Nuclear methods for astrophysical purposes

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Czech Republic contribution to SPIRAL2

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Overview

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Nucleosynthesis in our universe (apart from H, He and partly Li) mainly takes place inside stars. Stellar objects can be considered – in first approximation – as an isotropic system of self-gravitating particles, whose equilibrium can be described by the virial theorem:

$$< T >= -\frac{1}{2} < U >$$

Being T the total kinetic energy and U the gravitational potential.

To mantain its equilibrium, a star must than spend half of the gained gravitational energy (gained by contraction) to rise its temperature. The other half will be lost by radiation.

While the temperature increases, the contraction slows down due to the increasing internal pressure. This is caused by internal energy production, e.g, **nuclear fusion**

Overview (charged particles)

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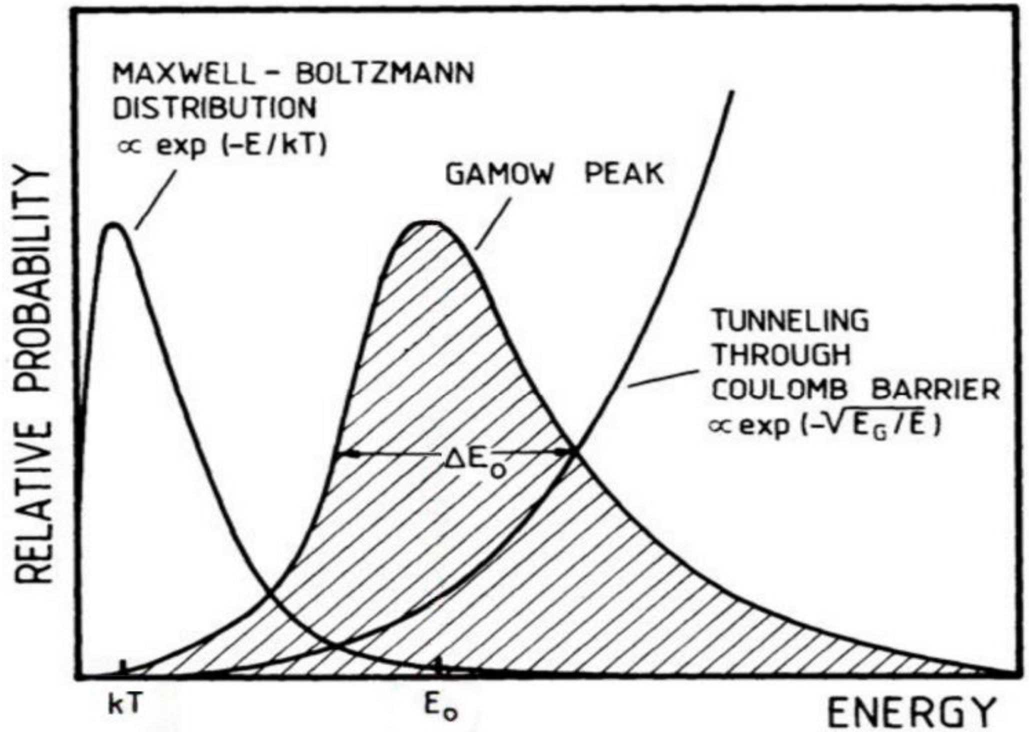
Particles inside stars can interact with each other due to the rise in temperature, and the probability is governed by the Maxwell-Boltzmann distribution, P ∝ *exp*(-E/kT).
The energies involved are of the order of hundreds of keV (tail of the MW distribution) or lower. This value is several orders of magnitude lower than the Coulomb barrier (usually some MeV).

Reacion are possible due to the tunnel effect: the convolution between the Maxwell-Boltzmann distribution and the tunneling probability, P ∝ *exp*$[-(E_G/E)^{1/2}]$, gives rise to the so-called **Gamow window**



[D. D. Clayton, *Principles of stellar evolution and nucleosynthesis, 1983*]

Overview (Nuclear quantities/1)

Reaction rate for two interacting particles (*a* and *A*)

$$r = (1 + \delta_{aA})^{-1} N_a N_A \left(\frac{8}{\pi\mu}\right)^{1/2} \frac{1}{(kT)^{3/2}} \int_0^\infty E\sigma(E)\exp\left(-\frac{E}{kT}\right) \mathrm{dE}$$

N_a and N_A = number of interacting particles

μ = reduced mass

k = Boltzmann constant

T = temperature

$\sigma(E)$ = cross-section as a function of the energy E

Overview (Nuclear quantities/2)

Cross section σ: probability for a certain reaction (scattering, direct capture, resonant capture, compound nucleus…) to occur

$$\sigma = \frac{J_{out}}{J_{inc}\rho_T}$$

being J_{inc} and J_{out} the flux of incoming and outgoing particles and ρ_T the surface density of target particles

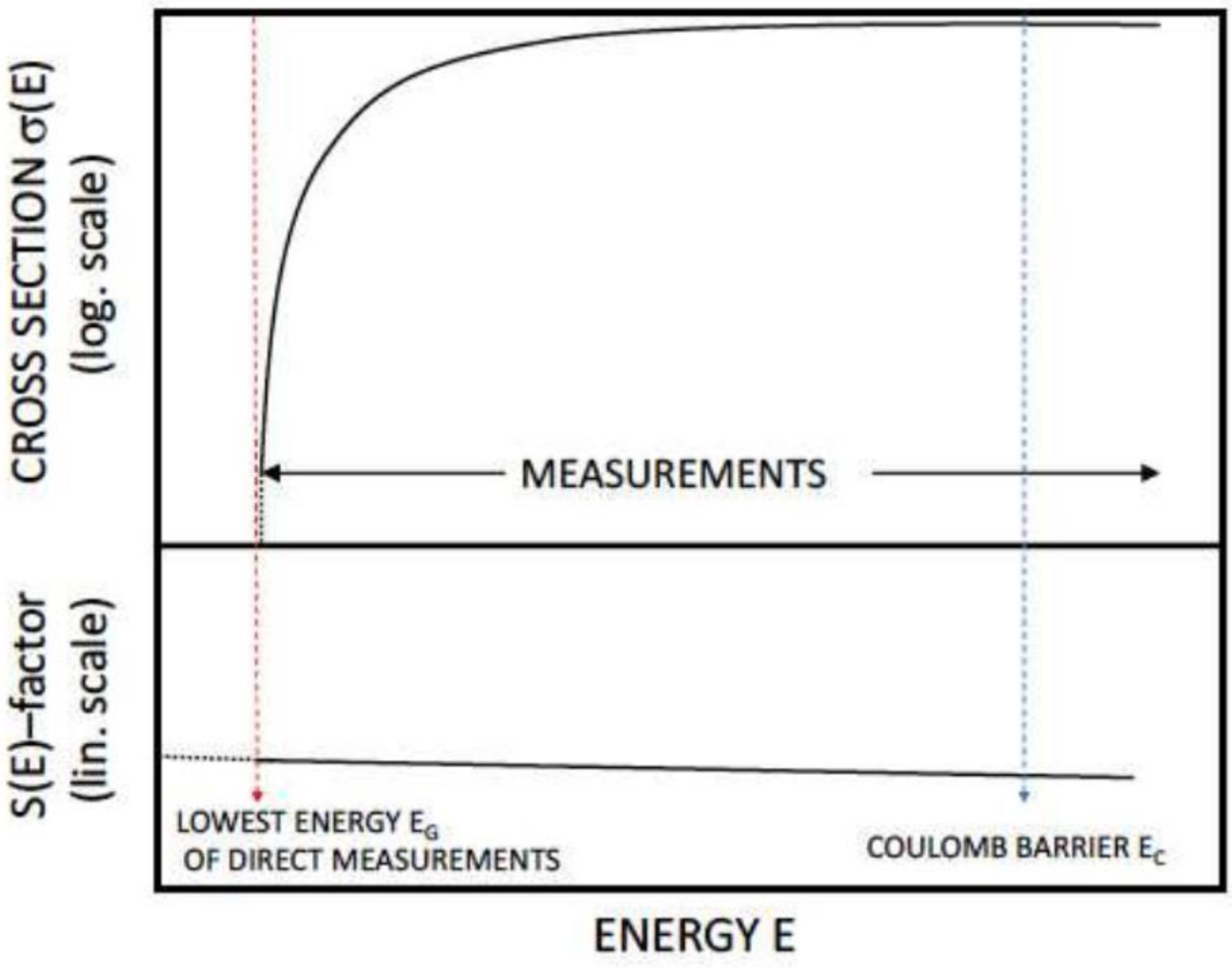
Astrophysical *S*-factor, *S(E)*

$$S(E) = E\sigma(E)\exp(2\pi\eta)$$

with $\eta = Z_1 Z_2 \alpha\beta$, being α the hyper-fine constant structure and β the relative velocity in units of *c*

The η parameter (Sommerfeld parameter) gives a measurement of the Coulomb interaction

Overview (Nuclear quantities/4)



Direct methods

At low energies, the presence of the Coulomb barrier strongly decreases the cross-section, and the presence of background noises makes the detection of the reaction really difficult, when not virtually impossible.

For example, the $p + p \rightarrow d + e^{+} + \nu_e$ typical of our sun (T = $1.5 \cdot 10^{7}$ K, E_B= 0,45 MeV, the weakest in the *pp*-chain) has total cross section $\sigma = 10^{-23}$ b, corresponding to one event every **1000000000 years**!

To gain information over such «unlikely» events it is possible:

- Increase the beam intensity
- Increase the density of the target
- Increase the solid range
- Enhance the signal-to-noise ratio reducing the backround noise (cosmic rays, environmental radioactivity, noise from electronic devices)

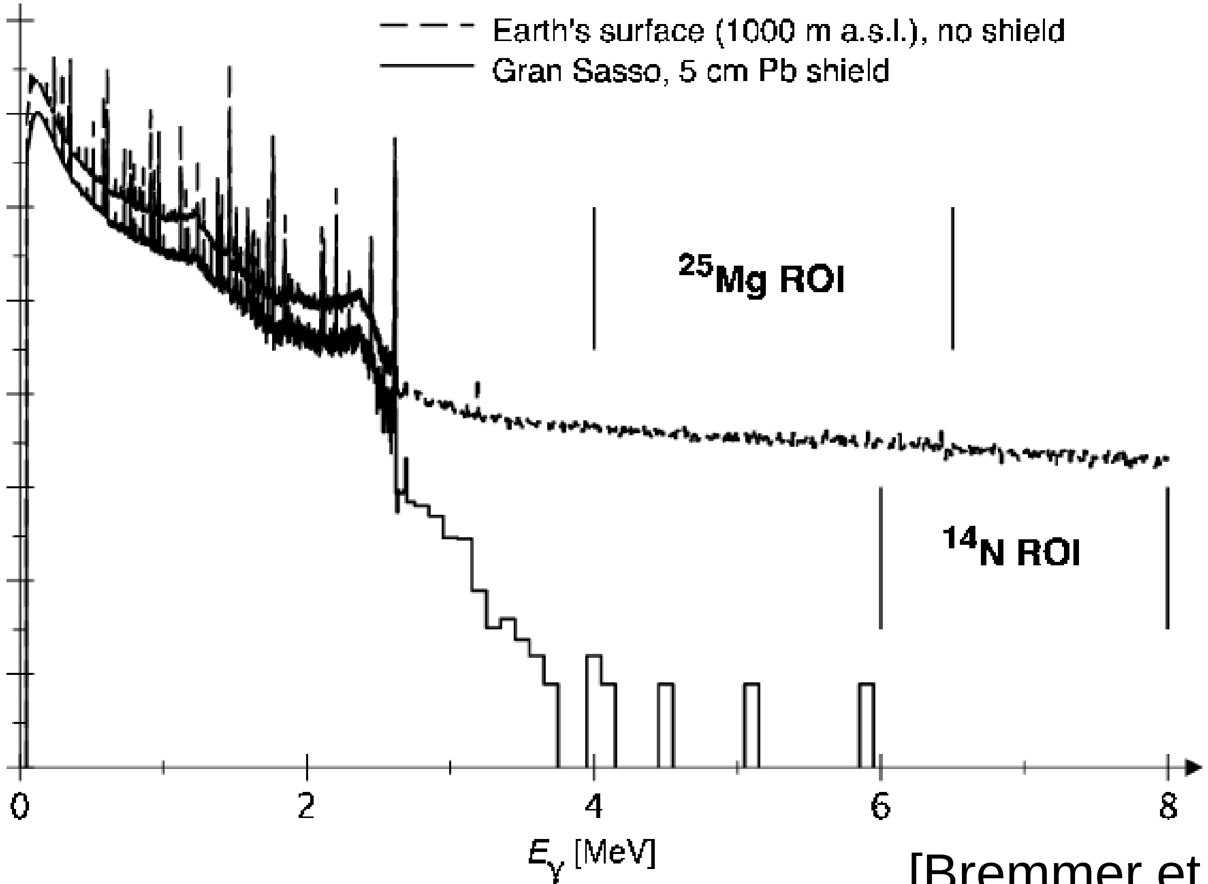
Direct methods/2

Any of the previous possibilities has its drawbacks:

