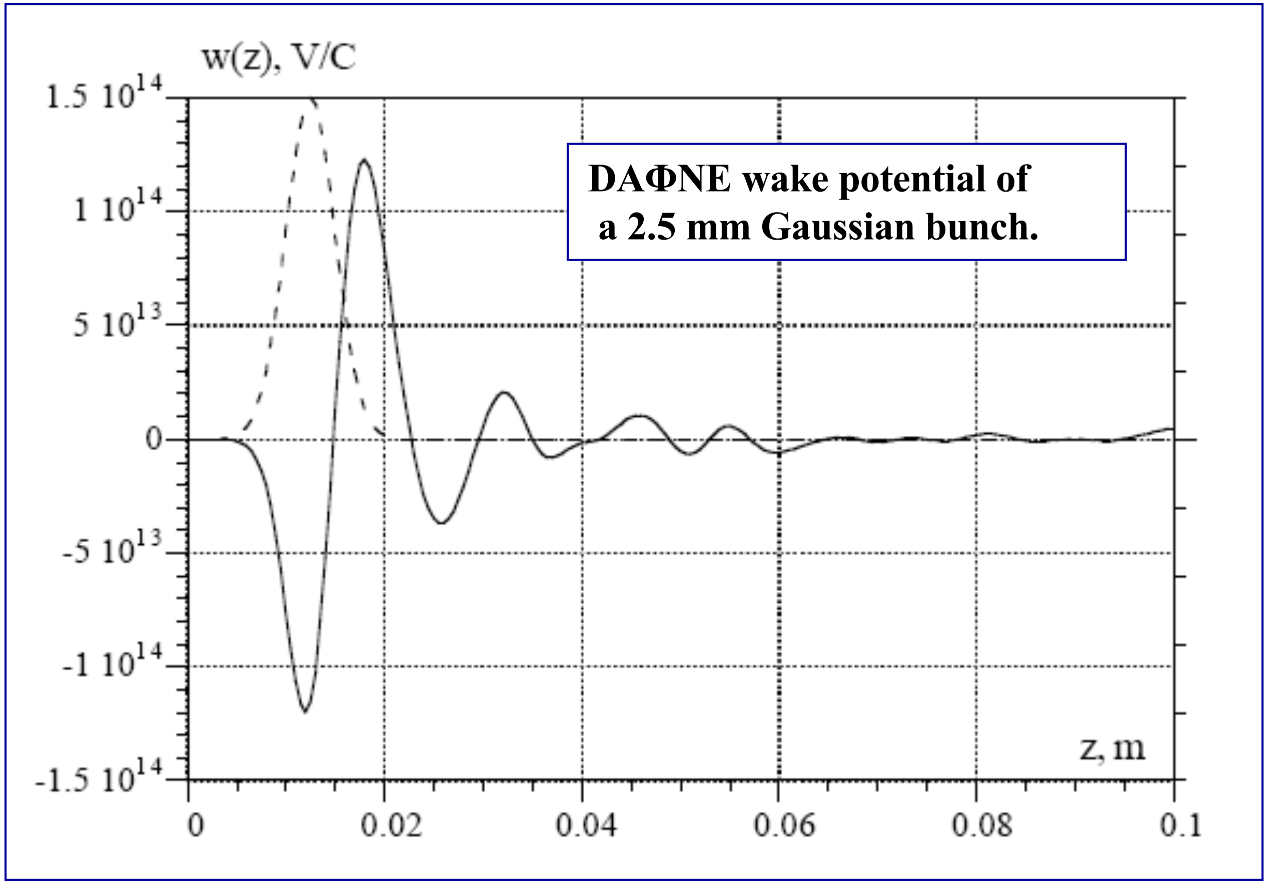




DAΦNE accumulator wake potential of a 2.5 mm Gaussian bunch.

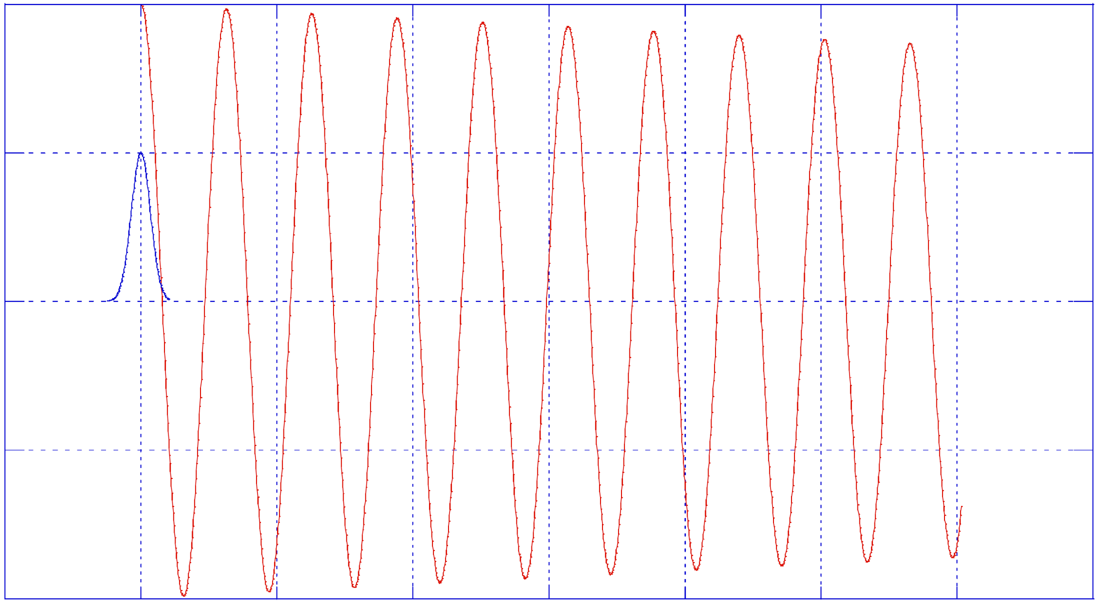
$$\frac{Z_{\|}(\omega)}{n} = \frac{Z_{\|}(\omega)}{\omega/\omega_o}$$

