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NLO+NLL QED corrections to electron PDFs

Based on: 1909.03886 (SF), 1911.12040 (Bertone, Cacciari, SF, Stagnitto)

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Goal: increase the accuracy in the computations of e^+e^- cross sections

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Framework: a factorisation formula

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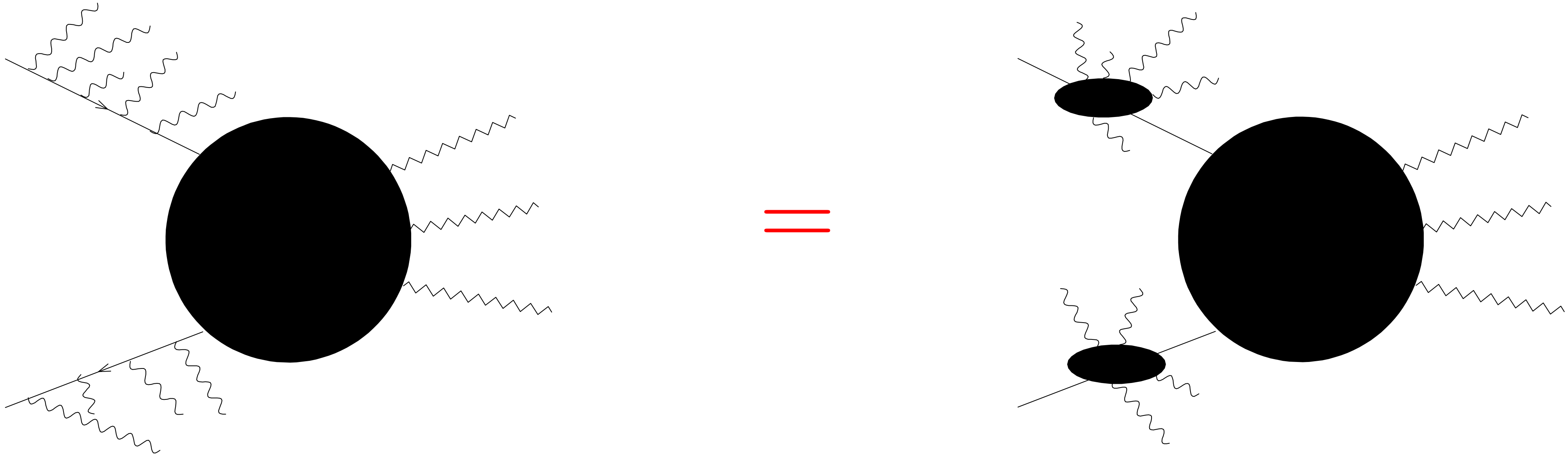
Factorisation



$$\sigma = \text{PDF} \star \text{PDF} \star \hat{\sigma}$$

PDFs collect (universal) small-angle dynamics

Goal: increase the accuracy in the computations of e^+e^- cross sections

Framework: a factorisation formula

- aka structure-function approach: best to *not* use this terminology

By means of: more accurate PDFs

- PDFs aka structure functions: best to *not* use this terminology
- improve the LL+LO accuracy, $(\alpha \log(E/m))^k$, by including NLL+NLO terms, $(\alpha \log(E/m))^k + \alpha\,(\alpha \log(E/m))^{k-1}$, in the PDFs
- the corresponding increased accuracy of short-distance cross sections is widely available, and is understood here

Current z-space LO+LL PDFs $(\alpha \log(E/m))^k$:

- $0 \leq k \leq \infty$ for $z \simeq 1$ (Gribov, Lipatov)
- $0 \leq k \leq 3$ for $z < 1$ (Skrzypek, Jadach; Cacciari, Deandrea, Montagna, Nicrosini; Skrzypek)
- matching between these two regimes

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- matching between these two regimes

Sought z-space NLO+NLL PDFs $(\alpha \log(E/m))^k + \alpha \, (\alpha \log(E/m))^{k-1}$:

- $0 \leq k \leq \infty$ for $z \simeq 1$
- $0 \leq k \leq \{3, 2\}$ for $z < 1 \quad \Longleftrightarrow \quad \mathcal{O}(\alpha^3)$
- matching between these two regimes
- for e^+, e^-, and γ
- both numerical and analytical

Main tool: the solution of PDFs evolution equations

Consider the production of a system X at an e^+e^- collider:

$$e^+(P_{e^+}) + e^-(P_{e^-}) \longrightarrow X$$

Its cross section is written as follows:

$$d\Sigma_{e^+e^-}(P_{e^+}, P_{e^-}) = \sum_{kl} \int dy_+ dy_- \, \mathcal{B}_{kl}(y_+, y_-) \, d\sigma_{kl}(y_+ P_{e^+}, y_- P_{e^-})$$

To be definite, let's stipulate that:

$$k \in \{e^+, \gamma\}, \qquad l \in \{e^-, \gamma\}$$

which is immediate to generalise, if need be. Then:

- $d\Sigma_{e^+e^-}$: the collider-level cross section
- $d\sigma_{kl}$: the particle-level cross section
- $\mathcal{B}_{kl}(y_+, y_-)$: describes beam dynamics
- e^+, e^- on the lhs: the beams
- e^+, e^-, γ on the rhs: the particles

I'll only talk about particles and particle-level cross sections

The parametrisation of beam dynamics is supposed to be given

I sum over polarisations

Write any particle cross section by means of a factorisation formula, quite similar to its QCD counterpart $\longrightarrow$

$$
\begin{aligned}
d\bar{\sigma}_{kl}(p_k, p_l) \quad &= \sum_{ij=e^+,e^-,\gamma} \int dz_+ dz_- \, \Gamma_{i/k}(z_+, \mu^2, m^2) \, \Gamma_{j/l}(z_-, \mu^2, m^2) \\
&\qquad \times d\hat{\sigma}_{ij}(z_+ p_k, z_- p_l, \mu^2) + \Delta
\end{aligned}
$$

with:

$$
d\bar{\sigma}_{kl} = d\sigma_{kl} + \mathcal{O}\left(\left(\frac{m^2}{s}\right)^p\right), \qquad s = (p_k + p_l)^2, \qquad p \geq 1
$$

- $d\bar{\sigma}_{kl}$: the particle-level cross section, with power-suppressed terms discarded
- $d\hat{\sigma}_{ij}$: the subtracted parton-level cross section. Independent of m
- e^+, e^-, γ on the lhs: the particles
- e^+, e^-, γ on the rhs: the partons
- $\Gamma_{i/k}$: the PDF of parton i inside particle k. It can be computed perturbatively
- μ: the hard scale, $m^2 \ll \mu^2 \sim s$

Differences wrt QCD:

- PDFs and power-suppressed terms can be computed perturbatively
- An object (e.g. e^-) may play the role of both particle and parton

As in QCD, a particle is a physical object, a parton is not

$$d\bar{\sigma}_{kl}(p_k, p_l) = \sum_{ij=e^+,e^-,\gamma} \int dz_+ dz_- \, \Gamma_{i/k}(z_+, \mu^2, m^2) \, \Gamma_{j/l}(z_-, \mu^2, m^2)$$

$$\times \, d\hat{\sigma}_{ij}(z_+ p_k, z_- p_l, \mu^2) + \Delta$$

This formula can be used in several ways:

A: to solve for the PDFs, given the particle and parton cross sections

B: for the computation of the particle cross section, given the parton cross section and the PDFs

C: for cross checks, given both cross sections and the PDFs

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\end{aligned}
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This formula can be used in several ways:

A: to solve for the PDFs, given the particle and parton cross sections

- Strategy used in 1909.03886 for the computation of NLO-accurate initial conditions (strict perturbative expansion)

B: for the computation of the particle cross section, given the parton cross section and the PDFs

- Being done, using the NLL-*evolved* PDFs obtained in 1911.12040

C: for cross checks, given both cross sections and the PDFs

- No phenomenological interest

Henceforth, I consider the dominant production mechanism at an e^+e^- collider, namely that associated with partons inside an electron*