- High intensity beams can gererate a spatial chage that will heat up the target, modifying its structure (density and chemical composition) or destroying it
- High density target will worsen the resolution of the outgoing particles
- Increasing the solid angle range (e.g. using wide detectors placed near the target) can be useful, but there are limitations regarding the beam direction and the high count rates.

Direct methods/3

Regarding the signal-to-noise ratio, it is possible to reduce it shielding the detectors. To do so, deep underground laboratories are necessary (cosmic rays), along with thick shields (natural radioactivity)

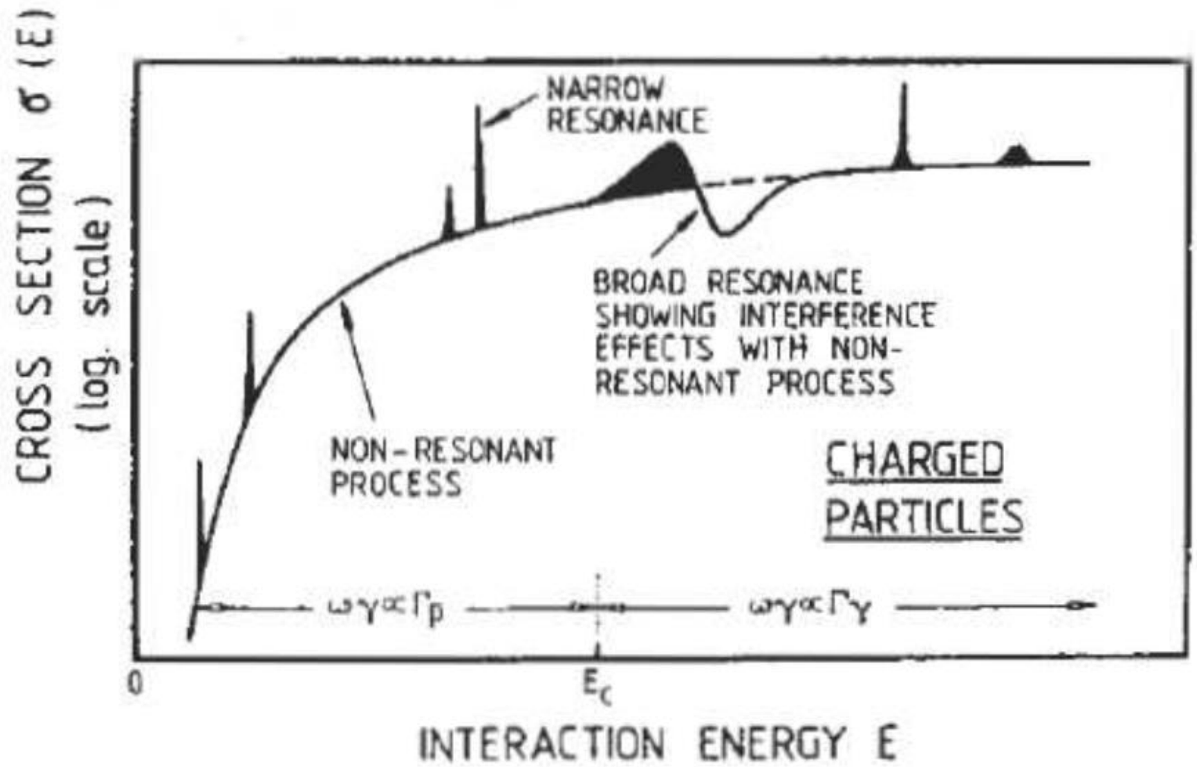


[Bremmer et.al, *EpjA*, *2005*]

Direct methods/4

Nonetheless, low energy measurements are time consuming and poses a technological challenge.

At low energies, cross-section is usually extrapolated



[Rolfs, *Cauldron in the cosmos, 1988*]

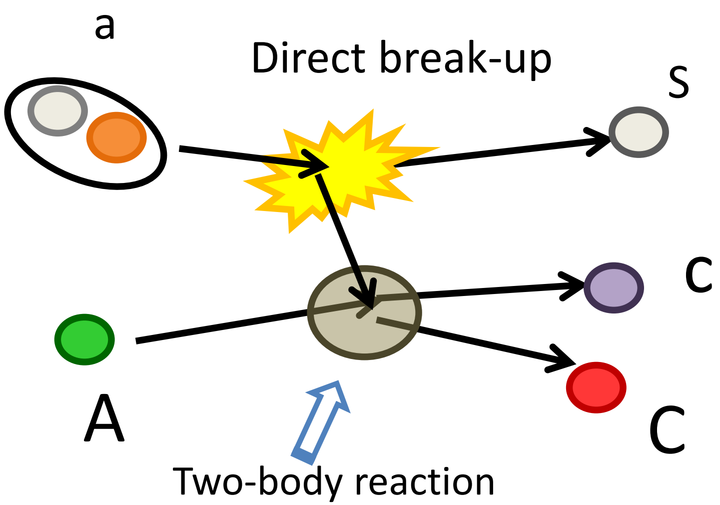
Indirect methods

When direct measurements are not feasible, indirect measurements can be used to calculate the cross-section and astrophysical *S(E)* factor at low energies (even zero).
In particular, our group is invoved in experiment regarding

- Trojan Horse Method (THM)
- Asymptotic Normalization Coefficient (ANC) Method

Such measurements are complementary to direct measurements: In particular the THM can retrieve the cross-section in arbitrary units, relying on normalization over direct measurements

Indirect methods: THM



Three-body reaction:
A + a → c + C + s
Used to trigger the two-body one *A* + *x* → *c* + *C*
a: x + *s* cluster
(RESONANT REACTION)

$E_A > E_{Coul}$

- NO coulomb suppression
- NO electron screening

Quasi-free Mechanism (QF):

- The x cluster (participant) interacts with the nucleus A.
- The *s* cluster acts as a spectator (p_s~0 in the cm).

Spectator momentum distribution

$$\frac{d^3\sigma}{dE_{C.M.}d\Omega_p d\Omega_{Ne}} \propto KF\left|\Phi\left(\vec{p}_s\right)\right|^2 \left(\frac{d\sigma}{d\Omega}\right)_l^{HOES}$$

2-body cross section

Indirect methods: THM/2

Using The Modified R-matrix [Lane & Thomas 1958, Rev. Mod. Phys; La Cognata et al. 2011,ApjL] formalism, In the case of a resonant THM reactionthe cross section takes the form

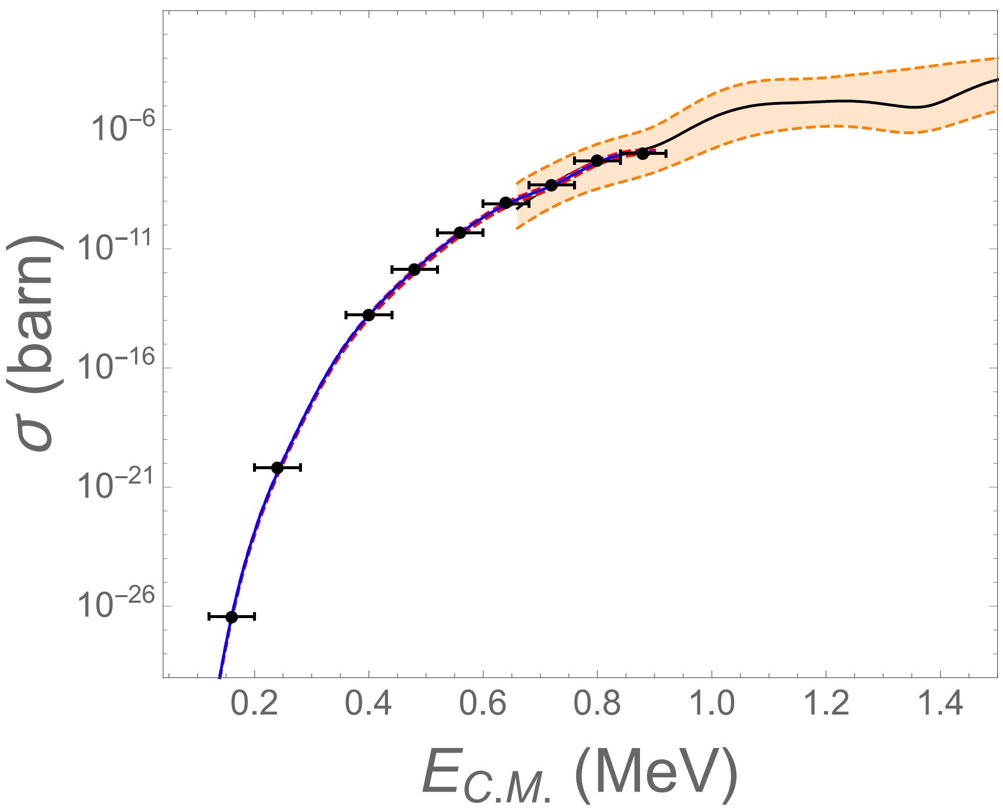
$$\frac{d^2\sigma}{dE_{xA}d\Omega_s} = NF \sum_i (2J_i + 1) \left| \sqrt{\frac{k_{f(E_{xA})}}{\mu_{cC}}} \frac{\sqrt{2P_{l_i}(k_{cC}R_{cC})}M_i(p_{xA}R_{xA})\gamma^i_{cC}\gamma^i_{xA}}{D_i(E_{xA})} \right|^2$$