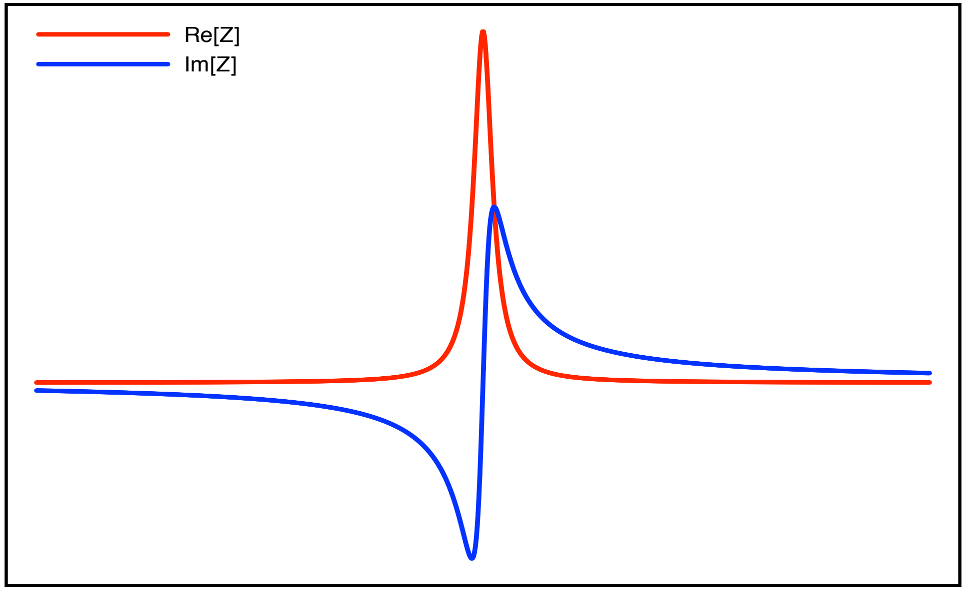
Short range wake field/potential acts over the bunch length



- Vanishes after a distance of few bunch lengths
- Influences the single bunch beam dynamics
- Poor frequency resolution of Fourier transform of coupling impedance => broad band impedance

Long range wake field/potential acts on many bunches/multi-turn

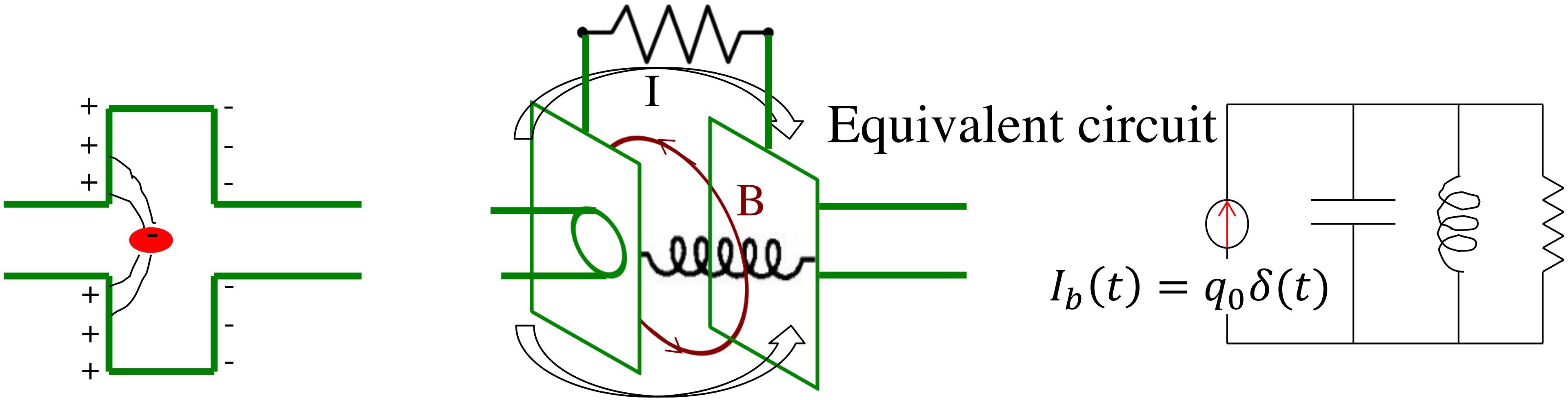




- Field oscillates over long distances
- High peak impedance
- Produced by high quality resonant modes
- Described by only 3 parameters: Q, ω_r and R_s

Longitudinal wake field of a resonant mode

When a charge crosses a resonant structure, as an RF cavity, it excites resonant modes (fundamental and HOMs).



Each mode can be treated as an electric RLC circuit loaded by an impulsive current. Just after the charge passage, the capacitor is charged with a voltage $V_o=q_o$ /C and the electric field is $E_{so}= V_o/l_o$.

The passage of the impulsive current charges only the capacitor, which changes its potential by an amount V_0. This potential will oscillate and decay producing a current flow in the resistor and inductance.

The time evolution of the electric field is governed by the same differential equation of the voltage

$$\ddot{V} + \frac{1}{RC}\dot{V} + \frac{1}{LC}V = 0$$

For $t > 0$ the potential satisfies the following equations and initial conditions:

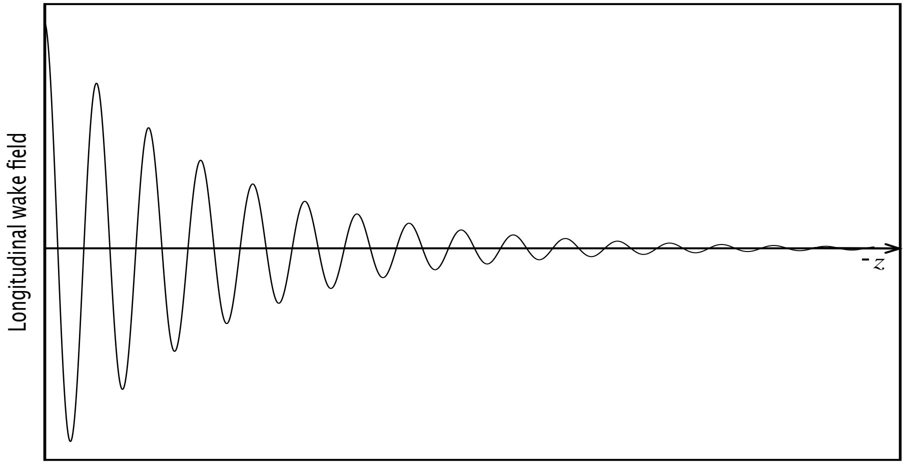
$$V(t) = V_0 e^{-\gamma t}\left[\cos(\omega_n t) - \frac{\gamma}{\omega_n}\sin(\omega_n t)\right]$$

$$V(t = 0^+) = \frac{q_0}{C} = V_0$$

$$\omega_n^2 = \omega_r^2 - \gamma^2 \qquad \omega_r = \frac{1}{\sqrt{LC}}$$

$$\gamma = \frac{1}{2RC}$$

$$\dot{V}(t = 0^+) = \frac{\dot{q}}{C} = -\frac{I(0^+)}{C} = -\frac{V_0}{RC}$$



putting z = -ct (z is negative behind the source charge),

$$w_0 = \frac{1}{C}$$

$$w_{\parallel}(z) = \frac{V(z)}{q_0} = w_0 e^{\frac{\gamma z}{c}}\left[\cos\left(\frac{\omega_n z}{c}\right) + \frac{\gamma}{\omega_n}\sin\left(\frac{\omega_n z}{c}\right)\right] H(-z)$$