Simplified notation:

$$\Gamma_i(z,\mu^2) \;\equiv\; \Gamma_{i/e^-}(z,\mu^2)$$

*The case of the positron is identical, at least in QED, and will be understood

NLO initial conditions (1909.03886)

Conventions for the perturbative coefficients:

$$\Gamma_i = \Gamma_i^{[0]} + \frac{\alpha}{2\pi}\,\Gamma_i^{[1]} + \mathcal{O}(\alpha^2)$$

Results:

$$\begin{aligned}
\Gamma_i^{[0]}(z,\mu_0^2) &= \delta_{ie^-}\delta(1-z) \\
\Gamma_{e^-}^{[1]}(z,\mu_0^2) &= \left[\frac{1+z^2}{1-z}\left(\log\frac{\mu_0^2}{m^2} - 2\log(1-z) - 1\right)\right]_+ + K_{ee}(z) \\
\Gamma_{\gamma}^{[1]}(z,\mu_0^2) &= \frac{1+(1-z)^2}{z}\left(\log\frac{\mu_0^2}{m^2} - 2\log z - 1\right) + K_{\gamma e}(z) \\
\Gamma_{e^+}^{[1]}(z,\mu_0^2) &= 0
\end{aligned}$$

Note:

- Meaningful only if $\mu_0 \sim m$
- In $\overline{\mathrm{MS}}$, $K_{ij}(z) = 0$; in general, these functions *define* an IR scheme

NLL evolution (1911.12040)

General idea: solve the evolution equations starting from the initial conditions computed previously

$$\frac{\partial \Gamma_i(z,\mu^2)}{\partial \log \mu^2} = \frac{\alpha(\mu)}{2\pi}\left[P_{ij} \otimes \Gamma_j\right](z,\mu^2) \quad \Longleftrightarrow \quad \frac{\partial \Gamma(z,\mu^2)}{\partial \log \mu^2} = \frac{\alpha(\mu)}{2\pi}\left[\mathbb{P} \otimes \Gamma\right](z,\mu^2)\,,$$

Done conveniently in terms of non-singlet, singlet, and photon

Two ways:

- Mellin space: suited to both numerical solution and all-order, large-z analytical solution (called *asymptotic solution*)
- Directly in z space in an integrated form: suited to fixed-order, all-z analytical solution (called *recursive solution*)

A technicality: owing to the running of α, it is best to evolve in t rather than in μ, with: (∼ Furmanski, Petronzio)

$$\begin{aligned} t &= \frac{1}{2\pi b_0} \log \frac{\alpha(\mu)}{\alpha(\mu_0)} \\ &= \frac{\alpha(\mu)}{2\pi} L - \frac{\alpha^2(\mu)}{4\pi} \left(b_0 L^2 - \frac{2b_1}{b_0} L \right) + \mathcal{O}(\alpha^3) \,, \qquad L = \log \frac{\mu^2}{\mu_0^2} \,. \end{aligned}$$

Note:

- $t \longleftrightarrow \mu$; notation-wise, the dependence on t is equivalent to the dependence on μ
- $t = 0 \iff \mu = \mu_0$
- L is my "large log"
- Tricky: fixed-α expressions are obtained with $t = \alpha L/(2\pi)$ (and not $t = 0$)

Mellin space

Introduce the evolution operator $\mathbb{E}_N$

$$\Gamma_N(\mu^2) = \mathbb{E}_N(t)\,\Gamma_{0,N}\,, \qquad \mathbb{E}_N(0) = I\,, \qquad \Gamma_{0,N} \equiv \Gamma_N(\mu_0^2)$$

The PDFs evolution equations are then re-expressed by means of an evolution equation for the evolution operator:

$$\begin{aligned}\frac{\partial \mathbb{E}_N(t)}{\partial t} &= \frac{b_0 \alpha^2(\mu)}{\beta(\alpha(\mu))} \sum_{k=0}^{\infty} \left(\frac{\alpha(\mu)}{2\pi}\right)^k \mathbb{P}_N^{[k]}\, \mathbb{E}_N(t) \\ &= \left[\mathbb{P}_N^{[0]} + \frac{\alpha(\mu)}{2\pi}\left(\mathbb{P}_N^{[1]} - \frac{2\pi b_1}{b_0}\,\mathbb{P}_N^{[0]}\right)\right] \mathbb{E}_N(t) + \mathcal{O}(\alpha^2)\end{aligned}$$