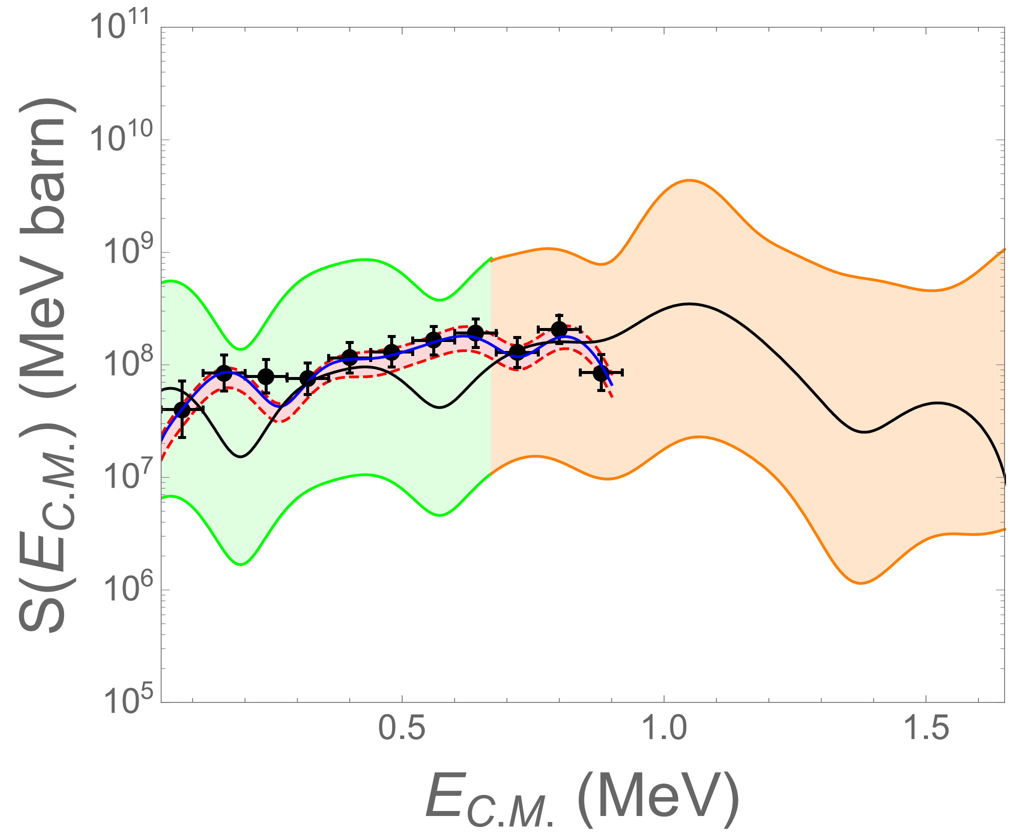
- NF is a normalization factor
- J_i the spin of the i−th resonance
- $k_f(E_{xA}) = \sqrt{2\mu_{cC}(E_{xA} + Q)}\hbar$ (Q is the reaction Q-value, E_{xA} the x − A relative energy)
- P_{li} the penetration factor
- R_{xA} and R_{cC} the channel radii
- $M_i(E)$ is the amplitude of the transfer reaction (upper vertex) that can be calculated
- $D_i(E_{xA})$ is the standard R-matrix denominator

THM – Cross-section and S(E)-factor for the $^{19}F(\alpha,p)^{22}Ne$

$$\sigma = \sum_l P_l \sigma^{THM}$$

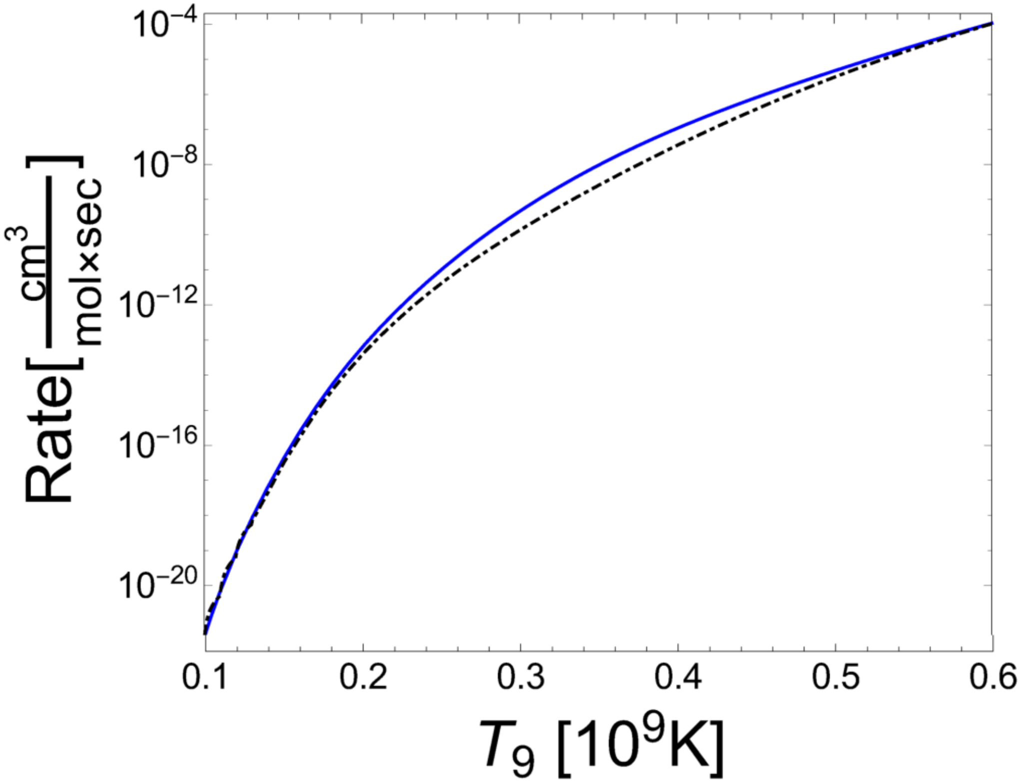
$$S(E) = E\sigma(E)e^{2\pi\eta}$$

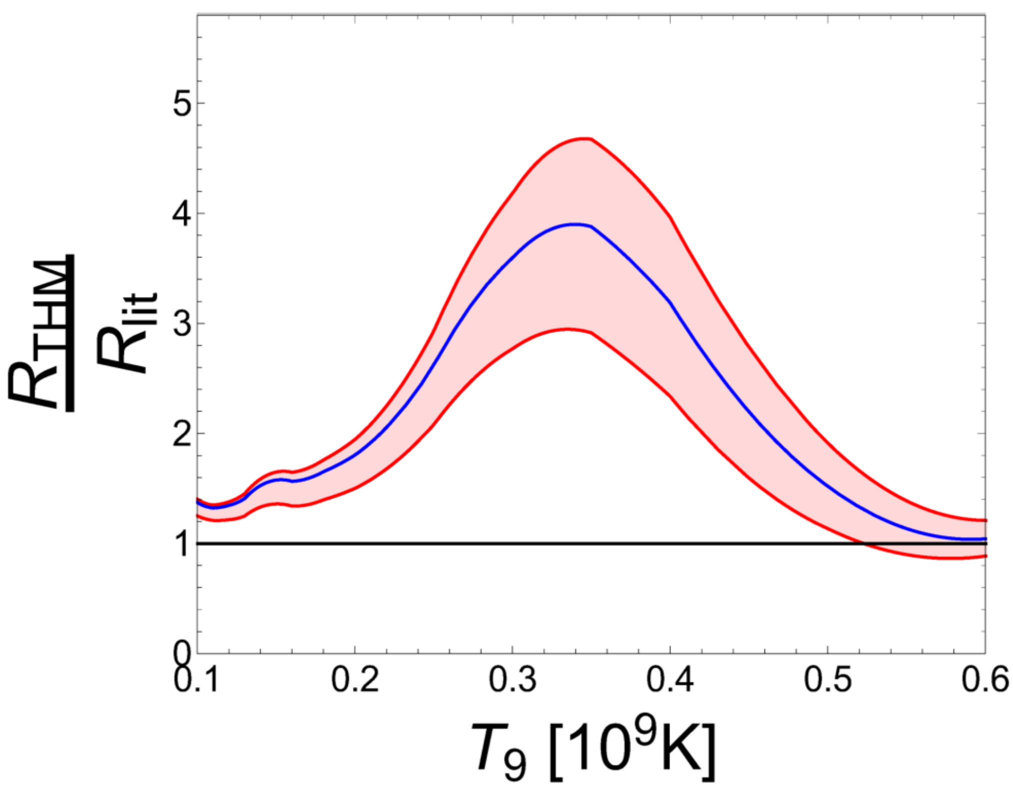




[D'Agata et al., *ApJ, 2018*]

THM – Reaction rate for the $^{19}F(\alpha,p)^{22}Ne$





[D'Agata et al., *ApJ, 2018*]

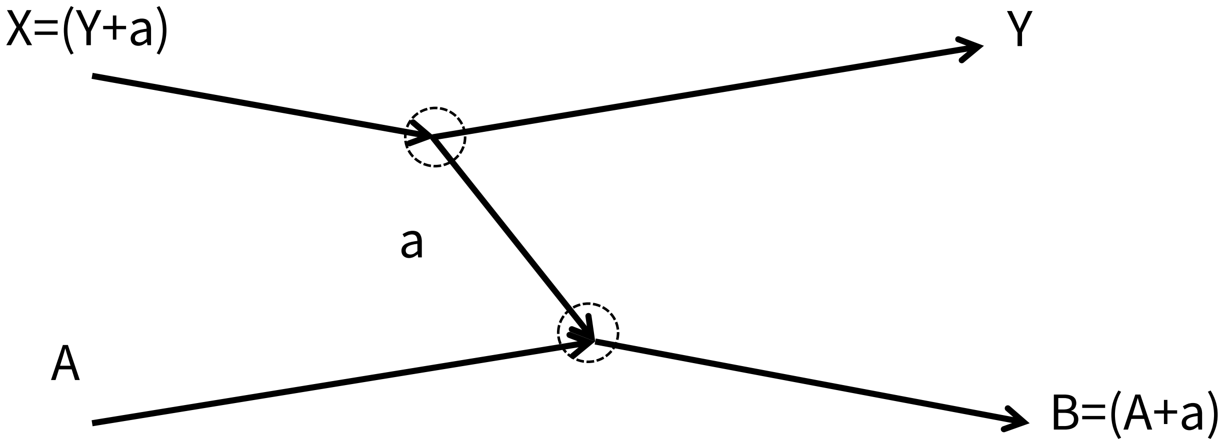
Reaction rate inside the Gamow window at AGB temperatures ($2 \cdot 10^8$ K< T < $4 \cdot 10^8$ K) is higher by a factor of 3÷5

Indirect methods: ANC

ANC:

- Indirect method
- Widely used to gain informations about **DIRECT RADIATIVE CAPTURE**

Studies performed by means of «simple» transfer reactions



Indirect methods: ANC/2

In Distorted Wave Born Approximation, the transition amplitude between the states before and after the reactions can be written as:

$$M(E_i, \vartheta_{c.m.}) = \sum_{M_a} \left\langle \chi_f^{(-)} \, \boldsymbol{I}_{Aa}^{B} \left| \Delta V \right| \boldsymbol{I}_{Ya}^{X} \, \chi_i^{(+)} \right\rangle$$

Here:

- $\chi_f^{(-)}$ and $\chi_f^{(+)}$ are the distorted wave for exit and entrance channels
- ΔV is the perturbative part of the complete potential
- I_{Aa}^{B} and I_{Ya}^{X} are the **radial overlap functions** of the two systems *Y*+*a* and *A*+*a* with the nuclei *B* and *X* respectively

Indirect methods: ANC/3

Both overlap functions (in general I_{ab}^{c}) can be approximated with the bound wave ones $\varphi_{n_c,l_c,j_c}(r_{ab})$ through the equation

$$\boldsymbol{I}_{a,b,l_c,j_c}^{c}(\boldsymbol{r}_{ab}) = S_{a,b,l_c,j_c}^{1/2}\varphi_{n_c,l_c,j_c}(\boldsymbol{r}_{ab})$$