Coupling impedances of a resonant mode

Longitudinal Impedance:

$$Z_{\parallel}(\omega) = \frac{R_s}{1 + iQ\left(\frac{\omega_r}{\omega} - \frac{\omega}{\omega_r}\right)}$$

The parameters R_s, Q and ω_r, that can be evaluated by computer codes, can be related to the parameters *RLC* of the parallel circuit

shunt impedance: $R_s = R = \frac{w_0}{2\gamma}$

quality factor: $Q = \frac{\omega_r}{2\gamma}$

Transverse wakefield and impedance of a resonant mode:

$$w_{\perp}(z) = \frac{R_{\perp}\omega_r}{Q} e^{\frac{\Gamma z}{c}} \sin\left(\frac{\overline{\omega} z}{c}\right)$$

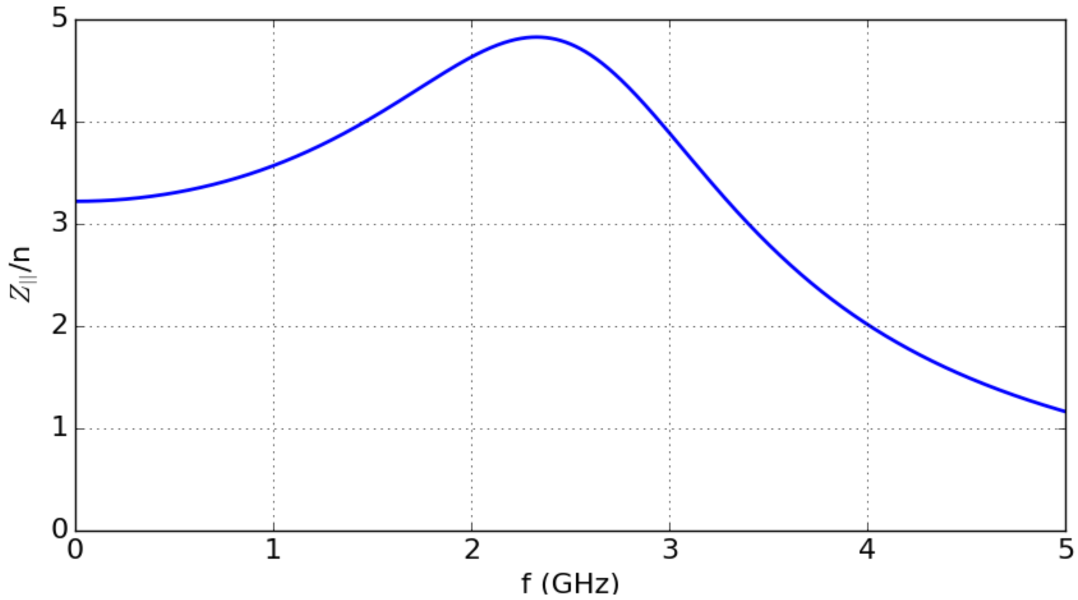
$$Z_{\perp}(\omega) = \frac{\overline{\omega}}{\omega} \frac{R_{\perp}}{1 + iQ\left(\frac{\omega_r}{\omega} - \frac{\omega}{\omega_r}\right)}$$

Some remarks on the longitudinal impedance of a resonant mode

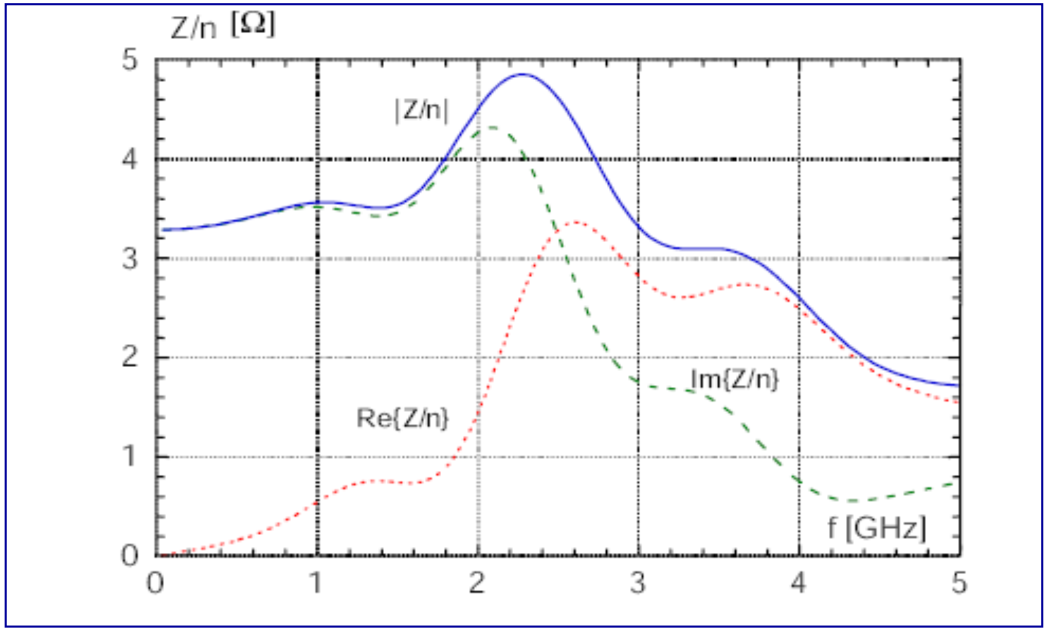
$$Z_{\parallel}(\omega) = \frac{R_s}{1 + iQ\left(\frac{\omega_r}{\omega} - \frac{\omega}{\omega_r}\right)}$$

This impedance can be also used as a simplified impedance model of a whole machine for the short range wake fields assuming Q ~ 1 (it is called **Broad Band Impedance Model**)

Broad Band Resonator Model



DAΦNE Accumulator Impedance



Another Broad Band Impedance model

Another simple **Broad Band Impedance Model** is obtained by a phenomenological expansion over $\sqrt{\omega}$ of the different contributions to a machine impedance. By considering only the first two terms of the expansion, we have the so called RL impedance model

$$Z_{\parallel}(\omega) = R - i\omega L$$

The resistive term R takes into account the losses of the beam, and the second term, which represents an inductive impedance, gives the low frequency behaviour typical of tapers, shielded bellows and vacuum ports, small discontinuities as slots, shallow cavities in flanges …