- Can be solved numerically
- Can be solved analytically in a closed form under simplifying assumptions. Chiefly: large-z is equivalent to large-N
- I'll show results for the non-singlet $\equiv$ singlet. The photon is feasible as well (see 1911.12040), but technically very involved

Show first that this formalism allows one to quickly re-obtain the known LL result:

$$\Gamma_{0,N}^{[0]} = 1 \quad \Longrightarrow \quad \Gamma_{\rm LL}(z,\mu^2) = M^{-1}\big[\exp\big(\log E_N\big)\big]$$

From the explicit expression of the AP ff kernel:

$$\log E_N = \frac{\alpha}{2\pi} P_N^{[0]} L \;\stackrel{N\to\infty}{\longrightarrow}\; -\eta_0 \left(\log \bar{N} - \lambda_0\right)$$

$$\eta_0 = \frac{\alpha}{\pi} L \,, \qquad \bar{N} = N\, e^{\gamma_{\rm E}} \,, \qquad \lambda_0 = \frac{3}{4}$$

The computation of the inverse Mellin transform is trivial:

$$\Gamma_{\rm LL}(z,\mu^2) = \frac{e^{-\gamma_{\rm E}\eta_0} e^{\lambda_0 \eta_0}}{\Gamma(1+\eta_0)}\, \eta_0 (1-z)^{-1+\eta_0}$$

The usual form, bar for the "-1" of soft origin (we're resumming collinear logs here)

The NLL case is only slightly more complicated; we use:

$$\Gamma_{\rm NLL}(z,\mu^2) = M^{-1}\big[\exp\big(\log E_N\big)\big] \otimes \Gamma_{\rm NLO}(z,\mu_0^2)$$

which is convenient because the form of the evolution operator is functionally the same as at the LL:

$$\log E_N \stackrel{N\to\infty}{\longrightarrow} -\xi_1 \log \bar{N} + \hat{\xi}_1$$

with:

$$\begin{aligned}
\xi_1 &= 2t - \frac{\alpha(\mu)}{4\pi^2 b_0}\left(1 - e^{-2\pi b_0 t}\right)\left(\frac{20}{9} n_{_F} + \frac{4\pi b_1}{b_0}\right) \\
&= 2t + \mathcal{O}(\alpha t) = \eta_0 + \ldots \\
\hat{\xi}_1 &= \frac{3}{2}\, t + \frac{\alpha(\mu)}{4\pi^2 b_0}\left(1 - e^{-2\pi b_0 t}\right)\left(\lambda_1 - \frac{3\pi b_1}{b_0}\right) \\
&= \frac{3}{2}\, t + \mathcal{O}(\alpha t) = \lambda_0 \eta_0 + \ldots \\
\lambda_1 &= \frac{3}{8} - \frac{\pi^2}{2} + 6\zeta_3 - \frac{n_{_F}}{18}(3 + 4\pi^2)
\end{aligned}$$

Thence:

$$\Gamma_{\rm NLL}(z,\mu^2) = \frac{e^{-\gamma_{\rm E}\xi_1}\, e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)}\, \xi_1 (1-z)^{-1+\xi_1}$$
$$\times \left\{ 1 + \frac{\alpha(\mu_0)}{\pi} \left[\left(\log\frac{\mu_0^2}{m^2} - 1 \right) \left(A(\xi_1) + \frac{3}{4} \right) - 2B(\xi_1) + \frac{7}{4} \right.\right.$$
$$\left.\left. + \left(\log\frac{\mu_0^2}{m^2} - 1 - 2A(\xi_1) \right) \log(1-z) - \log^2(1-z) \right] \right\}$$

where:

$$A(\kappa) = -\gamma_{\rm E} - \psi_0(\kappa)$$
$$B(\kappa) = \frac{1}{2}\,\gamma_{\rm E}^2 + \frac{\pi^2}{12} + \gamma_{\rm E}\,\psi_0(\kappa) + \frac{1}{2}\,\psi_0(\kappa)^2 - \frac{1}{2}\,\psi_1(\kappa)$$

z space

Use integrated PDFs (so as to simplify the treatment of endpoints)

$$\mathcal{F}(z,t) = \int_0^1 dy\, \Theta(y-z)\, \Gamma(y,\mu^2) \quad \Longrightarrow \quad \Gamma(z,\mu^2) = -\frac{\partial}{\partial z}\mathcal{F}(z,t)$$

in terms of which the formal solution of the evolution equation is:

$$\mathcal{F}(z,t) = \mathcal{F}(z,0) + \int_0^t du\, \frac{b_0 \alpha^2(u)}{\beta(\alpha(u))} \left[\mathbb{P}\,\overline{\otimes}\,\mathcal{F}\right](z,u)$$

By inserting the representation:

$$\mathcal{F}(z,t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(\mathcal{J}_k^{\mathrm{LL}}(z) + \frac{\alpha(t)}{2\pi}\, \mathcal{J}_k^{\mathrm{NLL}}(z) \right)$$

on both sides of the solution, one obtains recursive equations, whereby a $\mathcal{J}_k$ is determined by all $\mathcal{J}_p$ with $p < k$. The recursion starts from $\mathcal{J}_0$, which are the integrated initial conditions

For the record, the recursive equations are:

$$
\begin{aligned}
\mathcal{J}_k^{\rm LL} &= \mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1}^{\rm LL} \\
\mathcal{J}_k^{\rm NLL} &= (-)^k (2\pi b_0)^k \mathcal{F}^{[1]}(\mu_0^2) \\
&\quad + \sum_{p=0}^{k-1} (-)^p (2\pi b_0)^p \Bigg(\mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\rm NLL} + \mathbb{P}^{[1]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\rm LL} \\
&\qquad - \frac{2\pi b_1}{b_0} \mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\rm LL} \Bigg)
\end{aligned}
$$

We have computed these for $k \leq 3$ ($\mathcal{J}^{\rm LL}$) and $k \leq 2$ ($\mathcal{J}^{\rm NLL}$), ie to $\mathcal{O}(\alpha^3)$

Results in 1911.12040 and its ancillary files

A remarkable fact

Our asymptotic solutions, expanded in α, feature *all* of the terms:

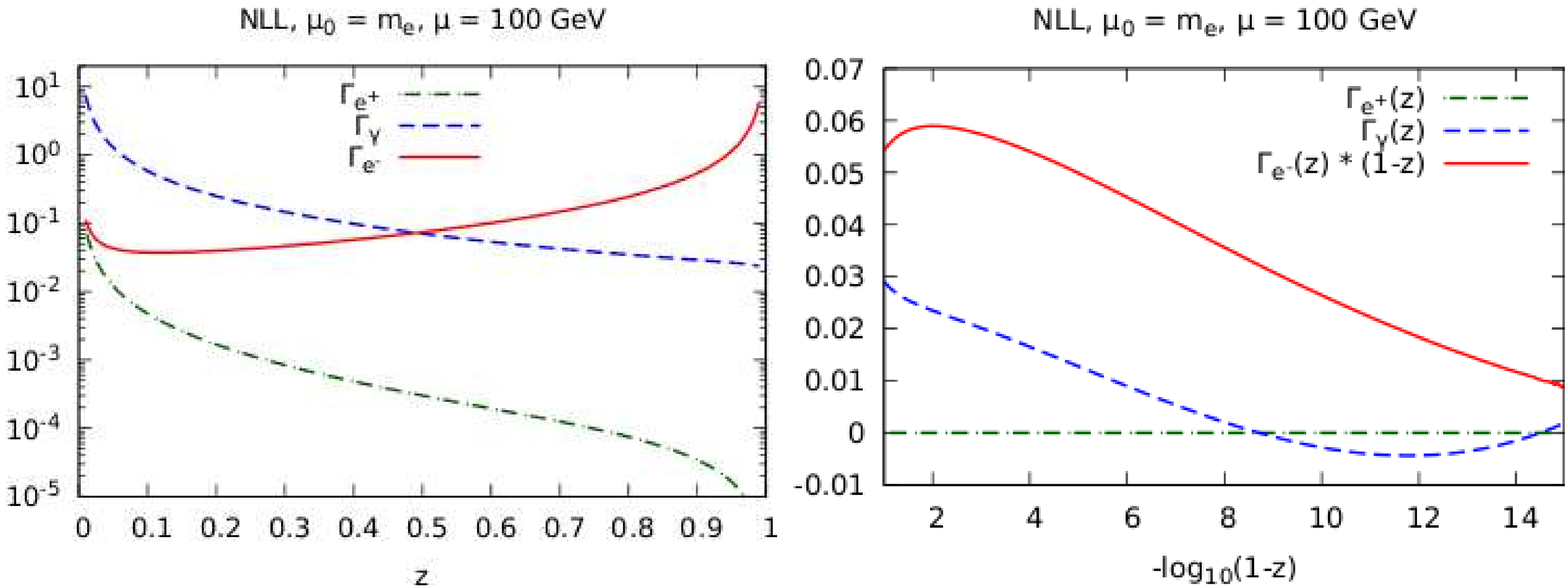
$$\frac{\log^q(1-z)}{1-z} \qquad \mathrm{singlet,\ non-singlet}$$
$$\log^q(1-z) \qquad \mathrm{photon}$$