The factor $S_{a,b,l_c,j_c}^{1/2}$ is called **spectroscopic factor**, that **depends on the potential**

Still, the experimental angular distribution $\frac{d\sigma}{d\Omega}$ can be reproduced by means of DWBA calculations

$$\frac{d\sigma}{d\Omega} = \sum_{j_B,j_x} S_{Aa,l_b,j_B} S_{Ya,l_x,j_x} \sigma_{l_B,j_b,l_x,j_x}^{DW}$$

The direct radiative capture cross-section at low energies contains the same radial overlap integral as the direct transfer reactions: at large distances from the nucleus,

$$\boldsymbol{I}_{ab,l_c,j_c}^{c}(\boldsymbol{r}_{ab}) \xrightarrow{r_{ab}>R_N} C_{ab,l_c,j_c}^{c}\frac{W_{-\eta_c,l_{c+1/2}}(2\kappa_{ab}r_{ab})}{r_{ab}}$$

$$\boldsymbol{\varphi}_{n_c,l_c,j_c}(\boldsymbol{r}_{ab}) \xrightarrow{r_{ab}>R_N} b_{ab,l_c,j_c}\frac{W_{-\eta_c,l_{c+1/2}}(2\kappa_{ab}r_{ab})}{r_{ab}}$$

Indirect methods: ANC/4

Coefficients b^2 and C^2 are called single particle ANC (SPANC) and ANC. If we use the approximations explained so far, the differential cross-section is equal to

$$\frac{d\sigma}{d\Omega} = \sum_{j_B,j_x} \left(C^B_{Aa,l_B,j_B}\right)^2 \left(C^X_{Ya,l_x,j_x}\right)^2 \frac{\sigma^{DWBA}_{l_B,j_B,l_x,j_x}}{b^2_{Aa,l_B,j_b}\, b^2_{Ya,j_x,j_x}} =$$

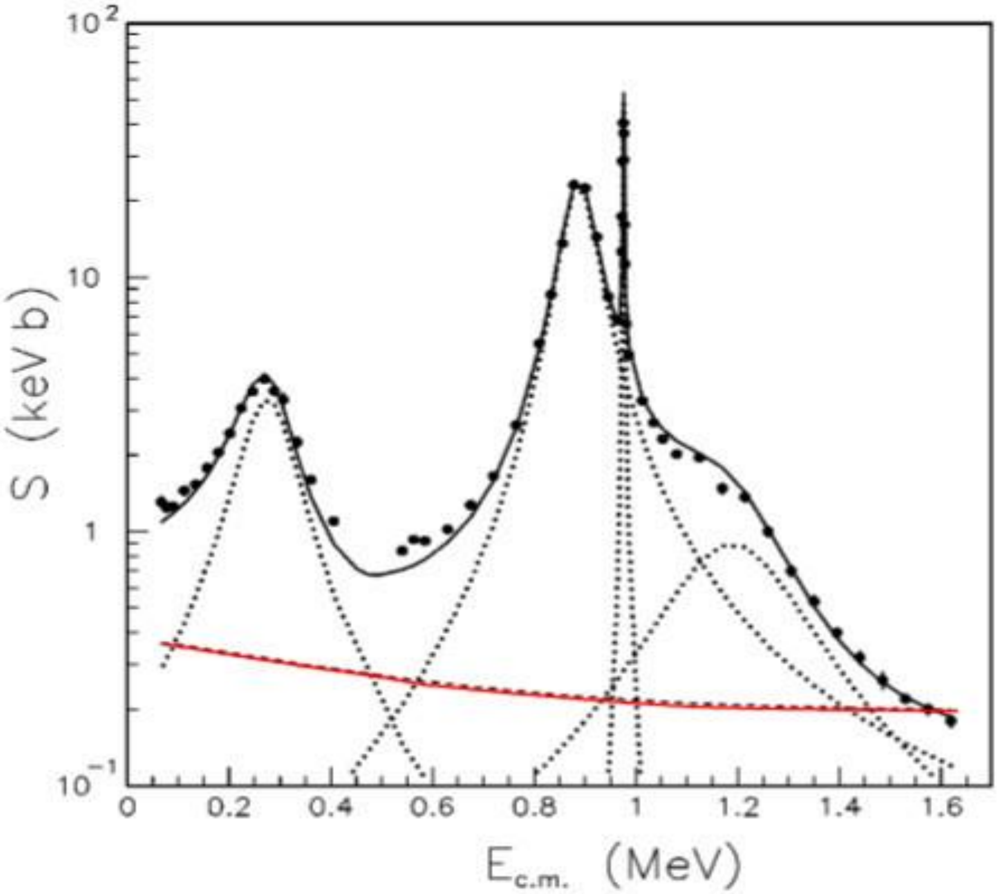
$$= \sum_{j_B,j_x} \left(C^B_{Aa,l_B,j_B}\right)^2 \left(C^X_{Ya,l_x,j_x}\right)^2 R_{l_B,j_B,l_x,j_x}$$

Using DWBA we were able to find the ANC’s coefficients from the spettroscopic factors. This gives us some advantages:

- For perihperal reactions, ANCs have small dependance from the potential
- R_{l_B,j_B,l_x,j_x} is nearly indipendent from b^2

ANC- Cross-section for the $^{9}Be(p,\gamma)^{10}B$

ANC values for 4 states



Agreement between values from $^{9}Be(^{10}B,^{9}Be)^{10}B$ and $^{9}Be(^{3}He,d)^{10}B$

R-Matrix fit [Sattarov et al., *Phys. Rev. C*, 1999] gave $S_0 = 3.98 \pm 0.12$keVb @269 keV

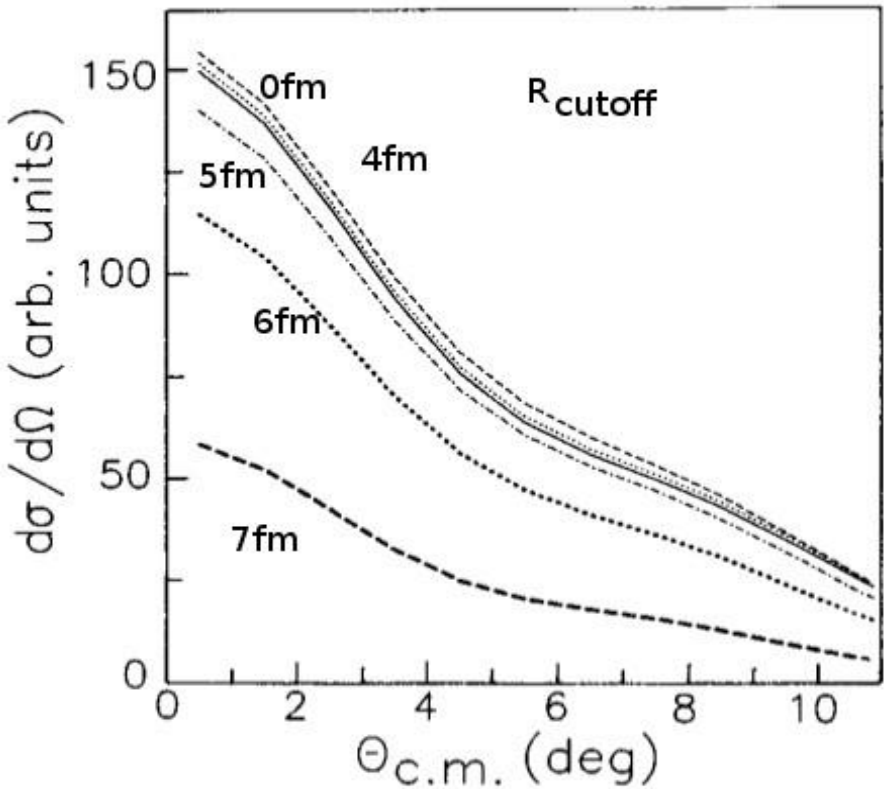
[Mukhamedzhanov et al, *Phys. Rev. C., 1997*]

Thanks for your attention!

ANC - Reaction peripherality for $^{9}Be(p,\gamma)^{10}B$

To verify the peripherality of the $^{9}Be(p,\gamma)^{10}B$ reaction, several checks are necessary:

1) Optical potential were deduced from angular distributions



Different values of R_{cutoff} are used to prove the small dependence of the internal part of the wave function

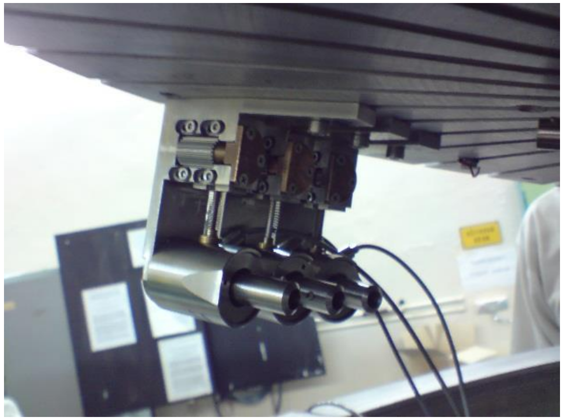
[Mukhamedzhanov et al, *Phys. Rev. C., 1997*]

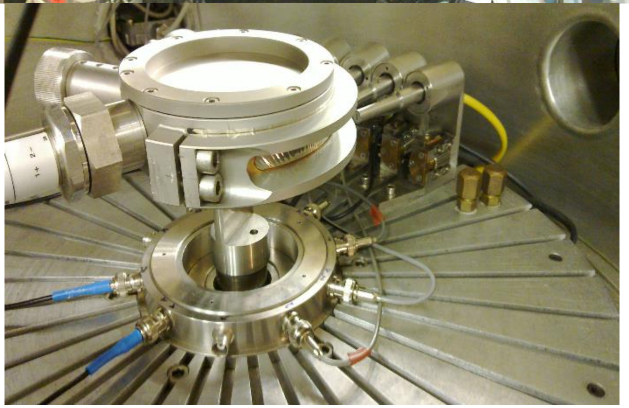
ANC experiments for nuclear astrophysics @ NPI-CAS – Experimental set-up

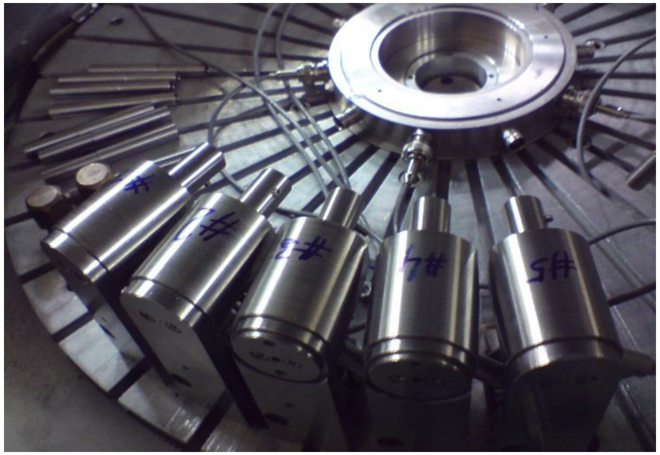
The cyclotrone U120M is able to provide ^{3}He and d beams – useful for (p,γ) and (n,γ) reactions via (^{3}He,d) and (d,p) - at circa 10-20 nA intensity in a various range of energy.