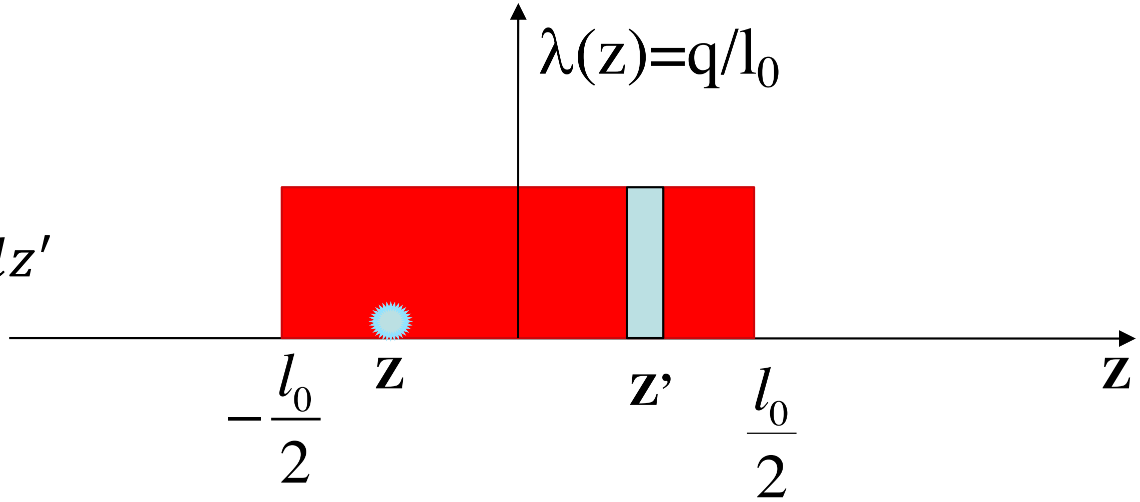
The wake field of the resistive impedance is just proportional to the Dirac delta function $w_{\parallel}(z) = cR\delta(z)$, while that of the inductance is similar to what we have obtained for the space charge.

Wake fields effects in LINACS

Example: Energy lost by a finite uniform beam due to a resonant mode

$$w_{\parallel}(z) = w_0 e^{\frac{\gamma z}{c}}\left[\cos\left(\frac{\omega_n z}{c}\right) + \frac{\gamma}{\omega_n}\sin\left(\frac{\omega_n z}{c}\right)\right] H(-z) \simeq w_0 \cos\left(\frac{\omega_r z}{c}\right) H(-z)$$

$$U(z) = -e\int_{-\infty}^{\infty} w_{\parallel}(z'-z)\lambda(z')dz'$$



$$U(z) = -\frac{eqw_0}{l_0}\int_z^{\frac{l_0}{2}} \cos\left[\frac{\omega_r}{c}(z'-z)\right] dz'$$

$$(z'-z) = x$$

$$U(z) = -\frac{eqw_0}{l_0}\int_0^{\frac{l_0}{2}-z} \cos\left(\frac{\omega_r}{c}x\right) dx =$$

$$= -\frac{eqw_0}{l_0}\left[\frac{\sin\left(\frac{\omega_r}{c}x\right)}{\left(\frac{\omega_r}{c}\right)}\right]_0^{\frac{l_0}{2}-z}$$

What is the wake potential?
What is the energy spread?

$$U(z) = -\frac{eqw_0}{2}\left[\frac{\sin\left[\frac{\omega_r}{c}\left(\frac{l_0}{2}-z\right)\right]}{\left(\frac{\omega_r}{c}\frac{l_0}{2}\right)}\right]$$

Energy loss

$$U_{bunch} = \frac{1}{e}\int_{-\infty}^{\infty} U(z)\lambda(z)dz \simeq \frac{-q^2 w_0}{2l_0\left(\frac{\omega_r}{c}\frac{l_0}{2}\right)}\int_{-\frac{l_0}{2}}^{\frac{l_0}{2}} \sin\left[\frac{\omega_r}{c}\left(\frac{l_0}{2} - z\right)\right] dz$$

$$U_{bunch} = \frac{-q^2 w_0 c}{\omega_r l_0^2}\left|\frac{-\cos\left[\frac{\omega_r}{c}\left(\frac{l_0}{2} - z\right)\right]}{-\frac{\omega_r}{c}}\right|_{-\frac{l_0}{2}}^{\frac{l_0}{2}}$$

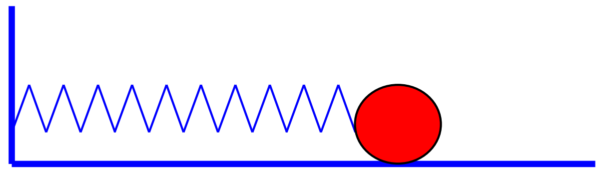
$$U_{bunch} = \frac{-q^2 w_0 c^2}{\omega_r^2 l_0^2}\left[1 - \cos\left(\frac{\omega_r l_0}{c}\right)\right] = -\frac{2q^2 w_0 c^2}{\omega_r^2 l_0^2}\sin^2\left(\frac{\omega_r l_0}{2c}\right)$$

$$U_{bunch} = -\frac{q^2 w_0}{2}\frac{\sin^2\left(\frac{\omega_r l_0}{2c}\right)}{\left(\frac{\omega_r l_0}{2c}\right)^2}$$

$$\lim_{l_0 \to 0} U_{bunch} = -q^2 k = ?$$

Instability: driven oscillator

Consider an harmonic oscillator with natural frequency ω and with an external excitation at frequency Ω. Instead of time, let us use, as independent variable, $s = ct$:



$$x'' + \frac{\omega^2}{c^2} x = A \cos\left(\frac{\Omega s}{c}\right)$$

General solution: ⇨

$$x(s) = x^{free}(s) + x^{driven}(s)$$

$$\cos\left(\frac{\Omega s}{c}\right) \Longrightarrow e^{i\frac{\Omega s}{c}}$$

$$x^{free}(s) = \tilde{x}_m^f e^{i\frac{\omega s}{c}}$$

$$x^{driven}(s) = \tilde{x}_m^d e^{i\frac{\Omega s}{c}}$$

substitution in the diff. equation:

$$(\omega^2 - \Omega^2)\, \tilde{x}_m^d e^{i\frac{\Omega s}{c}} = c^2 A e^{i\frac{\Omega s}{c}}$$

$$x^{driven}(s) = \frac{c^2 A}{(\omega^2 - \Omega^2)} e^{i\frac{\Omega s}{c}}$$

The general solution has to satisfy the initial conditions at s=0. In our case we assume that the oscillator is at rest for s=0:

$$x^{free}(s=0) = -x^{driven}(s=0)$$

$$\tilde{x}_m^f = -\frac{c^2 A}{(\omega^2 - \Omega^2)}$$

thus we get:

$$x(s) = \frac{c^2 A}{(\omega^2 - \Omega^2)}\left(e^{i\frac{\Omega s}{c}} - e^{i\frac{\omega s}{c}}\right)$$

taking only the real part:

$$x(s) = \frac{c^2 A}{(\omega^2 - \Omega^2)}\left[\cos\left(\frac{\Omega s}{c}\right) - \cos\left(\frac{\omega s}{c}\right)\right]$$