of our recursive solutions

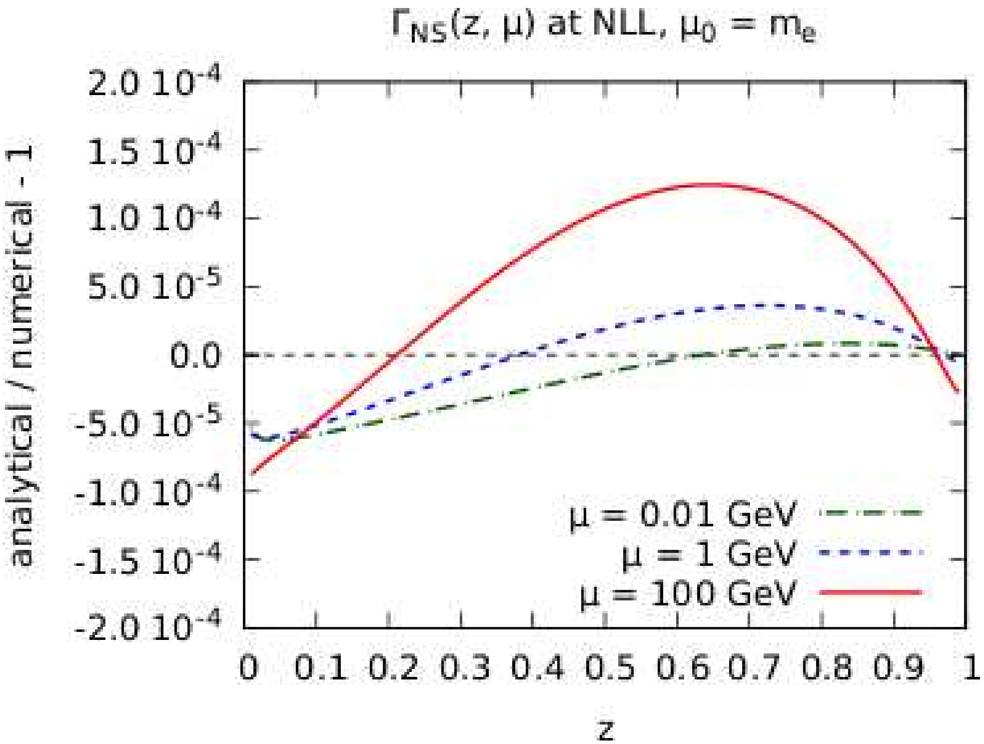
Non-trivial; stems from keeping subleading terms (at $z \to 1$) in the AP kernels