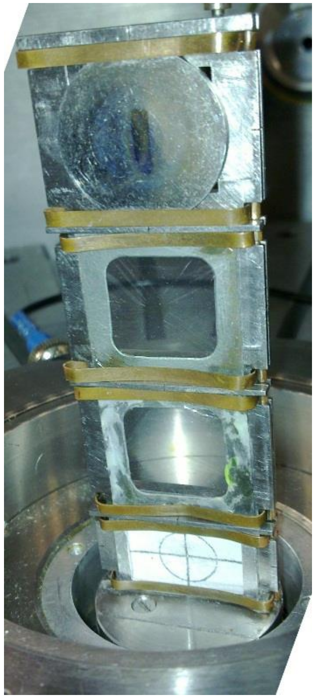
Our experimental chamber is the light blue one in the center

ANC experiments for nuclear astrophysics @ NPI-CAS – Experimental set-up

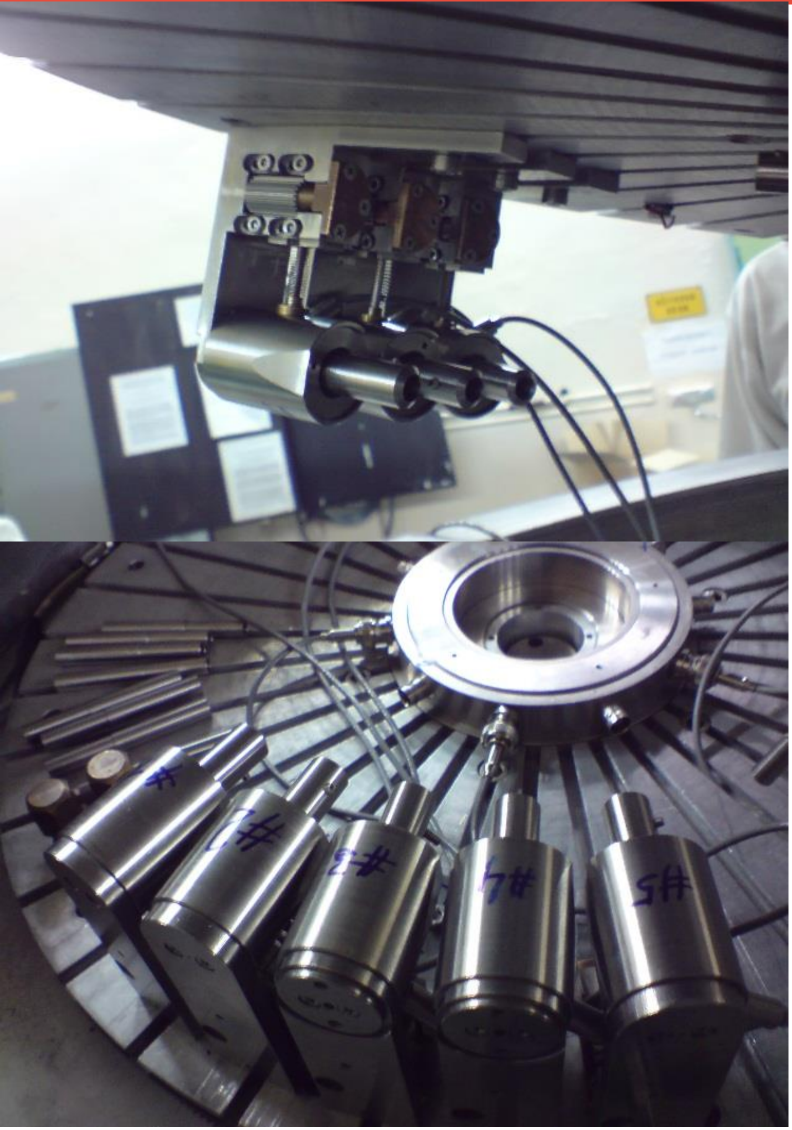








ANC experiments for nuclear astrophysics @ NPI-CAS – Experimental set-up



Experiments performed by means of several ΔE-E telescopes made by silicon detectors (30-5000μm, depending on the particle of interest)

ANC with mirror pairs

A relation between ANC of mirror pair nuclei has been established [Timofeyuk et al., *Phys. Rev. C,* 2003; Trache et al., *Phys. Rev. C*, 2003] and has been applied to many reaction (for example $^{8}B(p,\gamma)^{9}C$ [Guo et al., *Nucl. Phys*. A, 2005])

^{27}Mg and ^{27}P are mirror pairs, and the proton ANC can be extracted using the relation

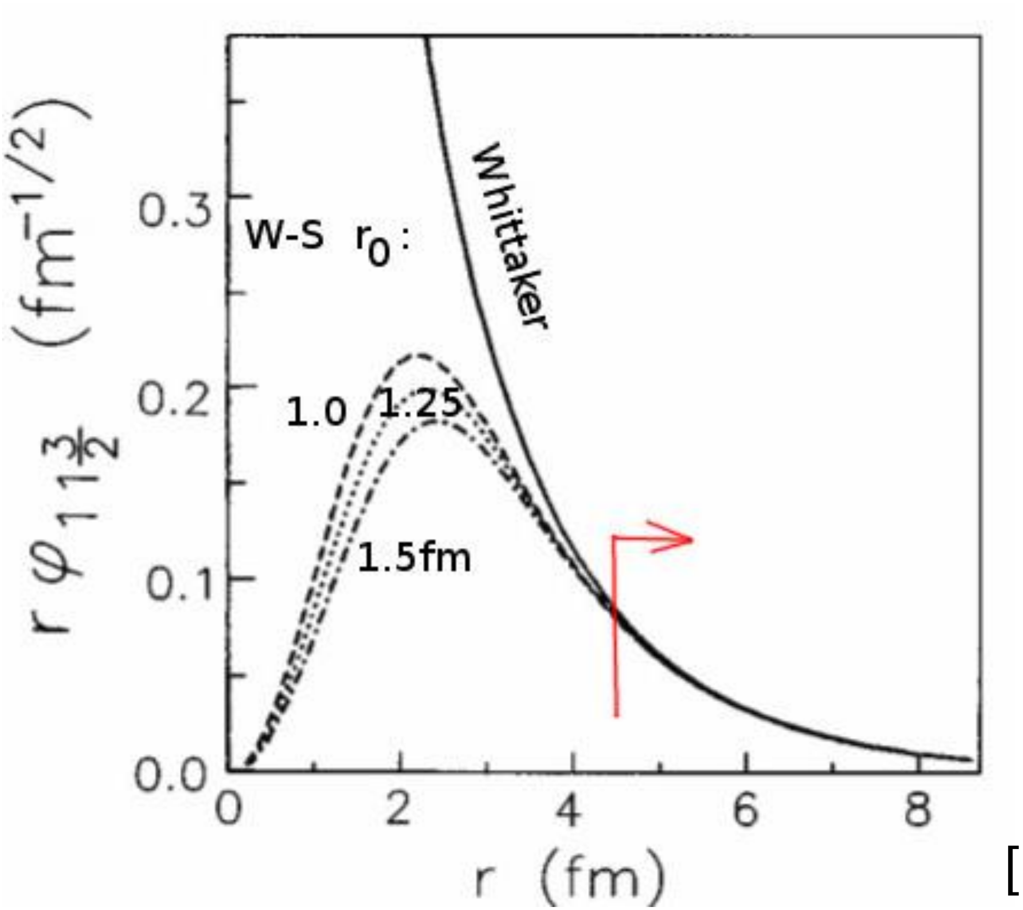
$$\left(C_{ij}^{^{27}P}\right)^2 = R\left(C_{ij}^{^{27}Mg}\right)^2, \text{with } R = \left|\frac{F_l(ik_pR_N)}{k_pR_Nj_l(ik_nR_N)}\right|^2$$

Here F_l is the regular Coulomb wave function, j_l the Bessel function, R_N the radius of the nuclear interior, while k_p and k_n are related to proton and neutron separation energies ϵ_p and ϵ_n via $k = \left(\frac{2\mu\epsilon}{\hbar^2}\right)^{1/2}$. Here a realtion between Γ_p and $|C_n|^2$ has been found [Timofeyuk et al., *Phys. Rev. C,* 2003]:

$$\frac{\Gamma_p}{|C_n|^2} = R_\Gamma \approx R_0^{res} = \frac{\kappa_p}{\mu}\left|\frac{F_l(\kappa_pR_N)}{\kappa_pR_Nj_l(i\kappa_nR_N)}\right|^2$$

ANC- Reaction peripherality for $^{9}Be(p,\gamma)^{10}B$

2) Study of the radial part of the wave function in the asymptotic region



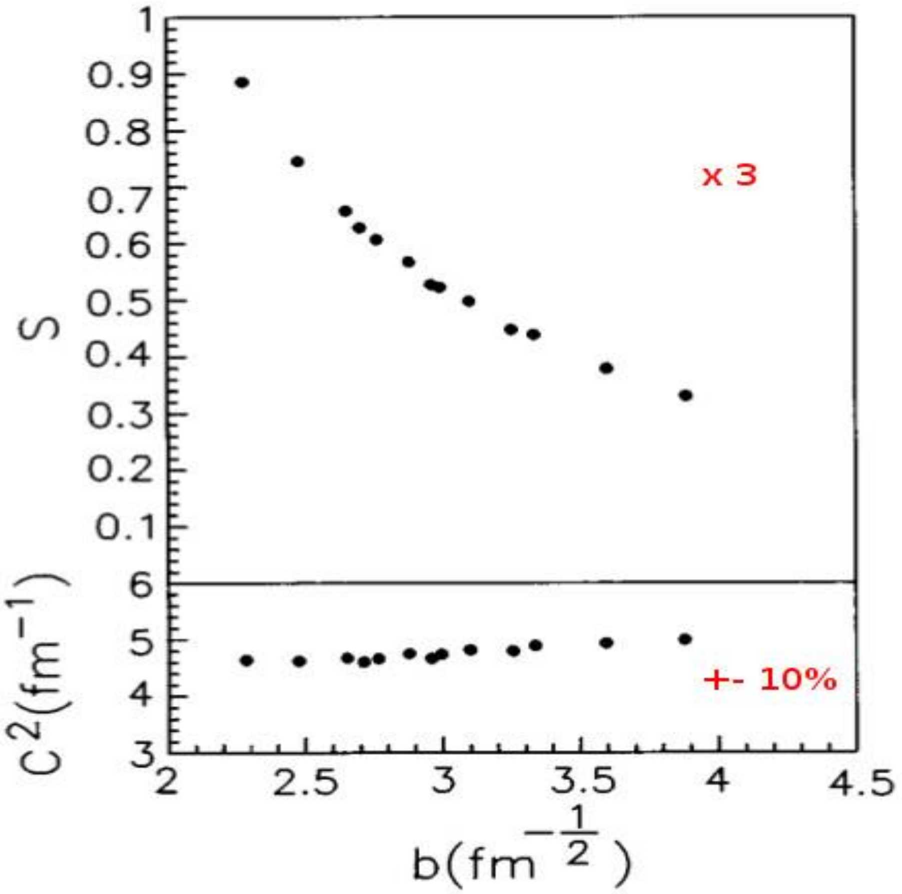
Asymptotic behaviour of the single-particle bound state of the wave function for different Wood-Saxon parameters

After 4fm the behaviour is identical to the Whittaker function

[Mukhamedzhanov et al, *Phys. Rev. C., 1997*]