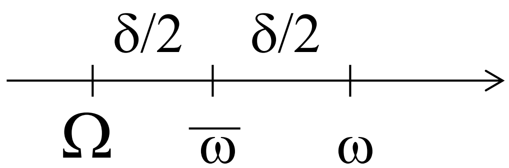
NB: if the initial conditions are different, we just need to add to our solution a sinusoidal term

$$x(s) = X_0 \cos\left(\frac{\omega s}{c} + \theta_0\right) + \frac{c^2 A}{(\omega^2 - \Omega^2)}\left[\cos\left(\frac{\Omega s}{c}\right) - \cos\left(\frac{\omega s}{c}\right)\right]$$

This expression is suitable for deriving the response of the oscillator driven at resonance or at frequency very close:

$$\omega = \Omega + \delta, \qquad \delta \to 0$$

$$\bar{\omega} = (\omega + \Omega)/2; \quad \omega = \bar{\omega} + \delta/2, \qquad \Omega = \bar{\omega} - \delta/2$$



$$\omega^2 - \Omega^2 = (\omega - \Omega)(\omega + \Omega) = \delta 2\bar{\omega}$$

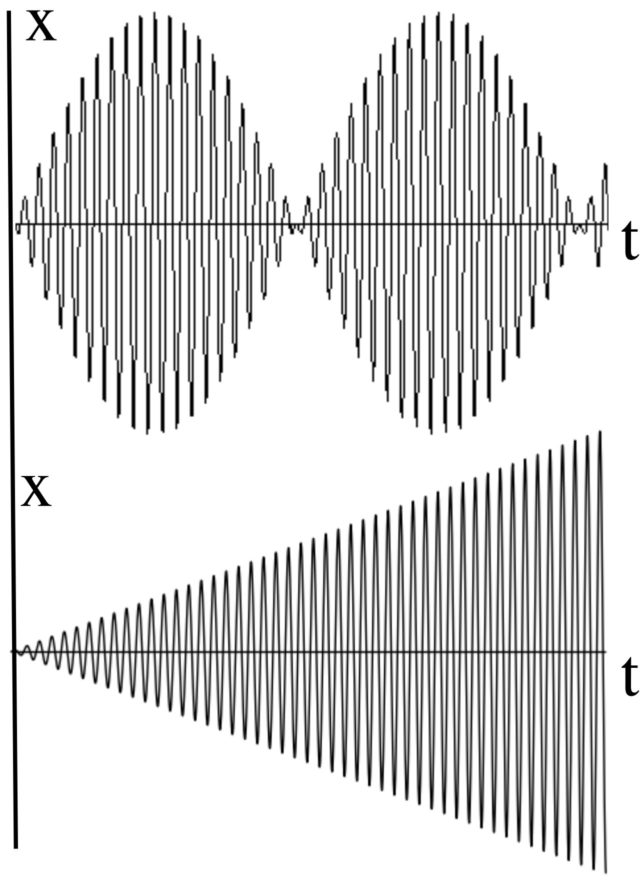
$$x(s) = \frac{c^2 A}{2\bar{\omega}\delta}\left[\cos\left(\frac{\bar{\omega}s}{c}\right)\cos\left(\frac{\delta s}{2c}\right) + \sin\left(\frac{\bar{\omega}s}{c}\right)\sin\left(\frac{\delta s}{2c}\right)\right] + $$

$$-\left[\cos\left(\frac{\bar{\omega}s}{c}\right)\cos\left(\frac{\delta s}{2c}\right) - \sin\left(\frac{\bar{\omega}s}{c}\right)\sin\left(\frac{\delta s}{2c}\right)\right]$$

amplitude modulation

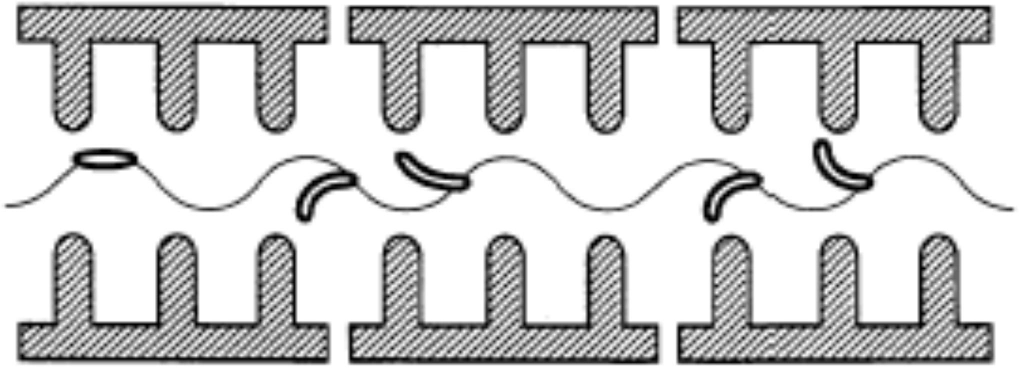
$$x(s) = \overbrace{\frac{c^2 A}{\bar{\omega}\delta}\sin\left(\frac{\delta s}{2c}\right)}\sin\left(\frac{\bar{\omega}s}{c}\right) = \frac{cAs}{2\bar{\omega}}\sin\left(\frac{\bar{\omega}s}{c}\right)\frac{\sin\left(\frac{\delta s}{2c}\right)}{\frac{\delta s}{2c}}$$

$$\lim_{\delta\to 0} x(s) = \frac{cAs}{2\bar{\omega}}\sin\left(\frac{\bar{\omega}s}{c}\right)$$