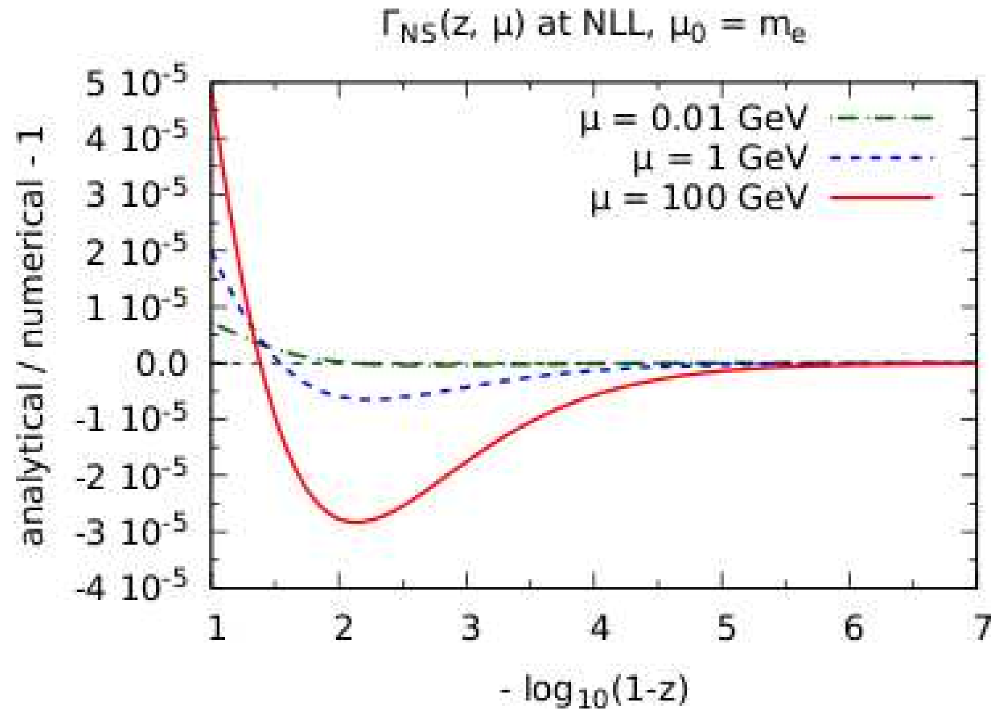
Illustrative results for PDFs

- Analytical results obtained by means of an additive matching between the recursive and the asymptotic solutions
- All are in $\overline{\mathrm{MS}}$
- Bear in mind that PDFs are unphysical quantities