ANC- Reaction peripherality for $^{9}Be(p,\gamma)^{10}B$

3) Different behaviour between ANC and SF



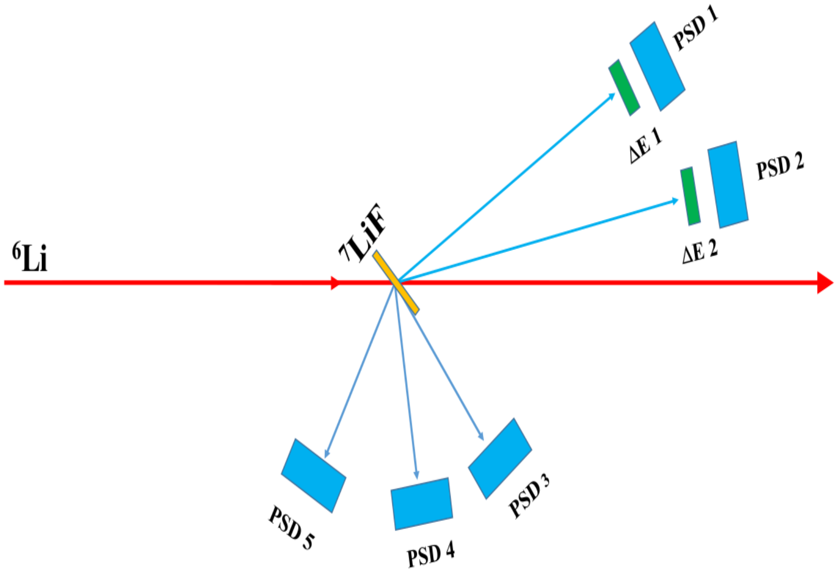
Comparison between the dependence of SF and ANC on SPANC

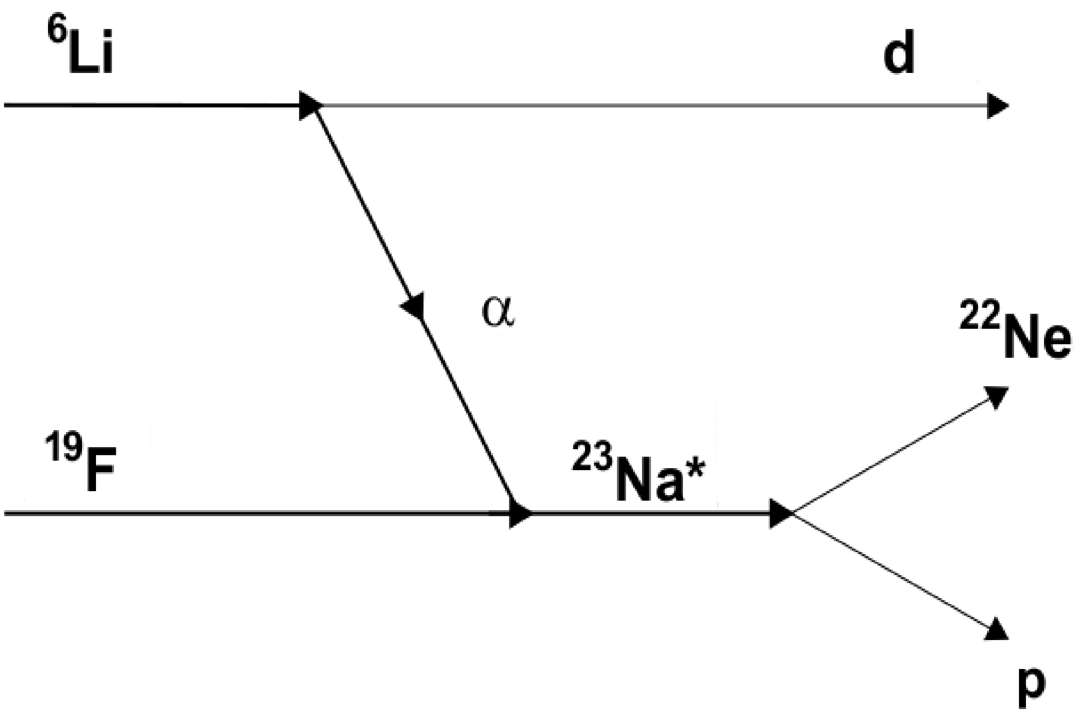
As can be seen C^2 is way less model-dependent

[Mukhamedzhanov et al, *Phys. Rev. C., 1997*]

Experimental Set-up

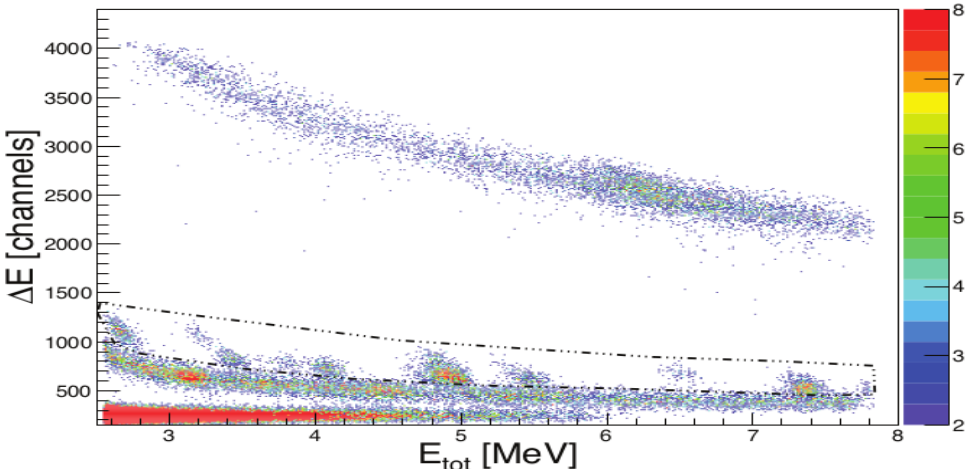
- ✓ $^{19}F(\alpha,p)^{22}Ne$ was studied using THM applied to $^{6}Li(^{19}F,p^{22}Ne)d$
- ✓ V_{coul}=5.41 MeV
- ✓ ^{6}Li has a cluster structure α+d, E_B=1,47 MeV

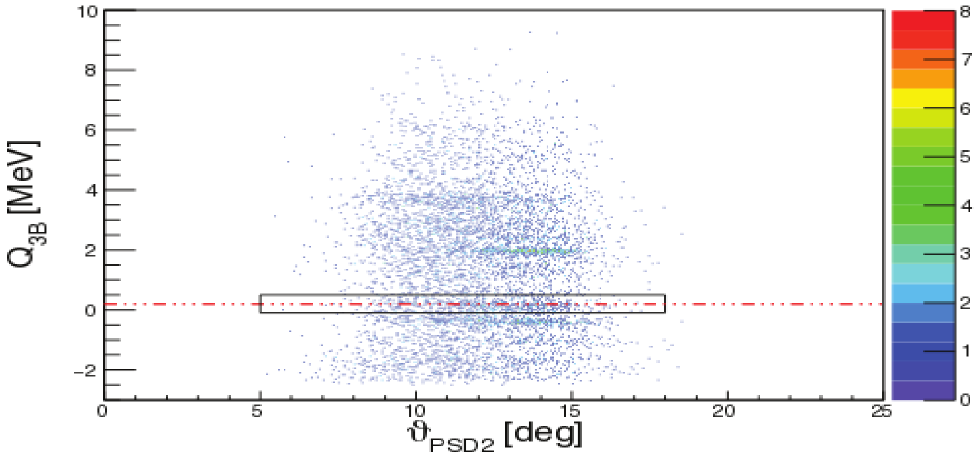


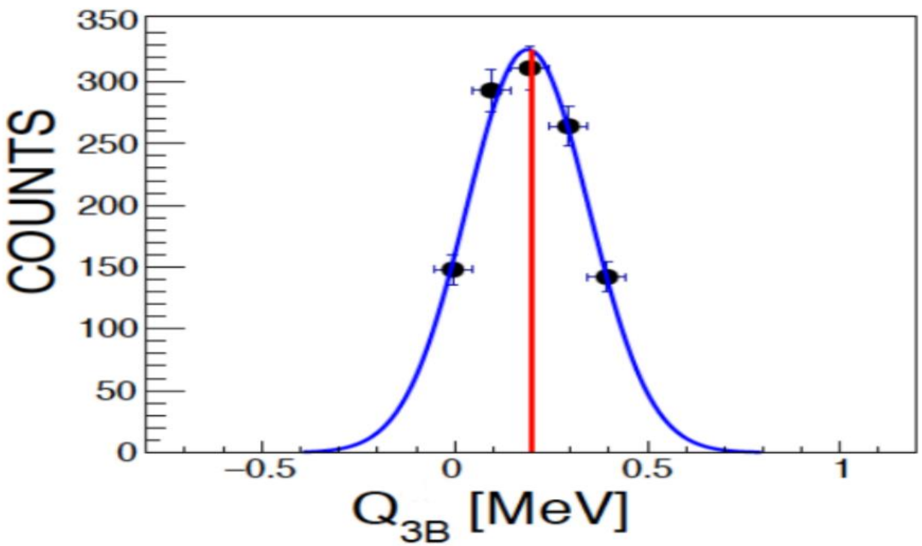


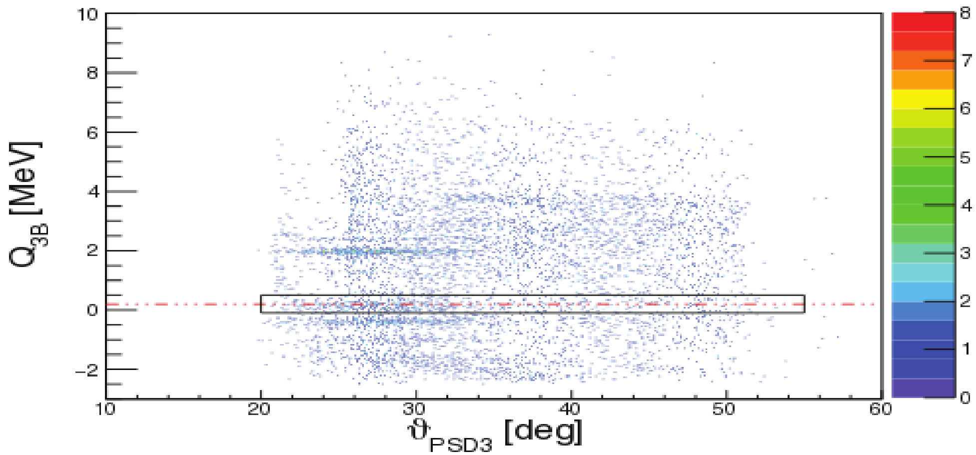
- ✓ Experiment @ Rudjer Boskovic Institute
- ✓ ^{6}Li beam, E_{beam}=6 MeV (i=5nA)
- ✓ Target ^{7}LiF 100 μg/cm^2
- ✓ Two ΔE-E telescopes (ΔE 9 μm, E 500 μm) + three PSDs (500 μm)

THM (channel selection) – channel selection for the $^{19}F(^{6}Li,pd)^{22}Ne$









[D'Agata et al., *ApJ, 2018*]

Q_{value} after selection. In blue a Gaussian fit, with centroid at 0.19 and σ=0.16 MeV

THM - Quasi-free selection for the $^{19}F(^{6}Li,pd)^{22}Ne$

Deuteron momentum distribution inside ^{6}Li follows the square modulus of a Hankel function (green line) [Pizzone et al., 2009 Phys. Rev. C, 80]:

$$|\Phi(\overrightarrow{p_s})| = N \frac{1}{\left(\frac{p^2}{\hbar^2} + \beta^2\right)^2} \left[\frac{\sin \frac{p_s}{\hbar} R_C}{\frac{p_s}{\hbar}} + \frac{\cos \frac{p_s}{\hbar} R_C}{\beta} \right]$$