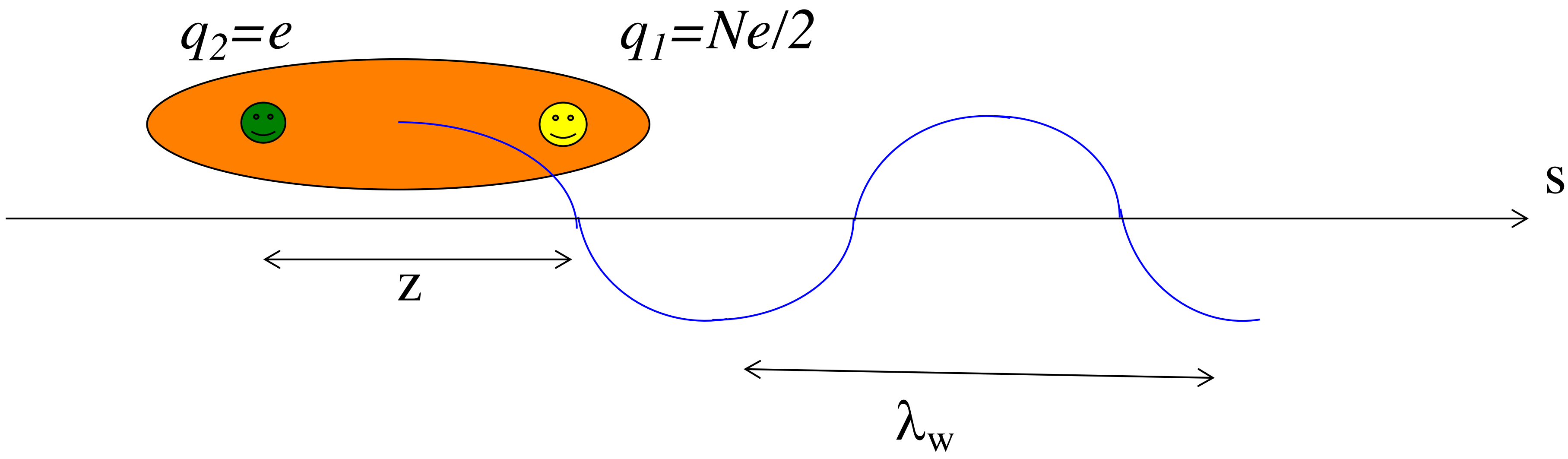


Single Bunch Beam Break Up in Linacs

A beam injected off-centre in a LINAC, because of the focusing quadrupoles, executes betatron oscillations. The displacement produces a transverse wake field in all the devices crossed during the flight, which deflects the trailing charges.



In order to understand the effect, we consider a simple model with only two charges $q_1=Ne/2$ (source charge = half bunch) and $q_2=e$ (test charge = single charge).



the source charge executes free betatron oscillations:

$$y_1(s) = \hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right); \quad \frac{\omega_y}{c} = \frac{2\pi}{\lambda_\beta} = \frac{Q_y}{\rho_x}$$

the test charge, at a distance z behind, over a length L_w experiences a deflecting force proportional to the displacement y_1, and dependent on the distance z:

$$w_\perp(z) = \frac{M(z)}{q^2}$$

$e \qquad Ne/2$

$$r_0 M(z) = \int_0^{L_w} F_\perp ds = \langle F_\perp(r_0, z)\rangle L_w \quad \Longrightarrow \quad \langle F_\perp(r_0, z)\rangle = \frac{Ne^2}{2L_w} w_\perp(z) y_1(s)$$

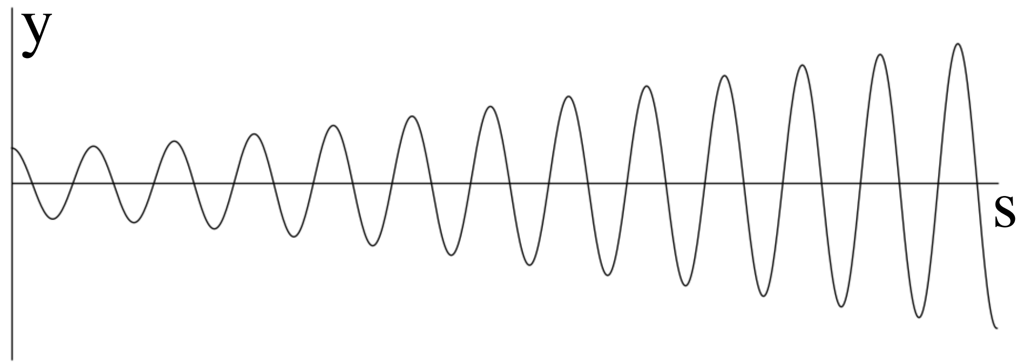
This force drives the motion of the test charge:

betatron equation of motion with coherent force

$$y_2'' + \left(\frac{\omega_y}{c}\right)^2 y_2 = \frac{1}{\beta^2 E_0}\langle F_\perp(r_0, z)\rangle = \frac{Ne^2}{2\beta^2 E_0 L_w} w_\perp(z)\hat{y}_1 \cos\left(\frac{\omega_y}{c}s\right)$$

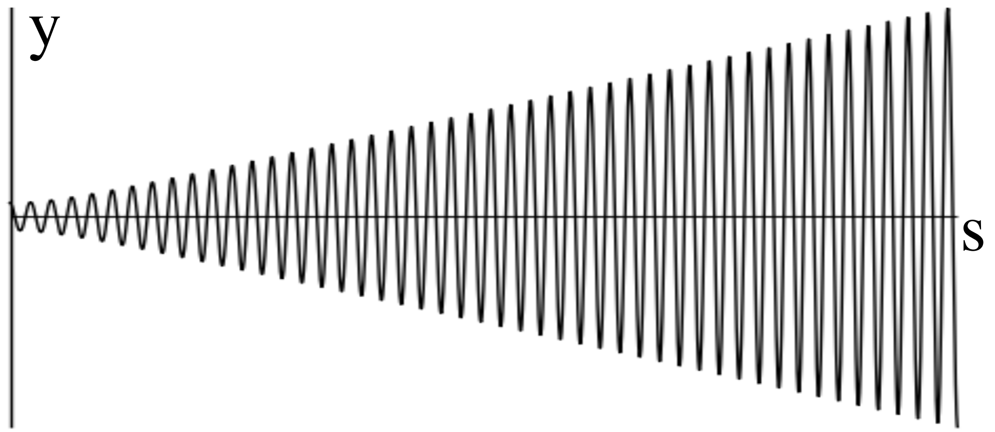
This is the typical equation of an harmonic oscillator driven at the resonant frequency. The solution is given by the superposition of the "free" oscillation and a "driven" oscillation, which, being driven at the resonant frequency, grows linearly with s.

$$y_2(s) = \hat{y}_2 \cos\left(\frac{\omega_y}{c}s\right) + y_2^{driven}$$



$$y_2^{driven} = \frac{cNe^2 w_\perp(z)s}{4\omega_y E_0 L_w}\hat{y}_1 \sin\left(\frac{\omega_y}{c}s\right)$$

$(\beta = 1)$



At the end of the LINAC of length L_L, the oscillation amplitude of the tail with respect to the head is grown by ($\hat{y}_1 = \hat{y}_2$)

$$\left[\frac{y_2(L_L) - y_1(L_L)}{\hat{y}_1}\right]_{max} = \frac{cNew_\perp(z)L_L}{4\omega_y(E_0/e)L_w}$$

Balakin-Novokhatsky-Smirnov Damping

The BBU instability can be quite harmful and hard to take under control even at high energy, with a strong focusing, and after a careful injection and steering.

A simple method to cure it has been proposed observing that the strong oscillation amplitude of the bunch tail is due to the **"resonant" driving force**.