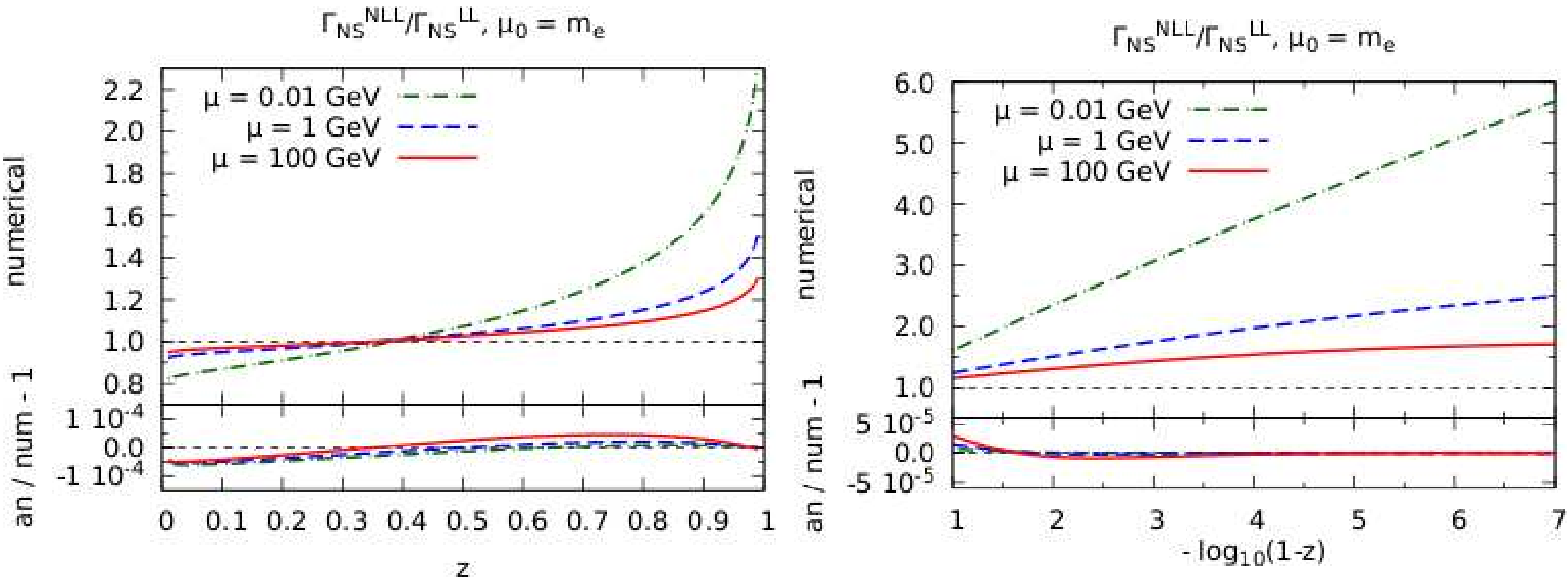


e^- vs γ vs e^+. Note that e^- in the right-hand panel is strongly damped





Numerical vs analytical, non-singlet



NLL vs LL, non-singlet. The insets show the double ratio, ie numerical vs analytical

In order to understand the large-z bit of the previous plots:

$$\Gamma_{\rm LL}(z,\mu^2) = \frac{e^{-\gamma_{\rm E}\eta_0} e^{\lambda_0\eta_0}}{\Gamma(1+\eta_0)}\, \eta_0 (1-z)^{-1+\eta_0}$$

$$\Gamma_{\rm NLL}(z,\mu^2) = \frac{e^{-\gamma_{\rm E}\xi_1} e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)}\, \xi_1 (1-z)^{-1+\xi_1}$$

$$\times \left\{1 + \frac{\alpha(\mu_0)}{\pi}\left[\left(\log\frac{\mu_0^2}{m^2} - 1\right)\left(A(\xi_1) + \frac{3}{4}\right) - 2B(\xi_1) + \frac{7}{4}\right.\right.$$

$$\left.\left. + \left(\log\frac{\mu_0^2}{m^2} - 1 - 2A(\xi_1)\right)\log(1-z) - \log^2(1-z)\right]\right\}$$

with:

$$\xi_1 \simeq \eta_0\,, \qquad \hat{\xi}_1 \simeq \lambda_0\eta_0$$

$$A(\kappa) \;=\; \frac{1}{\kappa} + \mathcal{O}(\kappa) \;\Longrightarrow\; \log(1-z) \text{ dominates}$$

$$B(\kappa) \;=\; -\frac{\pi^2}{6} + 2\zeta_3\kappa + \mathcal{O}(\kappa^2)$$

Conclusions

- We have computed all NLO initial conditions for PDFs and FFs (1909.03886), unpolarised

- We have NLL-evolved those relevant to the electron PDFs (1911.12040), both analytically and numerically

- These can be obtained at:

  `https://github.com/gstagnit/ePDF`

Many results are based on establishing a "dictionary" QCD $\longrightarrow$ QED, which works at any order in α_S and α

Being done/to be done

- Assess the impact of PDFs NLL effects on physical cross sections

- The inclusion of these results in MG5_aMC@NLO v3.X is the only missing ingredient in the latter for the computation of NLO QED corrections in e^+e^- collisions

  NLO QCD+EW in hh collisions and NLO QCD in e^+e^- collisions already OK

- γ PDFs; soft effects; alternative IR schemes; FFs

- Polarisations?