With

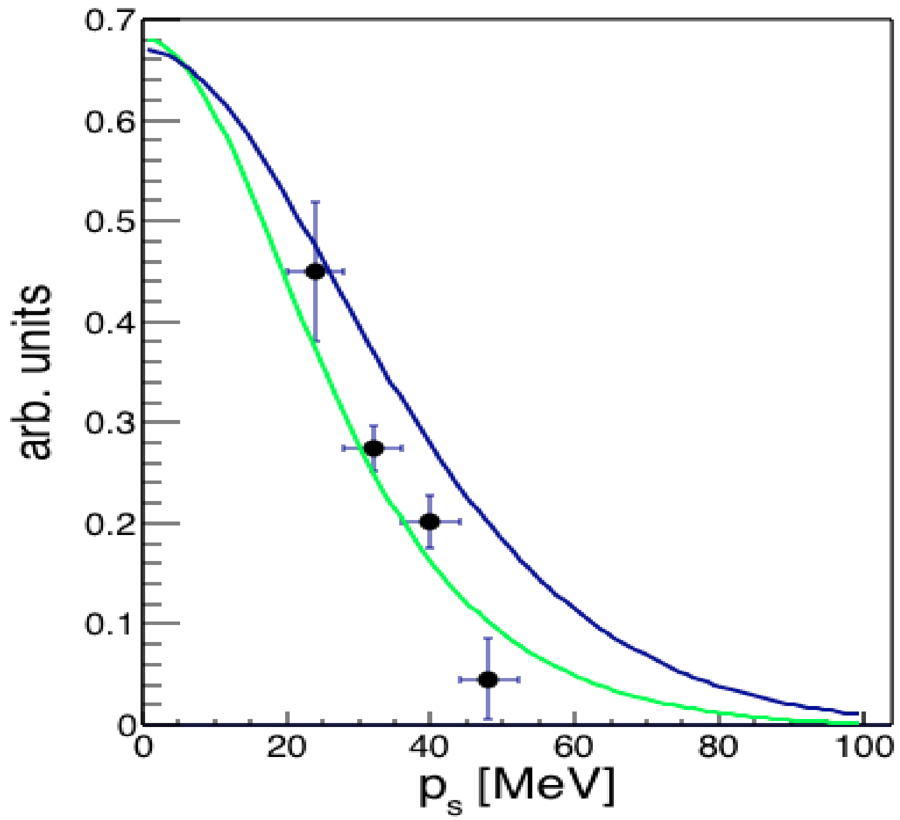
$$k_s = \frac{2\pi p_s}{h}$$

$$R_c = 1.11 \text{ fm (cutoff radius)}$$

$$\beta = \left(\frac{4\pi\mu E_B}{h}\right)^{1/2}$$

In this case the FWHM, $W(q_t)$, is equal to

$W(q_t) \approx 53 \pm 7$ MeV/c



[D'Agata et al., *ApJ, 2018*]

Quasi-free Selection/2

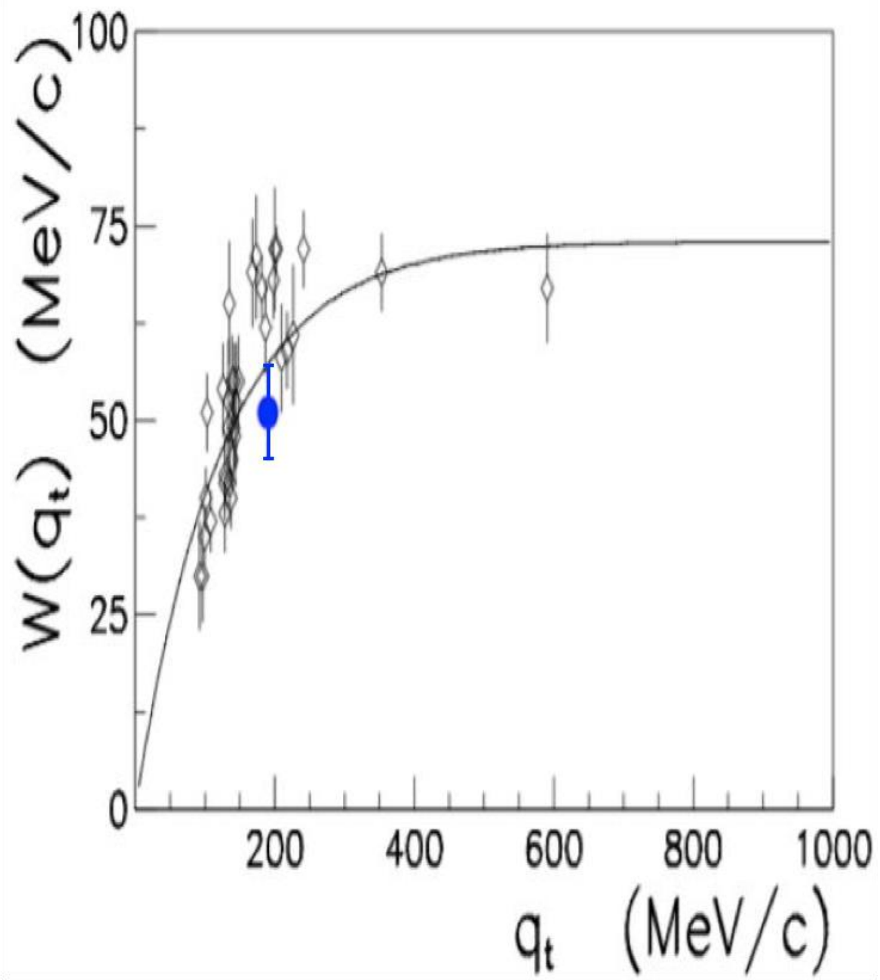
The $W(q_t)$ must follow the trend [Barbarino et al. 1980 PhRvC; Pizzone et al., 2009 Phys. Rev. C]:

$$W(q_t) = f_0\left(1 - e^{q_t/q_0}\right)$$

where

$$\vec{q}_t = \vec{p}_{Li} - \frac{\vec{p}_p + \vec{p}_{Ne}}{2}$$

In our case q_t=190 MeV/c



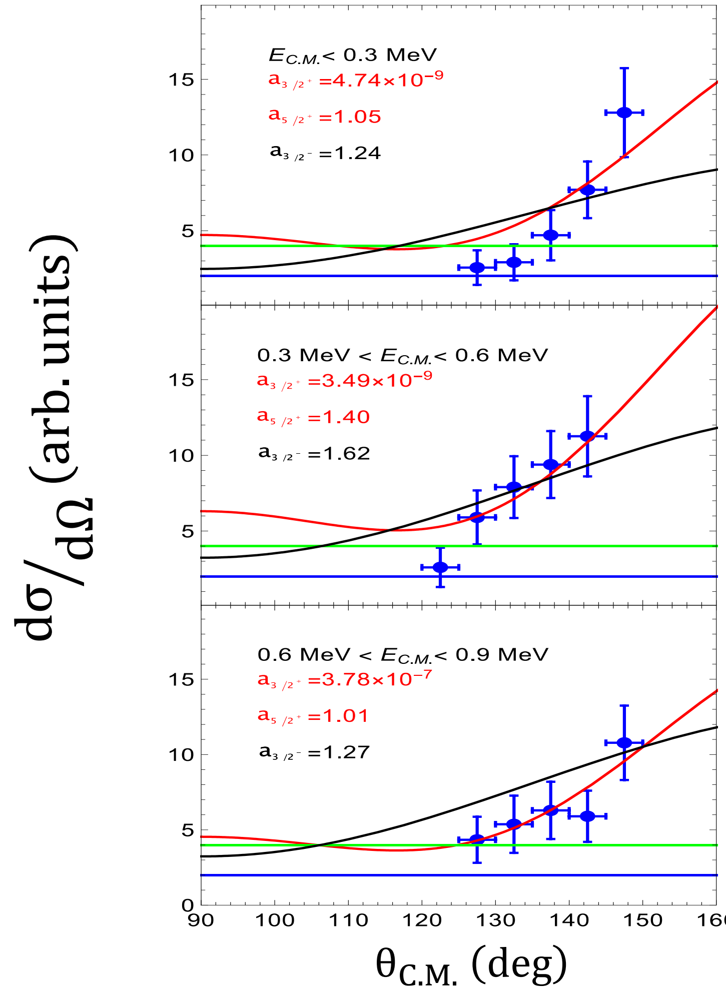
C.M. Angle

The experimental set-up allows to obtain data between 120° and 160° in the C.M. system

$$\theta_{C.M.} = \arccos \frac{(\vec{v}_F - \vec{v}_\alpha) \bullet (\vec{v}_{Ne} - \vec{v}_p)}{|\vec{v}_F - \vec{v}_\alpha||\vec{v}_{Ne} - \vec{v}_p|}$$

[Slaus et al, 1977, Nucl. Phys. A, 286]

Now it is possible to proceed with angular integration



[Pizzone, D'Agata et al. 2017, Apj]