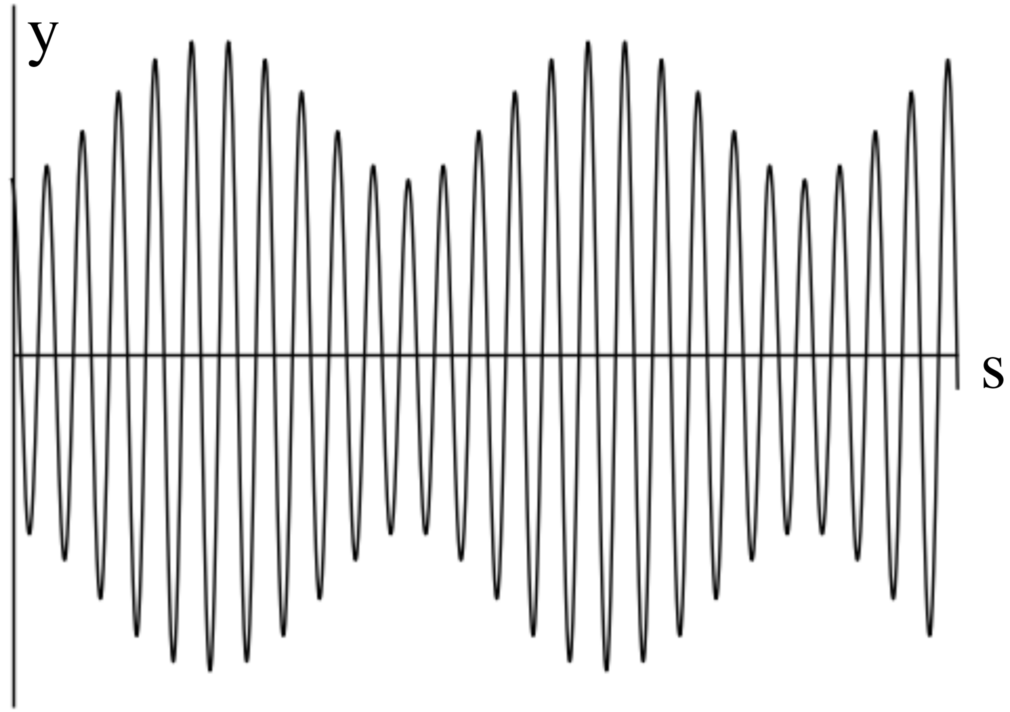
If the tail and the head of the bunch oscillate with different frequencies, this effect can be significantly removed.

Let us assume that the tail oscillates with a frequency $\omega_y + \Delta\omega_y$, the equation of motion becomes:

$$y_2'' + \left(\frac{\omega_y + \Delta\omega_y}{c}\right)^2 y_2 = \frac{Ne^2}{2\beta^2 E_0 L_w} w_\perp(z)\hat{y}_1 \cos\left(\frac{\omega_y}{c}s\right)$$

the solution of which is $(\hat{y}_1 = \hat{y}_2)$

$$y_2(s) = \hat{y}_1 \cos\left(\frac{\omega_y + \Delta\omega_y}{c} s\right) + \frac{c^2 N e^2 w_\perp(z)}{4\omega_y \Delta\omega_y E_0 L_w} \hat{y}_1 \left[\cos\left(\frac{\omega_y}{c} s\right) - \cos\left(\frac{\omega_y + \Delta\omega_y}{c} s\right)\right]$$



by a suitable choice of $\Delta\omega_y$, it is possible to fully depress the oscillations of the tail.

$$\frac{c^2 N e^2 w_\perp(z)}{4\omega_y \Delta\omega_y E_0 L_w} = 1 \quad \Rightarrow \quad \boxed{y_2(s) = \hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right) = y_1(s)}$$

$$\boxed{\Delta\omega_y = \frac{c^2 N e^2 w_\perp(z)}{4\omega_y E_0 L_w}}$$

The extra focusing at the tail can be obtained by:

- Using an RFQ, where head and tail see a different focusing strength.
- Creating a correlated energy distribution along the bunch which, because of the chromaticity, induces a spread in the betatron frequencies. An energy spread correlated with the longitudinal position is attainable with the external accelerating voltage, or with the longitudinal wake fields.

Instabilities in Circular Accelerators

Longitudinal effects on beam dynamics

Short range wake fields:

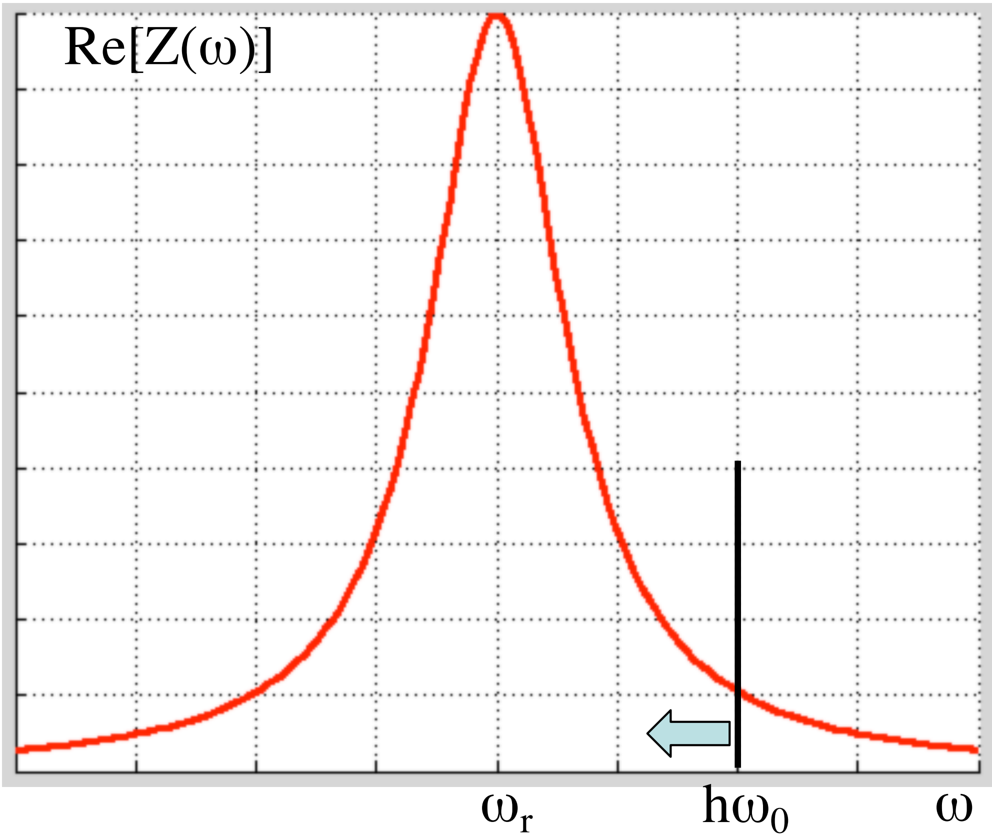
- Potential well distortion ➔ deformation of the longitudinal distribution
- Longitudinal emittance growth, microwave instability

Long range wake fields:

- Robinson instability (RF fundamental mode)
- Coupled bunch instability (HOMs)

Robinson instability of the RF fundamental mode

Let us consider the real part of the RF fundamental mode, and a bunch with revolution period T_0. The bunch spectrum has lines every ω_0 (we suppose the bunch as a point charge), and its lost energy due to the mode is proportional to the real part of the impedance at $h\omega_0$. If the bunch, during the synchrotron oscillations, has an increasing energy, and we are above transition, its revolution period increases and the frequency decreases.



If ($h\omega_0 > \omega_r$), as in the figure, the resistance found by the beam is higher, producing a higer energy loss, which reduces the energy increase giving a stabilizing effect.

Robinson instability of the RF fundamental mode

Longitudinal equations of motion of the bunch centre of mass, for constant energy in a circular machine, ignoring radiation damping

$$\frac{d\phi}{dt} = -\frac{h\eta}{R_0 p_0}\Delta E \qquad\qquad \frac{d\Delta E}{dt} = \frac{qV_{rf}}{T_0}(\sin\phi - \sin\phi_s)$$

Combining the two equations, for small oscillation amplitudes, we obtain a second order linear differential equation

$$\frac{d^2\Delta\phi}{dt^2} + \omega_s^2\Delta\phi = 0 \quad \text{with} \quad \omega_s^2 = \frac{qV_{rf}h\eta c^2\cos\phi_s}{2\pi R_0^2 E_0} \quad \text{and} \quad \eta\cos\phi_s > 0$$

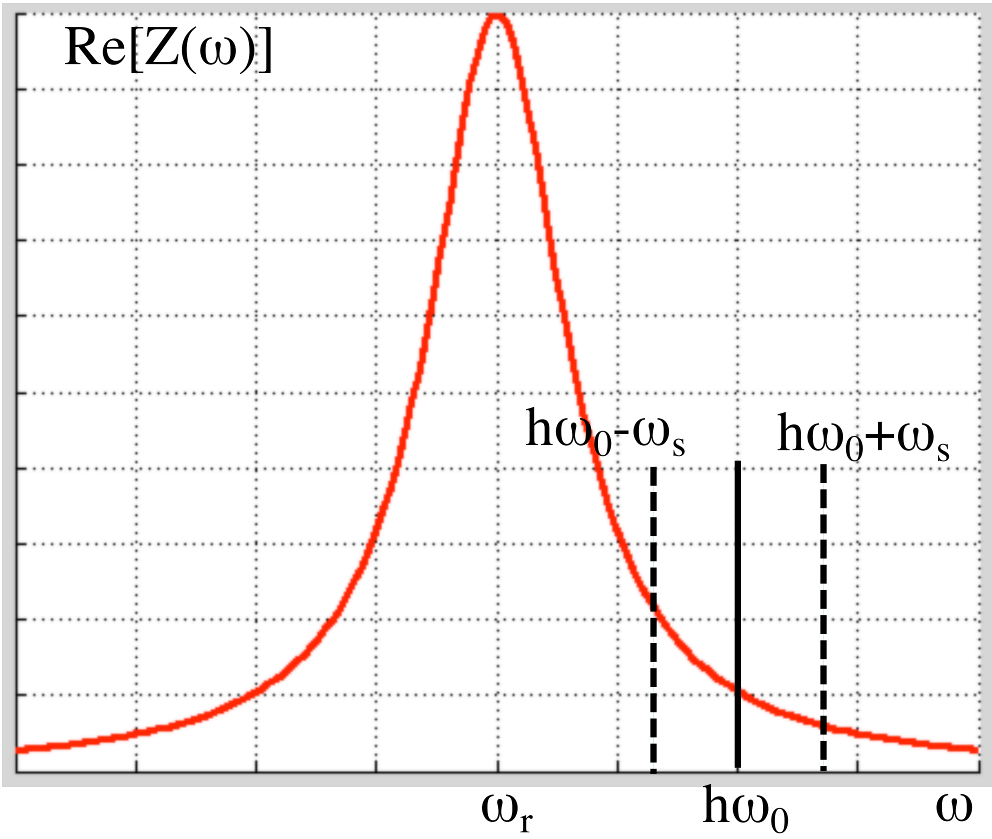
Solution $\quad \Delta\phi = \Delta\phi_{max}\cos(\omega_s t + \theta_0)$

Robinson instability of the RF fundamental mode

By including also the wake field of the fundamental resonant mode (beam loading effect) the equation of motion becomes

damping/exciting term due to the resonant mode

$$\frac{d^2\Delta\phi}{dt^2} + \boxed{2\alpha_r \frac{d\Delta\phi}{dt}} + \omega_s^2 \Delta\phi = 0$$



If $\alpha_r < \omega_s \rightarrow \omega_n = \sqrt{\omega_s^2 - \alpha_r^2}$

Solution

$$\Delta\phi = \Delta\phi_{max} \exp[-\alpha_r t] \cos(\omega_n t + \theta_0)$$

$$\alpha_r = \frac{eN_p \eta h \omega_0}{2\omega_s (E_0/e) T_0^2} \mathrm{Re}[\Delta Z] \qquad \mathrm{Re}[\Delta Z] = \mathrm{Re}[Z(h\omega_0 + \omega_s) - Z(h\omega_0 - \omega_s)]$$

LANDAU DAMPING

- There is a natural stabilising effect against the collective instabilities called “Landau Damping”. The basic mechanism relies on the fact that if the particles in a beam have a spread in their natural frequencies (synchrotron or betatron), their motion can’t be coherent for a long time.
- The mechanism is in general triggered when an infinite set of identical systems oscillates at different frequencies, spread over some range of values. Under these conditions, if any periodic force has its frequency within the considered range, the oscillation amplitude, averaged over all the systems, instead of growing as one should expect, remains constant.
- Even if a periodic force pumps energy into the system, this energy is not converted into an increase of the average oscillation amplitude: the number of particles in resonance with the external force decreases with time, so that the net contribution to the average oscillation amplitude remains constant.

Appendix

Relationship between transverse and longitudinal forces:

The transverse gradient of the longitudinal force is equal to the longitudinal gradient of the transverse force

"**Panofsky-Wenzel theorem**".

$$\nabla_{\perp} F_{//} = \frac{\partial}{\partial z} \boldsymbol{F}_{\perp}$$

$$\nabla_{\perp} w_{//} = \frac{\partial}{\partial z} \boldsymbol{w}_{\perp}$$

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