Cross section/1

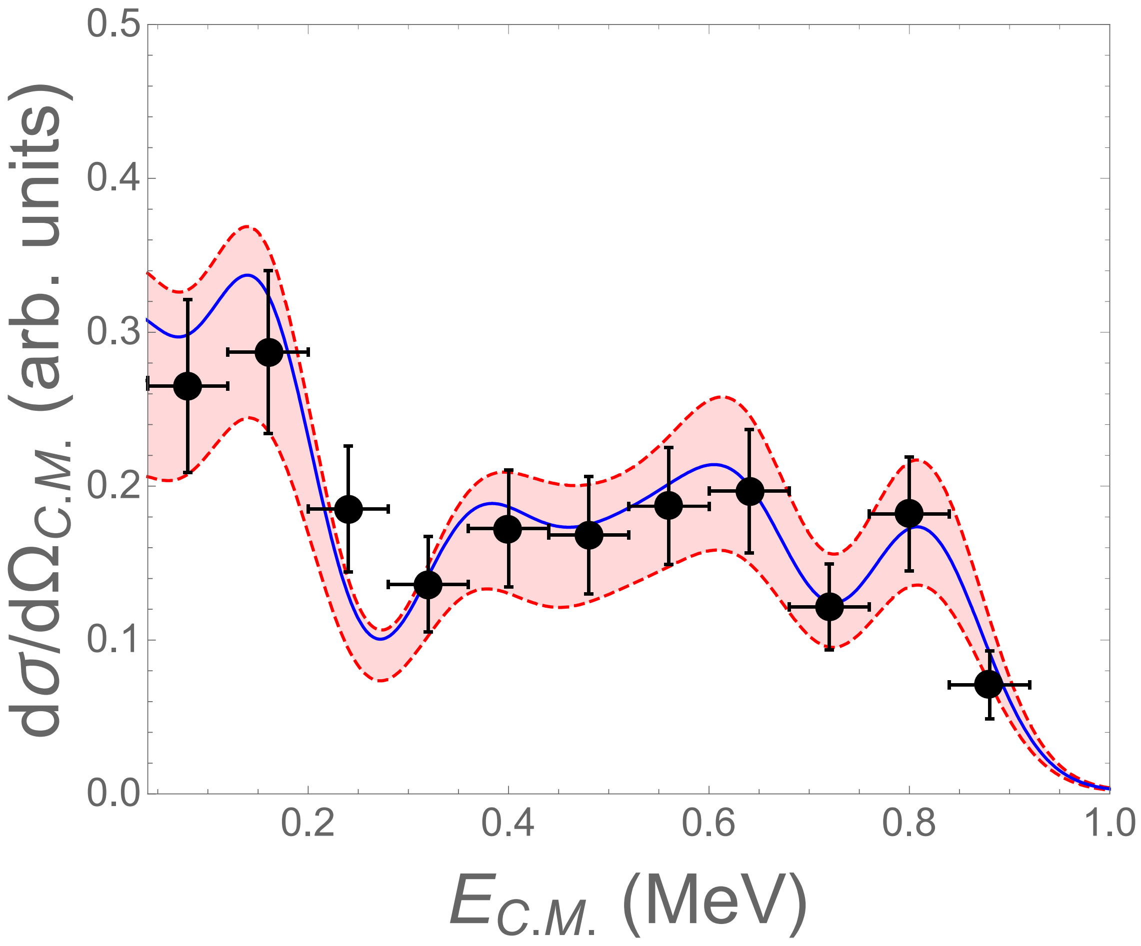


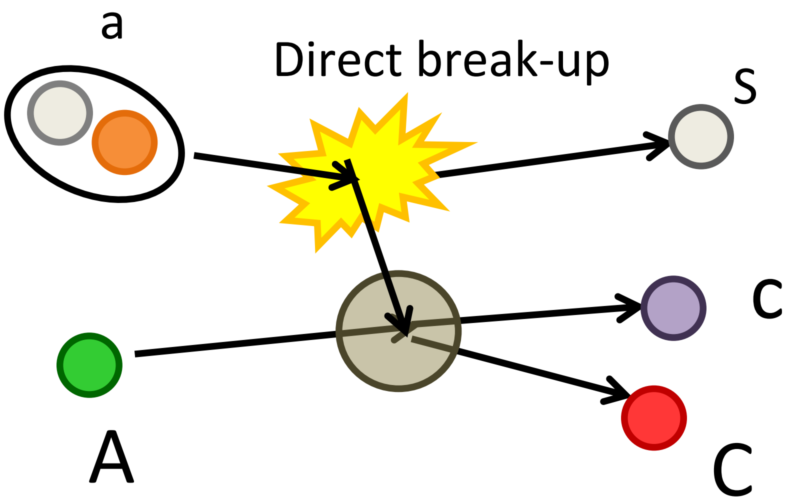
Table 2. Characteristics of the ^{23}Na states included in the present analysis. The measured central values, the J^π of the levels and the reduced widths involved in the *R-matrix* fit calculations discussed in Pizzone et al. 2017 are also reported. Values marked with and asterisk are taken from Ugalde et al. 2008.

E_R [MeV]	$E_{C.M.}$ [MeV]	J^π	γ_α [MeV$^{1/2}$]	γ_p [MeV$^{1/2}$]	$\gamma_{p'}$ [MeV$^{1/2}$]
10.477	0.01	$3/2^+$	$0.0010^{+0.0001}_{-0.0002}$	0.124	0.342
10.616	0.149	$5/2^+$	$0.0055^{+0.0002}_{-0.0080}$	0.087	0.327
10.823	0.356	$3/2^+$	$0.0070^{+0.0001}_{-0.0010}$	0.131	0.417
10.907	0.44	$5/2^+$	$0.0007^{+0.0001}_{-0.0002}$	0.054	0.350
10.972	0.505	$5/2^+$	$0.0090^{+0.0002}_{-0.0009}$	0.044	0.184
10.994	0.527	$3/2^+$	$0.0050^{+0.0002}_{-0.0010}$	0.011	0.079
11.038	0.571	$3/2^+$	$0.0027^{+0.0002}_{-0.0005}$	0.049	0.179
11.109	0.642	$5/2^+$	$0.0120^{+0.0015}_{-0.0015}$	0.016	0.096
11.273	0.806	$3/2^+$	0.003*	0.045	0.279
11.280	0.812	$3/2^+$	0.003*	0.127	0.320
11.303	0.836	$3/2^+$	0.003*	0.105	0.148

About QF reactions

Beam energy and target must be chosen taking into account that the center-of-mass energy of the two-body reaction in quasi-free condition must be as near as possible to the Gamow energy:

$$E_{qf} = E_A - B_{x-s} \cong E_{Gamow}$$

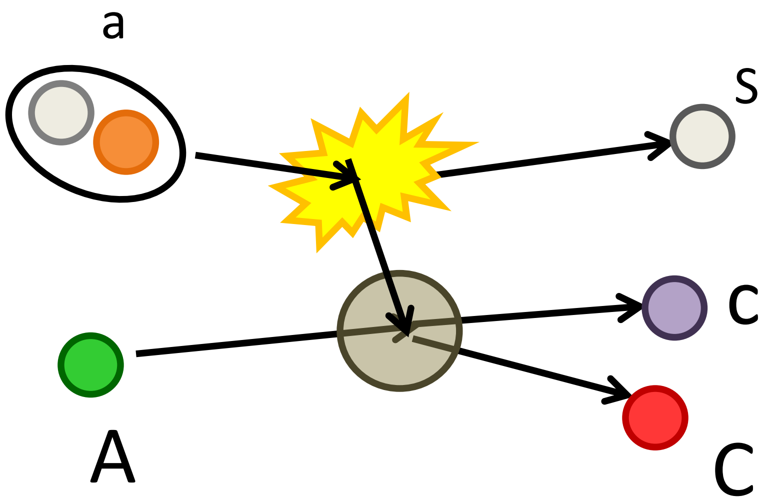


In this formula

- E_{Bx} is the energy of the projectile in the center-of-mass reference frame of the two-body reaction
- $B_{x\text{-}S}$ is the binding energy of the two components of the Trojan Horse nucleus *a*

THM conditions

- The *A* nucleus does not interact at the same times with the two components of the cluster: this is true if the De Broglie wavelength of the *A* particle is less than the average distance between *x* and *S*



- The probability for the inteaction between *A* and *x* is the same that should have in case of a free particle. This also means that *s* does non participate to the reaction

- The binding energy of the system *x*-*S* is negligible if compared with the interaction energy between *A* and *x*

About the cross-section

A precise measurement of the cross-section is mandatory to extract the reaction rate. In fact $\langle\sigma v\rangle = \left(\frac{8}{\mu\pi}\right)^{1/2} \frac{1}{(kT)^{3/2}} \int_0^\infty \sigma_{BW}(E) E exp\left(-\frac{E}{kT}\right) dE$, with $\sigma_{BW}(E)$ Breit-Wigner cross section $\sigma_{BW}(E) = \frac{\lambda^2 \omega}{2\pi} \frac{\Gamma_1\Gamma_2}{(E-E_R)^2+\left(\Gamma/_2\right)^2}$ and $\omega = \frac{2J_{C^*}+1}{(2J_a+1)(2J_X+1)}(1+\delta_{aX})$. The last is a statistical factor which depends on the spin of the compound nucleus J_{C^*}, of the target J_X and of the projectile J_a.

If the resonance is narrow and isolated $\langle\sigma v\rangle = \left(\frac{2\pi}{\mu kT}\right)^{3/2} \frac{h}{4\pi^2} \omega\gamma|_r \left(-\frac{E_R}{kT}\right)$.

The statistical factor ω times the width of the ratio $\gamma = \frac{\Gamma_1\Gamma_2}{\Gamma}$ is called *strenght* of a certain resonance, and it is equal to $\gamma\omega = \frac{2J_{C^*}+1}{(2J_a+1)(2J_X+1)}(1+\delta_{aX}) \frac{\Gamma_1\Gamma_2}{\Gamma}$, and represents the key parameter to extract the reaction rate

Gamow window

The reaction rate for a couple of particles is equal to:

$$\langle\sigma v\rangle = \left(\frac{8}{\mu\pi}\right)^{1/2} \frac{1}{(kT)^{3/2}} \int_0^\infty S(E)\, exp\left(-\frac{E}{kT} - bE^{-1/2}\right) dE$$

$$b = 31.28 \cdot Z_1 Z_2 A^{1/2}$$

In astrophysical environment such a window is really narrow

S(E)=S_0=constant

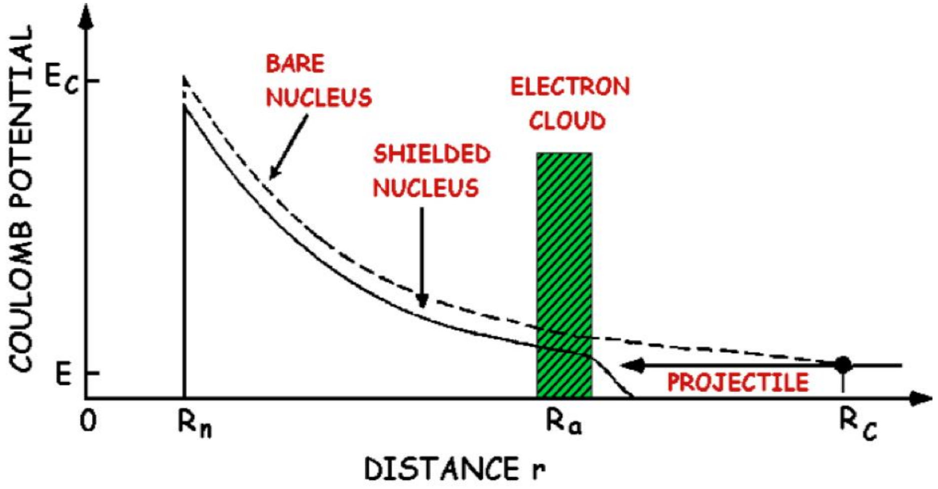
Can be approximated with a gaussian centered in E_0

$$\langle\sigma v\rangle = \left(\frac{8}{\mu\pi}\right)^{1/2} \frac{S_0}{(kT)^{3/2}} e^{-\tau} \int_0^\infty exp\left(-\frac{(E-E_0)^2}{2\Delta^2}\right) dE$$

$$\tau = \frac{3E_0}{kT}$$

Δ = Gamow window

Electron screening



The cross-section is enhanced for low energies of the projectile

$$f_{lab} = \frac{\sigma_s(E)}{\sigma_b(E)}$$

If the screening potential is bigger than the beam energy

$$f_{lab} = \frac{d\sigma_b(E+U_e)}{d\sigma_b(E)} = \frac{E}{E+U_e} exp\left(\frac{\pi\eta U_r}{E}\right)$$

Reaction	$U_e(keV)$	$f_{lab}(U_e/E = 0.1)$	$f_{lab}(U_e/E = 0.01)$	$f_{lab}(U_e/E = 0.001)$
$d + d$	0.027	16.5	1.10	1.003
$d + {}^3He$	0.11	20.9	1.11	1.003
${}^3He + {}^3He$	0.22	131	1.18	1.006
$p + {}^7Li$	0.24	14	1.09	1.003
$\alpha + {}^1 2C$	2.0	868	1.25	1.007

Table 2.3: Some values of f_{lab} for different reactions: even for $U_e/E = 0.01$ the discrepancy is relevant ($E \approx$ 3 - 30 keV)[Assenbaum et al., 1987]

The Electron screening in stellar environment

In stellar environment elecrons tend to cluster around positive ions. Such a region is called *Debye-Hückel sphere*, and its radius is equal to

$$R_D = \left(\frac{kT}{4\pi e^2 \rho N_A \xi}\right)^{1/2}$$

And so

$$f_{plasma} = exp\left(\frac{Z_a Z_A e^2}{kT R_D}\right)$$